ESAIM: M2AN 47 (2013) 33–55
DOI: 10.1051/m2an/2012018

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

MORTAR SPECTRAL METHOD IN AXISYMMETRIC DOMAINS

Saloua Mani Aouadi[1] and Jamil Satouri[1]

Abstract. We consider the Laplace equation posed in a three-dimensional axisymmetric domain. We reduce the original problem by a Fourier expansion in the angular variable to a countable family of two-dimensional problems. We decompose the meridian domain, assumed polygonal, in a finite number of rectangles and we discretize by a spectral method. Then we describe the main features of the mortar method and use the algorithm Strang Fix to improve the accuracy of our discretization.

Mathematics Subject Classification. 65N35, 65N55.

Received December 14, 2010. Revised January 20, 2012.
Published online July 31, 2012.

1. Introduction

We consider the Laplace equation

$$\begin{cases} -\Delta \breve{u} = \breve{f} & \text{in } \breve{\Omega}, \\ \breve{u} = \breve{g} & \text{on } \partial\breve{\Omega}, \end{cases} \tag{1.1}$$

where $\breve{\Omega}$ is a three-dimensional domain, $\breve{f}$ represents the density of forces and $\breve{g}$ the boundary data. This equation appears in many problems of physics such as the astronomy, the electrostatics, the fluid mechanics, the heat flow, diffusion... We suppose that the domain $\breve{\Omega}$ is axisymmetric, *i.e.* it is invariant by rotation around an axis. This hypothesis is realistic in many situations such as the description of the flow in a cylindrical pipe or around a spherical obstacle.

The advantage of working with such a domain is that the three-dimensional solution admits a Fourier expansion with respect to the angular variable and that each Fourier coefficient is the solution of a two-dimensional problem set in the meridian domain [2, 3]. The three-dimensional problem is then reduced to a sequence of uncoupled two-dimensional problems. One of the difficulties of this dimension reduction is that the Cartesian measure is replaced by a weighted measure due to the use of cylindrical coordinates. The variational formulations of the two-dimensional problems are thus written in weighted Sobolev spaces, as fully investigated in [4] in a general framework.

We begin the approximation of the three-dimensional solution by a Fourier truncation. Then we solve only a finite number of two-dimensional problems. The error corresponding to this truncation involves only the regularity of the data [2].

Keywords and phrases. Axisymmetric domains, mortar method, spectral methods, Laplace equation.

[1] Faculty of Sciences of Tunis, University Tunis El Manar, 2090 Tunis, Tunisia. saloua.mani@fst.rnu.tn

In a second step, we pass to the discretization of each two-dimensional problem. The bidimensional domain may have a complex geometry or can be physically heterogeneous. In order to avoid this geometric complexity or to separate heterogeneous domains into homogeneous regions, as well as to take advantage of parallelism, we consider a non conform domain decomposition method [13, 14]. The mortar method is then used to treat the non conformities on the interfaces and to transfer the information between sub-domains [1, 7, 15].

To discretize the local problems in the sub-domains, we use the spectral method [5].

Moreover, the presence of corners in the meridian domain induces some singularities on the solution. We then break up the solution into a regular part and a linear combination of singular functions as mentioned in [10, 12]. The algorithm of Strang and Fix is then used to improve the accuracy of the discretization [17].

The contribution of this work has two levels. First, it combines the mortar element method with domain reduction techniques and spectral approximation in weighted spaces. It justifies, from a theoretical point of view, the use of discretization strategies defined independently in subdomains. Second, it illustrates numerically the relevance of each one of our approximation tools.

The outline of the paper is as follows. In Section 2, we present the geometry and recall the weighted Sobolev spaces and the variational formulation of the two dimensional problems. Then Section 3 is devoted to the description and numerical analysis of the discrete problems in the case of axisymmetric data. Only the Fourier coefficient of order $k = 0$ is no null and so only one discrete problem has to be solved. In Section 4, the problem with general data is considered. In Section 5, we go back to the three dimensional problem and estimate the error between the exact solution and the solution constructed by a three-level approach, namely the truncation of Fourier series, the numerical integration and the spectral element approximation. Section 6 is devoted to the Strang and Fix algorithm. Finally, the numerical experiments are presented in Section 7.

2. The geometry and the continuous problem

2.1. Geometry

In $\mathbb{R}^3$, we will use the Cartesian coordinates (x, y, z) or the cylindrical ones (r, θ, z) with

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r \in \mathbb{R}_+ \ \text{ and } \ \theta \in [-\pi, \pi[\,.$$

We denote by $\mathbb{R}^2_+$ the half space $\mathbb{R}_+ \times \mathbb{R}$ of $\mathbb{R}^2$. Let Ω be a polygon in $\mathbb{R}^2_+$ with boundary $\partial\Omega = \overset{n}{\underset{i=1}{\cup}} \Gamma_i$ made of a finite number of segments Γ_i, $1 \le i \le n$. The finite endpoints of these segments are known as corners of Ω. We call $c_1, c_2, \dots c_p$ the corners which are on the axis $r = 0$, and $e_1, e_2, \dots e_j$ the other corners of Ω. Let Γ_0 the intersection of $\partial\Omega$ with the axis $r = 0$ and $\Gamma = \partial\Omega\backslash\Gamma_0$. Let $\breve{\Omega}$ be the domain of $\mathbb{R}^3$ obtained by rotation of Ω around the axis $r = 0$. The set Ω is called meridian domain and we have

$$\breve{\Omega} = \left\{(r, \theta, z) \in \mathbb{R}^3,\ \ (r, z) \in \Omega \cup \Gamma_0,\ -\pi \le \theta \le \pi\right\}.$$

In Figure 1, we illustrate some examples of domains $\breve{\Omega}$ which we will treat numerically.

2.2. Weighted Sobolev spaces

We define the Hilbert spaces $L^2_1(\Omega)$, $L^2_{-1}(\Omega)$ and $H^m_1(\Omega)$, for $m \in \mathbb{N}^*$, by:

$$L^2_{\pm 1}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{C} \text{ measurable },\|u\|_{L^2_{\pm 1}(\Omega)} = (_{\Omega}|u^2\,(r, z)\,|r^{\pm 1}\mathrm{d}r\mathrm{d}z)^{\frac{1}{2}} < +\infty\right\}$$

$$H^m_1(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{C} \text{ measurable},\ \|u\|_{H^m_1(\Omega)} = \left(\sum_{k=0}^{m}\sum_{\ell=0}^{k} ||\partial^\ell_r \partial^{k-\ell}_z u||^2_{L^2_1(\Omega)}\right)^{\frac{1}{2}} < +\infty\right\}.$$

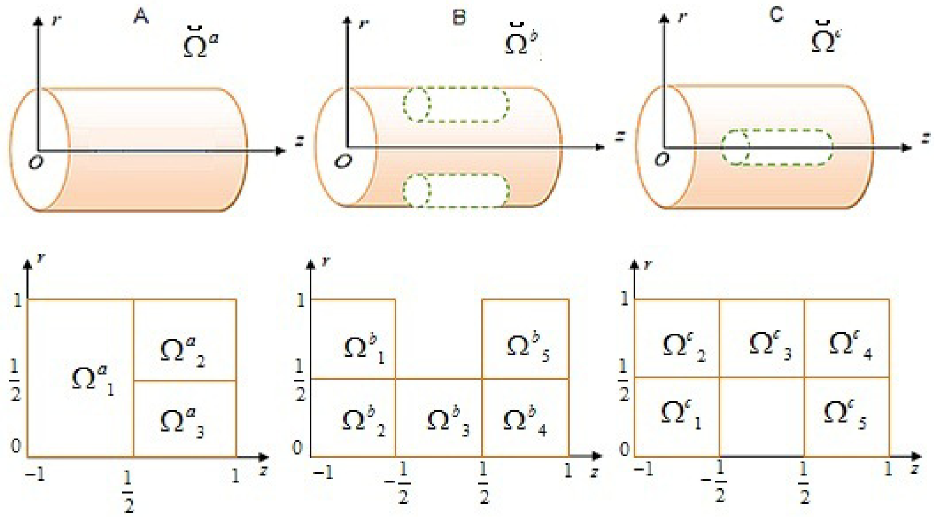

Figure 1. Domains of study.

We also define the Hilbert space $V_1^1(\varOmega) = H_1^1(\varOmega) \cap L_{-1}^2(\varOmega)$ endowed with the norm

$$||w||_{V_1^1(\varOmega)} = \left(||w||^2_{H_1^1(\varOmega)} + ||w||^2_{L^2_{-1}(\varOmega)}\right)^{\frac{1}{2}}.$$

To any $\breve{v} \in L^2(\breve{\varOmega})$, we associate its Fourier coefficients v^k, $k \in \mathbb{Z}$ given by

$$v^k\,(r,z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \breve{v}(r,\theta,z) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \tag{2.1}$$

which belongs to $L_1^2(\varOmega)$. For each vector field $\breve{v} \in L^2(\breve{\varOmega})$, we consider its associated Fourier coefficients (v^k). It is proved in [2] that the Fourier transformation: $\breve{v} \longmapsto (v^k)_{k\in\mathbb{Z}}$ is one to one from $H^1(\breve{\varOmega})$ onto $\varPi_{k\in Z} H^1_{(k)}(\varOmega)$ where:

$$H^1_{(k)}(\varOmega) = V_1^1(\varOmega) \text{ if } k \neq 0, H_1^1(\varOmega) \text{ if } k = 0.$$

Moreover, we endow $H^1_{(k)}(\varOmega)$ with the norm $||w||_{H^1_{(k)}(\varOmega)} = (||w||^2_{H_1^1(\varOmega)} + |k|^2||w||^2_{L^2_{-1}(\varOmega)})^{\frac{1}{2}}$ and we have the following equivalence of norms:

$$c||\breve{v}||_{H^1(\breve{\varOmega})} \leq \left(\sum_{k\in\mathbb{Z}} ||v^k||^2_{H^1_{(k)}(\varOmega)}\right)^{\frac{1}{2}} \leq c'||\breve{v}||_{H^1(\breve{\varOmega})}. \tag{2.2}$$

In order to take into account the boundary conditions, we introduce the spaces:

$$H^1_{1\diamond}(\varOmega) = \{v \in H_1^1(\varOmega); v = 0 \text{ on } \varGamma\},$$
$$V^1_{1\diamond}(\varOmega) = H^1_{1\diamond}(\varOmega) \cap V_1^1(\varOmega)$$
and
$$H^1_{(k)\diamond}(\varOmega) = H^1_{(k)}(\varOmega) \cap H^1_{1\diamond}(\varOmega).$$

More general results on the spaces $H^s_{(k)}(\varOmega)$, with a positive real number s, exist in [2].

Remark 2.1. In the one-dimensional case of an edge Λ of Ω, the spaces $L^2_{\pm 1}(\Lambda)$,$H^m_1(\Lambda)$, $V^1_1(\Lambda)$ and $H^1_{(k)}(\Lambda)$ are defined in the same way of the two-dimensional case by using the measure $\mathrm{d}\tau = rdr$ if Λ is perpendicular to the axis (Oz) and $\mathrm{d}\tau = \mathrm{d}z$ if not. For more details see [2].

2.3. Variational formulation

If $\breve{u}$ is the solution of problem (1.1), with $\breve{g} = 0$, the Fourier coefficients u^k are the solutions, for all $k \in \mathbb{Z}$, of the problems:

$$\begin{cases} -\partial_r^2 u^k - \frac{1}{r}\partial_r u^k - \partial_z^2 u^k + \frac{k^2}{r^2}u^k = f^k & \text{in } \Omega, \\ u^k = 0 & \text{on } \Gamma, \end{cases} \tag{2.3}$$

where f^k is the kth Fourier coefficients of $\breve{f}$. Moreover, if $\breve{f} \in L^2(\breve{\Omega})$, u^k is the solution of the variational problem:

$$\begin{aligned} &\text{Find } u^k \in H^1_{(k)\diamond}(\Omega) \text{ such that} \\ &A_k(u^k, v) = (f^k, v) \ \forall v \in H^1_{(k)\diamond}(\Omega), \end{aligned} \tag{2.4}$$

where

$$(f, v) =_\Omega (f(r,z).\bar{v}(r,z))r\mathrm{d}r\mathrm{d}z$$

is the Hermitian product,

$$A_k(u, v) = a(u, v) +_\Omega \frac{k^2}{r} u\bar{v}\mathrm{d}r\mathrm{d}z,$$

$$a(u, v) =_\Omega (\partial_r u \partial_r \bar{v} + \partial_z u \partial_z \bar{v})r\mathrm{d}r\mathrm{d}z =: (\nabla u, \nabla v).$$

It is readily checked, by the Lax Milgram Lemma and the weighted Poincaré–Friedrichs inequalities [16], Proposition 3, that for any data $f^k \in L^2_1(\Omega)$, problem (2.4) has a unique solution u^k which verifies

$$\left\| u^k \right\|_{H^1_{(k)\diamond}(\Omega)} \leq c \left\| f^k \right\|_{L^2_1(\Omega)}. \tag{2.5}$$

3. The discretization in the axisymmetric case

We assume here that the datum $\breve{f}$ is axisymmetric, *i.e.* independent of θ. Thus only its Fourier coefficient of order $k = 0$ is non zero and so only problem (2.4) for $k = 0$ has a non zero solution. The solution u is then real and $a(.,.)$ is given by $a(u, v) =_\Omega (\partial_r u \partial_r v + \partial_z u \partial_z v)r\mathrm{d}r\mathrm{d}z$.

3.1. The discrete spaces

We will consider a spectral discretization associated to a non-conforming domain decomposition method. Thus, we decompose Ω into L open rectangles Ω_ℓ, $1 \leq \ell \leq L$, such that

$$\bar{\Omega} = \bigcup_{\ell=1}^{L} \bar{\Omega}_\ell \ \text{ and } \ \Omega_\ell \cap \Omega_m = \varnothing, \quad 1 \leq \ell < m \leq L; \tag{3.1}$$

each edge of Ω_ℓ is either parallel or orthogonal to the axis (Oz) (see Fig. 1). For any nonnegative integer N and two-dimensional domain O, we denote by $\mathbb{P}_N(O)$ the space of polynomials on O with degree $\leq N$ with respect to each variable r and z. We define a family of L positive integers $\delta = (N_1, \ldots, N_L)$ and the skeleton $\mathcal{S}$ of the domain decomposition equal to $\bigcup_{\ell=1}^{L} \partial\Omega_\ell \backslash \partial\Omega$. It admits a partition without overlap into mortars

$$\bar{\mathcal{S}} = \bigcup_{\mu=1}^{M^+} \gamma_\mu^+ \quad \text{with} \quad \gamma_\mu^+ \cap \gamma_{\mu'}^+ = \varnothing, \quad 1 \leq \mu < \mu' \leq M^+.$$

Above each γ_μ^+ is a whole edge of one of Ω_ℓ, which is then denoted by Ω_μ^+. We notice that the choice of this decomposition is not unique, however it chosen for all the discretizations we use. Once we fix the skeleton, we have another partition of it into non-mortars:

$$\bar{\mathcal{S}} = \bigcup_{\mu=1}^{M^-} \gamma_m^- \quad \text{with} \quad \gamma_m^- \cap \gamma_{m'}^- = \varnothing, \quad 1 \le m < m^{'} \le M^-$$

where each γ_m^- is a whole edge of one of $\Omega_\ell \neq \Omega_\mu^+$, that we denote by Ω_m^-.

Then, we introduce the discrete space

$$\mathbb{Y}_\delta = \{v_\delta \in L_1^2(\Omega), v_{\delta|\Omega_\ell} = v_\ell \in \mathbb{P}_{N_\ell}(\Omega_\ell), \forall \ell = 1, \dots L\}.$$

To any v_δ in $\mathbb{Y}_\delta$, we associate the mortar function $\phi_{v_\delta} \in L_1^2(\mathcal{S})$ defined by $\phi_{v_\delta|\gamma_\mu^+} = (v_{\delta|\Omega_\mu^+})_{|\gamma_\mu^+}$, $1 \le \mu \le M^+$. We define our fundamental discrete space $\mathbb{X}_\delta$ by:

$$\mathbb{X}_\delta = \left\{ v_\delta \in \mathbb{Y}_\delta, \quad \int_{\gamma_m^-} (v_\delta - \phi_{v_\delta})(\tau)\psi(\tau)\mathrm{d}\tau = 0 \ \ \forall \psi \in \mathbb{P}_{N_m-2}(\gamma_m^-), \ \ \forall \gamma_m^-, \ 1 \le m \le M^- \right\} \tag{3.2}$$

where $\mathrm{d}\tau = r\mathrm{d}r$ if γ_m^- is parallel to the axis oz and $\mathrm{d}\tau = \mathrm{d}z$ otherwise.

We also introduce the spaces

$$\mathbb{X}_\delta^\diamond = \{v_\delta \in \mathbb{X}_\delta, v_\delta = 0 \text{ on } \Gamma\}$$
and
$$\mathbb{X}_\delta^\circ = \{v_\delta \in \mathbb{X}_\delta, v_\delta = 0 \text{ on } \Gamma \cup \Gamma_0\}.$$

3.2. Quadrature formulas

Quadrature formulas are a lot of the spectral method, we refer to [2] for a detailed discussion of these formulas in weighted spaces. We begin by classifying subdomains according to the intersection of their border with Γ_0. The formulas that we use change according to this intersection. Let $(\Omega_\ell)_{1\le\ell\le L_0}$ denote the rectangles such that $\partial\overline{\Omega}_\ell \cap \Gamma_0 \neq \varnothing$ and $(\Omega_\ell)_{L_0+1\le\ell\le L}$ the rest of the partition. We denote by (ξ_j, ρ_j), $0 \le j \le N$, the nodes and weights of the Gauss–Lobatto quadrature formulas on $[-1,1]$ for the measure $\mathrm{d}\zeta$ and (ζ_j, ω_i), $1 \le j \le N+1$, the corresponding ones for the measure $(1+\zeta)\mathrm{d}\zeta$. On the square $\Sigma = \left]-1,1\right[^2$, we use the following Gauss–Lobatto and weighted Gauss–Lobatto formulas:

$$\forall \phi \in \mathbb{P}_{2N-1}(\Sigma), \quad \int_\Sigma \phi(\zeta,\xi)(1+\zeta)\,\mathrm{d}\zeta\mathrm{d}\xi = \sum_{i=1}^{N+1}\sum_{j=0}^{N} \phi(\zeta_i,\xi_j)\,\omega_i\rho_j,$$
$$\forall \phi \in \mathbb{P}_{2N-1}(\Sigma), \quad \int_\Sigma \phi(\zeta,\xi)\,\mathrm{d}\zeta\mathrm{d}\xi = \sum_{i=0}^{N}\sum_{j=0}^{N} \phi(\xi_i,\xi_j)\,\rho_i\rho_j.$$

We transform the nodes and weights in Ω_ℓ as follows.

If $\Omega_\ell =]0, r'_\ell[\times]z_\ell, z'_\ell[$ for $1 \le \ell \le L_0$ and $N = N_\ell$ then $\zeta_i^\ell = \dfrac{r'_\ell}{2}(\zeta_i + 1)$, $\omega_i^\ell = \omega_i \dfrac{r'^2_\ell}{4}$, $1 \le i \le N_\ell + 1$.

If $\Omega_\ell =]r_\ell, r'_\ell[\times]z_\ell, z'_\ell[$ for $L_0 + 1 \le \ell \le L$ and $N = N_\ell$ then $\xi_i^{(r)\ell} = \dfrac{(r'_\ell - r_\ell)}{2}\xi_i + \dfrac{(r'_\ell + r^\ell)}{2}$, $\rho_i^{(r)\ell} = \rho_i \dfrac{r'_\ell - r_\ell}{2}$, $0 \le i \le N_\ell$.

If $\Omega_\ell =]r_\ell, r'_\ell[\times]z_\ell, z'_\ell[$ for $1 \le \ell \le L$ and $N = N_\ell$ then $\xi_i^\ell = \dfrac{(z'_\ell - z_\ell)}{2}\xi_i + \dfrac{(z'_\ell + z_\ell)}{2}$, $\rho_i^\ell = \rho_i \dfrac{z'_\ell - z_\ell}{2}$, $0 \le i \le N_\ell$. We finally define the discrete scalar product for $u, v \in C^0\left(\cup\bar{\Omega}_\ell\right)$ by:

$$(u,v)_\delta = \sum_{\ell=1}^{L_0}\sum_{i=1}^{N_\ell+1}\sum_{j=0}^{N_\ell} u_\ell(\zeta_j^\ell,\xi_i^\ell)v_\ell(\zeta_j^\ell,\xi_i^\ell)\omega_i^\ell\rho_j^\ell + \sum_{\ell=L_0+1}^{L}\sum_{i,j=0}^{N_\ell} u_\ell(\xi_i^{(r)\ell},\xi_j^\ell)v_\ell(\xi_i^{(r)\ell},\xi_j^\ell)\xi_i^{(r)\ell}\rho_i^{(r)\ell}\rho_j^\ell.$$

Let $\mathcal{I}_\ell^+$ and $\mathcal{I}_\ell$ be the Lagrange interpolation operators, with values in $\mathbb{P}_{N_\ell}(\Omega_\ell)$, associated respectively with the nodes $(\zeta_j^\ell, \xi_i^\ell)$ for $1 \leq \ell \leq L_0$ and with $(\xi_j^{(r)\ell}, \xi_i^\ell)$ for any $L_0 + 1 \leq \ell \leq L$. Let $\mathcal{I}_\delta$ defined by $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_\ell^+$ if Ω_ℓ intersects Γ_0 and $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_\ell$ if not.

3.3. The discrete problem

For a datum $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$, we define our discrete problem, associated with (2.3) for $k = 0$, by:

$$\begin{cases} \text{Find } u_\delta \in \mathbb{X}_\delta^\diamond \text{ such that} \\ \forall v_\delta \in \mathbb{X}_\delta^\diamond,\ a_\delta\left(u_\delta, v_\delta\right) = \left(\mathcal{I}_\delta f, v_\delta\right)_\delta, \end{cases} \tag{3.3}$$

where $a_\delta\left(u, v\right) = \left(\nabla u, \nabla v\right)_\delta$, $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_{N_\ell}^+$ if Ω_ℓ intersects Γ_0 and $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_{N_\ell}$ if not.

Let N_a be the maximum number of corners of $\bar{\Omega}_\ell$ which are inside one of the non-mortar γ_m^-, $1 \leq m \leq M^-$ and let $X(\Omega)$ be the space defined by:

$$\begin{aligned} X(\Omega) = \Big\{ & v \in L_1^2(\Omega),\ v_{|\Omega_\ell} \in V_1^1(\Omega_\ell),\ \text{such that } \forall\ 1 \leq m \leq M^-,\ \forall \psi \in \mathbb{P}_{N_a}(\gamma_m^-), \\ & \int_{\gamma_m^-} (v - \phi_v)\psi \mathrm{d}\tau = 0,\ v = 0 \text{ on } \Gamma \Big\}. \end{aligned}$$

We have the following result.

Proposition 3.1.

1. *There exists a positive constant c depending only on Ω such that:*

$$||v||^2_{L_1^2(\Omega)} \leq c|v|^2_{H_1^1(\cup\Omega_\ell)}\ \forall\ v \in X(\Omega). \tag{3.4}$$

2. *Problem (3.3) is well posed for any $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$.*

Proof.

1. Let $v \in X(\Omega)$ such that $|v|_{H_1^1(\cup\Omega_\ell)} = 0$ and $v_\ell = v_{|\Omega_\ell}$. Hence v_ℓ is constant on each Ω_ℓ and $v_\ell = 0$ in Ω_ℓ for all ℓ with $meas\left(\partial\Omega_\ell \cap \Gamma\right) > 0$. We cannot directly conclude that $v_\ell = 0$ for all ℓ, since we have not necessarily $v_\ell = v_m$ on $\gamma^{\ell m}$. So we fix m such that $1 \leq m \leq M^-$ and $meas\left(\partial\Omega_m^- \cap \Gamma\right) > 0$. According to the matching condition (3.2), we have:

$$\int_{\gamma_m^-} \left(v_{\gamma_m^-} - \phi\right)(\tau)\,\psi\left(\tau\right)\mathrm{d}\tau = 0, \quad \forall \psi \in \mathbb{P}_{N_m^- - 2}\left(\gamma_m^-\right).$$

Hence, we obtain:

$$\sum_{j \in J} \int_{\gamma^{jm}} \left(v_{\gamma_m^-} - v_j\right)\psi\left(\tau\right)\mathrm{d}\tau = 0, \quad \forall \psi \in \mathbb{P}_{N_m^- - 2}\left(\gamma_m^-\right)$$

where $\gamma^{jm} = \bar{\Omega}_j \cap \bar{\Omega}_m^-$ and $meas(\gamma^{jm}) > 0$. Since v_ℓ is constant this leads to:

$$\sum_{j \in J} \left(v_{\gamma_m^-} - v_j\right) \int_{\gamma^{jm}} \psi\left(\tau\right)\mathrm{d}\tau = 0.$$

We introduce the ends a_{j_0} and $a_{j_0 - 1}$ of the interface $\gamma^{j_0 m}$ and consider the polynomial χ of degree $N_m^- - 1$ defined on γ_m^- verifying:

$$\chi\left(a_0\right) = \chi\left(a_1\right) = \cdots = \chi\left(a_{j_0 - 1}\right) = 0, \quad \text{and} \quad \chi\left(a_{j_0}\right) = \chi\left(a_{j_0 + 1}\right) = \cdots = \chi\left(a_S\right) = 1.$$

Since $S \leq N_a$, we can choose $\psi_{j_0} = \chi'$ and we obtain:

$$\int_{\gamma^{jm}} \psi_{j_0}(\tau)\,\mathrm{d}\tau = \chi(a_j) - \chi(a_{j-1}) = \delta_{j_0}^j$$

where δ indicates the Kronecker symbol. We have thus:

$$\sum_{j\in J} \left(v_{\gamma_m^-} - v_j\right) \int_{\gamma^{jm}} \psi_{j_0}(\tau)\,\mathrm{d}\tau = v_{\gamma_m^-} - v_{j_0} = 0,$$

and then $v_{\gamma_m^-} = v_{j_0}$. We deduce that $v_\ell = 0$ for all ℓ such that Ω_ℓ is adjacent with a rectangle which intersects $\partial\Omega\backslash\Gamma_0$. By extension, we deduce that $v_\ell = 0\ \forall\ell$. We have then checked that $|.|_{H_1^1(\cup\Omega_\ell)}$ is a norm. By applying the Peetre–Tartar lemma [11], Chapter 1, Theorem 2.1, with $E_1 = H_1^1(\Omega)$, $E_2 = E_3 = L_1^2(\Omega)$, $A = \nabla \in (E_1, E_2)$ and $B = Id_{E_2}$, we obtain (3.4).

2. Using the Cauchy–Schwarz inequality and the exactitude of the Gauss–Lobatto formula with respect to each variable r and z, we obtain that for every $u_\delta, v_\delta \in X_\delta^\diamond(\Omega)$ we have

$$|a_\delta(u_\delta, v_\delta)| \leq c|u_\delta|_{H_1^1(\cup\Omega_\ell)}|v_\delta|_{H_1^1(\cup\Omega_\ell)},$$

and since $X_\delta^\diamond(\Omega) \subset X(\Omega)$ for each δ, the following coercivity inequality is true on $X_\delta^\diamond(\Omega)$ with a constant c independent of δ:

$$|a_\delta(u_\delta, u_\delta)| \geq c'|u_\delta|^2_{H_1^1(\cup\Omega_\ell)}. \tag{3.5}$$

Hence, for all $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$, problem (3.3) admits a unique solution $u_\delta \in X_\delta^\diamond(\Omega)$ verifying:

$$||u_\delta||_{H_1^1(\cup\Omega_\ell)} \leq c||\mathcal{I}_\delta f||_{L_1^2(\Omega)}.$$

□

From now on, we suppose that $N_\ell \geq N_a + 2$ for all $1 \leq \ell \leq L$.

3.4. Error estimates

Classical techniques in the approximation theory [17] lead to the following estimate:

$$\begin{aligned}\|u - u_\delta\|_{H_1^1(\cup\Omega_\ell)} \leq c\Bigg(&\inf_{v_\delta\in X_\delta^\diamond}\left\{\|u - v_\delta\|_{H_1^1(\cup\Omega_\ell)} + \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{|a(v_\delta, w_\delta) - a_\delta(v_\delta, w_\delta)|}{\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}}\right\} \\ &+ \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{\left|\sum_{\gamma_m^-\in\mathcal{S}}\int_{\gamma_m^-}(\frac{\partial u}{\partial n_m})[w_\delta]\mathrm{d}\tau\right|}{\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}} + \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{\left|\int_\Omega f w_\delta r\mathrm{d}r\mathrm{d}z - (\mathcal{I}_\delta f, w_\delta)_\delta\right|}{\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}}\Bigg),\end{aligned} \tag{3.6}$$

where the term $\frac{\partial u}{\partial n_m}$ refers to the normal derivative of u and $[w_\delta]$ the jump of w_δ through γ_m^-. We will study each term of this estimate.

Proposition 3.2. *For any solution u such that $u_{|\Omega_\ell} \in H_1^{s_\ell+1}(\Omega_\ell)$, with $s_\ell > \frac{1}{2}$ or $s_\ell > \frac{3}{2}$ if $\ell \leq L_0$, the approximate error verifies:*

$$\inf_{v_\delta\in X_\delta^\diamond}\|u - v_\delta\|_{H_1^1(\cup\Omega_\ell)} \leq c\lambda_\delta^{\frac{1}{2}}\sum_{\ell=1}^{L} N_\ell^{-s_\ell}||u_\ell||_{H_1^{s_\ell+1}(\Omega_\ell)} \tag{3.7}$$

where $\lambda_\delta = \max\left\{\frac{N_\mu^+}{N_m^-}, \frac{N_m^-}{N_\mu^+}\right\}$ for all mortars γ_μ^+, $1 \leq \mu \leq M^+$ and non-mortars γ_m^-, $1 \leq m \leq M^-$ such that $\gamma_\mu^+ \cap \gamma_m^-$ has a nonnegative measure.

Before proving the proposition, we recall [2], Remark IV.3.1, Proposition IV.3.4, that there exist projection operators:

$$\tilde{\pi}_N^1 : H^1(\Lambda) \longrightarrow \mathbb{P}_N(\Lambda) \text{ and } \tilde{\pi}_N^{+,1} : H_1^1(\Lambda) \longrightarrow \mathbb{P}_N(\Lambda),\ \Lambda =]-1,1[, \tag{3.8}$$

verifying, for $0 \le t \le 1 \le s$:

$$\left\|\tilde{\varphi} - \tilde{\pi}_N^1 \tilde{\varphi}\right\|_{H^t(\Lambda)} \le C N^{t-s} \left\|\tilde{\varphi}\right\|_{H^s(\Lambda)} \quad \text{and} \quad \left\|\tilde{\phi} - \tilde{\pi}_N^{+,1}\tilde{\phi}\right\|_{H_1^t(\Lambda)} \le C N^{t-s} \left\|\tilde{\phi}\right\|_{H_1^s(\Lambda)} \tag{3.9}$$

and for which it is easy to verify the following matching conditions:

$$\forall \psi \in \mathbb{P}_{N-2}(\Lambda),\ \forall \tilde{\varphi} \in H^1(\Lambda),\ \int_{-1}^{1} (\tilde{\varphi} - \tilde{\pi}_N^1 \tilde{\varphi})\psi \mathrm{d}\tau = 0, \tag{3.10}$$

$$\forall \psi \in \mathbb{P}_{N-2}(\Lambda),\ \forall \tilde{\phi} \in H_1^1(\Lambda),\ \int_{-1}^{1} (\tilde{\phi} - \tilde{\pi}_N^{+,1} \tilde{\phi})\psi \mathrm{d}\tau = 0 \tag{3.11}$$

where $\mathrm{d}\tau = \mathrm{d}z$ respectively $\mathrm{d}\tau = (1+\zeta)\mathrm{d}\zeta$.

We recall also [16], Lemma 2.3.3, the following lemma.

Lemma 3.3. *Let $(a_p)_{1\le p\le P}$ P distinct points in Λ. For each $N \ge P+2$ and each p, there exists a polynomial $\eta_p \in P_N(\Lambda)$ verifying $\eta_p(a_p) = 1$, $\eta_p(\pm 1) = 0$, $\eta_p(a_{p'\neq p}) = 0$ and satisfying:*

$$\left\|\eta_p\right\|_{L_1^2(\Lambda)} \le c N^{-\frac{1}{2}}, \quad \left\|\eta_p'\right\|_{L_1^2(\Lambda)} \le c N^{\frac{1}{2}}, \tag{3.12}$$

where the constant c depends only on the points a_p.

Proof of Proposition 3.2. We refer to [16], Proposition 2.3.5, for more details concerning this proof which is divided into three parts.

Part 1. We first construct v_δ^1 setting,

$$v_\ell^1 = \mathcal{I}_{N_\ell}^+ u \text{ in } \Omega_\ell \text{ if } 1 \le \ell \le L_0 \text{ and } \mathcal{I}_{N_\ell} u \text{ if } L_0 + 1 \le \ell \le L.$$

According to [2], (VI.3), we have for all $s_\ell > \frac{1}{2}$ and $s_\ell > \frac{3}{2}$ if $\ell \le L_0$:

$$\left\|u_{|\Omega_\ell} - v_\ell^1\right\|_{H_1^1(\Omega_\ell)} \le c N_\ell^{-s_\ell} \ \left\|u_{|\Omega_\ell}\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}. \tag{3.13}$$

And for every $1 \le \ell \le L$, we have:

$$\left\|u_{|\Omega_\ell} - v_\ell^1\right\|_{H_1^1(\Gamma)} + N_\ell \left\|u_{|\Omega_\ell} - v_\ell^1\right\|_{L_1^2(\Gamma)} \le c' N_\ell^{\frac{1}{2}-s_\ell} \ \left\|u_{|\Omega_\ell}\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}. \tag{3.14}$$

However, v_δ^1 does not verify the mortar matching condition across the interfaces and so we need to change its values on the non-mortars.

Part 2. Secondly, we construct v_δ^2: for every $1 \le \mu \le M^+$, we consider $\mathcal{C}_\mu^+$ the set of the corners of Ω_ℓ which are inside γ_μ^+. We set:

$$v_\delta^2 = \sum_{\mu=1}^{M^+} \sum_{e\in\mathcal{C}_\mu^+} \left(u - v_{\delta|\Omega_\mu^+}^1\right)(e)\tilde{\Phi}_{\mu,e}, \quad \text{where} \quad \tilde{\Phi}_{\mu,e} = \begin{cases} \Phi_{\mu,e} & \text{in } \Omega_\mu^+, \\ 0 & \text{in } \Omega\backslash\bar{\Omega}_\mu^+ \end{cases}$$

$\Phi(\zeta,\eta)=\eta_p(\zeta)(\frac{1-\eta}{2})^{N_\mu^+}$ and $\Phi_{\mu,e}$ is obtained from Φ by homothety and translation. It follows that $v_\delta^1+v_\delta^2=u$ at all the nodes $e\in\mathcal{C}_\mu^+$. Since $\|\Phi_{\mu,e}\|_{H_1^1(\Omega_\mu^+)}$ is bounded independently of N_μ^+, we have:

$$\sum_{\ell=1}^{L}\left\|v_\delta^2\right\|_{H_1^1(\Omega_\ell)}\leq c\sum_{\mu=1}^{M^+}\sum_{e\in\mathcal{C}_\mu}|\left(u-v^1_{\delta|\Omega_\mu^+}\right)(e)|, \tag{3.15}$$

with c independent of N. Applying a Galiardo-Niremberg inequality [8] on each γ_μ^+ and using (3.14), we obtain:

$$\left\|u-v^1_{\delta|\Omega_\mu^+}\right\|_{L^\infty(\gamma_\mu^+)}\leq c(N_\mu^+)^{-s_\mu^+}\left\|u\right\|_{H_1^{s_\mu^++1}(\Omega_\mu^+)}. \tag{3.16}$$

We deduce then from (3.15) and (3.16) that:

$$\sum_{\ell=1}^{L}\left\|v_\delta^2\right\|_{H_1^1(\Omega_\ell)}\leq c\sum_{\mu=1}^{M^+}(N_\mu^+)^{-s_\mu^+}\left\|u\right\|_{H_1^{s_\mu^++1}(\Omega_\mu^+)}.$$

Similarly, we derive from Lemma 3.3, and (3.16) that:

$$\left\|v_\delta^2\right\|_{H_1^1(\gamma_\mu^+)}\leq c(N_\mu^+)^{\frac12-s_\mu^+}\left\|u\right\|_{H_1^{s_\mu^++1}(\Omega_\mu^+)}\ \text{ and }\ \left\|v_\delta^2\right\|_{L_1^2(\gamma_m^-)}\leq c(N_\mu^+)^{-\frac12-s_\mu^+}\left\|u\right\|_{H_1^{s_\mu^++1}(\Omega_\mu^+)}.$$

Part 3. Construction of v_δ^3 : We set $v_\delta^{12}=v_\delta^1+v_\delta^2$. According to the first part, the trace $v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-}$ vanishes at the endpoints of all γ_m^-, $1\leq m\leq M^-$. We set respectively $\tilde\pi_{N_\ell}^{+,1,(r),\ell}$, $1\leq\ell\leq L_0$, $\tilde\pi_{N_\ell}^{1,(r),\ell}$, $L_0+1\leq\ell\leq L$ and $\tilde\pi_{N_\ell}^{1,(z),\ell}$, $1\leq\ell\leq L$, the corresponding projection operators respectively with $\tilde\pi_N^{+1}$ in the direction r and $\tilde\pi_N^1$ in the direction z. We define also the operator $\tilde\pi^{\gamma_m^-}$ by:

$$\tilde\pi^{\gamma_m^-}=\begin{cases}\tilde\pi_m^{+,1,(r)} & \text{if } \gamma_m^-//(Or) \text{ and } \gamma_m^-\cap(Oz)\neq\varnothing,\\ \tilde\pi_m^{1,(r)} & \text{if } \gamma_m^-//(Or) \text{ and } \gamma_m^-\cap(Oz)=\varnothing,\\ \tilde\pi_m^{1,(z)} & \text{if } \gamma_m^-\ //(Oz)\end{cases}$$

and set:

$$v_\delta^3=\sum_{m=1}^{M^-}\left[\tilde{\mathcal{R}}_\star^{\gamma_m^-}\circ\tilde\pi^{\gamma_m^-}\left(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-}\right)_{|\gamma_m^-}\right] \tag{3.17}$$

where γ_m^+ is the side γ_m^- seen in the other direction,

$$\tilde{\mathcal{R}}_\star^\gamma=\tilde{\mathcal{R}}_-^\gamma \text{ if } \gamma//(Or)\ \text{ and }\ \gamma\cap(Oz)\neq\varnothing \text{ and } \tilde{\mathcal{R}}^\gamma \text{ otherwise.}$$

We notice that $\mathcal{R}_-^{\gamma_m^-}$ (resp. $\mathcal{R}^{\gamma_m^-}$) is the lifting introduced in [6], Proposition 4.25 and $\tilde{\mathcal{R}}_-^{\gamma_m^-}$ (resp. $\tilde{\mathcal{R}}^{\gamma_m^-}$) is the lifting deduced from $\mathcal{R}_-^{\gamma_m^-}$ (resp. $\mathcal{R}^{\gamma_m^-}$) by dilatation and translation.

We use for each real s, the notation:

$$(H^{s,\gamma},V^{s,\gamma},L^{2,\gamma})=(H_1^s,V_1^s,L_1^2) \text{ if } \gamma//(Or)\ \text{ and }\ \gamma\cap(Oz)\neq\varnothing \text{ and } (H^s,V^s,L^2) \text{ otherwise.}$$

Then, we have:

$$\tilde{\mathcal{R}}_\star^{\gamma_m^-}\circ\tilde\pi^{\gamma_m^-}\left(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-}\right)=\tilde{\mathcal{R}}_\star^{\gamma_m^-}\left(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-}\right)-\tilde{\mathcal{R}}_\star^{\gamma_m^-}(Id-\tilde\pi^{\gamma_m^-})\left(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-}\right).$$

Using the fact that

$$\left\|\varphi-\tilde\pi^{\gamma_m^-}\varphi\right\|_{H^{r,\gamma_m^-}(\Omega_m^-)}\leq c(N_m^-)^{r-s}\left\|\varphi\right\|_{H^{s,\gamma_m^-}(\gamma_m^-)},\ \text{for } 0\leq r\leq1\leq s,$$

we obtain that:

$$\left\|\tilde{\mathcal{R}}_\star^{\gamma_m^-}\circ\tilde{\pi}^{\gamma_m^-}(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-})\right\|_{H^{1,\gamma_m^-}(\Omega_m^-)}\le c\left\|(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-})\right\|_{V^{\frac12,\gamma_m^-}(\gamma_m^-)}+N_m^{-\frac12}\left\|(v^{12}_{\delta|\gamma_m^+}-v^{12}_{\delta|\gamma_m^-})\right\|_{H^{1,\gamma_m^-}(\gamma_m^-)};$$

by summing, we obtain:

$$\sum_{\ell=1}^{L}\left\|v_\delta^3\right\|_{H_1^1(\Omega_\ell)}\le c\lambda_\delta^{\frac12}\sum_{\ell=1}^{L}N_\ell^{-s_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}.$$

Finally, the function $v_\delta^0=v_\delta^1+v_\delta^2+v_\delta^3$ satisfies the matching conditions, belongs to $X_\delta^\diamond$, and satisfies the desired estimate since

$$\inf_{v_\delta\in X_\delta^\diamond}\left\|u-v_\delta\right\|_{H_1^1(\cup\Omega_\ell)}\le\left\|u-v_\delta^0\right\|_{H_1^1(\cup\Omega_\ell)}\le\left\|u-v_\delta^1\right\|_{H_1^1(\cup\Omega_\ell)}+\left\|v_\delta^2\right\|_{H_1^1(\cup\Omega_\ell)}+\left\|v_\delta^3\right\|_{H_1^1(\cup\Omega_\ell)}.\qquad\square$$

Remark 3.4. We can replace the total term λ_δ by the local term λ_ℓ defined by:

$$\lambda_\ell=\max_m\max_{\mu\in\mathcal{K}_m^-}\left\{\frac{N_\mu^+}{N_m^-},\frac{N_m^-}{N_\mu^+}\right\},\tag{3.18}$$

where the first max is taken on m of non mortars γ_m^- which are edges of Ω_ℓ and we obtain:

$$\inf_{v_\delta\in X_\delta^\diamond}\left\|u-v_\delta\right\|_{H_1^1(\cup\Omega_\ell)}\le c\sum_{\ell=1}^{L}(1+\lambda_\ell)^{\frac12}N_\ell^{-s_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}.$$

In the following proposition, we are interested in errors due to non-conformities on the interfaces.

Proposition 3.5. *For any solution u that verifies $u_{|\Omega_\ell}\in H_1^{s_\ell+1}(\Omega_\ell)$, with $s_\ell>\frac12$ and $s_\ell>\frac32$ if $\ell\le L_0$, and for all $w_\delta\in X_\delta^\diamond$, the following estimate holds*

$$\left|\sum_{\gamma_m^-\in\mathcal{S}}\int_{\gamma_m^-}\left(\frac{\partial u}{\partial n_m}\right)[w_\delta]\mathrm{d}\tau\right|\le c\left[\sum_{\ell=1}^{L}N_\ell^{-s_\ell}(\log N_\ell)^{\varrho_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}\right]\left\|w_\delta\right\|_{H_1^1(\cup\Omega_\ell)},\tag{3.19}$$

where ϱ_ℓ is equal to 1 if one of the sides of Ω_ℓ is a γ_m^- and intersects at least two subdomains $\bar{\Omega}_{\ell'}$, $\ell'\neq\ell$ and 0 otherwise.

Proof. We assume that $\gamma_m^-\subset\cup_{1\le\mu\le I}\,\Omega_\mu$, where I is a nonnegative integer. For any $\varepsilon>0$, we have:

$$w_{\delta|\Omega_\mu}\in H_1^{\frac12-\varepsilon}(\gamma_m^-\cap\partial\Omega_\mu)\text{ if }\Omega_\mu\text{ touches the axis }\{r=0\}\text{ and }w_{\delta|\Omega_\mu}\in H^{\frac12-\varepsilon}(\gamma_m^-)\text{ otherwise.}$$

In an other hand, and according to [7], Remark 2.10, page 11, the extension by zero is continuous from $H^{\frac12-\varepsilon}(\gamma)$ onto $H^{\frac12-\varepsilon}(\gamma_m^-)$ for any part γ of γ_m^- and its norm verifies:

$$\left\|.\right\|_{H^{\frac12-\varepsilon}(\gamma_m^-)}\le c\varepsilon^{-1}\left\|.\right\|_{H^{\frac12-\varepsilon}(\gamma)}.\tag{3.20}$$

To unify the two cases $\ell>L_0$ and $\ell\le L_0$, we will use the fact that the norms $\|.\|_{H^{\frac12}(\gamma_m^-)}$ and $\|.\|_{H_1^{\frac12}(\gamma_m^-)}$, respectively the norms $\|.\|_{H^1(\Omega_\mu)}$ and $\|.\|_{H_1^1(\Omega_\mu)}$, are equivalent if γ_m^-, respectively Ω_μ, is far from the axis $\{r=0\}$. In the general case, we can consider that the constants depend on the diameter of Ω. Since $[w_\delta]=w_{\delta|\gamma_m^-}-\Phi_{|\gamma_m^-}$ where $\Phi_{|\gamma_m^-}=\sum_{1\le i\le I}\tilde{w}^i_{\delta|\gamma_m^-}$ and $\tilde{w}^i_{\delta|\gamma_m^-}$ is the extension of $w^i_{\delta|\partial\Omega_\mu\cap\gamma_m^-}$ on γ_m^-, and using (3.2), we obtain:

$$\begin{aligned}\left|\int_{\gamma_m^-}\frac{\partial u}{\partial n_m}\left[w_\delta\right](\tau)\,\mathrm{d}\tau\right|&=\int_{\gamma_m^-}\left(\frac{\partial u}{\partial n_m}-\psi^+)\right(\Phi-w_\delta\Bigg)\,\mathrm{d}\tau\\&\le c\left\|\frac{\partial u}{\partial n_m}\right\|_{H^{-\frac12+\varepsilon}(\gamma_m^-)}\left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^{\frac12-\varepsilon}(\gamma_m^-)}+\left\|\Phi\right\|_{H_1^{\frac12-\varepsilon}(\gamma_m^-)}\right),\end{aligned}\tag{3.21}$$

where $\psi^+ = \pi^+_{N_m-2}\left(\frac{\partial u}{\partial n_m}\right)$ and π^+_N is the projection operator from $L^2_1(\Lambda)$ onto $\mathbb{P}_N(\Lambda)$ defined in [2], Section IV.2.b. Applying the inequalities (3.20) and the fact that:

$$\|\Phi\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} \leq \sum_{1\leq i\leq I} \left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)},$$

we obtain:

$$\begin{aligned}\left|_{\gamma_m^-} \frac{\partial u}{\partial n_m}\left[w_\delta\right](\tau)\,\mathrm{d}\tau\right| &\leq C\left\|\tfrac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)}\left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} + c\varepsilon^{-1}\sum_{1\leq i\leq I}\left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_\mu)}\right)\\ &\leq C(1+c\varepsilon^{-1})\left\|\tfrac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)}\left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^1(\Omega_m)} + \sum_{1\leq i\leq I}\left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^1(\Omega_\mu)}\right).\end{aligned}$$

In addition, for $\varepsilon = 1/\log N_m$ we obtain:

$$\left\|\frac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)} \leq cN_m^{(\varepsilon-\frac{1}{2})-(s_m-\frac{3}{2})}\left\|\frac{\partial u}{\partial n_m}\right\|_{H_1^{s_m-\frac{3}{2}}(\gamma_m^-)} \leq ceN_m^{1-s_m}\|u\|_{H_1^{s_m}(\Omega_m)}.$$

It follows that:

$$\left|\int_{\gamma_m^-}\frac{\partial u}{\partial n_m}\left[w_\delta\right](\tau)\,\mathrm{d}\tau\right| \leq c(1+c\varepsilon^{-1})N_m^{1-s_m}\|u\|_{H_1^{s_m}(\Omega_m)}\|w_\delta\|_{H_1^1(\cup\Omega_\ell)} \tag{3.22}$$

and

$$\frac{\left|\int_{\gamma_m^-}\frac{\partial u}{\partial n_m}\left[w_\delta\right](\tau)\,\mathrm{d}\tau\right|}{\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}} \leq cN_m^{-s_m}(\log N_m)\|u\|_{H_1^{s_m+1}(\Omega_m)}.$$

Finally, by adding with respect to m, we deduce (3.19).

In the conforming case we eliminate the term $(\log N_m)$ since we have $w_{\delta|\Omega_\mu} \in H_1^{\frac{1}{2}}(\gamma_m^-)$. □

We are now able to state the following estimate error.

Proposition 3.6. *Let f such that $f_{|\Omega_\ell} \in H_1^{\sigma_\ell}(\Omega_\ell)$, $\sigma_\ell > 1$ ($\sigma_\ell > \frac{3}{2}$ if $\ell \leq L_0$). Let u be a solution of problem (2.4) with k=0, such that $u_{|\Omega_\ell} \in H_1^{s_\ell+1}(\Omega_\ell)$, $s_\ell > \frac{1}{2}$ ($s_\ell > \frac{3}{2}$ if $\ell \leq L_0$) and u_δ be the solution of problem (3.3), we have:*

$$\|u-u_\delta\|_{H_1^1(\cup\Omega_\ell)} \leq c\sum_{\ell=1}^{L}\left[(1+\lambda_\ell)^{\frac{1}{2}}N_\ell^{-s_\ell}(\log N_\ell)^{\varrho_\ell}\|u\|_{H_1^{s_\ell+1}(\Omega_\ell)} + N_\ell^{-\sigma_\ell}\left\|f_{|\Omega_\ell}\right\|_{H_1^{\sigma_\ell}(\Omega_\ell)}\right] \tag{3.23}$$

where c is a nonnegative constant and ϱ_ℓ is defined in Proposition 3.5.

Proof. We consider the inequality (3.6). The integration error on the external forces gives [2], Theorem VIII.2.6:

$$\left|\int_\Omega f w_\delta r\mathrm{d}r\mathrm{d}z - (\mathcal{I}_\delta f, w_\delta)_\delta\right| \leq \sum_{\ell=1}^{L}N_\ell^{-\sigma_\ell}\left\|f_{|\Omega_\ell}\right\|_{H_1^{\sigma_\ell}(\Omega_\ell)}. \tag{3.24}$$

For the consistence error, we set $\delta - 1 = (N_1-1, N_2-1, \ldots, N_L-1)$ and $x_{\delta-1}$ such that $x_{\delta-1}|_{\Omega_\ell} = \Pi^{+,1}_{N_\ell-1}u$ where $\Pi^{+,1}_{N_\ell-1}$ is the projection operator from $H_1^1(\Omega_\ell)$ into $\mathbb{P}_{N_\ell-1}(\Omega_\ell)$ defined in [2], Section V.3.b, and verifying:

$$\left\|u - \Pi^{+,1}_{N_\ell-1}u\right\|_{H_1^1(\Omega_\ell)} \leq N_\ell^{-s_\ell}\|u_\ell\|_{H_1^{s_\ell+1}(\Omega_\ell)}.$$

We have:

$$
\begin{aligned}
|a_\delta(v_\delta, w_\delta) - a(v_\delta, w_\delta)| &= |a(v_\delta - x_{\delta-1}, w_\delta) - a_\delta(v_\delta - x_{\delta-1}, w_\delta)| \\
&\le c\left\{\left\|u - \Pi^{+,1}_{N_\ell-1}u\right\|_{H^1_1(\cup\Omega_\ell)} + \|u - v_\delta\|_{H^1_1(\cup\Omega_\ell)}\right\}\|w_\delta\|_{H^1_1(\cup\Omega_\ell)} \\
&\le c\left\{\|u - v_\delta\|_{H^1_1(\cup\Omega_\ell)} + \sum_{\ell=1}^{L} N_\ell^{-s_\ell}\|u_\ell\|_{H_1^{s_\ell+1}(\Omega_\ell)}\right\}\|w_\delta\|_{H^1_1(\cup\Omega_\ell)}.
\end{aligned}
\tag{3.25}
$$

Finally by combining (3.6), (3.7), (3.19) with (3.24) and (3.25), we deduce (3.23). □

Now are we going to give a more explicit estimates of the errors when the singularities of the solution are taken into account. We recall that, since all the angles of Ω in the corners $c_i \in \Gamma_0$ are equal to $\frac{\pi}{2}$, these corners do not make appear any singular function. The angles ω_{e_i} in the corners e_i are equal to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. In a neighborhood of such corner, the solution admits the expansion:

$$
u = u_{reg} + \sum_{n\ge 0}\gamma_{e_i}^{(n)} S_{e_i}^{(n)},
$$

(see [2] for an explicit definition and properties of this expansion).

Theorem 3.7. *For any function $f \in H_+^{s-1}(\Omega)$, $s > \frac{5}{2}$, the following error estimates hold:*

1. $\|u - u_\delta\|_{H^1_1(\cup\Omega_\ell)} \le c(1+\lambda_\delta)^{\frac{1}{2}}\sup\{N_\delta^{1-s}, E_\delta\}\,\|f\|_{H_1^{s-1}(\Omega)}\,;$ (3.26)
2. $\|u - u_\delta\|_{L^2_1(\Omega)} \le c(1+\lambda_\delta)^{\frac{1}{2}}\sup\{N_\delta^{1-s}, N_\delta^{-1}(\log N_\delta)^{\varrho}E_\delta\}\,\|f\|_{H_1^{s-1}(\Omega)}$ (3.27)

where $N_\delta = \min\{N_\ell, 1\le \ell \le L\}$, λ_δ is given in Proposition 3.2, $E_\delta = \max\{E_\ell, 1\le\ell\le L\}$,

$$
E_\ell = \begin{cases} 0 & \text{if } \bar\Omega_\ell \text{ does not contain any } e_i, \\ N_{e_i}^{-4}(\log N_{e_i})^{\frac{3}{2}} & \text{if } \bar\Omega_\ell \text{ contains } e_i \text{ with } \omega_{e_i} = \frac{\pi}{2}, \\ N_{e_i}^{-\frac{4}{3}}(\log N_{e_i})^{\frac{1}{2}} & \text{if } \bar\Omega_\ell \text{ contains } e_i \text{ with } \omega_{e_i} = \frac{3\pi}{2}, \end{cases}
\tag{3.28}
$$

N_{e_i} is the minimum of the N_ℓ for the Ω_ℓ such that e_i is a vertex of Ω_ℓ and ϱ is zero in conforming decomposition and 1 otherwise.

Proof.
1. Writing any $v_\delta \in X_\delta^\diamond$ in the form:

$$
v_\delta = w_\delta + \gamma_e^{(0)}\chi_e(r_e)z_\delta + \sum_{n\ge 1}\gamma_e^{(0)n}\chi_e(r_e)z_\delta^n,
\tag{3.29}
$$

where w_δ, z_δ and z_δ^ℓ are in $X_\delta^\diamond$, we obtain:

$$
\begin{aligned}
\inf_{v_\delta\in X_\delta^\diamond}\|u - v_\delta\|_{H^1_1(\cup\Omega_\ell)} \le c\Bigg(&\inf_{w_\delta\in X_\delta^\diamond}\|u_{reg} - w_\delta\|_{H^1_1(\cup\Omega_\ell)} + \inf_{z_\delta\in X_\delta^\diamond}\sum_\ell\left|\gamma_e^{(0)}\right|\left\|S_e^{(0)} - z_\delta\right\|_{H^1_1(\Omega_\ell)} \\
&+ \inf_{z_\delta\in X_\delta^\diamond}\sum_\ell\sum_{n=1}\left|\gamma_e^{(0)n}\right|\left\|S_e^{(0)} - z_\delta^n\right\|_{H^1_1(\Omega_\ell)}\Bigg).
\end{aligned}
\tag{3.30}
$$

We have from [2] the following estimates:

$$\left\| S_e^{(0)} - z_\delta \right\|_{H_1^1(\cup\Omega_\ell)} \leq N_e^{-4} (\log N_e)^{\frac{3}{2}} \text{ if } \omega_e = \frac{\pi}{2}, \tag{3.31}$$

and

$$\left\| S_e^{(0)} - z_\delta \right\|_{H_1^1(\cup\Omega_\ell)} \leq N_e^{-\frac{4}{3}} (\log N_e)^{\frac{1}{2}} \text{ if } \omega_e = \frac{3\pi}{2}, \tag{3.32}$$

$$\left| \gamma^{(0)n} \right| \leq c \, \|f\|_{H_1^{s-1}(\Omega)} \text{ for } s > 2. \tag{3.33}$$

Finally, combining the inequalities (3.30)–(3.33) and Proposition 3.6, we obtain (3.26).

2. Using the Aubin–Nitsche method of duality, we obtain that:

$$\|u - u_\delta\|_{L_1^2(\Omega)} = \sup_{g \in L_1^2(\Omega)} \frac{{}_\Omega (u - u_\delta)(r,z)\, g(r,z)\, r \mathrm{d}r \mathrm{d}z}{\|g\|_{L_1^2(\Omega)}}.$$

For any function g in $L_1^2(\Omega)$ and ℓ, $1 \leq \ell \leq L$, we set χ_ℓ the solution in $H_{1\diamond}^1(\Omega_\ell)$ of the variational formulation associated to the problem $-\Delta\chi_\ell = g_\ell$ in Ω_ℓ with $\chi_\ell = 0$ on $\partial\Omega_\ell$ if $\ell \geq L_0$, and $\chi_\ell = 0$ on Γ_ℓ if $\ell \leq L_0$. Since Ω_ℓ is convex, $\chi_\ell \in H_1^2(\Omega_\ell)$ and verifies $\|\chi_\ell\|_{H_1^2(\Omega_\ell)} \leq c \, \|g_\ell\|_{L_1^2(\Omega_\ell)}$. We set χ such that $\chi_{|\Omega_\ell} = \chi_\ell$ and notice that $\chi \in H_{1\diamond}^1(\Omega)$. We define $\chi_{\delta-1} \in H_{1\diamond}^1(\cup\Omega_\ell)$ by:

$$\chi_{\delta-1|\Omega_\ell} = \tilde{\Pi}_{N_\ell-1}^{+,1,\diamond}\chi_\ell \text{ if } 1 \leq \ell \leq L_0 \text{ and } \chi_{\delta-1|\Omega_\ell} = \tilde{\Pi}_{N_\ell-1}^{-,1,\diamond}\chi_\ell \text{ if } L_0 \leq \ell \leq L$$

where:

$$\tilde{\Pi}_{N_\ell-1}^{+,1,\diamond} : H_{1\diamond}^1(\Omega_\ell) \to \mathbb{P}_{N_\ell-1}^\diamond(\Omega_\ell) = \{v \in \mathbb{P}_{N_\ell-1}(\Omega_\ell), v = 0 \text{ on } \partial\Omega_\ell \backslash (Oz)\}$$

and

$$\tilde{\Pi}_{N_\ell-1}^{-,1,\diamond} : V_{1\diamond}^1(\Omega_\ell) \to \mathbb{P}_{N_\ell-1}^0(\Omega_\ell) = \{v \in \mathbb{P}_{N_\ell-1}(\Omega_\ell), v = 0 \text{ on } \partial\Omega_\ell\}$$

are the projection operators defined in [2], Chapitre V, and which verify:

$$\|\chi - \chi_{\delta-1}\|_{H_1^1(\cup\Omega_\ell)} \leq c N_\delta^{-1} \|\chi\|_{H_1^2(\cup\Omega_\ell)} \leq c' N_\delta^{-1} \|g\|_{L_1^2(\Omega)}. \tag{3.34}$$

Such construction leads to:

$$\begin{aligned} {}_\Omega (u - u_\delta)\, g r \mathrm{d}r \mathrm{d}z &= \sum_{\ell=1}^L \int_{\Omega_\ell} \nabla\chi_\ell \nabla(u - u_\delta) \mathrm{d}\tau - \sum_{\gamma_m^- \in \mathcal{S}} \int_{\gamma_m^-} \left(\frac{\partial\chi}{\partial n_m}\right) [u - u_\delta] \mathrm{d}\tau \\ &= a(\chi - \chi_{\delta-1}, u - u_\delta) + \int_\Omega f\chi_{\delta-1}\mathrm{d}\tau - (\mathcal{I}_\delta f, \chi_{\delta-1})_\delta - \sum_{\gamma_m^- \in \mathcal{S}} \int_{\gamma_m^-} \left(\frac{\partial\chi}{\partial n_m}\right) [u - u_\delta] \mathrm{d}\tau. \end{aligned}$$

By combining the continuity of a, the estimates (3.19), (3.24) and (3.34), we deduce (3.27). □

4. The discretization in the general case

For each $k \neq 0$, the variational formulation of problem (2.3) is written:

$$\begin{cases} \text{Find } u^k \in V_{1\diamond}^1(\Omega) \text{ such that} \\ \forall v \in V_{1\diamond}^1(\Omega), \; a_k(u^k, v) = \sum\limits_{\ell=1}^L {}_{\Omega_\ell} f^k \bar{v} \; r \mathrm{d}r \mathrm{d}z, \end{cases} \tag{4.1}$$

where:

$$a_k(u^k, v) = \sum_{\ell=1}^{L} \int_{\Omega_\ell} \left\{\nabla u^k . \nabla \bar{v} \ r\mathrm{d}r\mathrm{d}z + k^2 u^k \bar{v} \ r^{-1}\right\} \mathrm{d}r\mathrm{d}z \text{ and } V^1_{1\diamond}(\Omega) = \{v \in V^1_1(\Omega);\ v = 0 \text{ on } \Gamma\}. \tag{4.2}$$

The bilinear form $a_k(.,.)$ is continuous and coercive on $V^1_{1\diamond}(\Omega)$ endowed with the norm $\|.\|_{H^1_{(k)}(\cup\Omega_\ell)} = \big(\sum_{\ell=1}^{L} \|.\|^2_{H^1_{(k)}(\Omega_\ell)}\big)^{\frac{1}{2}}$. The problem (4.1) admits a unique solution u^k which verifies:

$$\left\|u^k\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \le c \left\|f^k\right\|_{L^2_1(\Omega)}.$$

The discrete problem associated to problem (4.1) is:

$$\begin{cases} \text{Find } u^k_\delta \text{ in } X^\circ_\delta(\Omega) \text{such that} \\ \forall v_\delta \in X^\circ_\delta(\Omega), \quad a_{k,\delta}\left(u^k_\delta, v_\delta\right) = \left(\mathcal{I}_\delta f^k, v_\delta\right)_\delta, \end{cases} \tag{4.3}$$

where the form $a_{k,\delta}(.,.)$ is defined by:

$$a_{k,\delta}(u_\delta, v_\delta) = a_\delta(u_\delta, v_\delta) + k^2 \left(\frac{u_\delta}{r}, \frac{v_\delta}{r}\right)_\delta.$$

It is readily checked that problem (4.3) is well posed and we have the following proposition.

Proposition 4.1. *Let u^k be the solution of problem (4.1). We assume that $u^k_{|\Omega_\ell} \in H^{s_\ell+1}_1(\Omega_\ell)$ with $s_\ell > \frac{1}{2}$ ($s_\ell > \frac{5}{2}$ if $\ell \le L_0$). Then there exists a constant c independent of k such that:*

$$\inf_{v_\delta \in X^\circ_\delta} \left\|u^k - v_\delta\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \le c\lambda_\delta^{\frac{1}{2}} \sum_{\ell=1}^{L} N_\ell^{-s_\ell} ||u^k||_{H^{s_\ell+1}_{(k)}(\Omega_\ell)}, \tag{4.4}$$

where λ_δ is defined in Proposition 3.2 for all mortar γ^+_μ, $1 \le \mu \le M^+$ and non-mortar γ^-_m, $1 \le m \le M^-$ such that $\gamma^+_\mu \cap \gamma^-_m$ has a nonnegative measure.

Proof. We set $v^1_\ell = \mathcal{I}^{(k)}_{N_\ell} u^k_{|\Omega_\ell}$ in Ω_ℓ, v^1_δ such that $v^1_{\delta|\Omega_\ell} = v^1_\ell$ and $v^2_\delta = \sum_{\mu=1}^{M^+} \sum_{e\in\mathcal{C}_\mu} (u^k - v^1_{\delta|\Omega^+_\mu})(e)\tilde{\varPhi}_{\mu,e}$ where $\tilde{\varPhi}_{\mu,e}$ is defined in the Proof of Proposition 3.2 and where the interpolate operator $\mathcal{I}^{(k)}_{N_\ell}$ is defined in [2]. We set:

$$\begin{aligned}
\tilde{\pi}^{(k),\gamma^-_m}_\delta &= \tilde{\pi}^{(k),1,(r)}_{\delta,m} if \gamma^-_m \text{ is parallel to } (Or) \text{ and } \tilde{\pi}^{(k),\gamma^-_m}_\delta = \tilde{\pi}^{(k),1,(z)}_{\delta,m} \text{ if } \gamma^-_m \text{ is parallel to} (Oz), \\
v^{12}_\delta &= v^1_\delta + v^2_\delta, \\
v^{3*}_\delta &= \tilde{\pi}^{(k),\gamma^-_m}_\delta (v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m})(\tau)\tilde{\chi}_{N^-_m}(\sigma) \text{ in } \bar{\Omega}^-_m, \quad v^{3*}_\delta = 0 \text{ in } \Omega\backslash\bar{\Omega}^-_m, \\
v^3_\delta &= \sum_{m=1}^{M^-} v^{3*}_\delta
\end{aligned}$$

where τ resp. σ is the tangential resp. normal variable on γ^-_m and where $\tilde{\chi}_{N^-_m}$ is obtained from $\chi_{N^-_m}$ by homothety and translation ($\chi_{N^-_m}(\sigma) = (\frac{1-\sigma}{2})^{N^-_m}$); $\tilde{\pi}^{(k),1,(r)}_{\delta,m}$ and $\tilde{\pi}^{(k),1,(z)}_{\delta,m}$ are defined in [2]. The function $v_\delta = v^1_\delta + v^2_\delta + v^3_\delta$ belongs to discrete space X°_δ and verifies the inequality (4.4) (see [16] for details). $\square$

Remark 4.2. We Notice, that in the case of a conforming decomposition, we obtain the same estimate but with k and N_δ chosen arbitrarily.

In the same way as for the axisymmetric case, we can prove the following error estimates.

Theorem 4.3. *For any function $f^k \in H_-^{s-1}(\Omega)$, with $s > \frac{5}{2}$ the following error estimates hold:*

1. $\|u^k - u_\delta^k\|_{H^1_{(k)}(\cup\Omega_\ell)} \le c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, E_\delta\} \|f^k\|_{H^{s-1}_{(k)}(\Omega)}$;
2. $\|u^k - u_\delta^k\|_{L^2_1(\cup\Omega_\ell)} \le c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, N_\delta^{-1}\log(N_\delta)^\varrho E_\delta\} \|f^k\|_{H_1^{s-1}(\Omega)}$

where ϱ and E_δ are defined in Theorem 3.7 and c is a constant independent of k.

5. Tridimensional problem, Fourier truncation

Of course, we solve only a finite number of problems (2.3). So, we chose an integer K and define an approximation of the solution of the three-dimensional problem (1.1) by:

$$\breve{u}_K(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\le K} u^k(r,z)\, \mathrm{e}^{\mathrm{i}k\theta}. \tag{5.1}$$

The Fourier coefficients of the data are generally not known accurately and are calculated by using a quadrature formula. Then, we define their interpolate by:

$$f_K^k(r,z) = \frac{\sqrt{2\pi}}{2K+1} \sum_{|m|\le K} \breve{f}(r,\theta_m,z)\mathrm{e}^{-ik\theta_m},\ \theta_m = \frac{2m\pi}{2K+1}\cdot$$

After that, we define the approximate $\breve{u}^*_{K,\delta}$ setting:

$$\breve{u}^*_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\le K} u^k_{K,\delta}(r,z)\, \mathrm{e}^{\mathrm{i}k\theta}, \tag{5.2}$$

where $u^0_{K,\delta}(r,z)$ is the solution of problem (3.3) for datum f_K^0 and $u^k_{K,\delta}(r,z)$, $k \neq 0$, is a solution of problem (4.3) for datum f_K^k. The error between the exact solution $\breve{u}$ and the solution $\breve{u}^*_{K,\delta}$ obtained by applying successively a Fourier truncation a numerical integration then a spectral approximation is given in the following theorem.

Theorem 5.1. *We assume that $\breve{f} \in H^{s-1}(\breve{\Omega})$, $s > \frac{5}{2}$. We obtain that:*

$$1.\ \|\breve{u} - \breve{u}^*_{K,\delta}\|_{H^1(\cup\breve{\Omega}_\ell)} \le c(1+\lambda_\delta)^{\frac{1}{2}} \{\sup(N_\delta^{1-s}, E_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})}; \tag{5.3}$$

$$2.\ \|\breve{u} - \breve{u}^*_{K,\delta}\|_{L^2(\breve{\Omega})} \le c(1+\lambda_\delta)^{\frac{1}{2}} \{\sup(N_\delta^{1-s}, N_\delta^{-1}\log(N_\delta)^\varrho E_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})}, \tag{5.4}$$

where ϱ and E_δ are defined in Theorem 3.7.

Proof. Error processing is similar to that appearing in [2] associated with a conforming decomposition. Special care on the analysis of two-dimensional non-conformities is necessary. The basic formulas are the two-dimensional error estimates of the preceding paragraph and the formula of truncation on the exact solution [2], (VII.1.3) and (II.1.8). ∎

6. Strang and fix algorithm

6.1. Axisymmetric case

We raise to study the case of a singularity due to a convex and a nonconvex corner. We denote by S_1 the first singular function appearing in the solution of problem (2.3) with $k=0$, and consider the Hilbert space $\mathring{X}_\delta = X_\delta^\diamond + \mathbb{R}S_1$. We set $\mathring{u}_\delta = u_\delta + \lambda S_1$ and $\mathring{v}_\delta = v_\delta + \mu S_1$. The space $\mathring{X}_\delta$ is endowed with the norm

$$\|\mathring{v}_\delta\|_\circ = \sum_{\ell=1}^{L}\left(\|v_\ell\|^2_{H^1_1(\Omega_\ell)} + |\mu|^2\,\|S_1\|^2_{H^1_1(\Omega_\ell)}\right)^{\frac{1}{2}}. \tag{6.1}$$

We define the discrete bilinear form on $\mathring{X}_\delta(\Omega)$ by:

$$\mathring{a}_\delta\left(\mathring{u}_\delta, \mathring{v}_\delta\right) = a_\delta\left(u_\delta, v_\delta\right) + \sum_{\ell=1}^{L}\left(\lambda\int_{\Omega_\ell}\nabla S_1\nabla v_\ell r\mathrm{d}r\mathrm{d}z + \mu\int_{\Omega_\ell}\nabla u_\ell\nabla S_1 r\mathrm{d}r\mathrm{d}z + \lambda\mu\int_{\Omega_\ell}\left(\nabla S_1\right)^2 r\mathrm{d}r\mathrm{d}z\right).$$

Taking into account the singularities, problem (3.3) becomes:

$$\begin{cases}\text{Find } \mathring{u}_\delta \in \mathring{X}_\delta(\Omega) \text{such that}\\ \forall \mathring{u}_\delta \in \mathring{X}_\delta(\Omega)\ \mathring{a}_\delta\left(\mathring{u}_\delta, \mathring{v}_\delta\right) = \left(\mathcal{I}_\delta f, \mathring{v}_\delta\right)_\delta.\end{cases} \tag{6.2}$$

For any $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$, problem (6.2) has unique solution $\mathring{u}_\delta$ in $\mathring{X}_\delta$ verifying:

$$\|\mathring{u}_\delta\|_\circ \le c\,\|\mathcal{I}_\delta f\|_{L^2_1(\Omega)}.$$

Theorem 6.1. *We assume that $f \in H_+^{s-1}(\Omega)$, $s > \frac{5}{2}$, then the following error estimate holds between the solution u of problem (2.4), with $k=0$, and the solution $\mathring{u}_\delta$ of problem (6.2):*

$$\|u - \mathring{u}_\delta\|_\circ \le c(1+\lambda_\delta)^{\frac{1}{2}}\sup\{N_\delta^{1-s}, \mathring{E}_\delta\}\,\|f\|_{H_1^{s-1}(\Omega)}, \tag{6.3}$$

where $N_\delta = \min\{N_\ell, 1 \le \ell \le L\}$, $\mathring{E}_\delta = \max\left\{\mathring{E}_\ell, 1 \le \ell \le L\right\}$, N_{e_i} is defined in Theorem 3.7 and

$$\mathring{E}_\ell = \begin{cases} 0 & \text{if } \bar{\Omega}_\ell \text{ does not contain any } e_i,\\ N_{e_i}^{-8}(\log N_{e_i})^{\frac{3}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_j = \frac{\pi}{2},\\ N_{e_i}^{\frac{-8}{3}}(\log N_{e_i})^{\frac{1}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_j = \frac{3\pi}{2}.\end{cases} \tag{6.4}$$

Proof. We notice that $\mathring{a}_\delta\left(\mathring{v}_\delta, \mathring{w}_\delta\right) - \mathring{a}(\mathring{v}_\delta, \mathring{w}_\delta) = a_\delta\left(v_\delta, w_\delta\right) - a(v_\delta, w_\delta)$ and set $u = u_{reg} + \lambda S_1 + \mu S_2$ and $\mathring{v}_\delta = z_\delta + \lambda S_1 + \mu w_\delta$. This leads to

$$\inf_{\mathring{w}_\delta \in \mathring{X}_\delta}\|u - \mathring{v}_\delta\|_\circ \le \inf_{z_\delta \in X_\delta^\diamond}\|u_{reg} - z_\delta\|_\circ + \inf_{w_\delta \in X_\delta^\diamond}|\mu|\,\|S_2 - w_\delta\|_\circ + \ldots$$

The first term on the right side is estimated in Proposition 3.2. In order to estimate the second term, we use the definition $S_e^{(0)} = \chi_e(r_e^\lambda)r_e^\lambda(\log r_e)^q\varphi(\theta_e)$ and we obtain:

$$\inf_{w_\delta \in X_\delta^\diamond}\|S_2 - w_\delta\|_\circ \le N_\ell^{-8}(\log N_\ell)^{\frac{3}{2}} \tag{6.5}$$

for $\lambda = \frac{2\pi}{\omega_{e_j}}$, $\omega_{e_j} = \frac{\pi}{2}$ and $q=0$. The case with $\omega_{e_j} = \frac{3\pi}{2}$ and $q=0$ yields to:

$$\inf_{w_\delta \in X_\delta^\diamond}\|S_2 - w_\delta\|_\circ \le N_\ell^{-\frac{8}{3}}(\log N_\ell)^{\frac{1}{2}}. \tag{6.6}$$

We conclude using the fact that $\sup(|\lambda|, |\mu|) \le c\,\|f\|_{H_1^{s-1}(\Omega)}$ for $s > 2$ (see [16] for more details). $\square$

6.2. General case

As in the axisymmetric case, we define the space $\ddot{X}_\delta = X_\delta^\circ + \mathbb{R}S_1$ and set for u_δ^k and $v_\delta^k \in \ddot{X}_\delta$:

$$\mathring{u}_\delta^k = u_\delta^k + \lambda S_1 \text{ and } \mathring{v}_\delta = v_\delta + \mu S_1.$$

We notice that the first singularity is independent of k and that the singularity S_1 is the same of the axisymmetric case. We define the discrete bilinear form on $\ddot{X}_\delta$ by:

$$\begin{aligned}\mathring{a}_{k,\delta}\left(\mathring{u}_\delta^k, \mathring{v}_\delta\right) = a_{k,\delta}\left(u_\delta^k, v_\delta\right) + \sum_{\ell=1}^{L}(\lambda \int_{\Omega_\ell} \nabla S_1 \nabla v_\ell r\mathrm{d}r\mathrm{d}z + \mu \int_{\Omega_\ell} \nabla u_\ell^k \nabla S_1 r\mathrm{d}r\mathrm{d}z \\ + \lambda\mu \int_{\Omega_\ell} (\nabla S_1^2) r\mathrm{d}r\mathrm{d}z + \lambda k^2 \int_{\Omega_\ell} (S_1 v_\ell) r^{-1}\mathrm{d}r\mathrm{d}z \\ + \mu k^2 \int_{\Omega_\ell} (S_1 u_\ell^k) r^{-1}\mathrm{d}r\mathrm{d}z + \lambda\mu k^2 \int_{\Omega_\ell} (S_1^2) r^{-1}\mathrm{d}r\mathrm{d}z)\end{aligned} \tag{6.7}$$

and endow $\ddot{X}_\delta$ with the norm: $\|\mathring{v}_\delta\|_{\circ k} = \sum_{\ell=1}^{L}(\left\|v_{\delta|\Omega_\ell}\right\|^2_{H^1_{(k)}(\Omega_\ell)} + |\lambda|^2 \left\|S_{1|\Omega_\ell}\right\|^2_{H^1_{(k)}(\Omega_\ell)})^{\frac{1}{2}}$.

The discrete problem writes:

$$\begin{cases}\text{Find } \mathring{u}_\delta \in \ddot{X}_\delta \text{ such that} \\ \forall \mathring{v}_\delta \in \ddot{X}_\delta,\ \mathring{a}_{k,\delta}\left(\mathring{u}_\delta^k, \mathring{v}_\delta\right) = \left(\mathcal{I}_\delta f^k, \mathring{v}_\delta\right)_\delta\end{cases} \tag{6.8}$$

and has a unique solution $\mathring{u}_\delta^k$ in $\ddot{X}_\delta$ verifying: $\left\|\mathring{u}_\delta^k\right\|_{\circ k} \le C \left\|\mathcal{I}_\delta f^k\right\|_{L^2_1(\Omega)}$ where C is independent of k. Following the steps of Theorem 6.1, we have the following estimates.

Theorem 6.2. *Let u^k be the solution of problem (4.1) and $\mathring{u}_\delta^k$ the solution of problem (6.8). We assume that $f^k \in H_-^{s-1}(\Omega)$ with $s > \frac{5}{2}$. Then we have*

$$\left\|u^k - \mathring{u}_\delta^k\right\|_{\circ k} \le C(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, \mathring{E}_\delta\} \left\|f^k\right\|_{H_1^{s-1}(\Omega)}, \tag{6.9}$$

where N_δ and $\mathring{E}_\delta = \max\left\{\mathring{E}_\ell, 1 \le \ell \le L\right\}$ are defined in Theorem 6.1.

6.3. Return to the tridimensional problem

We set, for an integer K:

$$\mathring{u}_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\le K} \mathring{u}_\delta^k(r,z)\, \mathrm{e}^{\mathrm{i}k\theta},$$

where $\mathring{u}_\delta^0(r,z)$ is the solution of problem (6.2) for datum f^0 and $\mathring{u}_\delta^k(r,z)$, $(k \ne 0)$ is the solution of problem (6.8) for datum f^k. And we set:

$$\mathring{u}_{K,\delta}^*(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\le K} \mathring{u}_{K,\delta}^k(r,z)\, \mathrm{e}^{\mathrm{i}k\theta}$$

where $\mathring{u}_{K,\delta}^0(r,z)$ is the solution of problem (6.2) for datum f_K^0 and $\mathring{u}_{K,\delta}^k(r,z)$, $(k \ne 0)$ is the solution of problem (6.8) for datum f_K^k. We define also $\mathring{u}_{K,\delta}^*$ by:

$$\mathring{u}_{K,\delta}^*(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\le K} \mathring{u}_{K,\delta}^k(r,z)\, \mathrm{e}^{\mathrm{i}k\theta}. \tag{6.10}$$

Then, following the steps of Theorem 5.1 and using (6.3) and (6.9), we have when $\breve{f} \in H^{s-1}(\breve{\Omega})$, $s > \frac{5}{2}$

$$\left\|\breve{u} - \mathring{u}_{K,\delta}^*\right\|_{H^1(\cup\breve{\Omega}_\ell)} \le c(1+\lambda_\delta)^{\frac{1}{2}}\{\sup(N_\delta^{1-s}, \mathring{E}_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})} \tag{6.11}$$

where $\mathring{E}_\delta$ is defined in Theorem 6.1.

7. Numerical results

We present hereafter, numerical tests which would confirm our theoretical predictions in the axisymmetric and general cases. These tests are made on three types of domains: the convex Ω^a or the non-convex ones Ω^b and Ω^c (see Fig. 1). Each domain is broken up into convex subdomains. In each subdomain, we solve the final linear problem, resulting from the spectral discretization by using the iterative conjugate gradient method with diagonal preconditioning. This linear problem has the form

$$(Q^T A_k Q)u^k = Q^T F_k$$

where A_k has the form

$$A_k = \begin{pmatrix} A_{k,1} & 0 & \cdots & 0 & D_{k,1} \\ 0 & A_{k,2} & \cdots & 0 & D_{k,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{k,L} & D_{k,L} \\ C_{k,1} & C_{k,2} & \cdots & C_{k,L} & M_k \end{pmatrix}. \tag{7.1}$$

The matrix $A_{k,\ell}$ (which is symmetric and positive definite) acts on the internal nodes for each sub-domain Ω_ℓ, whereas $C_{k,\ell}$, $D_{k,\ell}$ and M_k represent the matrices which act on the skeleton $\mathcal{S}$ of domain Ω. Finally F_k is the matrix associated to the data and the matrix Q translates the conditions through the interfaces of subdomains. For more details one can consult [2].

In the aim to enrich our tests, we take in certain cases $\breve{g} \neq 0$. Our previous theoretical results remain valid if $\breve{g}$ is sufficiently regular. The functions u presented below are the solutions of the two-dimensional problems and the curves of errors represent the errors on the tridimensional problem with Fourier truncation.

All calculations are done on a personal computer using the MATLAB software.

7.1. Axisymmetric case

We consider here the problem (3.3) eventually with $g \neq 0$.

Nonconvex domain Ω^b: we consider the domain of Figure 1B broken up into 5 subdomains.

In a first test, we consider the functions

$$f = r^{1/2}\left(z^2 + \frac{8}{25}r^2 - 1\right) \text{ in } \Omega^b,\ g = 0 \text{ if } z = \pm 1 \text{ and } g = \frac{4}{25}r^{5/2}(1-z^2)|_\Gamma \text{ if not.}$$

We present, in Figure 2A, the layout of u_δ in parts Ω_1^b, Ω_3^b, Ω_4^b, Ω_5^b with $N = 24$ and in Ω_2^b with $N = 28$. The zoom of the encircled part of Figure 2A is presented in Figure 2B.

We make a second test with:

$$f = r^{1/2} z \text{ in } \Omega^b,\ g = 0 \text{ on } \Gamma.$$

We present in Figure 3. the layout of u_δ in parts Ω_1^b with $N = 28$, Ω_2^b, Ω_3^b, Ω_4^b with $N = 30$ and Ω_5^b with $N = 24$.

We notice that as long as N_i are chosen close to each other in the different subdomains, λ_δ defined in Proposition 3.2, is small and the layouts in different parts stick perfectly. Figure 3 shows this continuity through the interfaces.

The error measure in the domain $\breve{\Omega}^a$*:* We consider the singular bidimensional function:

$$u = r^{10/3}(z-1).$$

In Figure 4, we give the curves $\log_{10} \|\breve{u} - \breve{u}_\delta\|_{L^2(\breve{\Omega})}$ and $\log_{10} \|\breve{u} - \breve{u}_\delta\|_{H^1(\cup \breve{\Omega}_\ell)}$ as functions of $\log_{10}(N)$. We remark that the slopes of the errors are independent of N. This is in agreement with the estimates (3.26) and (3.27).

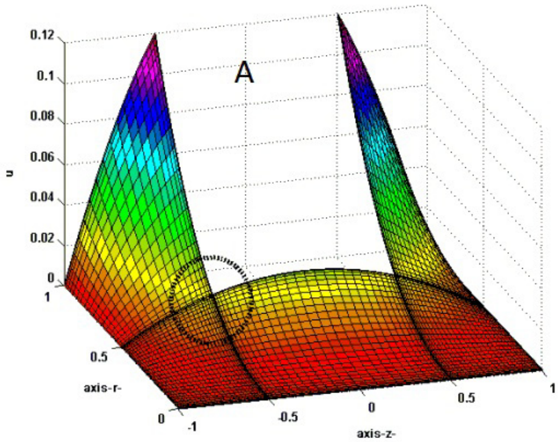

FIGURE 2. The layout of u_δ with axisymmetric data.

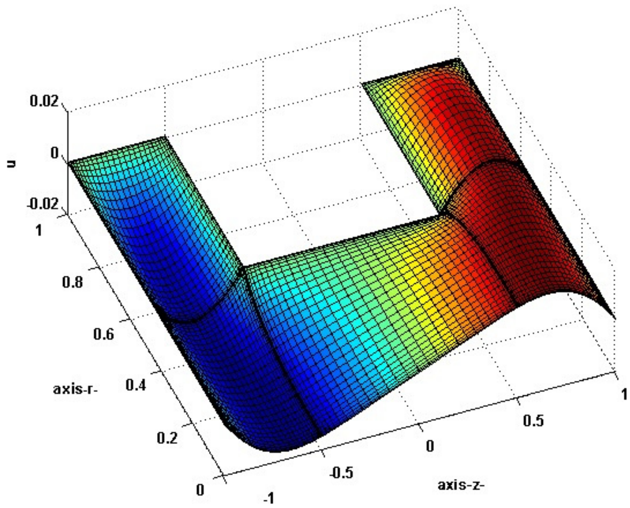

FIGURE 3. The layout of u_δ in the domain Ω^b.

7.2. General case

We consider here the problem (4.1) eventually with $g \neq 0$.

Case of the domain $\breve{\Omega}^a$: We consider the domain $\breve{\Omega}^a$ of Figure 1A. Let

$$\breve{f}(r,\theta,z) = r^{7/2} z \sin(r\cos\theta + r\sin\theta) \text{ in } \breve{\Omega}^a \text{ and } \breve{g} = 0 \text{ on } \partial\breve{\Omega}^a.$$

In Figure 5B, we represent the isovalues of $u^0_{K,\delta}$ with $N = 24$ in each subdomain and $K = 6$. In Figure 5A, we present the layout of $\mathrm{Re}(u^1_{K,\delta})$ with $K = 6$ and $N = 24$ in the part Ω^a_1 and $N = 28$ in the parts Ω^a_2, Ω^a_3.

We remark that the solution $u^{(k)}_{K,\delta}$, $k \neq 0$ becomes complex.

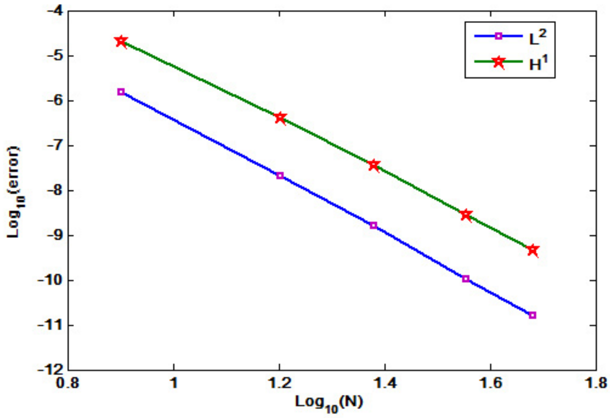

FIGURE 4. The error estimates for $u = r^{10/3}(z-1)$.

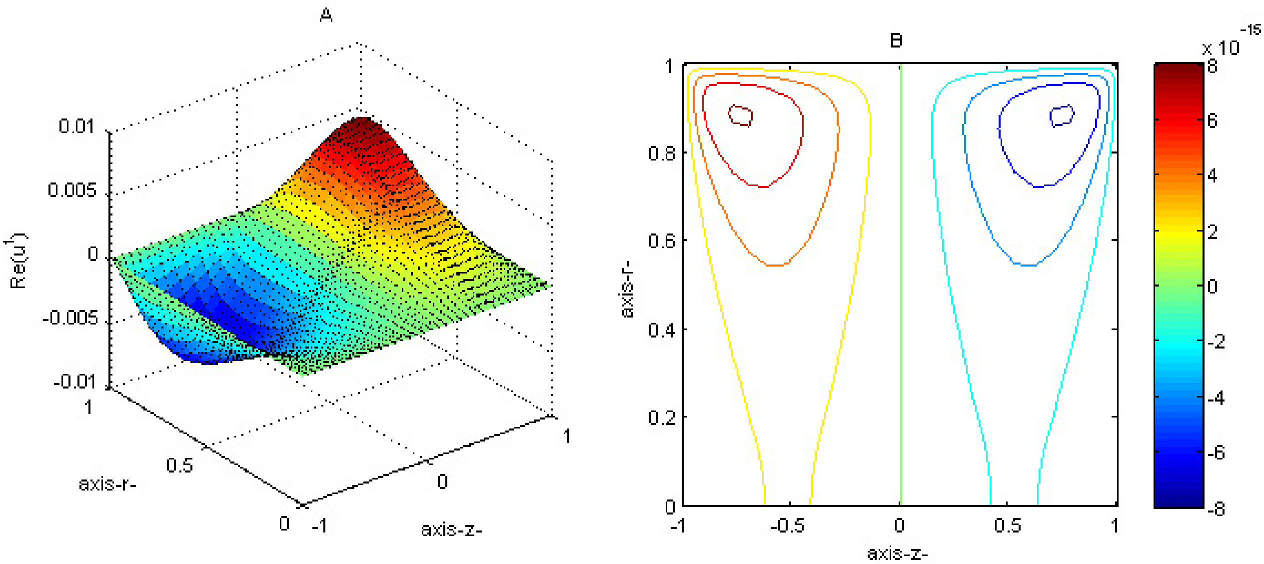

FIGURE 5. The isovalues of $u^0_{K,\delta}$ and the layout of $\mathrm{Re}(u^1_{K,\delta})$ in the domain Ω^a.

Case of the domain $\breve{\Omega}^b$: We consider the functions

$$\breve{f} = \cos(r\cos\theta + r\sin\theta + z) \text{ in } \breve{\Omega}^b \text{ and } \breve{g} = 0 \text{ on } \partial\breve{\Omega}^b. \tag{7.2}$$

We represent in Figure 6A the isovalues of $u^0_{K,\delta}$ with $N = 24$. In Figure 6B, we present the layout of $\mathrm{Re}(u^1_{K,\delta})$, with $K = 4$ everywhere and $N = 24$ in Ω^b_1, $N = 30$ in Ω^b_2, Ω^b_3, Ω^b_4 and $N = 20$ in Ω^b_5.

Case of the domain $\breve{\Omega}^c$: We present, in Table 1, the convergence rates obtained for different values of N by considering the solution:

$$\breve{u}(r,\theta,z) = -\cos(r\cos\theta + r\sin\theta + z) \text{ in } \breve{\Omega}^c$$

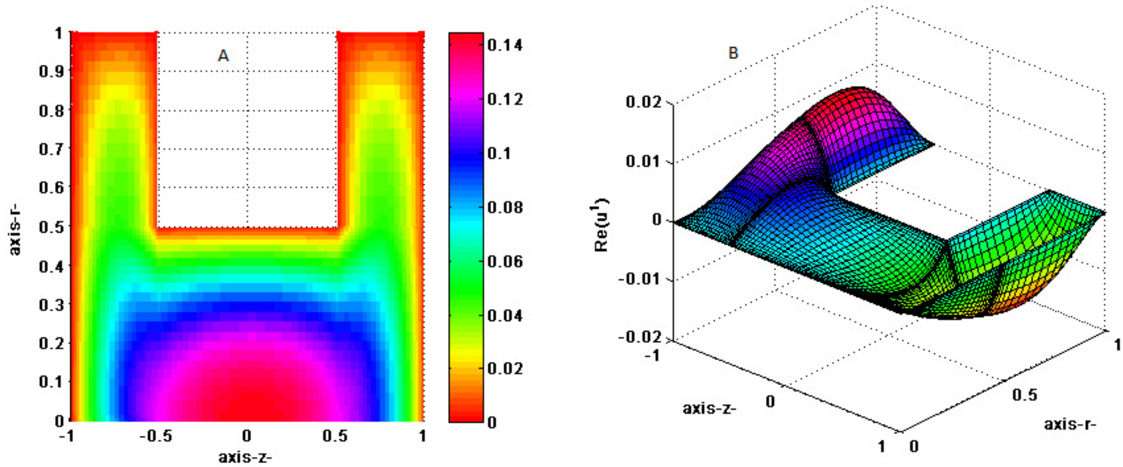

FIGURE 6. The isovalues of $u^0_{K,\delta}$ and the layout of $Re(u^1_{K,\delta})$ in the domain Ω^b.

TABLE 1.

N	$k=0$	$k=4$	Spectral-Fourier
16	1.3712	1.3812	1.3701
24	1.3706	1.3798	1.3643
32	1.3687	1.3611	1.3581
48	1.3533	1.3567	1.3402

TABLE 2.

N	5	10	15	20
CPU/s	4.0121	66.1430	339.90	1136.8
Nb of iterations	20	36	70	140

firstly with $k = 0$, secondly with $k = 4$ and last by Spectral-Fourier computations with a cut-off frequency $K = 6$.

We notice that this rate converges to $4/3$. This confirms the value of E_δ in Theorem 4.3.

We present in Table 2, the computation time and the number of iterations of the conjugate gradient algorithm with the data in (7.2). Here, N represents the degree of polynomials in all the subdomains of Ω^b.

7.3. Strang and Fix algorithm

We treat only the axisymmetric case. Let R be the matrix that comes from the terms $\chi_\ell \int_{\Omega_\ell} \nabla S_1 \nabla l^{(2)}_{i'} l_{j'}\, \mathrm{d}\tau$ with $\chi_\ell = 1$ if the singularity is contained in Ω_ℓ and 0 if not. Let M be the matrix that comes from the terms $\chi_\ell \int_{\Omega_\ell} \nabla S_1 \nabla l^{(2)}_i l_j\, \mathrm{d}\tau$ and let J be the matrix coming from the terms $\int_{\Omega_\ell} (\nabla S_1)^2\, \mathrm{d}\tau$ (see [9], Chap. III). The linear system which we solve has the form $(Q^T \mathring{A} Q)\mathring{u}^k = Q^T \mathring{F}$, where $\mathring{A} = \begin{pmatrix} A & R \\ M & J \end{pmatrix}$.

We consider the bidimensional solution $u = r^{2,1} z$ and we use the algorithm of Strang and Fix on the domain $\breve{\Omega}^b$. We present in Figures 7A and 7B the error curves, $\log_{10} \|\breve{u} - \breve{u}_\delta\|_{L^2(\breve{\Omega}^b)}$ and $\log_{10} \|\breve{u} - \breve{u}_\delta\|_{H^1(\cup \breve{\Omega}^b_\ell)}$ as functions of $\log_{10}(N)$, firstly without the Strang and Fix algorithm in Figure 7A and secondly using this

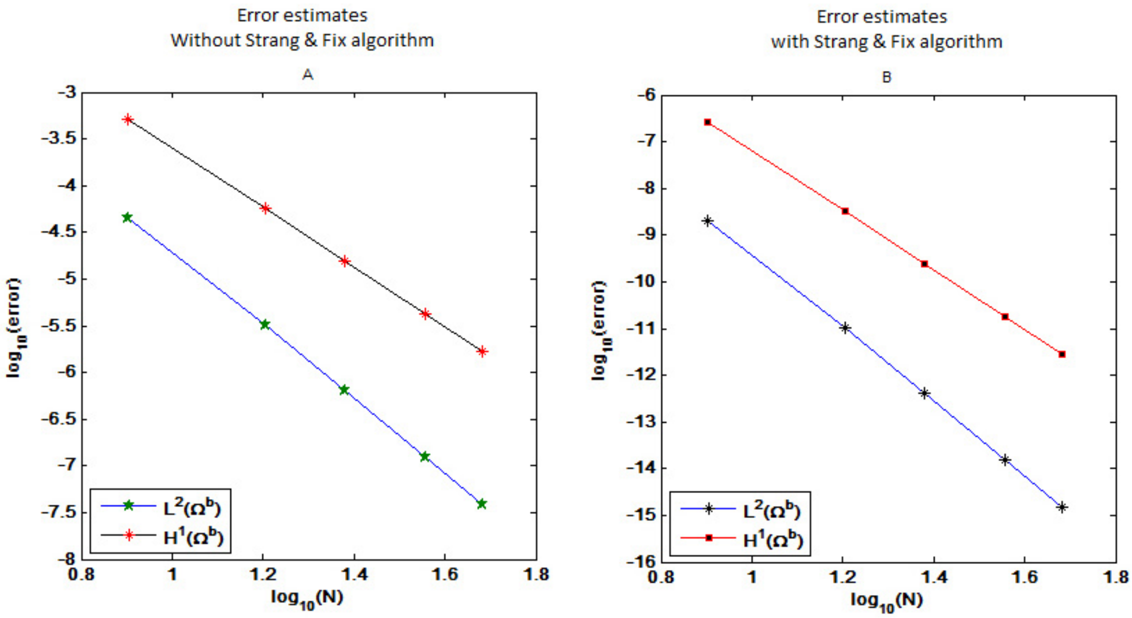

FIGURE 7. The error estimates show the contribution of the Strang and Fix algorithm.

algorithm in Figure 7B. We notice a clear improvement of the errors and remark that these results confirm the values of E_ℓ and $\mathring{E}_\ell$ in (3.28) and (6.4) in the case $\omega = 3\pi/2$.

8. Conclusion

In this work, we have shown that the mortar element method in the frame of axisymmetric geometries is very technical but very effective from a numerical point of view. The used techniques can be generalized to axisymmetric geometries complex and realistic where the issue of memory space and computing time is crucial.

References

[1] A.B. Abdallah, F.B. Belgacem, Y. Maday and F. Rapetti, Mortaring the two-dimensional edge finite elements for the discretization of some electromagnetic models. *Math. Mod. Methods Appl. Sci.* **14** (2004) 1635–1656.

[2] M. Azaïez, C. Bernardi, M. Dauge and Y. Maday, Spectral Methods for Axisymetric Domains. Series in *Appl. Math.* **3** (1999).

[3] F.B. Belgacem, C. Bernardi and F. Rapetti, Numerical analysis of a model for an axisymmetric guide for electromagnetic waves. Part I: The continuous problem and its Fourier expansion. *Math. Meth. Appl. Sci.* **28** (2005) 2007–2029.

[4] C. Bernardi and Y. Maday, Properties of some weighted Sobolev spaces and application to spectral approximations. *SIAM J. Numer. Anal.* **26** (1989) 769–829.

[5] C. Bernardi and Y. Maday, Approximations spectrales de problèmes aux limites elliptiques. *Math. Appl.* **10** (1992).

[6] C. Bernardi, M. Dauge and M. Azaïez, *Numerical Analysis and Spectral Methods in Axisymetric Problems.* Rapport Interne, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie (1995).

[7] S. Bertoluzza, S. Falletta and V. Perrier, The Mortar method in the wavelet context. *Model. Math. Anal. Numer.* **35** (2001) 647–673.

[8] H. Brezis, Analyse fonctionnelle, in *Théorie et Applications.* Masson, Paris (1983).

[9] N. Chorfi, *Traitement de singularités géométriques par méthode d'éléments spectraux avec joints.* Thèse de l'Université Pierre et Marie Curie, Paris VI (1998).

[10] M. Dauge, Elliptic Boundary Value Problems on Corner Domains. *Lect. Notes Math.* **1341** (1988).

[11] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations, in *Theory and Algorithms.* Springer-Verlag (1986).

[12] P. Grisvard, Singularities in boundary value problems, in *Collect. RMA* **22** (1992).

[13] P. Le Tallec, Domain decomposition methods in computational mechanics, in *Comput. Mech. Adv.* North-Holland (1994).

[14] R. Pasquetti, L.F. Pavarino, F. Rapetti and E. Zampieri, Overlapping Schwarz methods for Fekete and Gauss–Lobatto spectral elements. *SIAM J. Scient. Comput.* **29** (2007) 1073–1092.

[15] Y. Maday, C. Mavriplis and A.T. Patera, Nonconforming mortar element methods: application to spectral discretizations, in *Domain decomposition methods*. SIAM (1989) 392–418.

[16] J. Satouri, *Méthodes d'éléments spectraux avec joints pour des géométries axisymétriques*. Thèse de l'Université Pierre et Marie Curie, Paris VI (2010).

[17] G. Strang and G.J. Fix, An Analysis of the Finite Element Method, in *Automatic Computation*. Prentice Hall Serie (1973).