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CONVERGENT FINITE ELEMENT DISCRETIZATIONS OF THE NAVIER-STOKES-NERNST-PLANCK-POISSON SYSTEM

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Abstract. We propose and analyse two convergent fully discrete schemes to solve the incompressible Navier-Stokes-Nernst-Planck-Poisson system. The first scheme converges to weak solutions satisfying an energy and an entropy dissipation law. The second scheme uses Chorin's projection method to obtain an efficient approximation that converges to strong solutions at optimal rates.

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1. Introduction

We consider the following electrohydrodynamic model from [11,17,22]:

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain for $N \leq 3$ and $T > 0$. Find a velocity field $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^N$ and a corresponding pressure function $p : \Omega \times (0,T) \to \mathbb{R}$, concentrations of positive and negative charges $n^{\pm} : \Omega \times (0,T) \to \mathbb{R}_{\geq 0}$, and a quasi-electrostatic potential $\psi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{f}_C \qquad \text{in } \Omega_T := \Omega \times (0,T) \tag{1.1}$$
$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega_T \tag{1.2}$$
$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega_T := \partial\Omega \times (0,T) \tag{1.3}$$

$$\partial_t n^+ + \operatorname{div}\big(J_{n^+}\big) = 0 \qquad \text{in } \Omega_T \tag{1.4}$$
$$\langle J_{n^+}, \mathbf{n}\rangle = 0 \qquad \text{on } \partial\Omega_T \tag{1.5}$$
$$\partial_t n^- + \operatorname{div}\big(J_{n^-}\big) = 0 \qquad \text{in } \Omega_T \tag{1.6}$$
$$\langle J_{n^-}, \mathbf{n}\rangle = 0 \qquad \text{on } \partial\Omega_T \tag{1.7}$$

$$-\Delta\psi = n^+ - n^- \qquad \text{in } \Omega_T \tag{1.8}$$
$$\langle \nabla\psi, \mathbf{n}\rangle = 0 \qquad \text{on } \partial\Omega_T \tag{1.9}$$

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for

$$\mathbf{u}(\cdot,0)=\mathbf{u}_0, \qquad n^{\pm}(\cdot,0)=n_0^{\pm},$$
$$J_{n^{\pm}} := \mp n^{\pm}\nabla\psi - \nabla n^{\pm} + \mathbf{u}n^{\pm}, \quad \text{and} \quad \mathbf{f}_C := -\left(n^+ - n^-\right)\nabla\psi. \tag{1.10}$$

Well-posedness of this model has been shown in [22]: weak solutions to the system (1.1)–(1.10) are constructed by Schauder's fixed point theorem; the concentrations $n^{\pm}$ are non-negative and bounded in $L^{\infty}(\Omega_T)$ which follows from Moser's iteration technique; in addition, weak solutions satisfy an energy and an entropy law obtained by the use of special test functions. The local existence of strong solutions is also verified in [22].

The goal of this work is to recover these characteristic properties of weak solutions in a fully discrete setting by using finite elements. A first step in this direction is [20], where boundedness, non-negativity, an energy, and an entropy law of solutions for the Nernst-Planck-Poisson sub-system (1.4)–(1.10) (for $\mathbf{u}\equiv\mathbf{0}$) are transferred from the continuous setting to a spatio-temporal finite element based discretization. Here, we consider the whole system (1.1)–(1.10), which requires to properly account for the interaction of the given sub-system with the fluid flow part in the discretization scheme. As will turn out, a key issue for stability and convergence of the scheme is to establish lower and upper pointwise control for involved discrete charges.

Electrokinetic flows have many applications: An important class of microfluidic and nowadays especially nanofluidic systems aims to perform basic chemical analyses and other processing steps on a fluidic chip. Fluid motion in such chemical (bio)chip systems is often achieved by using electroosmotic flow which enjoys several advantages over pressure driven flows. Briefly, the electroosmotic flow produces a nearly uniform velocity profile which results in reduced sample species dispersion as compared to the velocity gradients associated with pressure-driven flows. This characteristic property enables such applications as fluid pumping, non-mechanical valves, mixing and molecular separation. Many of these systems also employ electrophoresis; this is another electrokinetic phenomena describing the Coulomb force driven motion of suspended molecular species in the solution. Since on a chip electroosmotic and electrophoretic systems grow in complexity, the need of a detailed understanding and computational validation by experimental comparison for such flow models becomes more and more critical. Therefore related reliable numerical schemes are of great importance for design optimization.

For an overview of the applied models describing the electrokinetic flows, it is often customary to distinguish between electroosmosis (no external driving force) and electrophoresis (arising by an external force). For a complete description of these two terms we refer to [11,17]. However, we briefly introduce principle ideas in the following.

We first consider the purely electroosmotic description. When an electrolyte is brought into contact with a solid surface, a spontaneous electrochemical reaction typically occurs between the two types of media resulting in a redistribution of charges. In the cases of interest, an electric double layer (EDL) is formed that consists of a charged solid surface and a region near the surface that supports a net excess of counter-ions. By the assumption that the concentration profile in the ionic region of the EDL can be described by the Boltzmann distribution, one obtains the Poisson-Boltzmann equation for the net charge density

$$-\Delta\psi = \frac{-F}{\epsilon}\sum_{i=1}^{L} z_i c_{\infty,i}\exp\Big(-\frac{z_i e\psi}{kT}\Big) =: -\frac{F}{\epsilon}\rho \tag{1.11}$$

which is usually considered only for one particle, *i.e.*, $L=1$. A second approximation is to consider only a symmetric electrolyte such that the right hand side in (1.11) reduces to a sinh function. In a third approximation, called the Debye-Hückel limit, one obtains the linear form of the Poisson-Boltzmann equation in case the term $\frac{ze\psi}{kT}$ is small enough to replace the sinh by its argument. Finally, the velocity field $\mathbf{u}$ is described in an arbitrarily shaped micro- or nano-channel for an incompressible liquid *via* the linear Stokes equation for the Coulomb driving force $\mathbf{f}_C$ on the right hand side. This linear description allows now to consider separately the velocity components due to the electric field $\mathbf{u}_\psi$ and the pressure gradient $\mathbf{u}_p$, where $\mathbf{u}=\mathbf{u}_\psi+\mathbf{u}_p$ solves

TABLE 1. Scheme A: (I) = $\{M$-matrix (strongly acute mesh, $h > 0$ small enough); existence *via* Banach's fixed point theorem $\big(k \le Ch^{\frac{N}{3}+\beta}, \beta > 0\big)\}$. Scheme B bases on Chorin's projection scheme.

Algorithm	**Convergence**	**Scheme**	**Convergence**	**System**
A_1	$\xrightarrow{(I),\,\theta\to 0}$	A	$\xrightarrow{h,\,k\to 0}$	weak solutions of (1.1)–(1.10)
		B	$\xrightarrow{h,\,k\to 0}$	strong solutions of (1.1)–(1.10)

the linear Stokes equation

$$
\begin{aligned}
-\Delta \mathbf{u} = \mathbf{f}_C - \nabla p \qquad &\text{in } \Omega\\
\operatorname{div} \mathbf{u} = 0 \qquad &\text{in } \Omega
\end{aligned}
\tag{1.12}
$$

for $\mathbf{f}_C = \rho\nabla\psi$ analogously to the electrohydrodynamic model (1.1)–(1.10) and hence $\rho := n^+ - n^-$. In the computational part (Sect. 6) we investigate in Examples 1 and 2 the purely electroosmotic behavior especially in view of the energy and entropy property (Sect. 3) proposed in this article.

These electrophoretic phenomena are induced by applying an electric field, and result in the motion of colloidal particles or molecules suspended in ionic solutions. The application of the Stokes law $\mathbf{f}_v = 3\pi\mu d\mathbf{u}$ allows to balance the electrostatic force $q\nabla\psi$ and the viscous drag $\mathbf{f}_v$ associated with its resulting motion. As a result we have

$$
\mathbf{u} = \frac{q\nabla\psi}{3\pi\mu d} \tag{1.13}
$$

where q is the total charge on the molecule, $\nabla\Psi$ is the applied field, and d is the diameter of the Stokes sphere in a continuum flow. Hence, we consider the species in the liquid to be of sphere shape as in [22]. The above considerations are made for a stationary liquid. The idea is that the electric double layer acts like a capacitor, and suggests that the dynamics can be described in terms of equivalent circuits, where the double layer remains in quasi equilibrium with the neutral bulk is discussed and validated in the thin double layer limit by asymptotic analysis of the Nernst-Planck-Poisson equations [3]. Moreover, in [12], the Nernst-Planck-Poisson equations are recently modified to account for the effect of steric constraints on the dynamics. A more detailed description of the physical derivation and motivations concerning the electrohydrodynamic model (1.1)–(1.10) is given in [21].

In this paper we investigate the incompressible Navier-Stokes-Nernst-Planck-Poisson system (1.1)–(1.10) which is a more general description of electrokinetic flows if compared to the above reduced models for electroosmosis and electrophoresis, see [11,17]. First, we introduce a fully implicit Scheme A which allows for non-negativity and a discrete maximum principle for the concentrations, and further validates a discrete energy and entropy law for solutions. All results for Scheme A are obtained *via* an implementable Algorithm A_1 which is proven to converge to Scheme A for $\theta \to 0$, where $\theta > 0$ defines the threshold parameter of the stopping criterion in the fixed point iteration, see Table 1. Hence, we verify existence of iterates for Scheme A by validating a contraction principle for Algorithm A_1, which requires $k \le Ch^{\frac{N}{3}+\beta}$ for any $\beta > 0$. Further, non-negativity and boundedness of iterates of the discrete Nernst-Planck equations in Algorithm A_1 are obtained *via* the M-matrix property, provided a compatibility constraint (see (2.6) below) for admissible finite element spaces is met, and used meshes are strongly acute. This latter compatibility requirement accounts for the coupling of the Nernst-Planck system with the incompressible Navier-Stokes system. Then, iterates of Scheme A converge towards weak solutions of the system (1.1)–(1.10) for $h, k \to 0$. Moreover, we verify a discrete energy law for solutions of Scheme A, and in two dimensions a discrete entropy dissipation property. The latter discrete (perturbed) entropy estimate is verified in two dimensions for the coupling $k \le Ch^2$ of the mesh parameters (h, k), and initial data satisfying $n_0^\pm \in H^1(\Omega)$. Hence, we have to require slightly more regularity on the initial concentrations $n_0^\pm$, and a dimensional restriction compared to the continuous setting.

Let us briefly mention why the energy based approach convinces more than an entropy based approach introduced in [20], where an (unperturbed) entropy law holds without any mesh constraint for the Nernst-Planck-Poisson sub-system (1.4)–(1.10), for $\mathbf{u} \equiv 0$. An entropy based approach does not allow a constructive existence and uniqueness proof *via* a fully practical fixed point algorithm, enables only quasi-non-negativity of concentrations, does not easily allow for a discrete maximum principle and requires a perturbation of the momentum equation by the entropy-provider $\boldsymbol{\mathcal{S}}_\varepsilon(\cdot)$ to guarantee a discrete energy law.

In the second part of this article we propose a Scheme B based on Chorin's projection method [5], which was independently proposed by Temam [24], to construct discrete approximations, where iterates converge to the strong solution of the system (1.1)–(1.10) with optimal rates, see [18,19]. The main advantage of Scheme B is its efficiency; it is, however, that solutions of Scheme B are not known to satisfy physically relevant properties, such as a discrete maximum principle for concentrations, a discrete energy, and an entropy law.

The results are given in Section 3; Section 2 introduces notation. The proofs are given in Section 4 for Scheme A, and in Section 5 for Scheme B. Comparative computational studies are reported in Section 6.

2. Preliminaries

2.1. Notation

We use the standard Lebesgue and Sobolev spaces [1]. To keep the notation simple, let $\|\cdot\| := \|\cdot\|_{L^2}$. The Poisson equation for homogeneous Neumann conditions, that is

$$-\Delta u = f \qquad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega, \tag{2.1}$$

is of special interest to our analysis and concerns the following regularity estimate for $1 < p < \infty$

$$\|u\|_{W^{2,p}} \leq C\|f\|_{L^p}, \tag{2.2}$$

which is known if we make the following assumption; *cf.* [8]:

(A1) Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with a $C^{1,1}$ boundary, or convex in the case of $N = 2$.

We frequently use the following spaces [15],

$$\begin{aligned}
\tilde{\boldsymbol{\mathcal{D}}}(\Omega) &= \{\mathbf{u} \in C_0^\infty(\Omega, \mathbb{R}^N): \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega\},\\
\mathbf{V}^{0,2}(\Omega) &= \text{the closure of } \tilde{\boldsymbol{\mathcal{D}}} \text{ in } L^2 = \overline{\tilde{\boldsymbol{\mathcal{D}}}}^{L^2},\\
\mathbf{V}^{1,2}(\Omega) &= \text{the closure of } \tilde{\boldsymbol{\mathcal{D}}} \text{ in } H_0^1 = \overline{\tilde{\boldsymbol{\mathcal{D}}}}^{H_0^1}.
\end{aligned}$$

We denote the dual space of $\mathbf{V}^{1,2}(\Omega)$ by $\mathbf{V}^{-1,2}(\Omega)$. Subsequently, $\mathcal{T}_h$ denotes a quasi-uniform triangulation [4] of $\Omega \subset \mathbb{R}^N$ for $N = 2, 3$. Let $\mathcal{N}_h = \{\mathbf{x}_\ell\}_{\ell \in L}$ denote the set of all nodes of $\mathcal{T}_h$. We define *strongly acute meshes* [6,16] as follows:

> The sum of the opposite angles to the common side of any two adjacent triangles is $\leq \pi - \theta$, with $\theta > 0$ independent of h.

This condition is sufficient to validate $k_{\beta\beta'} := (\nabla\varphi_\beta, \nabla\varphi_{\beta'}) \leq -C_\theta < 0$, for $\beta \neq \beta'$, for the stiffness matrix in three dimensions; here, φ_β is the nodal basis as introduced below. We make the assumption:

(A2) Let $\mathcal{T}_h$ be a strongly acute triangulation, or for $N = 2$ a Delaunay triangulation.

Let P_ℓ denote the set of all polynomials in two variables of degree $\leq \ell$. We introduce the following spaces

$$\begin{aligned}
\mathbf{Y}_h &= \big\{\mathbf{U} \in C_0(\overline{\Omega}, \mathbb{R}^N) \,:\, \mathbf{U}|_K \in P_1(K, \mathbb{R}^N) \quad \forall K \in \mathcal{T}_h\big\} \\
Y_h &= \big\{\varphi \in C(\overline{\Omega}) \,:\, \varphi|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\big\} && (2.3) \\
\mathbf{B}_h^\ell &= \big\{\mathbf{U} \in C_0(\overline{\Omega}, \mathbb{R}^N) \,:\, \mathbf{U}|_K \in P_\ell(K, \mathbb{R}^N) \cap H_0^1(K, \mathbb{R}^N) \quad \forall K \in \mathcal{T}_h\big\} \\
\mathbf{X}_h &= \mathbf{Y}_h \cup \mathbf{B}_h^3 && (2.4) \\
M_h &= \big\{Q \in L_0^2(\Omega) \cap C(\overline{\Omega}) \,:\, Q|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\big\}, && (2.5)
\end{aligned}$$

where

$$\begin{aligned}
C_0(\Omega, \mathbb{R}^N) &:= \big\{\mathbf{u} \in C(\overline{\Omega}, \mathbb{R}^N) \,:\, \mathbf{u} = 0 \text{ on } \partial\Omega\big\} \\
L_0^2(\Omega) &:= \big\{u \in L^2(\Omega) \,:\, (u, 1) = 0\big\}.
\end{aligned}$$

A well-known example [2] that satisfies the discrete inf-sup condition

$$\sup_{\mathbf{U} \in \mathbf{X}_h} \frac{(\operatorname{div} \mathbf{U}, Q)}{\|\nabla \mathbf{U}\|} \geq C\|Q\| \quad \forall Q \in M_h,$$

is the MINI-element defined by $\mathbf{X}_h$ in (2.4), and by M_h in (2.5). Let

$$\mathbf{V}_h = \big\{\mathbf{V} \in \mathbf{X}_h \,:\, (\operatorname{div} \mathbf{V}, Q) = 0 \quad \forall Q \in M_h\big\}.$$

The following compatibility condition of spaces

$$Y_h \cap L_0^2(\Omega) \subset M_h, \tag{2.6}$$

accounts for coupling effects in the electrohydrodynamical system (1.1)–(1.10). We use the nodal interpolation operator $\mathcal{I}_{Y_h} \,:\, C(\overline{\Omega}) \to Y_h$ such that

$$\mathcal{I}_{Y_h}(\psi) := \sum_{\mathbf{z} \in \mathcal{N}_h} \psi(\mathbf{z}) \varphi_{\mathbf{z}}$$

where $\big\{\varphi_{\mathbf{z}} \,:\, \mathbf{z} \in \mathcal{N}_h\big\} \subset Y_h$ denotes the nodal basis of Y_h, and $\psi \in C(\overline{\Omega})$. For functions $\phi,\, \psi \in C(\overline{\Omega})$, we define mass-lumping as

$$\begin{aligned}
(\phi,\, \psi)_h &:= \int_\Omega \mathcal{I}_{Y_h}\big(\phi\psi\big)\, \mathrm{d}\mathbf{x} = \sum_{\mathbf{z} \in \mathcal{N}_h} \beta_{\mathbf{z}} \phi(\mathbf{z}) \psi(\mathbf{z}), \\
\|\phi\|_h^2 &:= (\phi,\, \phi)_h,
\end{aligned}$$

where $\beta_{\mathbf{z}} = \int_\Omega \varphi_{\mathbf{z}}\, \mathrm{d}\mathbf{x}$ for $\mathbf{z} \in \mathcal{N}_h$. For all $\Phi,\, \Psi \in Y_h$ there holds [7]

$$\begin{aligned}
\|\Phi\| \leq \|\Phi\|_h &\leq (N+2)^{\frac{1}{2}} \|\Phi\|, \\
\big|(\Phi, \Psi)_h - (\Phi, \Psi)\big| &\leq Ch\|\Phi\|\|\nabla\Psi\|. && (2.7)
\end{aligned}$$

Moreover, in appropriate situations we use the convention with its induced norms

$$[\cdot, \cdot]_i := \begin{cases} (\cdot, \cdot) & \text{for } i = 1, \\ (\cdot, \cdot)_h & \text{for } i = 2, \end{cases} \qquad \|\Phi\|_i^2 := \begin{cases} (\Phi, \Phi) & \text{for } i = 1, \\ (\Phi, \Phi)_h & \text{for } i = 2. \end{cases}$$

We define the discrete Laplace operators $\mathcal{L}_h^{(i)} : H^1(\Omega) \to Y_h$ for $i = 1, 2$ by

$$\left[-\mathcal{L}_h^{(i)}\phi, \Phi\right]_i = (\nabla\phi, \nabla\Phi) \qquad \forall \Phi \in Y_h\,. \tag{2.8}$$

Note that there exists a constant $C > 0$, such that for all $\Phi \in Y_h$ and $i = 1$, 2 there holds

$$\|\mathcal{L}_h^{(i)}\Phi\| \leq Ch^{-2}\|\Phi\| \quad \text{and} \quad \|\mathcal{L}_h^{(i)}\Phi\|_{L^\infty} \leq Ch^{-2}\|\Phi\|_{L^\infty} \quad \forall \Phi \in Y_h. \tag{2.9}$$

The following discrete Sobolev inequalities generalize results in [10], Lemma 4.4, in the case $N = 3$, for $i = 1$, 2,

$$\begin{aligned} \|\nabla\Phi\|_{L^3} &\leq C\|\nabla\Phi\|^{\frac{6-N}{6}}\left(\|\mathcal{L}_h^{(i)}\Phi\| + \|\Phi\|_{H^1}\right)^{\frac{N}{6}} \quad \forall \Phi \in \mathbf{Y}_h, \\ \|\nabla\Phi\|_{L^6} &\leq C\left(\|\mathcal{L}_h^{(i)}\Phi\| + \|\Phi\|_{H^1}\right) \quad \forall \Phi \in Y_h. \end{aligned} \tag{2.10}$$

In the sequel, we use the L^2-orthogonal projections $J_{\mathbf{V}_h} : L^2(\Omega, \mathbb{R}^N) \to \mathbf{V}_h$, $J_{\mathbf{Y}_h} : L^2(\Omega, \mathbb{R}^N) \to \mathbf{Y}_h$, and $J_{M_h} : L_0^2(\Omega) \to M_h$ which satisfy for all $\mathbf{u} \in L^2(\Omega, \mathbb{R}^N)$ and $q \in L_0^2(\Omega)$

$$(\mathbf{u} - J_{\mathbf{V}_h}\mathbf{u}, \mathbf{V}) = 0 \qquad \forall \mathbf{V} \in \mathbf{V}_h \tag{2.11}$$

$$(\mathbf{u} - J_{\mathbf{Y}_h}\mathbf{u}, \boldsymbol{\Phi}) = 0 \qquad \forall \boldsymbol{\Phi} \in \mathbf{Y}_h, \tag{2.12}$$

$$(q - J_{M_h}q, Q) = 0 \qquad \forall Q \in M_h. \tag{2.13}$$

The following estimates can be found in [10]:

$$\|\mathbf{u} - J_{\mathbf{V}_h}\mathbf{u}\| + h\|\nabla(\mathbf{u} - J_{\mathbf{V}_h}\mathbf{u})\| \leq Ch^2\|D^2\mathbf{u}\| \quad \forall \mathbf{u} \in \mathbf{V}^{1,2}(\Omega) \cap H^2(\Omega, \mathbb{R}^N), \tag{2.14}$$

$$\|\mathbf{u} - J_{\mathbf{V}_h}\mathbf{u}\| \leq Ch\|\nabla\mathbf{u}\| \quad \forall \mathbf{u} \in \mathbf{V}^{1,2}(\Omega). \tag{2.15}$$

Corresponding approximation results also hold for J_{Y_h} and $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

2.2. Discrete time-derivatives and interpolations

Given a time-step size $k > 0$, and a sequence $\{U^j\}_{j=1}^J$ in some Banach space X, we set $d_t U^j := k^{-1}\{U^j - U^{j-1}\}$ for $j \geq 1$. Note that $(d_t U^j, U^j) = \frac{1}{2} d_t \|U^j\|^2 + \frac{k}{2}\|d_t U^j\|^2$, if X is a Hilbert space. Piecewise constant interpolations of $\{U^j\}_{j=1}^J$ are defined for $t \in [t_{j-1}, t_j)$, and $0 \leq j \leq J$ by

$$\underline{\mathcal{U}}(t) := U^{j-1} \quad \text{and} \quad \overline{\mathcal{U}}(t) := U^j,$$

and a piecewise affine interpolation on $[t_{j-1}, t_j)$ is defined by

$$\mathcal{U}(t) := \underline{\mathcal{U}} + \frac{\overline{\mathcal{U}} - \underline{\mathcal{U}}}{k}(t - t_{j-1}).$$

Further, we employ the spaces $\ell^p(0, t_J; X)$ for $1 \leq p \leq \infty$. These are the spaces of functions $\{\Phi^j\}_{j=0}^J$ with the bounded norms

$$\|\Phi^j\|_{\ell^p(0,t_J;X)} := \left(k\sum_{j=0}^J \|\Phi^j\|_X^p\right)^{\frac{1}{p}}, \qquad \|\Phi^j\|_{\ell^\infty(0,t_J;X)} := \max_{1\leq j\leq J}\|\Phi^j\|_X.$$

3. Main results

We recall the notion of weak solutions for (1.1)–(1.10), *cf.* [22].

Definition 3.1 (weak solution)**.** Assume (A1), $N \leq 3$, and $0 < T < \infty$. We call $(\mathbf{u}, n^+, n^-, \psi)$ a weak solution of (1.1)–(1.10), if

(i) it satisfies for $p = 2$, if $N = 2$, or for $p = \frac{4}{3}$, if $N = 3$, that

$$\begin{aligned}\mathbf{u} &\in L^2(0,T;\mathbf{V}^{1,2}(\Omega)) \cap L^\infty(0,T;\mathbf{V}^{0,2}(\Omega)) \cap W^{1,p}(0,T;\mathbf{V}^{-1,2}(\Omega)),\\ n^\pm &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T) \cap W^{1,\frac{6}{5}}(0,T;(H^1(\Omega))^*),\\ \psi &\in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega));\end{aligned}$$

(ii) it solves the equations (1.1)–(1.8) in the weak sense for the initial data

$$\mathbf{u}_0 \in \mathbf{V}^{0,2}(\Omega), \qquad n_0^\pm \in L^\infty(\Omega, \mathbb{R}_{\geq 0}), \tag{3.1}$$

where for $t \to 0$ holds

$$\mathbf{u}(\cdot,t) \rightharpoonup \mathbf{u}_0 \quad \text{in } L^2(\Omega,\mathbb{R}^N), \qquad n^\pm(\cdot,t) \rightharpoonup n_0^\pm \quad \text{in } L^2(\Omega); \tag{3.2}$$

(iii) it satisfies the following boundary conditions in the trace sense for *a.e.* $t \in [0,T]$, *i.e.*,

$$\langle J_{n^\pm}, \mathbf{n}\rangle\big|_{\partial\Omega} = 0, \qquad \text{and } \langle \nabla\psi, \mathbf{n}\rangle\big|_{\partial\Omega} = 0, \tag{3.3}$$

where $\mathbf{n} : \partial\Omega \to \mathbb{R}^N$ is the unit normal on the boundary of Ω,

The weak solution satisfies for a.e. $t \in [0,T]$ the energy and entropy inequalities

$$E(t) + \int_0^t \big(e(s) + d(s)\big)\,\mathrm{d}s \leq E(0) \tag{3.4}$$

$$W(t) + \int_0^t \big(I^+(s) + I^-(s)\big)\,\mathrm{d}s \leq W(0), \tag{3.5}$$

where $W(t) := W_{NPP}(t) + W_{INS}(t)$, for

$$\begin{aligned}W_{NPP} &:= \int_\Omega n^+\left(\log\left(n^+\right) - 1\right) + n^-\left(\log\left(n^-\right) - 1\right) + \frac{1}{2}\left|\nabla\psi\right|^2 + 2\,\mathrm{d}\mathbf{x}\\ W_{INS} &:= \int_\Omega \frac{1}{2}\left|\mathbf{u}\right|^2\,\mathrm{d}\mathbf{x} + \int_0^t\int_\Omega \left|\nabla\mathbf{u}\right|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\\ I^\pm &:= \int_\Omega n^\pm\left[\nabla\left(\log\left(n^\pm\right) - \psi\right)\right]^2\,\mathrm{d}\mathbf{x}\\ E &:= \frac{1}{2}\left[\|\mathbf{u}\|^2 + \|\nabla\psi\|^2\right]\\ e &:= \|\nabla\mathbf{u}\|^2 + \|\Delta\psi\|^2\\ d &:= \int_\Omega \left(n^+ + n^-\right)\left|\nabla\psi\right|^2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

The term $E(t)$ in the above Definition 3.1 contains the physically motivated kinetic energy $E^1(\mathbf{u}) := \frac{1}{2}\|\mathbf{u}\|^2$, and the energy density of the electric field $E^2(\psi) := \frac{1}{2}\|\nabla\psi\|^2$ at time $t \geq 0$; furthermore, the term $d(t)$ denotes the total electrical energy of the system at time $t \geq 0$.

To construct discrete approximations of the weak solutions given in Definition 3.1, we propose the following Scheme A. The main interest and hence the reason for the fully implicit character of the subsequently proposed Scheme A is to preserve most of the characteristic properties of weak solutions given in the above Definition 3.1.

Scheme A.

(1) Set $\mathbf{U}^0 = J_{\mathbf{V}_h}\mathbf{u}_0$, and $\big((N^+)^0,(N^-)^0\big) := \big(J_{Y_h}n_0^+, J_{Y_h}n_0^-\big)$.
(2) For $j=1,\ldots,J$, let $\mathbf{F}_C^j := -((N^+)^j-(N^-)^j)\nabla\Psi^j$. Find $(\,\mathbf{U}^j,\,(N^\pm)^j,\,\Psi^j) \in \mathbf{V}_h\times[Y_h]^3$ such that for all $(\,\mathbf{V},\,\Phi^\pm,\,\Phi\,)\in\mathbf{V}_h\times[Y_h]^3$,

$$(\mathrm{d}_t\mathbf{U}^j,\mathbf{V}) + (\nabla\mathbf{U}^j,\nabla\mathbf{V}) + \epsilon(\nabla d_t\mathbf{U}^j,\nabla\mathbf{V}) + \big((\mathbf{U}^{j-1}\cdot\nabla)\mathbf{U}^j,\mathbf{V}\big) + \frac{1}{2}\big((\mathrm{div}\,\mathbf{U}^{j-1})\mathbf{U}^j,\mathbf{V}\big) = (\mathbf{F}_C^j,\mathbf{V}), \quad (3.6)$$

$$[\mathrm{d}_t(N^\pm)^j,\Phi^\pm]_i + (\nabla(N^\pm)^j,\nabla\Phi^\pm) \pm \big((N^\pm)^j\nabla\Psi^j,\nabla\Phi^\pm\big) - (\mathbf{U}^j(N^\pm)^j,\nabla\Phi^\pm) = 0, \quad (3.7)$$

$$(\nabla\Psi^j,\nabla\Phi) = [(N^+)^j-(N^-)^j,\Phi]_i, \quad (3.8)$$

where $\epsilon := h^\alpha$ with $0<\alpha<\frac{6-N}{3}$.

Scheme A incorporates two discretization strategies: One approach uses exact integration $[\cdot,\cdot]_1 := (\cdot,\cdot)$, while the second strategy uses mass lumping $[\cdot,\cdot]_2 := (\cdot,\cdot)_h$ which is needed below to validate non-negativity and an entropy law of concentration iterates without any mesh-constraint involving $k,h>0$ to hold. In both versions of Scheme A, the stabilization term $\epsilon\big(\nabla d_t\mathbf{U}^j,\nabla\mathbf{V}\big)$ with $\epsilon = h^\alpha$ is introduced to conclude an M-matrix property for the sub-system (3.7) below, and hence accounts for the problematic nature of the coupled overall system. It is, however, that this term in turn introduces another perturbation error to the problem which dominates the consistency error related to space discretization. Since the Scheme A is fully implicit leading to a coupled nonlinear system, the use of an iterative solver is required; its implicit character allows to recover the properties of solutions from the continuous setting. See also the discussion in Section 4.

Remark 3.2. Let us recall the entropy based scheme for the Nernst-Planck-Poisson equations (1.4)–(1.10) (with $\mathbf{u}\equiv\mathbf{0}$) introduced in [20]. We use the notion of an entropy-provider: For any $\varepsilon\in(0,1)$, we call $\boldsymbol{\mathcal{S}}_\varepsilon : Y_h \to \big[L^\infty(\Omega)\big]^{d\times d}$ an entropy-provider if for all $\Phi\in Y_h$

(i) $\boldsymbol{\mathcal{S}}_\varepsilon(\Phi)$ is symmetric and positive definite,
(ii) $\boldsymbol{\mathcal{S}}_\varepsilon(\Phi)\nabla\mathcal{I}_h\big[F_\varepsilon'(\Phi)\big] = \nabla\Phi$,

where

$$F_\varepsilon'(x) = \begin{cases} x\varepsilon^{-1}+\ln\varepsilon-1, & \text{if } x\le\varepsilon,\\ \ln x, & \text{if } \varepsilon\le x\le 2,\\ \frac{x}{2}+\ln 2-1, & \text{if } 2\le x.\end{cases}$$

This entropy based approach, which allows for an unperturbed entropy law in [20], leads to:

Scheme A'. Fix $0<\varepsilon<1$, and let $\big((N^+)^0,(N^-)^0\big)\in\big[Y_h\big]^2$, such that $\big((N^+)^0-(N^-)^0,1\big)=0$. For every $j\ge1$, find iterates $\big((N^+)^j,(N^-)^j,\Psi^j\big)\in\big[Y_h\big]^3$, where $(\Psi^j,1)=0$ such that for all $\big(\Phi_1,\Phi_2,\Phi_3\big)\in\big[Y_h\big]^3$ holds

$$\big(d_t(N^+)^j,\Phi_1\big)_h + \Big(\nabla\Psi^j,\boldsymbol{\mathcal{S}}_\varepsilon\big((N^+)^j\big)\nabla\Phi_1\Big) + \big(\nabla(N^+)^j,\nabla\Phi_1\big) = 0, \quad (3.9)$$

$$\big(d_t(N^-)^j,\Phi_2\big)_h - \Big(\nabla\Psi^j,\boldsymbol{\mathcal{S}}_\varepsilon\big((N^-)^j\big)\nabla\Phi_2\Big) + \big(\nabla(N^-)^j,\nabla\Phi_2\big) = 0, \quad (3.10)$$

$$\big(\nabla\Psi^j,\nabla\Phi_3\big) = \big((N^+)^j-(N^-)^j,\Phi_3\big)_h. \quad (3.11)$$

In contrast, the extension of this Scheme A' to the electro-hydrodynamic model (1.1)–(1.10) requires to apply the entropy provider in the Coulomb force term $\mathbf{F}_C^j$ in the way

$$-\big(\boldsymbol{\mathcal{S}}_\varepsilon\big(N^+\big) - \boldsymbol{\mathcal{S}}_\varepsilon\big((N^-)^j\big)\big)\nabla\Psi^j$$

to verify a discrete energy law, and compensate for the lack of a discrete maximum principle in this scheme. This, together with the weaker results mentioned in the introduction (Sect. 1) motivate to follow the energy based approach as realized in Scheme A.

With the kinetic energy $E^1(\mathbf{U}^j) := \frac{1}{2}\|\mathbf{U}^j\|^2$ and the electric energy density $E^2(\Psi^j) := \frac{1}{2}\|\nabla\Psi^j\|^2$ we define the energy of the electro-hydrodynamic system to be

$$E(\mathbf{U}^j, \Psi^j) := \frac{1}{2}\Big[\|\mathbf{U}^j\|^2 + \|\nabla\Psi^j\|^2\Big].$$

In below, the compatibility condition (2.6) is needed to validate an $L^\infty(\Omega_T)$-bound for discrete concentrations. We state the first main result that is verified in Section 4.

Theorem 3.3 (properties of solutions for Scheme A)**.** *Assume the initial conditions of Definition* 3.1(ii), *and* (A1). *Let* (A2) *and* (2.6) *be valid, and $h \le h_0(\Omega)$ be small enough, such that $k \le Ch^{N/3+\beta}$ for any $\beta > 0$, and moreover assume $h^2 \le Ck$ in the case $i = 1$. Let $0 \le (N^\pm)^0 \le 1$. Then for every $j \ge 1$, there exists a solution $(\,\mathbf{U}^j, \Pi^j, (N^\pm)^j, \Psi^j\,) \in \mathbf{V}_h \times M_h \times \big[Y_h\big]^3$, such that* (3.6)–(3.8) *hold. Furthermore,*

$$0 \le (N^\pm)^j \le 1 \qquad (1 \le j \le J),$$

and for $i = 1, 2$ it holds

$$\text{(i)} \qquad E(\mathbf{U}^J, \Psi^J) + \frac{\epsilon}{2}\|\nabla\mathbf{U}^J\|^2 + k^2\sum_{j=1}^{J}\Big\{E(d_t\mathbf{U}^j, d_t\Psi^j) + \frac{\epsilon}{2}\|d_t\nabla\mathbf{U}^j\|^2\Big\}$$

$$+\,k\sum_{j=1}^{J}\Big[\|\nabla\mathbf{U}^j\|^2 + \|(N^+)^j - (N^-)^j\|_h^2\Big] + k\sum_{j=1}^{J}\Big(\big((N^+)^j + (N^-)^j\big), |\nabla\Psi^j|$$

$$= E(\mathbf{U}^0, \Psi^0) + \frac{\epsilon}{2}\|\nabla\mathbf{U}^0\|^2,$$

$$\text{(ii)} \qquad \frac{1}{2}\Big\{\|\mathbf{U}^J\|^2 + \epsilon\|\nabla\mathbf{U}^J\|^2\Big\} + \frac{k^2}{2}\sum_{j=1}^{J}\Big\{\|d_t\mathbf{U}^j\|^2 + \epsilon\|\nabla d_t\mathbf{U}^j\|^2\Big\} + k\sum_{j=1}^{J}\|\nabla\mathbf{U}^j\|^2$$

$$\le CE(\mathbf{U}^0, \Psi^0) + \frac{\epsilon}{2}\|\nabla\mathbf{U}^0\|^2,$$

$$\text{(iii)} \qquad \frac{1}{2}\Big[\|(N^+)^J\|_i^2 + \|(N^-)^J\|_i^2\Big] + \frac{k^2}{2}\sum_{j=1}^{J}\Big[\|d_t(N^+)^j\|_i^2 + \|d_t(N^-)^j\|_i^2\Big] + \frac{k}{2}\sum_{j=1}^{J}\Big[\|\nabla(N^+)^j\|^2 + \|\nabla(N^-)^j\|^2\Big]$$

$$\le CE(\mathbf{U}^0, \Psi^0) + \frac{1}{2}\Big[\|(N^+)^0\|_i^2 + \|(N^-)^0\|_i^2\Big],$$

$$\text{(iv)} \qquad k\sum_{j=1}^{J}\Big[\|d_t(N^+)^j\|_{(H^1)^*}^2 + \|d_t(N^-)^j\|_{(H^1)^*}^2\Big] \le C\Big\{E(\mathbf{U}^0, \Psi^0) + \Big[\|(N^+)^0\|^2 + \|(N^-)^0\|^2\Big]\Big\},$$

$$\text{(v)} \qquad k\sum_{j=\ell}^{J}\|\mathbf{U}^j - \mathbf{U}^{j-\ell}\|^2 \le C(\ell k)^{1/4} \quad \forall 0 \le \ell \le J.$$

The estimates (i)–(iii) provide uniform bounds in time which control the long time behavior of iterates, whereas (iv) and (v) are necessary for a compactness property needed in the convergence proof.

We introduce a practical Algorithm A_1 in Section 4 that is a simple fixed-point scheme, together with a suitable stopping criterion to verify the statements of Theorem 3.3.

Motivated by the entropy estimate for (1.1)–(1.10), we recover the proof from there in a fully discrete setting. Therefore, we introduce the entropy functional

$$J \mapsto W^J := E\big(\mathbf{U}^J, \Psi^J\big) + \frac{\epsilon}{2}\|\nabla \mathbf{U}^J\|^2 + \int_\Omega \Big\{ \mathcal{I}_{Y_h}\Big[F\Big((N^+)^J\Big) + F\Big((N^-)^J\Big)\Big] + 2\Big\}\, \mathrm{d}\mathbf{x}, \tag{3.12}$$

where $F(x) := x(\ln x - 1)$, and herewith we extend the version in [20].

Theorem 3.4 (entropy law for Scheme A)**.** *Let $n_0^\pm \in H^1(\Omega)$, (A2), (2.6), $N = 2$, $i = 2$, and $k \le Ch^2$ be valid for some $T := t_J > 0$. Suppose that $\delta \le (N^\pm)^0 \le 1$ for some $0 < \delta < \frac{1}{2}$, and let $\big\{\big(\mathbf{U}^j,\, (N^\pm)^j,\, \Psi^j\big)\big\}_{j=1}^J$ solve the Scheme A. Then, for all $0 \le j < j' \le J$,*

$$\begin{aligned} W^{j'} &+ \frac{k^2}{2}\sum_{l=j+1}^{j'} \Big[\|\nabla d_t \Psi^l\|^2 + \epsilon \|d_t \nabla \mathbf{U}^l\|^2\Big] + k \sum_{l=j+1}^{j'} \Big[\Big((N^+)^l, \big|\nabla\big\{\Psi^l + \mathcal{I}_h\big[F'((N^+)^l)\big]\big\}\big|^2\Big) \\ &+ \|\nabla \mathbf{U}^l\|^2 + \Big((N^-)^l, \big|\nabla\big\{\Psi^l - \mathcal{I}_h\big[F'((N^-)^l)\big]\big\}\big|^2\Big)\Big] \\ &\le W^j + Ch\delta^{-4}\Big[E(\mathbf{U}^0, \Psi^0) + \|\nabla (N^+)^0\|_h^2 + \|\nabla (N^-)^0\|_h^2\Big]^2. \end{aligned} \tag{3.13}$$

Asymptotically, the dissipation of $W^{j'}$ in (3.13) is then guaranteed for $\delta := h^{\frac{1}{4}-\epsilon}$ and $\frac{1}{4} > \epsilon > 0$ arbitrarily small.

The main convergence result concerning Scheme A is:

Theorem 3.5 (convergence of Scheme A)**.** *Assume the initial conditions of Definition 3.1(ii). Suppose (A1), (A2), (2.6), and $0 < t_J < \infty$. Let $0 \le (N^+)^0, (N^-)^0 \le 1$, $(n_0^+, n_0^-) \in \big[L^\infty(\Omega)\big]^2$, as well as*

$$\mathbf{U}^0 \rightharpoonup \mathbf{u}_0 \text{ in } L^2(\Omega, \mathbb{R}^N), \qquad (N^+)^0 \rightharpoonup n_0^+,\ (N^-)^0 \rightharpoonup n_0^- \text{ in } L^2(\Omega),$$
$$\lim_{h\to 0} h^{\alpha/2}\, \|\nabla J_{\mathbf{V}_h} \mathbf{u}_0\| = 0.$$

Let $(\boldsymbol{\mathcal{U}}, \boldsymbol{\Pi}, \mathcal{N}^\pm, \boldsymbol{\Psi})$ be constructed from the solution $\big\{\big(\mathbf{U}^j, \Pi^j, (N^\pm)^j, \Psi^j\big)\big\}_{j=1}^J \subset \mathbf{V}_h \times M_h \times \big[Y_h\big]^3$ of Scheme A by piecewise affine interpolation as outlined in Section 2.2. Then, for $h, k \to 0$ such that $h^2 \le Ck$ in the case $i = 1$, or $k \le Ch^{\frac{N}{3}+\beta}$ for some $\beta > 0$ in the case $i = 2$, there exists a convergent subsequence $\big\{\big(\boldsymbol{\mathcal{U}}, \boldsymbol{\Pi}, \mathcal{N}^\pm, \boldsymbol{\Psi}\big)\big\}_{k,h}$ whose limit is a weak solution of (1.1)–(1.10).

In Section 5, we analyze a time-splitting scheme based on Chorin's projection method [5,25]; we refer to [18], and [9] for a recent review of this and related methods to solve the incompressible Navier-Stokes equations. In this scheme, the computation of iterates is fully decoupled in every time-step, which leads to significantly reduced computational effort. However, this strategy sacrifices the discrete energy and entropy inequalities, which are relevant tools to characterize long-time asymptotics and convergence towards weak solutions. Therefore, instead, the related numerical analysis requires the existence of (local) strong solutions which is verified for the system (1.1)–(1.10) in [22].

Definition 3.6 (strong solution)**.** Let $0 < T \leq \infty$. The weak solutions $(\mathbf{u}, n^+, n^-, \psi)$ are called strong solutions of (1.1)–(1.10), if

(i)

$$\begin{array}{rcl} \mathbf{u} & \in & L^\infty(0,T;\mathbf{V}^{1,2}(\Omega)\cap \mathbf{H}^{2,2}(\Omega)) \cap W^{1,2}(0,T;\mathbf{V}^{1,2}(\Omega)) \\ n^\pm & \in & L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;H^1(\Omega)) \\ \psi & \in & C([0,T];H^2(\Omega)) \cap W^{1,\infty}(0,T;W^{1,2}(\Omega)) \\ p & \in & L^\infty(0,T;(H^1\cap L^2_0)(\Omega)), \end{array}$$

(ii) the initial conditions

$$\mathbf{u}_0 \in \mathbf{V}^{1,2}(\Omega)\cap \mathbf{H}^2(\Omega), \qquad n_0^\pm \in H^2(\Omega), \tag{3.14}$$

are attained for $t \to 0$,

$$\mathbf{u}(\cdot,t) \to \mathbf{u}_0 \quad \text{in } H^1(\Omega,\mathbb{R}^N), \qquad n^\pm(\cdot,t) \to n_0^\pm \quad \text{in } H^1(\Omega). \tag{3.15}$$

A slightly weakened notion of strong solution is studied in [22], and (local) existence of strong solutions as defined here then follows by a simple bootstrapping argument. Recall that in dimension $N = 3$ the time $T = T(\mathbf{u}_0) > 0$ is in general finite, see [15,22,25] for example. Moreover, for $N \leq 3$ and $t \in [0,T]$, the following energy and entropy identities hold for strong solutions,

$$E(t) + \int_0^t e(s) + d(s)\,\mathrm{d}s = E(0), \tag{3.16}$$

$$W(t) + \int_0^t I^+(s) + I^-(s)\,\mathrm{d}s = W(0), \tag{3.17}$$

see [22].

For convenience, we say that a quadruple $(\mathbf{u}, p, n^\pm, \psi)$ is in $\mathbf{S}$, if it satisfies the regularity properties i) of Definition 3.6. To approximate the strong solutions of Definition 3.6, we propose the following time-splitting:

Scheme B. Let $j \geq 1$ and $\{\mathbf{u}^{j-1}, (n^\pm)^{j-1}\}$, determine $\{\mathbf{u}^j, (n^\pm)^j, \psi^j\}$ as follows:

1. Start with $\mathbf{u}^0 = \mathbf{u}_0$, and $(n^\pm)^0 = n_0^\pm$.
2. Let $j \geq 1$. Compute $\psi^{j-1} \in H^1(\Omega)$ from

$$\begin{array}{rcl} -\Delta\psi^{j-1} & = & (n^+)^{j-1} - (n^-)^{j-1} \quad \text{in } \Omega \\ \langle \nabla\psi^{j-1}, \mathbf{n}\rangle & = & 0 \quad \text{on } \partial\Omega. \end{array}$$

3. Compute $(n^\pm)^j \in H^1(\Omega)$ *via*

$$\begin{array}{rcl} \frac{1}{k}\big\{(n^\pm)^j - (n^\pm)^{j-1}\big\} - \Delta(n^\pm)^j \pm \operatorname{div}((n^\pm)^j\nabla\psi^{j-1}) + (\mathbf{u}^{j-1}\cdot\nabla)(n^\pm)^j & = & 0 \quad \text{in } \Omega \\ \langle \nabla(n^\pm)^j \pm (n^\pm)^j\nabla\psi^{j-1}, \mathbf{n}\rangle & = & 0 \quad \text{on } \partial\Omega. \end{array}$$

4. Find $\tilde{\mathbf{u}}^j \in H^1_0(\Omega,\mathbb{R}^N)$ by solving

$$\begin{array}{rcl} \frac{1}{k}\big\{\tilde{\mathbf{u}}^j - \mathbf{u}^{j-1}\big\} - \Delta\tilde{\mathbf{u}}^j + (\mathbf{u}^{j-1}\cdot\nabla)\tilde{\mathbf{u}}^j & = & -\Big((n^+)^j - (n^-)^j\Big)\nabla\psi^{j-1} \\ \tilde{\mathbf{u}}^j & = & 0 \quad \text{on } \partial\Omega. \end{array}$$

5. Determine the tuple $\{\mathbf{u}^j, p^j\} \in \mathbf{V}^{0,2} \times H^1 \cap L_0^2$ that solves the system

$$\frac{1}{k}\{\mathbf{u}^j - \tilde{\mathbf{u}}^j\} + \nabla p^j = 0, \qquad \operatorname{div} \mathbf{u}^j = 0 \quad \text{on } \Omega, \tag{3.18}$$

$$\langle \mathbf{u}^j, \mathbf{n}\rangle \quad = \quad 0 \quad \text{on } \partial\Omega. \tag{3.19}$$

Step 5 is known as Chorin's projection step. Using the div-operator in (3.18) amounts to solving a Laplace-Neumann problem for the pressure iterate,

$$-\Delta p^j = -\frac{1}{k}\operatorname{div} \tilde{\mathbf{u}}^j \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}} p^j| = 0 \quad \text{on } \partial\Omega, \tag{3.20}$$

followed by an algebraic update for the present solenoidal velocity field,

$$\mathbf{u}^j = \tilde{\mathbf{u}}^j - k\nabla p^j \quad \text{in } \Omega. \tag{3.21}$$

The goal of the second part (Sect. 5) of this paper is to analyze Scheme B by investigating its stability and approximation properties. Therefore we propose a series of auxiliary problems to separately account for inherent time-discretization, decoupling effects, and those attributed to the quasi-compressibility constraint (3.20). For this purpose, the following notation is useful.

We say that

1. the quadruple $\Big(\mathbf{u},\, p,\, n^\pm,\, \psi\Big) := \{\xi_i\}_{i=1}^4 \in \mathbf{S}$ satisfies property $(P1)$, if the following is satisfied for $i \in \{1,3\}$,

$$k\sum_{j=1}^J\Big\{\|d_t\xi_i^j\|_{H^1}^2 + \|d_t\xi_4^j\|_{H^2}^2\Big\} + \max_{1\le j\le J}\Big\{\|\mathrm{d}_t\xi_i^j\|^2 + \|\xi_i^j\|_{H^2}^2 + \|\xi_2^j\|_{H^1}^2 + \|\xi_4^j\|_{H^2}^2\Big\} \le C;$$

2. the quadruple $\{\xi_i^j\}_{i=1}^4 \in \mathbf{S}$ satisfies property $(P2)_l$, for $l \in \{0,1\}$, if the following approximation properties are satisfied:

$$\max_{0\le j\le J}\Big\{\sqrt{\tau^j}\|\mathbf{u}(t_j) - \xi_1^j\| + \tau_l^j\|p(t_j) - \xi_2^j\|_{H^{-1}} + \|\psi(t_j) - \xi_4^j\| + \|\psi(t_j) - \xi_4^j\|_{H^1} + \|n^\pm(t_j) - \xi_3^j\|$$
$$+ \sqrt{k}\Big(\|\mathbf{u}(t_j) - \xi_1^j\|_{H^1} + \sqrt{\tau_l^j}\|p(t_j) - \xi_2^j\| + \|n^\pm(t_j) - \xi_3^j\|_{H^1}\Big)\Big\} \le Ck,$$

where

$$\tau_l^j := \begin{cases} 1, & \text{if } l = 0,\\ \min\{1, t_j\}, & \text{if } l = 1.\end{cases}$$

With a slight abuse of notation, iterates $\{\xi_i\}_{i=1}^4$ are here considered as continuous piecewise affine, continuous interpolants in time of corresponding time iterates. The property $(P2)_0$ is used in the analysis of Scheme B. The generic constant C is independent of k, and depends only on the given data. In the following theorem, we state the main result concerning optimal convergence behaviour of the solution obtained *via* Scheme B.

Theorem 3.7 (convergence of Scheme B)**.** *Suppose* (A1)*, the initial and boundary conditions from Definition* 3.6*, let* $\mathbf{u}_0 \in \mathbf{V}^{1,2} \cap \mathbf{H}^2$*,* $n_0^\pm \in H^2(\Omega)$*, and* $0 \le t_J \le T$*. Then the solution* $\{\mathbf{u}^j, (n^+)^j, (n^-)^j, \psi^j\}_{j=1}^J \subset \mathbf{S}$ *of Scheme* B *satisfies the properties* $(P1)$ *and* $(P2)_1$ *for sufficiently small time-steps* $k \le k_0(t_J)$*.*

The proof of this result is based on optimal estimates for Chorin's scheme in the context of incompressible Navier-Stokes equations in [18], and a combining analysis of further splitting, regularization, and perturbation effects in Scheme B. If we additionally include the error effects of a corresponding space discretization which uses the setup of (2.3)–(2.5), we arrive at the following:

Theorem 3.8 (convergence of Scheme B)**.** *Let* $\big\{\mathbf{U}^j,\, P^j,\, (N^{\pm})^j,\, (\Psi)^j\big\}_{j=1}^J \subset \mathbf{V}_h \times M_h \times \big[Y_h\big]^3$ *be the solution of a fully discrete version of Scheme* B *(see* (5.57) *of Sect.* 5.4*), and* $(\mathbf{u},\, p,\, n^{\pm},\, \psi) \in \mathbf{S}$ *be the strong solution of* (1.1)–(1.10) *under the additional requirement* $\mathbf{u}_0 \in \mathbf{V}^{1,2} \cap \mathbf{H}^2$*, and* $n_0^{\pm} \in H^2(\Omega)$*. Then for all times* $0 \leq t_J \leq T$*,*

$$\max_{1\leq j\leq J} \Big\{\sqrt{\tau^j}\|\mathbf{u}(t_j) - \mathbf{U}^j\| + \|\psi(t_j) - \Psi^j\| + \|n^{\pm}(t_j) - (N^{\pm})^j\|\Big\} \leq C\left(k + h^2\right)$$

$$\max_{1\leq j\leq J} \Big\{\|\mathbf{u}(t_j) - \mathbf{U}^j\|_{H^1} + \sqrt{\tau_l^j}\|p(t_j) - P^j\| + \|n^{\pm}(t_j) - (N^{\pm})^j\|_{H^1}\Big\} \leq C\left(\sqrt{k} + h\right).$$

This result is a simple consequence from the obtained stability properties given in Theorem 3.7, and we only sketch necessary arguments in Section 5.4.

4. Proof of the results for Scheme A

4.1. **Existence of solutions for Scheme A, Theorem 3.3**

The M-matrix property of the system matrix for the subsystem (3.7), and (2.6) are key tools to guarantee solvability of Scheme A, and non-negativity and boundedness of the iterates $\big\{\big((N^+)^j,\, (N^-)^j\big)\big\}_{j\geq 0}$. For the subsequent proof of Theorem 3.3, we propose the practical Algorithm A_1.

Algorithm $\mathbf{A}_1$**.** 1. Let $\Big(\mathbf{U}^0, (N^{\pm})^0, \Psi^0\Big) \in \mathbf{V}_h \times [Y_h]^3$, such that $\Big[(N^+)^0 - (N^-)^0, 1\Big]_i = 0$. For $j \geq 1$, set $\Big((N^+)^{j,0}, (N^-)^{j,0}, \Psi^{j,0}\Big) := \Big((N^+)^{j-1}, (N^-)^{j-1}, \Psi^{j-1}\Big)$, and $\ell := 0$.
2. For $\ell \geq 1$, compute $\Big(\mathbf{U}^{j,\ell}, (N^{\pm})^{j,\ell}, \Psi^{j,\ell}\Big) \in \mathbf{V}_h \times [Y_h]^3$ that solve for all $\Big(\mathbf{V}, \Phi^{\pm}, \Phi\Big) \in \mathbf{V}_h \times [Y_h]^3$, $i = 1, 2$, and $\mathbf{F}_C^{j,\ell-1} := -\big((N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}\big)\nabla\Psi^{j,\ell-1}$,

$$\Big(\nabla\Psi^{j,\ell}, \nabla\Phi\Big) = \Big[(N^+)^{j,\ell} - (N^-)^{j,\ell}, \Phi\Big]_i, \tag{4.1}$$

$$\begin{aligned}\frac{1}{k}\Big(\mathbf{U}^{j,\ell}, \mathbf{V}\Big) &+ \frac{h^\alpha}{k}\Big(\nabla\mathbf{U}^{j,\ell}, \nabla\mathbf{V}\Big) + \Big(\nabla\mathbf{U}^{j,\ell}, \nabla\mathbf{V}\Big) + \Big((\mathbf{U}^{j-1}\cdot\nabla)\mathbf{U}^{j,\ell}, \mathbf{V}\Big)\\ &+ \frac{1}{2}\Big((\mathrm{div}\mathbf{U}^{j-1})\mathbf{U}^{j,\ell}, \mathbf{V}\Big) = \Big(\mathbf{F}_C^{j,\ell-1}, \mathbf{V}\Big) + \frac{1}{k}\Big(\mathbf{U}^{j-1}, \mathbf{V}\Big) + \frac{h^\alpha}{k}\Big(\nabla\mathbf{U}^{j-1}, \nabla\mathbf{V}\Big),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{1}{k}\Big[(N^{\pm})^{j,\ell}, \Phi^{\pm}\Big]_i &\pm \Big((N^{\pm})^{j,\ell}\nabla\Psi^{j,\ell-1}, \nabla\Phi^{\pm}\Big) + \Big(\nabla(N^{\pm})^{j,\ell}, \nabla\Phi^{\pm}\Big)\\ &- \Big(\mathbf{U}^{j,\ell-1}(N^{\pm})^{j,\ell}, \nabla\Phi^{\pm}\Big) = \frac{1}{k}\Big[(N^{\pm})^{j-1}, \Phi^{\pm}\Big]_i.\end{aligned} \tag{4.3}$$

3. Stop, if for fixed $\theta > 0$ we have

$$\|\mathbf{U}^{j,\ell} - \mathbf{U}^{j,\ell-1}\| + \|\nabla\{\Psi^{j,\ell} - \Psi^{j,\ell-1}\}\| + \Big(\|(N^+)^{j,\ell} - (N^+)^{j,\ell-1}\|_{L^\infty} + \|(N^-)^{j,\ell} - (N^-)^{j,\ell-1}\|_{L^\infty}\Big) \leq \theta \tag{4.4}$$

and go to 4.; set $\ell \leftarrow \ell + 1$ and continue with 2. otherwise.
4. Stop, if $j + 1 = J$; set $j \leftarrow j + 1$ and go to 1. otherwise.

We first achieve $0 \leq (N^{\pm})^{1,\ell} \leq 1$ for $\ell \geq 1$ in Step 4; after the verification of a contraction property for iterates, we can identify

$$\big(\mathbf{U}^1,\, (N^{\pm})^1,\, \Psi^1\big) := \lim_{\ell\to\infty}\big(\mathbf{U}^{1,\ell},\, (N^{\pm})^{1,\ell},\, \Psi^{1,\ell}\big) \in \mathbf{V}_h \times [V_h]^3 \tag{4.5}$$

as a solution of Scheme A for $j = 1$. Key tools to accomplish this goal are an inductive argument (for $\ell \geq 1$) in Step 1 to 4, and the verification of the M-matrix property for subsystem (4.3) in Step 4 of the proof below; those results are then easily extended to $1 \leq j \leq J$.

Proof of Theorem 3.3. **Step 1** (stability for $\Psi^{j,\ell-1}$)**.** Let $0 \le (N^+)^{j,\ell-1}$, $(N^-)^{j,\ell-1} \le 1$. The solution $\Psi^{j,\ell-1} \in Y_h$ of (4.1) may be interpreted as the Ritz projection of $\psi^{j,\ell-1} \in (H^1 \cap L^2_0)(\Omega)$, *i.e.*, $\Psi^{j,\ell-1} = P_1 \psi^{j,\ell-1}$, such that $\big(\Psi^{j,\ell-1}, 1\big) = 0$, and

$$\big(\nabla \psi^{j,\ell-1}, \nabla \phi\big) = \big[(N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}, \phi\big]_i \qquad \forall \phi \in H^1(\Omega) \cap C(\overline{\Omega}),$$

for $i = 1, 2$. By the $W^{1,\gamma'}(\Omega)$-stability of P_1, *cf.* [4], Theorem 8.5.3, there holds $\left\|\Psi^{j,\ell-1}\right\|_{W^{1,\gamma'}} \le C \left\|\psi^{j,\ell-1}\right\|_{W^{1,\gamma'}}$. By Sobolev embedding, the right-hand side is bounded by $C\left\|(N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}\right\|_{L^{\gamma'}}$, for $1 \le \gamma' < \infty$, for $N = 2$, and $1 \le \gamma' \le 6$ in the case of $N = 3$.

Step 2 (*a priori* estimates for Algorithm A_1)**.** After testing equation (4.2) with $k\mathbf{U}^{j,\ell}$ we obtain

$$\left\|\mathbf{U}^{j,\ell}\right\|^2 + h^\alpha \left\|\nabla \mathbf{U}^{j,\ell}\right\|^2 + k \left\|\nabla \mathbf{U}^{j,\ell}\right\|^2 \le \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}, \tag{4.6}$$

where

$$\begin{aligned}
\mathrm{(I)} &:= \left|\big(\mathbf{U}^{j-1}, \mathbf{U}^{j,\ell}\big)\right| \le \frac{1}{2}\left\|\mathbf{U}^{j-1}\right\|^2 + \frac{1}{2}\left\|\mathbf{U}^{j,\ell}\right\|^2,\\
\mathrm{(II)} &:= h^\alpha \left|\big(\nabla\mathbf{U}^{j-1}, \nabla\mathbf{U}^{j,\ell}\big)\right| \le \frac{h^\alpha}{2}\left\|\nabla\mathbf{U}^{j-1}\right\|^2 + \frac{h^\alpha}{2}\left\|\nabla\mathbf{U}^{j,\ell}\right\|^2,\\
\mathrm{(III)} &:= k\left|\Big(\big\{(N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}\big\}\nabla\Psi^{j,\ell-1}, \mathbf{U}^{j,\ell}\Big)\right|\\
&\le kC\left\{\left\|(N^+)^{j,\ell-1}\right\| + \left\|(N^-)^{j,\ell-1}\right\|\right\}\left\|\nabla\Psi^{j,\ell-1}\right\|_{L^3}\left\|\nabla\mathbf{U}^{j,\ell}\right\|\\
&\le kC\left\|\nabla\Psi^{j,\ell-1}\right\|^2_{L^3}\left\{\left\|(N^+)^{j,\ell-1}\right\|^2 + \left\|(N^-)^{j,\ell-1}\right\|^2\right\} + \frac{k}{2}\left\|\nabla\mathbf{U}^{j,\ell}\right\|^2.
\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
\frac{1}{2}\|\mathbf{U}^{j,\ell}\|^2 + \frac{h^\alpha}{2}\|\nabla\mathbf{U}^{j,\ell}\|^2 + \frac{k}{2}\|\nabla\mathbf{U}^{j,\ell}\|^2 \le C\Big\{&\|\mathbf{U}^{j-1}\|^2 + h^\alpha\|\nabla\mathbf{U}^{j-1}\|^2\\
&+ k\left\|\nabla\Psi^{j,\ell-1}\right\|^2_{L^3}\Big[\|(N^+)^{j,\ell-1}\|^2 + \|(N^-)^{j,\ell-1}\|^2\Big]\Big\}.
\end{aligned} \tag{4.7}$$

Note that the right-hand side depends on $\ell \ge 1$. Therefore, in the following steps, we use an inductive argument, by showing that the right-hand side of (4.7) is in fact uniformly bounded with respect to $\ell \ge 1$, which follows from the uniform boundedness of $\{(N^\pm)^{j,\ell}\}$ from Step 4.

Step 3 (M-matrix property)**.** To establish the M-matrix property for a sub-system of Algorithm A_1, let $\mathcal{A}$ be the system matrix corresponding to the equations (4.3), with the convective term depending on $\mathbf{U}^{j,\ell-1}$, *i.e.*,

$$\Big(\mathbf{U}^{j,\ell-1}\varphi_\beta, \nabla\varphi_{\beta'}\Big) =: \big\{\mathcal{D}(\mathbf{U}^{j,\ell-1})\big\}_{\beta\beta'} =: d_{\beta\beta'},$$

where $\{\varphi_\beta\}_{\beta=1}^L$ is the canonical basis of Y_h. Correspondingly, we define for $i = 1$, 2

$$\begin{aligned}
&\Big(\varphi_\beta\nabla\Psi^{j,\ell-1}, \nabla\varphi_{\beta'}\Big) =: \big\{\mathcal{C}(\Psi^{j,\ell-1})\big\}_{\beta\beta'} =: c_{\beta\beta'}\\
&\Big(\nabla\varphi_\beta, \nabla\varphi_{\beta'}\Big) =: \{\mathcal{K}\}_{\beta\beta'} =: k_{\beta\beta'}\\
&\Big[\varphi_\beta, \varphi_{\beta'}\Big]_i =: \Big\{\mathcal{M}^{(i)}\Big\}_{\beta\beta'} =: m^{(i)}_{\beta\beta'}.
\end{aligned} \tag{4.8}$$

Here, $\mathcal{M}^{(1)}$ is the mass and $\mathcal{M}^{(2)}$ the lumped mass matrix. Hence, the system matrix $\{\mathcal{A}\}_{\beta\beta'} := a_{\beta\beta'}$ for (4.3) becomes

$$\mathcal{A} := \begin{pmatrix} \mathcal{A}^+ & \mathbf{0} \\ \mathbf{0} & \mathcal{A}^- \end{pmatrix}, \tag{4.9}$$

for $\mathcal{A}^\pm := \frac{1}{k}\mathcal{M}^{(i)} \pm \mathcal{C}(\Psi^{j,\ell-1}) + \mathcal{K} - \mathcal{D}(\mathbf{U}^{j,\ell-1})$ such that $\mathcal{A}[\mathbf{x}^{j,\ell}, \mathbf{y}^{j,\ell}]^\top = \mathbf{f}^{j,\ell}$, where

$$(N^+)^{j,\ell} := \sum_{\beta=1}^{L} x_\beta^{j,\ell}\varphi_\beta, \qquad (N^-)^{j,\ell} := \sum_{\beta=1}^{L} y_\beta^{j,\ell}\varphi_\beta, \tag{4.10}$$

with the right-hand sides $f_\beta^{j,\ell} := \frac{1}{k}\Big((N^+)^{j-1}, \varphi_\beta\Big)$, and $f_{L+\beta}^{j,\ell} := \frac{1}{k}\Big((N^-)^{j-1}, \varphi_\beta\Big)$, for $1 \le \beta \le L$.

Since the matrix $\mathcal{K}$ is already an M-matrix, we guarantee its dominating influence as part of $\mathcal{A}^\pm$ by a dimensional argument. In the following, we partly benefit from ideas developed in [20].

(a) Non-positivity of off-diagonal entries, *i.e.*, $a_{\beta\beta'} \le 0$ for all $\beta \ne \beta'$: Since $\mathcal{T}_h$ satisfies **(A2)**, there exists C_{θ_0}, such that $k_{\beta\beta'} \le -C_{\theta_0}h^{N-2} < 0$ uniformly for $h > 0$, for any pair of adjacent nodes. The remaining entries are bounded as follows,

$$\Big|\Big(\mathbf{U}^{j,\ell-1}\varphi_\beta, \nabla\varphi_{\beta'}\Big)\Big| \le \|\mathbf{U}^{j,\ell-1}\|_{L^\infty}\|\varphi_\beta\nabla\varphi_{\beta'}\|_{L^1} \le Ch^{N-1-\frac{N}{6}}\|\mathbf{U}^{j,\ell-1}\|_{L^6} \le Ch^{\frac{5N}{6}-\frac{\alpha}{2}-1}, \tag{4.11}$$

because of (4.7). Hence, we require $N-2 < \frac{5N}{6} - 1 - \frac{\alpha}{2}$ by a dimensional argument between $k_{\beta\beta'}$ and $d_{\beta\beta'}$, which amounts to $0 < \alpha < \frac{6-N}{3}$. We proceed similarly with $\mathcal{C}(\Psi^{j,\ell-1})$ for γ' as in Step 1, with $\gamma^{-1} + \gamma'^{-1} = 1$

$$\Big|\Big(\varphi_\beta\nabla\Psi^{j,\ell-1}, \nabla\varphi_{\beta'}\Big)\Big| \le \|\nabla\Psi^{j,\ell-1}\|_{L^{\gamma'}}\|\varphi_\beta\nabla\varphi_{\beta'}\|_{L^\gamma} \le Ch^{\frac{N}{\gamma}-1}. \tag{4.12}$$

Repeating the dimensional argument from above between $k_{\beta\beta'}$ and $c_{\beta\beta'}$ provides $N-2 < \frac{N}{\gamma} - 1$. Hence, $N < \frac{\gamma}{\gamma-1} = \gamma' \sim \frac{2N}{N-2}$, where " $\sim$ " = " $<$ " if $N = 2$ and " $\sim$ " = " $\le$ " if $N = 3$. In the case $i = 1$, we additionally have to control $\frac{1}{k}m_{\beta\beta'}^{(1)}$, *i.e.*

$$\frac{1}{k}\left|(\varphi_\beta, \varphi_{\beta'})\right| \le Ck^{-1}h^N.$$

Therefore (a) holds for $h \le h_0(\Omega)$ small enough if $i = 2$, and if $i = 1$, we additionally have to require $h^2 \le Ck$.

(b) Strict positivity of the diagonal entries of $\mathcal{A}$: We have to verify that

$$\frac{1}{k}m_{\beta\beta}^{(i)} + k_{\beta\beta} \pm c_{\beta\beta}(\Psi^{1,\ell-1}) - d_{\beta\beta}(\mathbf{U}^{1,\ell-1}) > 0.$$

We know that $\frac{1}{k}m_{\beta\beta}^{(i)} \ge c_{\theta_0}h^N$, and $k_{\beta\beta} \ge c_{\theta_0}h^{N-2}$, for some $c_{\theta_0} > 0$. Moreover, from (4.11) and (4.12) we obtain

$$|c_{\beta\beta}| + |d_{\beta\beta}| \le Ch^{\frac{5N}{6}-1-\frac{\alpha}{2}} + Ch^{\frac{N}{\gamma}-1} =: \eta(h). \tag{4.13}$$

Hence $c_{\theta_0}h^{N-2} - \eta(h) > 0$ is guaranteed by the same dimensional argument as in (a) for small enough $h \le h_0(\Omega)$.

(c) $\mathcal{A}$ strictly diagonal dominant, *i.e.*, $\sum_{\beta'\ne\beta}|a_{\beta\beta'}| < a_{\beta\beta}$: We use the fact that the number of neighboring nodes $\mathbf{x}_{\beta'} \in \mathcal{N}_h$ for each $\mathbf{x}_\beta$ is bounded independently of $h > 0$. Hence, there exists a constant

$\overline{C} := \overline{C}\big(\{\#\beta' : k_{\beta\beta'} \neq 0\}\big) > 0$, such that for k, $h > 0$ sufficiently small

$$\begin{aligned} a_{\beta\beta} &\geq \frac{1}{k} c_{K_\beta} h^N + c_{K_\beta} h^{N-2} - \eta(h) > \overline{C} \max_{\beta \neq \beta'} |a_{\beta\beta'}| \\ &= \overline{C} \Big| -C_{\theta_0} h^{N-2} - \eta(h) \Big| \geq \sum_{\beta \neq \beta'} |a_{\beta\beta'}|, \end{aligned} \tag{4.14}$$

where we used (b) for the first inequality and (a) and for the second inequality, and in both cases (4.13). Hence assertion (c) is verified for small enough $h \leq h_0(\Omega)$ and $k \leq k_0(\Omega)$.

The verification of (a)–(c) guarantees the M-matrix property of $\mathcal{A}$ for small enough $h \leq h_0(\Omega)$ and $k \leq k_0(\Omega)$. This property additionally implies the non-negativity of $((N^+)^{1,\ell}, (N^-)^{1,\ell})$.

Step 4 (boundedness of $0 \leq (N^\pm)^{1,\ell} \leq 1$)**.** Under the assumption $(N^\pm)^0 \leq 1$ and $(N^\pm)^{1,\ell-1} \leq 1$, we have $(\overline{N}^\pm)^{1,\ell-1} \leq 0$ for $(\overline{N}^\pm)^{1,\ell-1} := (N^\pm)^{1,\ell-1} - 1$, and also $(\overline{N}^\pm)^0 \leq 0$. Then for every $\Phi \in Y_h$, we conclude

$$\begin{aligned} \frac{1}{k}\Big[(\overline{N}^\pm)^{1,\ell}, \Phi\Big]_i + \Big(\nabla(\overline{N}^\pm)^{1,\ell}, \nabla\Phi\Big) \pm \Big(\{(\overline{N}^\pm)^{1,\ell} + 1\}\nabla\Psi^{1,\ell-1}, \nabla\Phi\Big) \\ + \Big(\mathbf{U}^{1,\ell-1}\{(\overline{N}^\pm)^{1,\ell} + 1\}, \nabla\Phi\Big) = \frac{1}{k}\Big[(\overline{N}^\pm)^0, \Phi\Big]_i. \end{aligned} \tag{4.15}$$

We use the M-matrix property of $\mathcal{A}$ for the equation (4.3) to find

$$\begin{aligned} &\frac{1}{k}\Big[\Phi - [\Phi]_+, [\Phi]_+\Big]_i \pm \Big([\Phi]_- \nabla\Psi^{1,\ell-1}, \nabla[\Phi]_+\Big) + \Big(\nabla\{\Phi - [\Phi]_+\}, \nabla[\Phi]_+\Big) + \Big(\mathbf{U}^{j,\ell-1}[\Phi]_-, \nabla[\Phi]_+\Big) \\ &\quad \geq \frac{1}{k}\big[[\Phi]_-, [\Phi]_+\big]_i \pm \Big([\Phi]_- \nabla\Psi^{1,\ell-1}, \nabla[\Phi]_+\Big) + \big(\nabla[\Phi]_-, \nabla[\Phi]_+\big) + \Big(\mathbf{U}^{j,\ell-1}[\Phi]_-, \nabla[\Phi]_+\Big) \\ &\quad \geq \sum_{\beta,\beta'} a_{\beta\beta'}[\Phi]_+(\mathbf{x}_\beta)[\Phi]_-(\mathbf{x}_{\beta'}) \geq 0, \end{aligned}$$

where $[\cdot]_- := \mathcal{I}_{Y_h}\min\{\cdot, 0\}$, $[\cdot]_+ := \mathcal{I}_{Y_h}\max\{\cdot, 0\}$, and $\Phi \in Y_h$. Since $\Phi = [\Phi]_+ + [\Phi]_-$ for all $\Phi \in Y_h$, the definition (4.9) of $a_{\beta\beta'}$ then directly implies

$$\begin{aligned} \frac{1}{k}\|[\Phi]_+\|_i^2 + \|\nabla[\Phi]_+\|^2 \leq \frac{1}{k}\Big[[\Phi]_+, \Phi\Big]_i + \Big(\nabla[\Phi]_+, \nabla\Phi\Big) \\ \pm \Big([\Phi]_- \nabla\Psi^{1,\ell-1}, \nabla[\Phi]_+\Big) + \Big(\mathbf{U}^{1,\ell-1}[\Phi]_-, \nabla[\Phi]_+\Big). \end{aligned} \tag{4.16}$$

Testing equation (4.15) with $\Phi = \big[(\overline{N}^\pm)^{1,\ell}\big]_+$ implies with (4.16) the inequality

$$\begin{aligned} \frac{1}{k}\Big\|[(\overline{N}^\pm)^{1,\ell}]_+\Big\|_i^2 + \Big\|\nabla[(\overline{N}^\pm)^{1,\ell}]_+\Big\|^2 \leq \Big|\Big(\{[(\overline{N}^\pm)^{1,\ell}]_+ + 1\}\nabla\Psi^{1,\ell-1}, \nabla[(\overline{N}^\pm)^{1,\ell}]_+\Big)\Big| \\ + \Big|\Big(\mathbf{U}^{1,\ell-1}\{[(\overline{N}^\pm)^{1,\ell}]_+ + 1\}, \nabla[(\overline{N}^\pm)^{1,\ell}]_+\Big)\Big|, \end{aligned} \tag{4.17}$$

where we already skipped $\frac{1}{k}\Big[(\overline{N}^\pm)^0, \Big[(\overline{N}^\pm)^{1,\ell}\Big]_+\Big]_i \leq 0$ on the right-hand side, *i.e.*, we use the interpolation of L^3 between L^2 and H^1, and

$$\Big(\mathrm{div}(\mathbf{U}^{1,\ell-1}), [(\overline{N}^\pm)^{1,\ell}]_+\Big) = \Big(\mathrm{div}(\mathbf{U}^{1,\ell-1}), [(\overline{N}^\pm)^{1,\ell}]_+ - \lambda\Big) = 0 \quad \text{where } \lambda = \frac{1}{|\Omega|}\int_\Omega [(\overline{N}^\pm)^{1,\ell}]_+ \, \mathrm{d}\mathbf{x},$$

which uses zero Dirichlet data for the velocity field, and compatibility condition (2.6) to estimate the last term (4.17) as follows

$$\begin{aligned}\Big|\Big(\mathbf{U}^{1,\ell-1}\{[(\overline{N}^{\pm})^{1,\ell}]_+ + 1\}, \nabla[(\overline{N}^{\pm})^{1,\ell}]_+\Big)\Big| &\le \|\mathbf{U}^{1,\ell-1}\|_{L^6}\|[(\overline{N}^{\pm})^{1,\ell}]_+\|_{L^3}\|\nabla[(\overline{N}^{\pm})^{1,\ell}]_+\| \\ &\le C\|\nabla\mathbf{U}^{1,\ell-1}\|\|[(\overline{N}^{\pm})^{1,\ell}]_+\|^{\theta}\|\nabla[(\overline{N}^{\pm})^{1,\ell}]_+\|^{2-\theta} \\ &\le \Big[Ch^{-\frac{\alpha}{2}}\Big(h^{\frac{\alpha}{2}}\|\nabla\mathbf{U}^{1,\ell-1}\|\Big)\Big]^{\frac{2}{\theta}}\Big\|[(\overline{N}^{\pm})^{1,\ell}]_+\Big\|^2 + \frac{1}{4}\Big\|\nabla[(\overline{N}^{\pm})^{1,\ell}]_+\Big\|^2,\end{aligned} \tag{4.18}$$

for $\theta = \frac{6-N}{6}$. The first term on the right-hand side of (4.17) we control with the help of

$$\begin{aligned}\Big(\nabla\Psi^{1,\ell-1}, \nabla[(\overline{N}^{\pm})^{1,\ell}]_+\Big) &= \Big((\overline{N}^{+})^{1,\ell-1} - (\overline{N}^{-})^{1,\ell-1}, [(\overline{N}^{\pm})^{1,\ell}]_+\Big) \\ &\le \frac{1}{2}\Big\|(\overline{N}^{+})^{1,\ell-1} - (\overline{N}^{-})^{1,\ell-1}\Big\|_i^2 + \frac{1}{2}\Big\|(\overline{N}^{\pm})^{1,\ell-1}\Big\|_i^2\end{aligned} \tag{4.19}$$

and the interpolation of L^3 between L^2 and H^1 to obtain the bound ($\theta = \frac{6-N}{6}$)

$$\Big|\Big([(\overline{N}^{\pm})^{1,\ell}]_+\nabla\Psi^{1,\ell-1}, \nabla[(\overline{N}^{\pm})^{1,\ell}]_+\Big)\Big| \le C\|[(\overline{N}^{\pm})^{1,\ell}]_+\|_i^2\|\nabla\Psi^{1,\ell-1}\|_{L^6}^{\frac{2}{\theta}} + \frac{1}{4}\|\nabla[(\overline{N}^{\pm})^{1,\ell}]_+\|_{L^2}^2. \tag{4.20}$$

Inserting now the bounds (4.18) and (4.20) into (4.17) results in

$$\begin{aligned}&\frac{1}{k}\Big\{\frac{1}{2} - Ck\Big(\|\nabla\Psi^{1,\ell-1}\|_{L^6}^{\frac{2}{\theta}} + \Big\|(\overline{N}^{+})^{1,\ell-1} - (\overline{N}^{-})^{1,\ell-1}\Big\|^2\Big) \\ &\quad - Ckh^{\alpha\frac{N-6}{6}}\Big(h^{\alpha}\left\|\nabla\mathbf{U}^{1,\ell-1}\right\|^2\Big)^{\frac{1}{\theta}}\Big\}\|([\overline{N}^{\pm})^{1,\ell}]_+\|_i^2 + \frac{1}{2}\|\nabla([\overline{N}^{\pm})^{1,\ell}]_+\|^2 \le 0.\end{aligned} \tag{4.21}$$

Hence, only $k \le Ch^{\delta}$ for $\delta > 0$ and because of Step 3, $h \le h_0(\Omega)$ small enough is required to validate the assertion which is by induction valid for all $\ell \ge 1$. The estimate (4.21) provides the boundedness of $(\overline{N}^{\pm})^{1,\ell}$ and hence allows the induction step $\ell - 1 \mapsto \ell$. The proof of this Step 4 bases on the results from Step 1 to 3 and allows to inductively obtain a uniformly bounded right hand side in (4.7), *i.e.*

$$\frac{1}{2}\left\|\mathbf{U}^{j,\ell}\right\|^2 + \frac{h^{\alpha}}{2}\left\|\nabla\mathbf{U}^{j,\ell}\right\|^2 + \frac{k}{2}\left\|\nabla\mathbf{U}^{j,\ell}\right\|^2 \le C\Big\{\left\|\mathbf{U}^{j-1}\right\|^2 + h^{\alpha}\left\|\nabla\mathbf{U}^{j-1}\right\|^2 + kC\Big\}. \tag{4.22}$$

Step 5 (contraction property)**.** We define $\mathbf{e}_{\mathbf{u}}^{\ell} := \mathbf{U}^{j,\ell} - \mathbf{U}^{j,\ell-1}$, and for $N^{\pm}$ and Ψ correspondingly $e_{n^{\pm}}^{\ell}$ and e_{ψ}^{ℓ}. First, we consider the terms

$$\left(\mathrm{NP}^{\pm}\right)^{j,\ell} := \Big(\mathbf{U}^{j,\ell-1}(N^{\pm})^{j,\ell}, \nabla e_{n^{\pm}}^{\ell}\Big). \tag{4.23}$$

We control the error term $e_{n^+}^{\ell}$ arising from $\left(\mathrm{NP}^{+}\right)^{j,\ell}$ *via* Hölder's inequalities and inverse estimates by

$$\begin{aligned}\left|\left(\mathrm{NP}^{+}\right)^{j,\ell} - \left(\mathrm{NP}^{+}\right)^{j,\ell-1}\right| &\le \Big|\Big(\mathbf{e}_{\mathbf{u}}^{\ell-1}(N^{+})^{j,\ell}, \nabla e_{n^+}^{\ell}\Big)\Big| + \Big|\Big(\mathbf{U}^{j,\ell-2}e_{n^+}^{\ell}, \nabla e_{n^+}^{\ell}\Big)\Big| \\ &\le C\big\|\mathbf{e}_{\mathbf{u}}^{\ell-1}\big\|^2 + \frac{1}{10}\big\|\nabla e_{n^+}^{\ell}\big\|^2 + Ch^{-\frac{N}{3}}\big\|\mathbf{U}^{j,\ell-2}\big\|_{L^6}^2\|e_{n^+}^{\ell}\|^2 \\ &\le C\|\mathbf{e}_{\mathbf{u}}^{\ell-1}\|^2 + \frac{1}{10}\|\nabla e_{n^+}^{\ell}\|^2 + Ch^{-\frac{N}{3}-\alpha}\|e_{n^+}^{\ell}\|^2,\end{aligned} \tag{4.24}$$

for some $0 < \alpha < \frac{6-N}{3}$, thanks to (4.22). In the same way we treat (NP^-). Next, we estimate errors arising from

$$\Big(\mathrm{NL}^{\pm}\Big)^{j,\ell} := \Big((N^{\pm})^{j,\ell}\nabla\Psi^{j,\ell-1}, \nabla e^{\ell}_{n^{\pm}}\Big)$$

by

$$\begin{aligned}\Big|\Big(\mathrm{NL}^{+}\Big)^{j,\ell} - \Big(\mathrm{NL}^{+}\Big)^{j,\ell-1}\Big| &\le \big|(e^{\ell}_{n^+}\nabla\Psi^{j,\ell-1}, \nabla e^{\ell}_{n^+})\big| + \Big|\Big((N^+)^{j,\ell-1}\nabla e^{\ell-1}_{\psi}, \nabla e^{\ell}_{n^+}\Big)\Big| \\ &\le C\Big[\|e^{\ell-1}_{n^+}\|^2 + \|e^{\ell-1}_{n^-}\|^2\Big] + C\Big[\|\nabla\Psi^{j,\ell-1}\|^4_{L^6} + \frac{1}{2}\Big]\|e^{\ell}_{n^+}\|^2_i + \frac{1}{10}\|\nabla e^{\ell}_{n^+}\|^2 \end{aligned} \tag{4.25}$$

and in the same way for (NL^-). Hence, we obtain for the Nernst-Planck-Poisson system

$$\begin{aligned}[1 - Ck - Ckh^{-\frac{N}{3}-\alpha}]\Big\{\|e^{\ell}_{n^+}\|^2 + \|e^{\ell}_{n^-}\|^2\Big\} + \frac{4k}{5}\Big\{\|\nabla e^{\ell}_{n^+}\|^2 + \|\nabla e^{\ell}_{n^-}\|^2\Big\} &\le Ck\Big\{\|e^{\ell-1}_{n^+}\|^2 + \|e^{\ell-1}_{n^-}\|^2\Big\} \\ &\quad + kC\|\mathbf{e}^{\ell-1}_{\mathbf{u}}\|^2.\end{aligned}$$

It leaves to control the error $\mathbf{e}^{\ell}_{\mathbf{u}}$ governed by the momentum equation (4.2);

$$\begin{aligned}\frac{1}{k}\Big|\Big(\mathbf{e}^{\ell}_{\mathbf{u}}, \mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| + \frac{h^{\alpha}}{k}\Big|\Big(\nabla\mathbf{e}^{\ell}_{\mathbf{u}}, \nabla\mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| + \Big|\Big(\nabla\mathbf{e}^{\ell}_{\mathbf{u}}, \nabla\mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| &\le \Big|\Big(\{e^{\ell-1}_{n^+} - e^{\ell-1}_{n^-}\}\nabla\Psi^{j,\ell-1}, \mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| \\ &\quad + \Big|\Big(\{(N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}\}\nabla e^{\ell-1}_{\psi}, \mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big|, \end{aligned}\tag{4.26}$$

where the convective term disappears by the skew symmetricity. We use Step 4 and (4.1) to control the last term on the right-hand side in (4.26) as follows

$$\begin{aligned}\Big|\Big(\{(N^+)^{j,\ell-1} - (N^-)^{j,\ell-1}\}\nabla e^{\ell-1}_{\psi}, \mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| &\le C\|\mathbf{e}^{\ell}_{\mathbf{u}}\|^2 + \|\nabla e^{\ell-1}_{\psi}\|^2 \\ &\le C\|\mathbf{e}^{\ell}_{\mathbf{u}}\|^2 + C\Big[\|e^{\ell-1}_{n^+}\|^2 + \|e^{\ell-1}_{n^-}\|^2\Big].\end{aligned}\tag{4.27}$$

The following control of the remaining term in (4.26) for $\theta = \frac{6-N}{6}$

$$\begin{aligned}\Big|\Big(\{e^{\ell-1}_{n^+} - e^{\ell-1}_{n^-}\}\nabla\Psi^{j,\ell-1}, \mathbf{e}^{\ell}_{\mathbf{u}}\Big)\Big| &\le \frac{1}{10}\left(\theta\left[\left\|e^{\ell-1}_{n^+}\right\|^2 + \left\|e^{\ell-1}_{n^-}\right\|^2\right] + (1-\theta)\left[\left\|\nabla e^{\ell-1}_{n^+}\right\|^2 + \left\|\nabla e^{\ell-1}_{n^-}\right\|^2\right]\right) \\ &\quad + C\left\|\nabla\Psi^{j,\ell-1}\right\|^2_{L^6}\|\mathbf{e}^{\ell}_{\mathbf{u}}\|^2 \\ &\le \frac{1}{10}\left[\left\|e^{\ell-1}_{n^+}\right\|^2 + \left\|e^{\ell-1}_{n^-}\right\|^2\right] + \frac{N}{60}\left[\left\|\nabla e^{\ell-1}_{n^+}\right\|^2 + \left\|\nabla e^{\ell-1}_{n^-}\right\|^2\right] \\ &\quad + C\left\|\nabla\Psi^{j,\ell-1}\right\|^2_{L^6}\left\|\mathbf{e}^{\ell}_{\mathbf{u}}\right\|^2,\end{aligned}$$

finally implies thanks to (4.22) the inequality

$$\begin{aligned}\left[1 - Ck - Ckh^{-\frac{N}{3}-\alpha}\right]\Big\{\|\mathbf{e}^{\ell}_{\mathbf{u}}\|^2 + \|e^{\ell}_{n^+}\|^2_i + \|e^{\ell}_{n^-}\|^2_i\Big\} + \frac{4k}{5}\Big\{\|\nabla\mathbf{e}^{\ell}_{\mathbf{u}}\|^2 + \|\nabla e^{\ell}_{n^+}\|^2 + \|\nabla e^{\ell}_{n^-}\|^2\Big\} \\ \le kC\Big\{\|\mathbf{e}^{\ell-1}_{\mathbf{u}}\|^2 + \|e^{\ell-1}_{n^+}\|^2_i + \|e^{\ell-1}_{n^-}\|^2_i\Big\} + \frac{3k}{5}\Big\{\|\nabla\mathbf{e}^{\ell-1}_{\mathbf{u}}\|^2 + \|\nabla e^{\ell-1}_{n^+}\|^2 + \|\nabla e^{\ell-1}_{n^-}\|^2\Big\}.\end{aligned}\tag{4.28}$$

Hence, we have the contraction for $k \le k_0(\Omega)$ small enough satisfying the mesh constraint $k \le Ch^{\frac{N}{3}+\alpha}$ for arbitrarily small chosen $0 < \alpha < \frac{6-N}{3}$ due to Step 3.

Step 6 (convergence of Algorithm A_1)**.** Fix $j \ge 1$. In the following, we denote the step that reaches the fixed point for the first time in Algorithm A_1 with $\overline{\ell}$.

Lemma 4.1. *Assume the initial conditions of Definition 3.1*(ii)*. Suppose* (A1), (A2), (2.6)*, fix* $T = t_J > 0$*, and let* $k \leq k_0(\Omega)$ *and* $h \leq h_0(\Omega)$ *be sufficiently small such that* $k \leq Ch^{\frac{N}{3}+\beta}$ *for any* $\beta > 0$*, and additionally* $h^2 \leq Ck$ *if* $i = 1$*. Then for every* $0 \leq j \leq J$*, there exists a unique solution* $\big(\mathbf{U}^{j,\overline{\ell}}, (N^{\pm})^{j,\overline{\ell}}, \Psi^{j,\overline{\ell}}\big) \in \mathbf{V}_h \times \big[Y_h\big]^3$ *of Algorithm* A_1*, such that* $0 \leq (N^{\pm})^{j,\overline{\ell}} \leq 1$*. Moreover,* $\big\{\mathbf{U}^{j,\overline{\ell}}, (N^{\pm})^{j,\overline{\ell}}, \Psi^{j,\overline{\ell}}\big\}_{1\leq j\leq J}$ *satisfies the assertions* (i)–(v) *of Theorem 3.3, where each of the right-hand sides is increased by* $C\theta^2 t_J$*. In addition,* $\big(\mathbf{U}^{j,\overline{\ell}}, (N^{\pm})^{j,\overline{\ell}}, \Psi^{j,\overline{\ell}}\big) \to \big(\mathbf{U}^j, (N^{\pm})^j, \Psi^j\big)$ *as* $\theta \to 0$ *for every* $j \geq 1$*, and the limit solves Scheme* A.

Proof of Lemma 4.1. We first restate the nonlinear terms in Algorithm A_1, *i.e.*, (4.1)–(4.3). First we consider (4.3). The term depending on $\Psi^{j,\ell-1}$ may be restated as

$$\Big((N^{\pm})^{j,\overline{\ell}}\nabla\Psi^{j,\overline{\ell}-1}, \nabla\Phi^{\pm}\Big) = \Big((N^{\pm})^{j,\overline{\ell}}\nabla\Psi^{j,\overline{\ell}}, \nabla\Phi^{\pm}\Big) - \Big((N^{\pm})^{j,\overline{\ell}}\big\{\nabla\Psi^{j,\overline{\ell}} - \nabla\Psi^{j,\overline{\ell}-1}\big\}, \nabla\Phi^{\pm}\Big),$$

where the last term can be controlled by

$$\leq \|(N^{\pm})^{j,\overline{\ell}}\|_{L^\infty}\|\nabla\Psi^{j,\overline{\ell}} - \nabla\Psi^{j,\overline{\ell}-1}\|\|\nabla\Phi^{\pm}\| \leq \|\nabla\Phi^{\pm}\|\theta. \tag{4.29}$$

The second relevant term in (4.3) is rewritten in the following way,

$$-\Big(\mathbf{U}^{j,\overline{\ell}-1}(N^{\pm})^{j,\overline{\ell}}, \nabla\Phi^{\pm}\Big) = -\Big(\mathbf{U}^{j,\overline{\ell}}(N^{\pm})^{j,\overline{\ell}}, \nabla\Phi^{\pm}\Big) + \Big(\big\{\mathbf{U}^{j,\overline{\ell}} - \mathbf{U}^{j,\overline{\ell}-1}\big\}(N^{\pm})^{j,\overline{\ell}}, \nabla\Phi^{\pm}\Big),$$

where the last term, which contains the error $\mathbf{U}^j - \mathbf{U}^{j,\overline{\ell}-1}$, is estimated as

$$\leq \|\mathbf{U}^j - \mathbf{U}^{j,\overline{\ell}-1}\|\|(N^{\pm})^{j,\overline{\ell}}\|_{L^\infty}\|\nabla\Phi^{\pm}\| \leq \|\nabla\Phi^{\pm}\|\theta. \tag{4.30}$$

Consider the equation (4.2). The only relevant term is the Coulomb force $\mathbf{F}_C^{j,\overline{\ell}-1}$ rewritten as

$$\begin{aligned} -\Big(\big((N^+)^{j,\overline{\ell}-1} - (N^-)^{j,\overline{\ell}-1}\big)\nabla\Psi^{j,\overline{\ell}-1}, \mathbf{V}\Big) = &-\Big(\big((N^+)^{j,\overline{\ell}} - (N^-)^{j,\overline{\ell}}\big)\nabla\Psi^{j,\overline{\ell}}, \mathbf{V}\Big)\\ &+\Big(\big\{(N^+)^{j,\overline{\ell}} - (N^+)^{j,\overline{\ell}-1}\big\} - \big\{(N^-)^{j,\overline{\ell}} - (N^-)^{j,\overline{\ell}-1}\big\}\nabla\Psi^{j,\overline{\ell}}, \mathbf{V}\Big)\\ &+\Big(\big\{(N^+)^{j,\overline{\ell}-1} - (N^-)^{j,\overline{\ell}-1}\big\}\nabla\big\{\Psi^{j,\overline{\ell}} - \Psi^{j,\overline{\ell}-1}\big\}, \mathbf{V}\Big), \end{aligned} \tag{4.31}$$

which may be controlled by

$$\begin{aligned} &\leq \Big[\|(N^+)^{j,\overline{\ell}} - (N^-)^{j,\overline{\ell}-1}\|_{L^\infty} + \|(N^-)^{j,\overline{\ell}} - (N^-)^{j,\overline{\ell}-1}\|_{L^\infty}\Big]\|\nabla\Psi^{j,\overline{\ell}}\|\|\mathbf{V}\|\\ &\quad + \|(N^+)^{j,\overline{\ell}-1} - (N^-)^{j,\overline{\ell}-1}\|_{L^\infty}\|\nabla(\Psi^{j,\overline{\ell}} - \Psi^{j,\overline{\ell}-1})\|\|\mathbf{V}\| \leq C\|\mathbf{V}\|\theta. \end{aligned} \tag{4.32}$$

As a consequence, $\Big(\mathbf{U}^{j,\overline{\ell}}, (N^{\pm})^{j,\overline{\ell}}\Big)$ solves Scheme A, with perturbed right-hand sides controllable through (4.4). After passing to the limit $\theta \to 0$, iterates of Algorithm A_1 solve Scheme A. □

Step 7 (properties (i)–(v))**.** For assertion (i) we test equation (3.6) with $\mathbf{V} = \mathbf{U}^j$ and sum it with the Nernst-Planck-Poisson equation (3.7)–(3.8) tested with $\Big(\Phi^+, \Phi^-, \Phi\Big) = \Big(\Psi^j, -\Psi^j, (N^+)^j - (N^-)^j\Big)$. The second assertion uses Step 4 for $\mathbf{V} := \mathbf{U}^j$. The assertion (iii) is verified by testing with $\Phi^{\pm} := (N^{\pm})^j$. To control the discrete time derivatives (iv) for $i = 2$, we use the $H^1(\Omega)$-stability of the L^2-projection $J_{Y_h} : L^2(\Omega) \to Y_h$ and its orthogonality property $\left(\varphi - J_{Y_h}\varphi, \Phi\right) = 0$ for all $\Phi \in Y_h$ and Step 4 to conclude

$$\begin{aligned}
\|d_t(N^+)^j\|_{(H^1)^*} &\leq \sup_{\varphi\in H^1} \frac{\left(d_t(N^+)^j, J_{Y_h}\varphi\right)_h}{\|\varphi\|_{H^1}} + \sup_{\varphi\in H^1} \frac{\left|\left(d_t(N^+)^j, J_{Y_h}\varphi\right) - \left(d_t(N^+), J_{Y_h}\varphi\right)_h\right|}{\|\varphi\|_{H^1}} \\
&\leq C\Big[\|\nabla\Psi^j\| + \|\nabla(N^+)^j\| + \|\mathbf{U}^j\| + Ch\|d_t(N^+)^j\|\Big]. \qquad (4.33)
\end{aligned}$$

Moreover, we conclude from (3.7) and (2.7) that

$$\begin{aligned}
\|d_t(N^+)^j\|_h^2 &\leq \Big[\|\nabla\Psi^j\| + \|\nabla(N^+)^j\| + \|\mathbf{U}^j\|\Big]\|\nabla d_t(N^+)^j\| \\
&\leq C\left[\|\nabla\Psi^j\| + \|\nabla(N^+)^j\| + \|\mathbf{U}^j\|\right] h^{-1}\|d_t(N^+)^j\|. \qquad (4.34)
\end{aligned}$$

Putting (4.33), (4.34) together yields the assertion (iv). Finally, we verify (v); by summing in (3.6) up to times t_j and $t_{j-\ell}$, and subtracting resulting identities leads to

$$\begin{aligned}
\big(\mathbf{U}^j - \mathbf{U}^{j-\ell}, \mathbf{V}\big) + k\sum_{r=0}^{\ell-1}\big(\nabla\mathbf{U}^{j-r}, \nabla\mathbf{V}\big) + \epsilon\big(\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}], \nabla\mathbf{V}\big)& \\
+ k\sum_{r=0}^{\ell-1}\Big\{\Big([\mathbf{U}^{j-1-r}\cdot\nabla]\mathbf{U}^{j-r}, \mathbf{V}\Big) + \frac{1}{2}\Big([\operatorname{div}\mathbf{U}^{j-1-r}]\mathbf{U}^{j-r}, \mathbf{V}\Big)\Big\}& \\
= k\sum_{r=0}^{\ell}\Big([(N^+)^{j-r} - (N^-)^{j-r}]\nabla\psi^{j-r}, \mathbf{V}\Big).&
\end{aligned}$$

Upon choosing $\mathbf{V} = \mathbf{U}^j - \mathbf{U}^{j-\ell}$ for $\ell \geq 1$ leads to

$$k\sum_{j=\ell}^{J}\Big(\|\mathbf{U}^j - \mathbf{U}^{j-\ell}\|^2 + \epsilon\|\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}]\|^2\Big) \leq I + II + III + IV,$$

where

$$\begin{aligned}
I &:= Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}]\|k\sum_{r=0}^{\ell-1}\|\nabla\mathbf{U}^{j-r}\| \\
&\leq Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}]\|\Big(k\sum_{r=0}^{\ell-1}1\Big)^{1/2}\left(k\sum_{r=0}^{\ell-1}\|\nabla\mathbf{U}^{j-r}\|^2\right)^{1/2} \leq C\sqrt{\ell k}.
\end{aligned}$$

Similarly, because of $\|\cdot\|_{L^3} \leq C\|\cdot\|^{1/2}\|\nabla\cdot\|^{1/2}$, we find

$$\begin{aligned}
II &:= Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}]\|k\sum_{r=0}^{\ell-1}\|\mathbf{U}^{j-1-r}\|^{1/2}\|\nabla\mathbf{U}^{j-1-r}\|^{1/2}\|\nabla\mathbf{U}^{j-r}\| \\
&\leq Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j - \mathbf{U}^{j-\ell}]\|\left(k\sum_{r=0}^{\ell-1}1\right)^{1/4}\left(k\sum_{r=0}^{\ell-1}\|\nabla\mathbf{U}^{j-1-r}\|^2\right)^{3/4} \leq C(\ell k)^{1/4},
\end{aligned}$$

thanks to assertions (i), (ii) of Theorem 3.3. Correspondingly,

$$III := Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j-\mathbf{U}^{j-\ell}]\|k\sum_{r=0}^{\ell-1}\|\nabla\mathbf{U}^{j-1-r}\|\|\mathbf{U}^{j-r}\|_{L^3}\leq C(\ell k)^{1/4}.$$

In order to bound IV, we employ L^∞-bounds for charges,

$$\begin{aligned} IV &:= Ck^2\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j-\mathbf{U}^{j-\ell}]\|\sum_{r=0}^{\ell-1}\|\nabla\psi^{j-r}\| \\ &\leq Ck\sum_{j=\ell}^{J}\|\nabla[\mathbf{U}^j-\mathbf{U}^{j-\ell}]\|\left(k\sum_{r=0}^{\ell}1\right)^{1/2}\left(k\sum_{r=0}^{\ell}\|\psi^{j-r}\|^2\right)^{1/2}\leq C(\ell k)^{1/4}. \end{aligned}$$

Putting things together then implies assertion (v). □

The proof of Theorem 3.3 is constructive in the sense that it is achieved by the introduction of the practically valuable Algorithm A_1 that terminates by the contraction property. In the following Section 4.2, we establish the entropy property of Scheme A.

4.2. **Proof of the entropy estimate, Theorem 3.4**

We need the following preliminary estimates which only hold in dimension $N=2$.

Lemma 4.2. *Suppose $n_0^{\pm}\in H^1(\Omega)$, (A1), $N=2$, $i=2$, and $k\leq\tilde{C}h^2$ for $\tilde{C}>0$ sufficiently small. Let $0\leq(N^{\pm})^0\leq 1$ and assume the same requirements as in Theorem 3.3. Then the solution $\big\{\big(\mathbf{U}^j,\,(N^{\pm})^j,\Psi^j\big)\big\}_{j=1}^{J}$ of Scheme A satisfies for every $T=t_J>0$*

$$\begin{aligned} \text{(i)}\quad &\max_{1\leq j\leq J}\left(\frac{1}{2}-Ckh^{-2}\right)\Big[\|\nabla(N^+)^j\|^2+\|\nabla(N^-)^j\|^2\Big]+\frac{k^2}{2}\sum_{j=1}^{J}\Big[\|\nabla d_t(N^+)^j\|^2+\|\nabla d_t(N^-)^j\|^2\Big] \\ &\quad+\frac{k}{2}\sum_{j=1}^{J}\Big[\|\mathcal{L}_h^{(2)}(N^+)^j\|_h^2+\|\mathcal{L}_h^{(2)}(N^-)^j\|_h^2\Big]\leq C\Big[E(\mathbf{U}^0,\Psi^0)+\|\nabla(N^+)^0\|^2+\|\nabla(N^-)^0\|^2\Big], \\ \text{(ii)}\quad &k\sum_{j=1}^{J}\Big[\|d_t(N^+)^j\|^2+\|d_t(N^-)^j\|^2\Big]\leq C\Big[E(\mathbf{U}^0,\Psi^0)+\|\nabla(N^+)^0\|^2+\|\nabla(N^-)^0\|^2\Big], \end{aligned}$$

where (i) *is only uniformly controlled in t_J for $N=2$.*

Proof. (i) Choose $\Phi^{\pm}=-\mathcal{L}_h^{(2)}(N^{\pm})^j$ in (3.7). We compute with Hölder's inequality for the exponents $p_1=2$, $p_2=p_3=4$ and $N=2$

$$\begin{aligned} &\big|\big(\mathbf{U}^j(N^{\pm})^j,-\nabla\mathcal{L}_h^{(2)}(N^{\pm})^j\big)\big|\leq\Big|\Big(\big(\mathrm{div}\mathbf{U}^j\big)\,(N^{\pm})^j,\mathcal{L}_h^{(2)}(N^{\pm})^j\Big)\Big|+\Big|\Big(\mathbf{U}^j\cdot\nabla(N^{\pm})^j,\mathcal{L}_h^{(2)}(N^{\pm})^j\Big)\Big| \\ &\quad\leq C\left\|\nabla\mathbf{U}^j\right\|^2+\frac{1}{4}\left\|\mathcal{L}_h^{(2)}(N^{\pm})^j\right\|_h^2+C\left\|\mathbf{U}^j\right\|\left\|\nabla\mathbf{U}^j\right\|\left\|\nabla(N^{\pm})^j\right\|\left\|\mathcal{L}_h^{(2)}(N^{\pm})^j\right\|+\frac{1}{8}\left\|\mathcal{L}_h^{(2)}(N^{\pm})^j\right\|_h^2 \\ &\quad\leq C\left\|\nabla\mathbf{U}^j\right\|^2+CE(\mathbf{U}^0,\Psi^0)\left\|\nabla\mathbf{U}^j\right\|^2\left\|\nabla(N^{\pm})^j\right\|^2+\frac{1}{2}\left\|\mathcal{L}_h^{(2)}(N^{\pm})^j\right\|_h^2, \end{aligned}$$

where the interpolation inequality

$$\|\boldsymbol{\varphi}\|_{L^4}\leq C\left\|\boldsymbol{\varphi}\right\|^{\frac{1}{2}}\left\|\nabla\boldsymbol{\varphi}\right\|^{\frac{1}{2}}\qquad\text{for all }\boldsymbol{\varphi}\in H^1(\Omega,\mathbb{R}^2)$$

enters. As in [20], Lemma 3.1, we obtain the bound

$$\Big((N^{\pm})^j\nabla\Psi^j, \nabla\mathcal{L}_h^{(2)}(N^{\pm})^j\Big) \le C\left[\big\|(N^+)^j-(N^-)^j\big\|_h^2+\big\|\nabla(N^+)\big\|^4\right]+\frac{1}{4}\left\|\mathcal{L}_h^{(2)}(N^+)^j\right\|_h^2.$$

Adding up everything results in

$$\begin{aligned}&\Big(\frac{1}{2}-Ckh^{-2}\Big)\Big[\|\nabla(N^+)^J\|^2+\|\nabla(N^-)^J\|^2\Big]+\frac{k^2}{2}\sum_{j=1}^{J}\Big[\|\nabla d_t(N^+)^j\|^2+\|\nabla d_t(N^-)^j\|^2\Big]\\&\quad+\frac{k}{2}\sum_{j=1}^{J}\Big[\|\mathcal{L}_h^{(2)}(N^+)^j\|_h^2+\|\mathcal{L}_h^{(2)}(N^-)^j\|_h^2\Big]\\&\quad\le Ck\sum_{j=1}^{J-1}\Big\{\|\mathbf{U}^j\|^2+\|\nabla\mathbf{U}^j\|^2+CE(\mathbf{U}^0,\Psi^0)\|\nabla\mathbf{U}^j\|^2\left[\big\|\nabla(N^+)^j\big\|^2+\big\|\nabla(N^-)^j\big\|^2\right]\\&\quad+\|(N^+)^j-(N^-)^j\|_h^2+\|\nabla(N^+)^j\|^4+\|\nabla(N^-)^j\|^4\Big\}+\frac{1}{2}\left\{\|\nabla(N^+)^0\|^2+\|\nabla(N^-)^0\|^2\right\}\\&\quad\le C\exp\Big[Ck\sum_{j=1}^{J-1}\left\{\|\nabla(N^+)^j\|^2+\|\nabla(N^-)^j\|^2\right\}\Big]\Big\{E(\mathbf{U}^0,\Psi^0)+\|\nabla(N^+)^0\|^2+\|\nabla(N^-)^0\|^2\Big\},\end{aligned}$$

where the last inequality follows from Theorem 3.3, (iii), together with the discrete Gronwall inequality. Moreover, the right hand side is uniformly bounded in time due to assertion (iii) of Theorem 3.3.
(ii) Choose $\Phi^+ = d_t(N^+)^j$ in (3.7) and then treat the terms on the right hand side as in (i). □

Now, we can give the proof for the entropy inequality. For this purpose, we borrow arguments from [20]. Similar to Step 4 in the proof of Theorem 3.3 we obtain the following assertion:

Claim. $(N^{\pm})^0 \ge \delta$ implies $(N^{\pm})^j \ge \delta$ for all $j \ge 0$.

Proof. Let $(\overline{N}^{\pm})^j := (N^{\pm})^j - \delta$. We obtain from (3.7)

$$\begin{aligned}&\left[d_t(\overline{N}^{\pm})^j,\Phi^{\pm}\right]_i+\left(\nabla(\overline{N}^{\pm})^j,\nabla\Phi^{\pm}\right)\pm\left(\{(\overline{N}^{\pm})^j+\delta\}\nabla\Psi^j,\nabla\Phi^{\pm}\right)\\&-\Big(\mathbf{U}^j\{(\overline{N}^{\pm})^j+\delta\},\nabla\Phi^{\pm}\Big)=0\qquad\forall\Phi^{\pm}\in Y_h.\end{aligned}$$

Because of (2.6), there holds $(\mathbf{U}^j,\nabla\Phi^{\pm}) = 0$ for all $\Phi^{\pm}\in Y_h$. Moreover, (3.8) implies

$$\left(\nabla\Psi^j,\nabla\Phi^{\pm}\right)=\left[(\overline{N}^+)^j-(\overline{N}^-)^j,\Phi^{\pm}\right]_i.$$

As a consequence,

$$\begin{aligned}&\frac{1}{k}\left[(\overline{N}^{\pm})^j,\Phi^{\pm}\right]_i+\left(\nabla(N^{\pm})^j,\nabla\Phi^{\pm}\right)+\Big((\overline{N}^{\pm})^j\nabla\Psi^j,\nabla\Phi^{\pm}\Big)+\Big(\mathbf{U}^j\cdot(\overline{N}^{\pm})^j,\nabla\Phi^{\pm}\Big)\\&\qquad=\frac{1}{k}\left[(\overline{N}^{\pm})^{j-1},\Phi^{\pm}\right]_i\pm\delta\left[(\overline{N}^-)^j-(\overline{N}^+)^j,\Phi^{\pm}\right]_i.\end{aligned}$$

Let now $\Phi^{\pm} = \left[(\overline{N}^{\pm})^j\right]_+$, and note that $\{\mathbf{U}^j\}_{j\ge0}$, $\{\Psi^j\}_{j\ge0}$ are already known to exist and satisfy the properties listed in Theorem 3.3. We may then follow the (linear algebra) arguments below (4.16) to show the claim. □

Now, we may choose $\Phi^+ = \mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j$ in (3.7),

$$\Big[d_t(N^+)^j, F'((N^+)^j)\Big]_2 + \big[d_t(N^+)^j, \Psi^j\big]_2 + \big(\mathbf{U}^j(N^+)^j, \nabla\Phi^\pm\big)$$
$$= -\Big((N^+)^j\nabla\Psi^j, \nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\Big) - \Big(\nabla(N^+)^j, \nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\Big). \quad (4.35)$$

Afterwards we can repeat the same steps by choosing $\Phi^- = \mathcal{I}_h\big[F'((N^-)^j)\big] + \Psi^j$ in (3.7). We use the identity $(N^+)^j\nabla F'((N^+)^j) = \nabla(N^+)^j$ to estimate the right hand side of (4.35)

$$\begin{aligned} &= -\Big((N^+)^j\nabla\big\{F'((N^+)^j) + \Psi^j\big\}, \nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\Big) \\ &\leq -\Big((N^+)^j, \big|\nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\big|^2\Big) \\ &\quad + \|\nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\|_{L^2}\Big[\|\nabla\big\{F'((N^+)^j) - \mathcal{I}_h\big[F'((N^+)^j)\big]\big\}\|_{L^2}\Big]. \end{aligned}$$

We employ $W^{1,2}$-stability of the interpolation operator to bound the first factor of the last term by $2\big[E(\Psi^0) + \delta^{-2}\|\nabla(N^+)^j\|^2\big]$. For the second factor, we use standard interpolation estimates for each element $K \in \mathcal{T}_h$, and $D^2(N^+)^j\big|_K = 0$ for all $K \in \mathcal{T}_h$,

$$\begin{aligned} \Big(\sum_{K\in\mathcal{T}_h}\|\nabla\big\{F'((N^+)^j) - \mathcal{I}_h\big[F'((N^+)^j)\big]\big\}\|^2_{L^2(K)}\Big)^{1/2} &\leq Ch\Big(\sum_{K\in\mathcal{T}_h}\|D^2F'((N^+)^j)\|^2_{L^2(K)}\Big)^{1/2} \\ &\leq Ch\delta^{-2}\|\nabla(N^+)^j\|^2_{L^4}. \end{aligned} \quad (4.36)$$

The remaining term in (4.35) is controlled as follows,

$$\begin{aligned} &\Big(\mathbf{U}^j(N^+)^j, \nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] + \Psi^j\big\}\Big) + \Big(\mathbf{U}^j(N^-)^j, \nabla\big\{\mathcal{I}_h\big[F'((N^-)^j)\big] - \Psi^j\big\}\Big) \\ &\quad - \Big(\big((N^+)^j - (N^-)^j\big)\nabla\Psi^j, \mathbf{U}^j\Big) \\ &= \Big(\mathbf{U}^j(N^+)^j, \nabla\mathcal{I}_h\big[F'((N^+)^j)\big]\Big) + \Big(\mathbf{U}^j(N^-)^j, \nabla\mathcal{I}_h\big[F'((N^-)^j)\big]\Big) \\ &= \Big(\mathbf{U}^j(N^+)^j, \nabla\big\{\mathcal{I}_h\big[F'((N^+)^j)\big] - F'((N^+)^j)\big\}\Big) \\ &\quad + \Big(\mathbf{U}^j(N^-)^j, \nabla\big\{\mathcal{I}_h\big[F'((N^-)^j)\big] - F'((N^-)^j)\big\}\Big) \\ &\quad + \Big(\mathbf{U}^j(N^+)^j, \nabla F'((N^+)^j)\Big) + \Big(\mathbf{U}^j(N^-)^j, \nabla F'((N^-)^j)\Big) \\ &\leq \|(N^+)^j\|_{L^\infty}\|\mathbf{U}^j\|Ch\delta^{-2}\|\nabla(N^+)^j\|^2_{L^4} + \|(N^-)^j\|_{L^\infty}\|\mathbf{U}^j\|Ch\delta^{-2}\|\nabla(N^-)^j\|^2_{L^4}. \end{aligned} \quad (4.37)$$

Here, again (4.36) enters in the last inequality, and we use $(N^\pm)^j\nabla F'((N^\pm)^j) = \nabla(N^\pm)^j$ together with

$$\Big(\mathbf{U}^j, \nabla\big((N^+)^j + (N^-)^j\big)\Big) = -\Big(\operatorname{div}\mathbf{U}^j, \big((N^+)^j + (N^-)^j\big)\Big) = 0, \quad (4.38)$$

which employs the compatibility property (2.6) of M_h and Y_h. The control on the norms $\|\nabla(N^\pm)^j\|^2_{L^4}$ is given by Lemma 4.2 and the discrete Sobolev estimate $\big\|\nabla(N^\pm)^j\big\|_{L^6} \leq C\Big(\Big\|\mathcal{L}_h^{(2)}(N^\pm)^j\Big\| + \big\|(N^\pm)^j\big\|_{H^1}\Big)$, see (2.10). Now putting (4.35) and (4.37) together, summing up over iteration steps yields the entropy law (3.13).

4.3. **Proof of the convergence of Scheme A, Theorem 3.5**

Step 1 (extraction of convergent subsequences)**.** The *a priori* estimates achieved in Theorem 3.3 allow to apply well-established standard results to conclude convergence of a subsequence to a weak solution in the sense of Definition 3.1. For $k, h \to 0$ as given in Theorem 3.3, there exist convergent subsequences such that

$$\begin{aligned}
&\overline{\mathcal{N}}^{\pm}, \underline{\mathcal{N}}^{\pm}, \mathcal{N}^{\pm} \rightharpoonup \hat{n}^{\pm} && \text{in } L^2\Big(0,T;H^1(\Omega)\Big) \cap W^{1,2}\Big(0,T;\big(H^1(\Omega)\big)^*\Big),\\
&\overline{\mathcal{N}}^{\pm}, \underline{\mathcal{N}}^{\pm}, \mathcal{N}^{\pm} \overset{*}{\rightharpoonup} \hat{n}^{\pm} && \text{in } L^\infty\big(0,T;L^\infty(\Omega)\big),\\
&\overline{\mathcal{N}}^{\pm}, \underline{\mathcal{N}}^{\pm}, \mathcal{N}^{\pm} \to \hat{n}^{\pm} && \text{in } L^2(\Omega_T),\\
&\nabla\overline{\mathbf{\Psi}},\, \nabla\underline{\mathbf{\Psi}},\, \nabla\mathbf{\Psi} \overset{*}{\rightharpoonup} \nabla\hat{\psi} && \text{in } L^\infty\Big(0,T;L^2(\Omega,\mathbb{R}^N)\Big),\\
&\overline{\boldsymbol{\mathcal{U}}}, \underline{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{U}} \overset{*}{\rightharpoonup} \hat{\mathbf{u}} && \text{in } L^\infty(0,T;L^2(\Omega,\mathbb{R}^N)),\\
&\overline{\boldsymbol{\mathcal{U}}}, \underline{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{U}} \rightharpoonup \hat{\mathbf{u}} && \text{in } L^2(0,T;H^1(\Omega,\mathbb{R}^N)),\\
&\overline{\boldsymbol{\mathcal{U}}}, \underline{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{U}} \to \hat{\mathbf{u}} && \text{in } L^2(0,T;L^2(\Omega,\mathbb{R}^N)),
\end{aligned} \tag{4.39}$$

where the property $(4.39)_3$ is a consequence of Aubin-Lions' compactness result applicable because of assertions (iii), (iv) of Theorem 3.3. Further, since for $t \in [t_{j-1}, t_j]$

$$\boldsymbol{\mathcal{U}} - \overline{\boldsymbol{\mathcal{U}}} = \boldsymbol{\mathcal{U}} - \mathbf{U}^j = \frac{t-t_j}{k}\Big(\mathbf{U}^j - \mathbf{U}^{j-1}\Big),$$

we have the relation

$$\begin{aligned}
\|\boldsymbol{\mathcal{U}} - \overline{\boldsymbol{\mathcal{U}}}\|^2_{L^2(0,T;L^2)} &= \sum_{j=1}^J \left\{ \|\mathbf{U}^j - \mathbf{U}^{j-1}\|^2 \int_{t_{j-1}}^{t_j} \Big(\frac{t-t_j}{k}\Big)^2 \,\mathrm{d}t \right\}\\
&= \frac{k}{3}\sum_{j=1}^J \|\mathbf{U}^j - \mathbf{U}^{j-1}\|^2 = \frac{k^3}{3}\sum_{j=1}^J \|d_t \mathbf{U}^j\|^2,
\end{aligned} \tag{4.40}$$

which tends to zero for $k \to 0$ thanks to Theorem 3.3, (ii). Hence, sequences $\{\boldsymbol{\mathcal{U}}\}$, $\{\overline{\boldsymbol{\mathcal{U}}}\}$, and $\{\underline{\boldsymbol{\mathcal{U}}}\}$ converge to the same limit as $h,\, k \to 0$. Statement $(4.39)_7$ is a consequence of Theorem 3.3, (i), (v), according to which there exists $\kappa \in (0,1)$ such that

$$\int_\delta^T \|\boldsymbol{\mathcal{U}}(t,\cdot) - \boldsymbol{\mathcal{U}}(t-\delta,\cdot)\|^2 \,\mathrm{d}t \le C\delta^\kappa \qquad \forall \delta \in [0,T].$$

A result of Lions [13] and Lions and Magenes [14,23] then implies compactness of this sequence in $L^2(\Omega_T, \mathbb{R}^N)$ as stated in $(4.39)_7$.

Step 2 (passing to the limit)**.** We may restate (3.6) for any $\mathbf{v} \in \tilde{\boldsymbol{\mathcal{D}}}$, with $\mathbf{V} := J_{\mathbf{V}_h}\mathbf{v} \in \mathbf{V}_h$, which satisfies $\mathbf{V} \to \mathbf{v}$ in $\mathbf{W}^{1,p}(\Omega)\,(h \to 0)$, for all $p \ge 1$, and and $\omega(t)$ a continuously differentiable function on $[0,T]$ with $\omega(T) = 0$ in the following way: For every $t > 0$, find $\boldsymbol{\mathcal{U}}(t,\cdot) \in \mathbf{V}_h$ such that

$$\Big((\boldsymbol{\mathcal{U}})_t, \omega(t)\mathbf{V}\Big) + \Big(\nabla\overline{\boldsymbol{\mathcal{U}}}, \omega(t)\nabla\mathbf{V}\Big) + h^\alpha\Big(\nabla(\boldsymbol{\mathcal{U}})_t, \omega(t)\nabla\mathbf{V}\Big) + \Big((\underline{\boldsymbol{\mathcal{U}}}\cdot\nabla)\overline{\boldsymbol{\mathcal{U}}}, \omega(t)\mathbf{V}\Big) = \Big(\overline{\mathbf{F}}_C, \omega(t)\mathbf{V}\Big), \tag{4.41}$$

where $\overline{\mathbf{F}}_C := -\left(\overline{\mathcal{N}}^+ - \overline{\mathcal{N}}^-\right)\nabla\overline{\boldsymbol{\Psi}}$. Integrate (4.41) in t, and integrate the first and third term by parts to get

$$\int_0^T \Big\{ -\Big(\boldsymbol{\mathcal{U}}, \omega'(t)\mathbf{V}\Big) + \Big(\nabla\overline{\boldsymbol{\mathcal{U}}}, \omega(t)\nabla\mathbf{V}\Big) - h^\alpha\Big(\nabla\boldsymbol{\mathcal{U}}, \omega'(t)\nabla\mathbf{V}\Big) + \Big((\underline{\boldsymbol{\mathcal{U}}}\cdot\nabla)\overline{\boldsymbol{\mathcal{U}}}, \omega(t)\mathbf{V}\Big)\Big\}\,\mathrm{d}t$$
$$= \Big(\boldsymbol{\mathcal{U}}(0), \omega(0)\mathbf{V}\Big) + h^\alpha\Big(\nabla\boldsymbol{\mathcal{U}}(0), \omega(0)\nabla\mathbf{V}\Big) + \int_0^T \Big(\overline{\mathbf{F}}_C, \omega(t)\mathbf{V}\Big)\,\mathrm{d}t. \quad (4.42)$$

We now pass to the limit in (4.42) with the sequence $h, k \to 0$ using essentially (4.39). In the limit we find

$$\int_0^T \Big\{ -\Big(\mathbf{u}, \omega'(t)\mathbf{v}\Big) + \Big(\nabla\mathbf{u}, \omega(t)\nabla\mathbf{v}\Big) + \Big((\mathbf{u}\cdot\nabla)\mathbf{u}, \omega(t)\mathbf{v}\Big)\Big\}\,\mathrm{d}t = \Big(\mathbf{u}_0, \omega(0)\mathbf{v}\Big) + \int_0^T \Big(\mathbf{F}_C, \omega(t)\mathbf{v}\Big)\,\mathrm{d}t, \quad (4.43)$$

provided that $\lim_{h\to 0} h^{\alpha/2}\,\|\nabla\boldsymbol{\mathcal{U}}(0)\| = 0$. Now writing, in particular, (4.43) with $\omega \in C_0^\infty([0,T))$ we see that $\mathbf{u}$ satisfies (1.1) in the sense of distributions and by density also in the weak sense. We remark that passing to the limit in $(\mathrm{div}\boldsymbol{\mathcal{U}}, Q) = 0$ for all $Q \in M_h$ is possible due to $(4.39)_6$ such that for $h \to 0$ we obtain with (2.11), and an approximation result

$$(\mathrm{div}\boldsymbol{\mathcal{U}}, J_{M_h}q) \qquad \to \qquad (\mathrm{div}\,\mathbf{u}, q) \qquad \forall q \in L_0^2(\Omega).$$

Finally, it remains to prove that $\mathbf{u}(0) = \mathbf{u}_0$. For this we multiply (1.1) by $\omega\mathbf{v}$ and integrate. After integrating the first term by parts, we get

$$\int_0^T \Big\{ -\Big(\mathbf{u}, \omega'(t)\mathbf{v}\Big) + \Big(\nabla\mathbf{u}, \omega(t)\nabla\mathbf{v}\Big) + \Big((\mathbf{u}\cdot\nabla)\mathbf{u}, \omega(t)\mathbf{v}\Big)\Big\}\,\mathrm{d}t = \Big(\mathbf{u}(0), \omega(0)\mathbf{v}\Big) + \int_0^T \Big(\mathbf{F}_C, \omega(t)\mathbf{v}\Big)\,\mathrm{d}t.$$

By comparison with (4.43),

$$(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v})\,\omega(0) = 0.$$

We can choose ω with $\omega(0) = 1$; thus

$$(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) = 0 \qquad \text{for all } \mathbf{v} \in \mathbf{V}^{1,2}.$$

The convergence of the Nernst-Planck-Poisson part may be verified as in [20] where the additional convective term satisfies for $k,\ h \to 0$

$$\int_0^T \left(\overline{\boldsymbol{\mathcal{U}}}\overline{\mathcal{N}}^\pm, \nabla J_{Y_h}\phi^\pm\right)\,\mathrm{d}t \to \int_0^T \left(\mathbf{u}n^\pm, \nabla\phi^\pm\right)\,\mathrm{d}t \qquad \text{for all } \phi^\pm \in H^1(\Omega),$$

by $(4.39)_3$ and $(4.39)_7$. Finally, in the sense of an overall-convergence in the Algorithm A_1 we may let $\theta, h, k \to 0$. As a consequence, the solutions of Algorithm A_1 converge to weak solutions of the system (1.1)–(1.10).

5. Analysis of Scheme B and proofs

5.1. Semi-discretization in time, Theorem 3.7

Since each step of Scheme B introduces different discretization, splitting, and perturbation errors, we introduce suitable auxiliary problems to analyze the ongoing error behavior of the proposed scheme, and we verify the properties $(P1)$ and $(P2)_l$, $l \in \{0,1\}$ (see Sect. 3) for each of it.

Auxiliary Problem *A***.** We analyze the error originating from the fully implicit time discretization.

Let the initial data $\big(\mathbf{u}_A^0,\,(n^\pm)_A^0\big)$ be given by (3.14), determine $\big\{\mathbf{u}_A^j,\,p_A^j,\,(n^\pm)_A^j,\,\psi_A^j\big\}_{j=1}^J \subset \mathbf{S}$ that solves

$$d_t\mathbf{u}_A^j - \Delta\mathbf{u}_A^j + (\mathbf{u}_A^j\cdot\nabla)\mathbf{u}_A^j + \nabla p_A^j = -\big((n^+)_A^j - (n^-)_A^j\big)\nabla\psi_A^j, \tag{5.1}$$

$$\operatorname{div}\mathbf{u}_A^j = 0, \tag{5.2}$$

$$d_t(n^\pm)_A^j \mp \operatorname{div}\big((n^\pm)_A^j\nabla\psi_A^j\big) - \Delta(n^\pm)_A^j + \mathbf{u}_A^j\cdot\nabla(n^\pm)_A^j = 0, \tag{5.3}$$

$$-\Delta\psi_A^j = (n^+)_A^j - (n^-)_A^j. \tag{5.4}$$

We gather the results concerning this auxiliary Problem A in Lemma 5.3.

Auxiliary Problem B. This auxiliary problem analyses the error caused by the semi-implicit coupling of the Coulomb force term in the Navier-Stokes equation (5.5) and the concentration equations (5.7), as well as a semi-implicit treatment of the convective terms.

Let the initial data $\big(\mathbf{u}_B^0,\,(n^\pm)_B^0\big)$ be given by (3.14), determine $\big\{\mathbf{u}_B^j,\,p_B^j,\,(n^\pm)_B^j,\,\psi_B^j\big\}_{j=1}^J \subset \mathbf{S}$ that solves

$$d_t\mathbf{u}_B^j - \Delta\mathbf{u}_B^j + (\mathbf{u}_B^j\cdot\nabla)\mathbf{u}_B^{j-1} + \nabla p_B^j = -\big((n^+)_B^j - (n^-)_B^j\big)\nabla\psi_B^{j-1}, \tag{5.5}$$

$$\operatorname{div}\mathbf{u}_B^j = 0, \tag{5.6}$$

$$d_t(n^\pm)_B^j \mp \operatorname{div}\big((n^\pm)_B^j\nabla\psi_B^{j-1}\big) - \Delta(n^\pm)_B^j + \mathbf{u}_B^{j-1}\cdot\nabla(n^\pm)_B^j = 0, \tag{5.7}$$

$$-\Delta\psi_B^{j-1} = (n^+)_B^{j-1} - (n^-)_B^{j-1}. \tag{5.8}$$

The results on convergence and stability behavior are collected in Lemma 5.4.

Auxiliary Problem C. This problem investigates the influence of Chorin's projection scheme.

Let the initial data $\big(\mathbf{u}_C^0,\,(n^\pm)_C^0\big)$ be given by (3.14), and let $\big\{(n^\pm)_C^j,\,\psi_C^{j-1}\big\}_{j=1}^J$ be given by Problem B, compute the iterates $\big(\mathbf{u}_C^j,\,p_C^j\big) \in H_0^1(\Omega,\mathbb{R}^N)\times(H^1\cap L_0^2)(\Omega)$ that solve

$$d_t\mathbf{u}_C^j - \Delta\mathbf{u}_C^j + (P_{\mathbf{V}^{0,2}}\mathbf{u}_C^{j-1}\cdot\nabla)\mathbf{u}_C^j + \nabla p_C^{j-1} = -\big((n^+)_B^j - (n^-)_B^j\big)\nabla\psi_B^{j-1}, \tag{5.9}$$

$$\operatorname{div}\mathbf{u}_C^j - k\Delta p_C^j = 0, \quad \partial_{\mathbf{n}}p_C^j|_{\partial\Omega} = 0, \tag{5.10}$$

where $P_{\mathbf{V}^{0,2}}$ denotes the L^2-projection onto the space $\mathbf{V}^{0,2}$.

Results concerning the analysis of Problem C are presented in Lemma 5.6.

Auxiliary Problem D. Chorin's projection method causes some recoupling effects which originate from a semi-explicit treatment of concentrations and velocity field. We remark that the notation $\mathbf{u}_D^j$ corresponds to $\tilde{\mathbf{u}}^j$ used in Scheme B.

For initial data $\big(\mathbf{u}_D^0,\,(n^\pm)_D^0\big)$ given by (3.14), determine $\mathbf{u}_D^j \in H_0^1(\Omega,\mathbb{R}^N)$ and $\big\{p_D^j,\,(n^\pm)_D^j,\,\psi_D^j\big\} \subset \mathbf{S}$ that solve

$$d_t\mathbf{u}_D^j - \Delta\mathbf{u}_D^j + (P_{\mathbf{V}^{0,2}}\mathbf{u}_D^{j-1}\cdot\nabla)\mathbf{u}_D^j + \nabla p_D^{j-1} = -\big((n^+)_D^j - (n^-)_D^j\big)\nabla\psi_D^{j-1}, \tag{5.11}$$

$$\operatorname{div}\mathbf{u}_D^j - k\Delta p_D^j = 0, \quad \partial_{\mathbf{n}}p_D^j|_{\partial\Omega} = 0, \tag{5.12}$$

$$d_t(n^\pm)_D^j \pm \operatorname{div}\big((n^\pm)_D^j\nabla\psi_D^{j-1}\big) - \Delta(n^\pm)_D^j + (P_{V^{0,2}}\mathbf{u}_D^{j-1})\cdot\nabla(n^\pm)_D^j = 0, \tag{5.13}$$

$$-\Delta\psi_D^{j-1} = (n^+)_D^{j-1} - (n^-)_D^{j-1}. \tag{5.14}$$

Lemma 5.7 provides both, stability and convergence results concerning Problem D.

Chorin's projection method has been analyzed in [18,19]. The right hand side of equation (1.1) satisfies $\mathbf{f}_C := \big(n^+ - n^-\big)\nabla\psi \in W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^N))$; *cf.* Definition 3.6. This allows to apply results for Chorin's projection scheme to solve the incompressible Navier-Stokes equations from [18]. The proof of the following result can be found in [18], Chapter 6, which requires $\mathbf{f}_C \in W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^N))$.

Lemma 5.1. *Assume* (A1)*, the initial and boundary conditions of Definition* 3.6*,* $\mathbf{u}_0 \in \mathbf{V}^{1,2}(\Omega, \mathbb{R}^N) \cap H^2(\Omega, \mathbb{R}^N)$*,* $n_0^\pm \in H^2(\Omega)$*, and* $\mathbf{f}_C \in W^{1,\infty}(0, T; L^2(\Omega, \mathbb{R}^N))$*. Let* $\{\tilde{\mathbf{u}}^j, p^j\}_{j=1}^J$ *be the (semi-)discrete solution of Chorin's method,* i.e.*, Steps* 4 *and* 5 *of Scheme* B *are accordingly adjusted, and* $\{\mathbf{u}(t_j), p(t_j)\}_{j=1}^J$ *is the strong solution of the Navier-Stokes equations* (1.1)*, for times* $0 < t_j < t_J$*. Then, for sufficiently small time-steps* $k \le k_0(t_J)$ *and* $\tau^j := \min\{1, t_j\}$*, there exists a constant* C *which only depends on the data of the problem, such that the following holds:*

1. *convergence estimates*

$$\max_{1\le j\le J}\left\{\sqrt{\tau^j}\|\mathbf{u}(t_j) - \tilde{\mathbf{u}}^j\| + \tau^j\|p(t_j) - p^j\|_{H^{-1}}\right\} \le Ck, \tag{5.15}$$

$$\max_{1\le j\le J}\left\{\|\mathbf{u}(t_j) - \tilde{\mathbf{u}}^j\|_{H^1} + \sqrt{\tau^j}\|p(t_j) - p^j\|\right\} \le C\sqrt{k}; \tag{5.16}$$

2. *stability result*

$$\max_{1\le j\le J}\left\{\|d_t\tilde{\mathbf{u}}^j\| + \|\tilde{\mathbf{u}}^j\|_{H^2} + \|p^j\|_{H^1}\right\} + k\sum_{j=1}^J \|d_t\tilde{\mathbf{u}}^j\|_{H^1}^2 \le C. \tag{5.17}$$

5.2. *A priori* estimates for the continuous problem (1.1)–(1.10)

The results on strong solutions in [22] immediately imply:

Lemma 5.2. *Let* $\{\mathbf{u}, p, n^\pm, \psi_0\} \in \mathbf{S}$ *be the strong solution of* (1.1)–(1.10) *for initial and boundary data required in Definition* 3.6*, and* $\mathbf{u}_0 \in \mathbf{V}^{1,2} \cap H^2(\Omega, \mathbb{R}^N)$*,* $n_0^\pm \in H^2(\Omega)$*. Then we have the following* a priori *bounds,*

$$\sup_{(0,t_J]}\left\{\|\mathbf{u}_t\|^2 + \|n_t^\pm\|^2 + \|\mathbf{u}\|_{H^2}^2 + \|n^\pm\|_{H^2}^2\right\} + \int_0^{t_J}\left\{\|\nabla\mathbf{u}_t\|^2 + \|\nabla n_t^\pm\|^2\right\}\mathrm{d}s \le C. \tag{5.18}$$

The analysis in the next Section 5.3 requires higher time-derivatives of $\mathbf{u}$ and $n^\pm$. We refer to the cited literature for the needed standard arguments which yield to

$$\int_0^{t_J} \|\mathbf{u}_{tt}\|_{\mathbf{V}^{-1,2}}^2 + \|n_{tt}^\pm\|_{(H^1)^*}^2\,\mathrm{d}s \le C. \tag{5.19}$$

The following sections provide main arguments which validate property $(P1)$ for every auxiliary Problem A through D and $(P2)_0$ for the Problems A and B, and $(P2)_1$ for C and D.

5.3. Properties of the auxiliary Problems *A* through *D*

It is well-understood how to treat the convective terms in the convergence analysis of splitting strategies based on Chorin's projection method, see [18], Chapter 6, for the sake of better readability of the proofs we will skip the convective term.

Lemma 5.3 (Problem A)**.** *The solution to Problem* A *satisfies the properties* $(P1)$ *and* $(P2)_0$ *for sufficiently small time-steps* $k \le k_0(t_J)$*.*

Proof. The property $(P1)$ is immediately verified by means of arguments that are used for the *a priori* estimates, see Lemma 5.2. Moreover, the *a priori* bounds for the auxiliary problem are obtained as in Theorem 3.3 due to its fully implicit structure. We introduce the abbreviations

$$\mathbf{e}^j := \mathbf{u}(t_j) - \mathbf{u}_A^j, \qquad \pi^j := p(t_j) - p_A^j,$$
$$(\eta^\pm)^j = n^\pm(t_j) - (n^\pm)_A^j, \qquad \zeta^j := \psi(t_j) - \psi_A^j.$$

The corresponding error equations are

$$d_t\mathbf{e}^j - \Delta\mathbf{e}^j + \nabla\pi^j = R^j(\mathbf{u}) - \big((\eta^+)^j - (\eta^-)^j\big)\nabla\psi(t_j) - \big((n^+)^j_A - (n^-)^j_A\big)\nabla\zeta^j, \tag{5.20}$$

$$\operatorname{div}\mathbf{e}^j = 0, \tag{5.21}$$

$$d_t(\eta^\pm)^j - \Delta(\eta^\pm)^j \mp \operatorname{div}\big((\eta^\pm)^j\nabla\psi(t_j)\big) \mp \operatorname{div}\big((n^\pm)^j_A\nabla\zeta^j\big) = R^j(n^\pm), \tag{5.22}$$

$$-\Delta\zeta^j = (\eta^+)^j - (\eta^-)^j, \tag{5.23}$$

where for $\varphi = n^\pm$ or $\mathbf{u}$, we set

$$R^j(\varphi) := -\frac{1}{k}\int_{t_{j-1}}^{t_j}(s-t_j)\varphi_{tt}(s)\,\mathrm{d}s. \tag{5.24}$$

If we test (5.20) with $\mathbf{e}^j$, (5.22) with $(\eta^\pm)^j$, and (5.23) with ζ^j, we obtain for $\theta = \frac{6-N}{6}$

$$\begin{aligned}
&d_t\Big\{\|\mathbf{e}^j\|^2 + \|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\Big\} + k\Big\{\|d_t\mathbf{e}^j\|^2 + \|d_t(\eta^+)^j\|^2 + \|d_t(\eta^-)^j\|^2\Big\}\\
&\quad + \frac{2}{5}\Big\{\|\nabla\mathbf{e}^j\|^2 + \|\nabla(\eta^+)^j\|^2 + \|\nabla(\eta^-)^j\|^2\Big\}\\
&\le C\Big\{\Big[\|R^j(\mathbf{u})\|^2_{\mathbf{V}^{-1,2}} + \|R^j(\eta^+)\|^2_{(H^1)^*} + \|R^j(\eta^-)\|^2_{(H^1)^*}\Big]\\
&\quad + \Big[\|\psi(t_j)\|^2_{H^2} + \|\psi(t_j)\|^{\frac{2}{\theta}}_{H^2} + \|(n^+)^j_A\|^2_{H^1} + \|(n^-)^j_A\|^2_{H^1} + 1\Big]\big(\|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\big)\Big\}
\end{aligned} \tag{5.25}$$

by the Sobolev embedding $\|\mathbf{u}(t_j)\|_{L^\infty} \le C\|\mathbf{u}(t_j)\|_{H^2}$, the Hölder inequality for the exponents $p_1 = 2$, $p_2 = 3$, $p_3 = 6$ and the Sobolev inequalities. We only give the estimate of the special term that requires the Gagliardo-Nirenberg inequality for $\theta = \frac{6-N}{6}$. We have

$$\begin{aligned}
\Big|\Big((\eta^\pm)^j\nabla\psi(t_j), \nabla(\eta^\pm)^j\Big)\Big| &\le C\|\psi(t_j)\|_{H^2}\|(\eta^\pm)^j\|^\theta\|\nabla(\eta^\pm)^j\|^{2-\theta}\\
&\le C\|\psi(t_j)\|^{\frac{12}{6-N}}_{H^2}\|(\eta^\pm)^j\|^2 + \frac{1}{10}\|\nabla(\eta^\pm)^j\|^2,
\end{aligned} \tag{5.26}$$

where we applied Young's inequality for the conjugate exponents $p = \frac{12}{6-N}$ and $p' = \frac{12}{6+N}$ in the last line of (5.26). The regularity of strong solutions results in

$$k\sum_{j=0}^{J}\|R^j(\varphi)\|^2_X \le Ck^{-1}\sum_{j=0}^{J}\int_{t_{j-1}}^{t_j}(s-t_j)^2\,\mathrm{d}s\int_{t_{j-1}}^{t_j}\|\varphi_{tt}(s)\|^2_X\,\mathrm{d}s \le Ck^2, \tag{5.27}$$

where

$$X := \begin{cases}\mathbf{V}^{-1,2} & \text{for } \varphi = \mathbf{u},\\ \big(H^1\big)^* & \text{for } \varphi = n^\pm.\end{cases}$$

Thus the discrete version of Gronwall's inequality finalizes the error property $(P2)_0$. □

The latter proof shows optimal rates of convergence for the auxiliary Problem A that satisfies a discrete energy law which implies property $(P2)$ for its iterates. The following Problem B involves a splitting strategy preventing a discrete energy law; in order to cope with this deficiency effectively, we apply an inductive argument and rely on regularity properties for iterates of Problem A.

Lemma 5.4 (Problem B)**.** *The solution to Problem B satisfies the properties $(P1)$ and $(P2)_0$, provided again that the time-step size $k \le k_0(t_J)$ is chosen sufficiently small.*

Proof. As in Lemma 5.3, we introduce

$$\mathbf{e}^j := \mathbf{u}_A^j - \mathbf{u}_B^j, \qquad \pi^j := p_A^j - p_B^j,$$
$$(\eta^\pm)^j := (n^\pm)_A^j - (n^\pm)_B^j, \qquad \zeta^j := \psi_A^j - \psi_B^j,$$

with the corresponding error equations

$$d_t\mathbf{e}^j - \Delta\mathbf{e}^j + \nabla\pi^j = k\Big(\big((n^+)_A^j - (n^-)_A^j\big)\nabla d_t\psi_A^j\Big) + \Big(\big((\eta^+)^j - (\eta^-)^j\big)\nabla\psi_A^{j-1}\Big) + \Big(\big((n^+)_B^j - (n^-)_B^j\big)\nabla\zeta^{j-1}\Big) \tag{5.28}$$

$$\operatorname{div}\mathbf{e}^j = 0, \tag{5.29}$$

$$d_t(\eta^\pm)^j - \Delta(\eta^\pm)^j \mp k\operatorname{div}\big((n^\pm)_A^j d_t\nabla\psi_A^j\big) \mp \operatorname{div}\big((\eta^\pm)^j\nabla\psi_A^{j-1}\big) \mp \operatorname{div}\big((n^\pm)_B^j\nabla\zeta^{j-1}\big) = 0, \tag{5.30}$$

$$-\Delta\zeta^{j-1} = (\eta^+)^{j-1} - (\eta^-)^{j-1}. \tag{5.31}$$

The verification of $(P1)$ is done by property $(P2)_0$, which is shown next. We test equation equation (5.30) with $(\eta^\pm)^j$, *i.e.*,

$$\frac{1}{2}d_t\|(\eta^\pm)^j\|^2 + \frac{k}{2}\|d_t(\eta^\pm)^j\|^2 + \|\nabla(\eta^\pm)^j\|^2 \le (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \tag{5.32}$$

where we estimate the terms on the right hand side in the following using the $(L^2,\, L^3,\, L^6)$-decomposition

$$(\mathrm{I}) := k\left|\Big((n^\pm)_A^j d_t\nabla\psi_A^j, \nabla(\eta^\pm)^j\Big)\right| \le k^2C\|d_t\psi_A^j\|_{H^2}^2\|(n^\pm)_A^j\|_{H^1}^2 + \frac{1}{4}\|\nabla(\eta^\pm)^j\|^2$$
$$(\mathrm{II}) := \left|\Big((\eta^\pm)^j\nabla\psi_A^{j-1}, \nabla(\eta^\pm)^j\Big)\right| \le C\|\psi_A^{j-1}\|_{H^2}^{\frac{12}{6-N}}\|(\eta^\pm)^j\|^2 + \frac{1}{8}\|\nabla(\eta^\pm)^j\|^2$$
$$(\mathrm{III}) := \left|\Big((n^\pm)_A^j\nabla\zeta^{j-1}, \nabla(\eta^\pm)^j\Big)\right| + \left|\big((\eta^\pm)^j\nabla\zeta^{j-1}, \nabla(\eta^\pm)^j\big)\right| \le C\left\|(n^\pm)_A^j\right\|_{H^2}^2\left\|\nabla\zeta^{j-1}\right\|^2 + \frac{1}{4}\left\|\nabla(\eta^\pm)^j\right\|^2 + (\mathrm{P1})$$

where

$$(\mathrm{P1}) := C\left[\left\|(\eta^+)^{j-1}\right\|^{\frac{12}{6-N}} + \left\|(\eta^-)^{j-1}\right\|^{\frac{12}{6-N}}\right]\left\|(\eta^\pm)^j\right\|^2$$

will be treated below. It leaves to test (5.28) with $\mathbf{e}^j$, *i.e.*,

$$\frac{1}{2}d_t\left\|\mathbf{e}^j\right\|^2 + \frac{k}{2}\left\|d_t\mathbf{e}^j\right\|^2 + \left\|\nabla\mathbf{e}^j\right\|^2 \le (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \tag{5.33}$$

where the terms on the right hand side are estimated thanks to (5.31) by

$$(\mathrm{I}) := k\left|\Big(\Big\{(n^+)_A^j - (n^-)_A^j\Big\}\nabla d_t\psi_A^j, \mathbf{e}^j\Big)\right| \le k^2C\left\|d_t\psi_A^j\right\|_{H^2}^2\left[\left\|(n^+)_A^j\right\|^2 + \left\|(n^-)_A^j\right\|^2\right] + \frac{1}{8}\left\|\nabla\mathbf{e}^j\right\|^2$$
$$(\mathrm{II}) := \left|\Big(\big\{(\eta^+)^j - (\eta^-)^j\big\}\nabla\psi_A^{j-1}, \mathbf{e}^j\Big)\right| \le C\left\|\psi_A^{j-1}\right\|_{H^2}^2\left[\|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\right] + \frac{1}{8}\left\|\nabla\mathbf{e}^j\right\|^2$$
$$(\mathrm{III}) := \left|\Big(\Big\{(n^+)_A^j - (\eta^+)^j - (n^-)_A^j + (\eta^-)^j\Big\}\nabla\zeta^{j-1}, \mathbf{e}^j\Big)\right| \le C\left[\left\|(n^+)_A^j\right\|_{H^2}^2 + \left\|(n^-)_A^j\right\|_{H^2}^2\right]\left\|\mathbf{e}^j\right\|^2$$
$$+ \frac{1}{8}\left[\left\|(\eta^+)^{j-1}\right\|^2 + \left\|(\eta^-)^{j-1}\right\|^2\right] + (\mathrm{P2}).$$

On the term

$$
\begin{aligned}
(\mathrm{P2}) := \left|\left(\left\{-(\eta^+)^j + (\eta^-)^j\right\}\nabla\zeta^{j-1}, \mathbf{e}^j\right)\right| \le C\left[\|(\eta^+)^{j-1}\|^2 + \|(\eta^-)^{j-1}\|^2\right]\|\mathbf{e}^j\|^2 \\
+ \frac{1}{8}\left[\|(\eta^+)^j\|^2 + \|\nabla(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2 + \|\nabla(\eta^-)^j\|^2\right]
\end{aligned}
$$

in (III) together with (P1) we employ the following inductive argument:

Claim. For $T = t_J$, there exist constants $C_i(\Omega, t_J)$, $i = 1, 2$, such that for $0 \le \ell \le J$ and $k \le k_0(C_1, C_2, \Omega_T)$, we have

$$
\begin{aligned}
&\frac{1}{2}\left[\|\mathbf{e}^\ell\|^2 + \|(\eta^+)^\ell\|^2 + \|(\eta^-)^\ell\|^2\right] + \beta k^2 \sum_{j=1}^{\ell}\left[\|d_t\mathbf{e}^j\|^2 + \|d_t(\eta^+)^j\|^2 + \|d_t(\eta^-)^j\|^2\right] \\
&\qquad + \alpha k \sum_{j=1}^{\ell}\left[\|\nabla\mathbf{e}^j\|^2 + \|\nabla(\eta^+)^j\|^2 + \|\nabla(\eta^-)^j\|^2\right] \le C_1 k^2 \exp\left(C_2 t_\ell\right). \qquad (5.34)
\end{aligned}
$$

The constant $C_1 = C_1(\Omega_T) > 0$ bounds the solutions of Problem A; $C_2 = C_2(C_1, \Omega_T) > 0$ will be chosen sufficiently large for the following argument. First, we verify (5.34) for $\ell = 1$: Summation of (5.32) and (5.33) by setting $j = 1$ and using $\eta^0 = 0$, $\mathbf{e}^0 = \mathbf{0}$ easily validates (5.34).

We come to the induction step $\ell - 1 \to \ell$: Adding (5.32) and (5.33) and a subsequent summation over $1 \le j \le \ell$ verifies (5.34) again by using Gronwall's inequality for $k \le k_0(C_1, C_2, t_J)$ sufficiently small, since by $(P2)_0$ for iterates of Problem A,

$$
\begin{aligned}
&\frac{1}{2}\left[\|\mathbf{e}^\ell\|^2 + \|(\eta^+)^\ell\|^2 + \|(\eta^-)^\ell\|^2\right] + \beta k^2 \sum_{j=1}^{\ell}\left[\|d_t\mathbf{e}^j\|^2 + \|d_t(\eta^+)^j\|^2 + \|d_t(\eta^-)^j\|^2\right] \\
&\quad + \alpha k \sum_{j=1}^{\ell}\left[\|\nabla\mathbf{e}^j\|^2 + \|\nabla(\eta^+)^j\|^2 + \|\nabla(\eta^-)^j\|^2\right] \le k^3 C_1 \sum_{j=1}^{\ell}\left\|d_t\psi_A^j\right\|_{H^2}^2\left[\left\|(n^+)_A^j\right\|_{H^1}^2 + \left\|(n^-)_A^j\right\|_{H^1}^2\right] \\
&\quad + kC_1 \sum_{j=1}^{\ell}\left[1 + \left\|(n^\pm)_A^j\right\|^2\right]\left[\|(\eta^+)^{j-1}\|^2 + \|(\eta^-)^{j-1}\|^2\right] \\
&\quad + kC_1 \sum_{j=1}^{\ell}\left[\left\|\psi_A^{j-1}\right\|_{H^2}^{\frac{12}{6-N}} + \left\|\psi_A^{j-1}\right\|_{H^2}^2 + \left\|(n^+)_A^j\right\|_{H^2}^2 + \left\|(n^-)_A^j\right\|_{H^2}^2\right]\left[\|\mathbf{e}^j\|^2 + \|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\right] \\
&\quad + kC_1 \sum_{j=1}^{\ell}\left[\|(\eta^+)^{j-1}\|^2 + \|(\eta^-)^{j-1}\|^2\right]\|\mathbf{e}^j\|^2 + \frac{1}{2}\left[\|\mathbf{e}^0\|^2 + \|(\eta^+)^0\|^2 + \|(\eta^-)^0\|^2\right] \\
&\quad + kC_1 \sum_{j=1}^{\ell}\left[\|(\eta^+)^{j-1}\|^{\frac{12}{6-N}} + \|(\eta^-)^{j-1}\|^{\frac{12}{6-N}}\right]\left[\|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\right] \\
&\le k^2 C_1^2 + C_1^2 t_\ell k^2 \exp\left(C_2 t_\ell\right) + kC_1^2 \sum_{j=1}^{\ell}\left[\|\mathbf{e}^j\|^2 + \|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\right] + kC_1^2 k^2 \exp\left(C_2 t_\ell\right)\sum_{j=1}^{\ell}\|\mathbf{e}^j\|^2 \\
&\quad + 2k\left[C_1^2 t_\ell k^2 \exp\left(C_2 t_\ell\right)\right]^{\frac{6}{6-N}} \sum_{j=1}^{\ell}\left[\|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2\right] + \frac{1}{2}\left[\|\mathbf{e}^0\|^2 + \|(\eta^+)^0\|^2 + \|(\eta^-)^0\|^2\right],
\end{aligned}
$$

which results with Gronwall's inequality and $\mathbf{e}^0 = \mathbf{0}$, $(\eta^\pm)^0 = 0$ to

$$\left(\frac{1}{2} - C_2 k\right)\left[\left\|\mathbf{e}^\ell\right\|^2 + \left\|(\eta^+)^\ell\right\|^2 + \left\|(\eta^-)^\ell\right\|^2\right] \le C_1 k^2 \exp\left(3\left[k^2 C_1^2 t_\ell \exp\left(C_2 t_\ell\right)\right]^3\right). \tag{5.35}$$

Hence, for $k \le k_0(C_1, C_2, \Omega_T)$ small enough, we have for all $0 \le \ell \le J$

$$3\left[C_1 k^2 \exp\left(C_2 t_J\right)\right]^3 \le C_2.$$

Therefore, the right hand side in (5.35) becomes $C_1 k^2 \exp\left(C_2 t_\ell\right)$ and the induction is verified. □

Remark 5.5. This inductive argument corresponds to the one for iterates of Algorithm A_1 in Step 2 of the proof of Theorem 3.3 to compensate for the lack of a discrete energy law. In contrast, the present argument here relies on the higher regularity given by strong solutions of the system (1.1)–(1.10).

The error estimates for Problem C concern only errors occurring due to Chorin's projection scheme for which results are given in Lemma 5.1. Since $\mathbf{f}_C \in C([0,T]; L^2(\Omega, \mathbb{R}^N))$, we immediately obtain with Lemma 5.1:

Lemma 5.6 (Problem C)**.** *The solution to Problem C satisfies the properties $(P1)$ and $(P2)_1$, provided time-steps $k \le k_0(t_J)$ are chosen sufficiently small.*

The last step to complete the error analysis of Scheme B is:

Lemma 5.7 (Problem D)**.** *The solution to Problem D satisfies the properties $(P1)$ and $(P2)_1$, provided time-steps $k \le k_0(t_J)$ are chosen sufficiently small.*

Proof. This proof is now done in more details. We introduce the shorthand notations

$$\mathbf{e}^j := \mathbf{u}_C^j - \mathbf{u}_D^j, \qquad \pi^j := p_C^j - p_D^j, \tag{5.36}$$

$$(\eta^\pm)^j := (n^\pm)_C - (n^\pm)_D^j, \qquad \zeta^j := \psi_C^j - \psi_D^j, \tag{5.37}$$

which induce the following error equations

$$d_t\mathbf{e}^j - \Delta\mathbf{e}^j + \nabla\pi^{j-1} = \big((\eta^+)^j - (\eta^-)^j\big)\nabla\psi_C^{j-1} + \big((n^+)_D^j - (n^-)_D^j\big)\nabla\zeta^{j-1}, \tag{5.38}$$

$$\operatorname{div}\mathbf{e}^j - k\Delta\pi^j = 0, \tag{5.39}$$

$$\partial_{\mathbf{n}}\pi^j|_{\partial\Omega} = 0, \tag{5.40}$$

$$d_t(\eta^\pm)^j - \Delta(\eta^\pm)^j \mp \operatorname{div}\big((\eta^\pm)^j\nabla\psi_C^{j-1}\big) \mp \operatorname{div}\big((n^\pm)_D^j\nabla\zeta^{j-1}\big) = 0, \tag{5.41}$$

$$-\Delta\zeta^{j-1} = (\eta^+)^{j-1} - (\eta^-)^{j-1}. \tag{5.42}$$

Step 1 ($\mathbf{e}^j$ and $(\eta^\pm)^j$ satisfy $(P2)_1$)**.** We test (5.38) with $\mathbf{e}^j$, and the corresponding pressure equation (5.39) with π^j. The following two properties,

$$2(a-b,a) = |a|^2 - |b|^2 + |b-a|^2 \tag{5.43}$$

and

$$\begin{aligned} k(\nabla\pi^{j-1}, \nabla\pi^j) &= k\|\nabla\pi^j\|^2 - k^2(\nabla\pi^j, \nabla \mathrm{d}_t\pi^j) \\ &= k\|\nabla\pi^j\|^2 - k(\mathrm{d}_t\mathbf{e}^j, \nabla\pi^j) \\ &\ge \frac{k}{2}\big\{\|\nabla\pi^j\|^2 - \|\mathrm{d}_t\mathbf{e}^j\|^2\big\} \end{aligned} \tag{5.44}$$

provide by repeating the techniques from Lemma 5.3 and 5.4 the estimate

$$\frac{1}{2}d_t\|\mathbf{e}^j\|^2 + \frac{1}{2}\|\nabla \mathbf{e}^j\|^2 + \frac{k}{2}\|\nabla \pi^j\|^2 \leq \text{(I)} + \text{(II)} \tag{5.45}$$

with the right hand sides

$$\begin{aligned}
\text{(I)} &:= \left|\left(\left\{(\eta^+)^j - (\eta^-)^j\right\}\nabla\psi_C^{j-1}, \mathbf{e}^j\right)\right| \leq C\left[\left\|(n^+)_C^{j-1}\right\|^2 + \left\|(n^-)_C^{j-1}\right\|^2\right]\|\mathbf{e}^j\|^2 \\
&\quad + \frac{1}{8}\left[\|(\eta^+)^j\|^2 + \|(\eta^-)^j\|^2 + \|\nabla(\eta^+)^j\|^2 + \|\nabla(\eta^-)^j\|^2\right] \\
\text{(II)} &:= \left|\left(\left\{(n^+)_C^j - (\eta^+)^j - (n^-)_C^j + (\eta^-)^j\right\}\nabla\zeta^{j-1}, \mathbf{e}^j\right)\right|.
\end{aligned}$$

Similarly, we obtain by testing (5.41) with $(\eta^\pm)^j$ the inequality

$$\frac{1}{2}d_t\|(\eta^\pm)^j\|^2 + \frac{k}{2}\|d_t(\eta^\pm)^j\|^2 + \frac{1}{2}\|\nabla(\eta^\pm)^j\|^2 \leq \widetilde{\text{(I)}} + \widetilde{\text{(II)}} \tag{5.46}$$

where

$$\begin{aligned}
\widetilde{\text{(I)}} &:= \left|\left((\eta^\pm)^j\nabla\psi_C^{j-1}, \nabla(\eta^\pm)^j\right)\right| \leq C\left[\left\|(n^+)_C^{j-1}\right\|^2 + \left\|(n^-)_C^{j-1}\right\|^2\right]\|\nabla(\eta^\pm)^j\|^2 + \frac{1}{8}\left\|(\eta^\pm)^j\right\|_{L^3}^2 \\
&\leq \left[\left\|(n^+)_C^{j-1}\right\|^2 + \left\|(n^-)_C^{j-1}\right\|^2\right]\|\nabla(\eta^\pm)^j\|^2 + \frac{1}{8}\left[\|(\eta^\pm)^j\|^2 + \|\nabla(\eta^\pm)^j\|^2\right] \\
\widetilde{\text{(II)}} &:= \left|\left(\left\{(n^+)_C^j - (\eta^+)^j - (n^-)_C^j + (\eta^-)^j\right\}\nabla\zeta^{j-1}, \nabla(\eta^\pm)^j\right)\right|.
\end{aligned}$$

The term (II) in (5.45) may now be bounded as the term (III) in (5.33) for the lower index A instead of C and also the term $\widetilde{\text{(II)}}$ in (5.46) corresponds to the term (III) in (5.32), and we can repeat the same inductive argument as in Lemma 5.4.

Step 2 (π^j enjoys property $(P2)_1$)**.** Step 1 and equation (5.12) enable the estimate

$$k\|\nabla\pi^j\| \leq \|\mathbf{e}^j\| \leq Ck, \tag{5.47}$$

and since $\pi^j = p_C^j - p_D^j$ and correspondingly for $(\eta^\pm)^j$ and ζ^j controlled by (5.46), we obtain

$$\max_{0\leq j\leq J}\|\nabla p_D^{j-1}\|^2 + \|\nabla(n^\pm)^{j-1}\|^2 + \|\psi_D^{j-1}\|_{H^2}^2 \leq C. \tag{5.48}$$

Hence, from equations (5.11), (5.48) and (5.45) we get further *a priori* bounds of the form

$$k\sum_{j=0}^{J}\left\{\|\Delta\mathbf{u}_D^j\|^2 + \|\mathrm{d}_t\mathbf{u}_D^j\|^2\right\} \leq C. \tag{5.49}$$

The error bound on the pressure function, as it is given in $(P2)_1$, we achieve by the estimate

$$\begin{aligned}
\|\pi^{j-1}\|_{H^{-1}} \leq \sup_{\chi\in H_0^1\cap H^2}\Bigg\{\frac{1}{\|\chi\|_{H^2}}\Bigg[&\left|\left(d_t\mathbf{e}^j, \chi\right)\right| + \left|\left(\mathbf{e}^j, \Delta\chi\right)\right| \\
&+ \left|\left(((\eta^+)^{j-1} - (\eta^-)^{j-1})\nabla\psi_C^{j-1}, \chi\right)\right| + \left|\left(((n^+)_C^{j-1} - (n^-)_C^{j-1})\nabla\zeta^{j-1}, \chi\right)\right|\Bigg]\Bigg\}.
\end{aligned} \tag{5.50}$$

Then (5.45), (5.49), and Lemma 5.6 allow to control the right hand side of (5.50) in the following way

$$\leq C\Big\{\|d_t\mathbf{e}^j\| + \|\mathbf{e}^j\| + \Big[\|(\eta^+)^{j-1}\| + \|(\eta^-)^{j-1}\|\Big]\Big\}\|\nabla\psi_C^{j-1}\|$$
$$+ \Big[\|(n^+)_C^{j-1}\| + \|(n^-)_C^{j-1}\|\Big]\|\nabla\zeta^{j-1}\| \leq C\Big\{k + \|d_t\mathbf{e}^j\|\Big\}. \tag{5.51}$$

Hence, it leaves to control $\sum_{j=1}^J \|d_t\mathbf{e}^j\|^2$. For this purpose, we test (5.38) with $d_t\mathbf{e}^j$. First, observe the identity

$$\Big(\nabla\pi^{j-1}, d_t\mathbf{e}^j\Big) = k\Big(\nabla d_t\pi^j, \nabla\pi^{j-1}\Big) = -\frac{1}{2}\Big(\|\nabla\pi^{j-1}\|^2 - \|\nabla\pi^j\|^2 + \|\nabla\pi^j - \nabla\pi^{j-1}\|^2\Big)$$
$$= \frac{k}{2}\Big\{d_t\|\nabla\pi^j\|^2 - k\|\nabla d_t\pi^j\|^2\Big\} \geq \frac{k}{2}d_t\|\nabla\pi^j\|^2 - \frac{1}{2}\|d_t\mathbf{e}^j\|^2, \tag{5.52}$$

where we used equation (5.39), the identity (5.43) and in the last line again (5.39) tested with $\nabla d_t\pi^j$. Now, we test equation (5.38) with $d_t\mathbf{e}^j$ that results in

$$\frac{1}{2}\|d_t\mathbf{e}^j\|^2 + \frac{1}{2}d_t\|\nabla\mathbf{e}^j\|^2 + k\frac{2}{5}\|d_t\nabla\mathbf{e}^j\|^2 + \frac{k}{2}d_t\|\nabla\pi^j\|^2 \leq C\|\psi_C^{j-1}\|_{H^2}^2\Big(\|(\eta^+)^j\|_{H^1}^2 + \|(\eta^-)^j\|_{H^1}^2\Big)$$
$$+ C\Big[\|(n^+)_D^j\|_{H^1}^2 + \|(n^-)_D^j\|_{H^1}\Big]\Big(\|(\eta^+)^{j-1}\|^2 + \|(\eta^-)^{j-1}\|^2\Big). \tag{5.53}$$

Define π^0 by continuation of (5.39) as a solution of

$$-\Delta\pi^0 = 0 \qquad \text{on } \Omega, \qquad \partial_{\mathbf{n}}\pi^0 = 0 \qquad \text{on } \partial\Omega,$$

and hence $\pi^0 = 0$. To complete, we also test (5.13) with $d_t(\eta^\pm)^j$. Then we add the resulting estimate up with (5.53) to apply Gronwall's inequality at the end. This provides after summation over $0 \leq j \leq J$ the error controls

$$k\sum_{j=1}^J\Big\{\|d_t\mathbf{e}^j\|^2 + \|d_t(\eta^\pm)^j\|^2\Big\} + \Big\{\|\nabla\mathbf{e}^J\|^2 + \|\nabla(\eta^\pm)^J\|^2\Big\} + \frac{k}{2}\|\nabla\pi^J\|^2 \leq Ck^{\frac{3}{2}}. \tag{5.54}$$

Step 3 (further *a priori* bounds for $(P1)$)**.** With the estimate (5.54) in Step 2 we obtain directly with the *a priori* results stated in Lemma 5.6 the bounds

$$\max_{1\leq j\leq J}\Big\{\|d_t\mathbf{u}_D^j\| + \|d_t(n^\pm)_D^j\| + \|\nabla\mathbf{u}_D^j\| + \|\nabla(n^\pm)_D^j\| + \|\psi^j\|_{H^2} + \|\nabla p_D^j\|\Big\} \leq C. \tag{5.55}$$

The latter bound (5.55) together with (5.11), and (5.54) enable the estimate

$$\|\Delta\mathbf{u}_D^j\| \leq \|d_t\mathbf{u}_D^j\| + \|\nabla p_D^{j-1}\| + C\|\nabla\mathbf{u}_D^j\|_{L^3}\|\mathbf{u}_D^{j-1}\|_{L^6} + C\Big[\|(n^+)_D^j\|_{L^3} + \|(n^-)_D^j\|_{L^3}\Big]\|\psi_D^{j-1}\|_{H^2}.$$

Interpolation of L^3 between L^2 and H^1 then yields to $\max_{1\leq j\leq J}\|\Delta\mathbf{u}_D^j\| \leq C$. In the same way we can control $\Delta(n^\pm)_D^j$, such that we end up with

$$\max_{0\leq j\leq J}\Big\{\|\mathbf{u}_D^j\|_{H^2} + \|(n^\pm)_D^j\|_{H^2}\Big\} \leq C.$$

Step 4 (optimal pressure bounds)**.** Therefore we first apply d_t to (5.38) and then test the resulting equation with $\tau^j d_t e^j$, *i.e.*,

$$\frac{1}{2}\tau^j d_t\|d_t\mathbf{e}^j\|^2 + \tau^j\|d_t\nabla\mathbf{e}^j\|^2 + k\tau^j\|d_t\nabla\pi^j\|^2 \leq \Big|F1 + F2\Big|, \tag{5.56}$$

where we use the identity

$$\tau^j\Big(d_t\nabla\pi^j, d_t\mathbf{e}^j\Big) = \frac{\tau^j}{k}\Big\{\Big(\nabla\pi^j, d_t\mathbf{e}^j\Big) - \Big(\nabla\pi^{j-1}, d_t\mathbf{e}^j\Big)\Big\}$$
$$= \frac{\tau^j}{k}\Big\{k\Big(d_t\nabla\pi^j, \nabla\pi^j\Big) - k\Big(d_t\nabla\pi^j, \nabla\pi^{j-1}\Big)\Big\} = k\tau^j\|d_t\nabla\pi^j\|^2,$$

which is obtained by testing equation (5.39) with $d_t\pi^j$. In the following, we control the nonlinear terms on the right hand side of (5.56). The two last terms originating from the Coulomb force are again controlled by the same Hölder inequality, *i.e.*,

$$\begin{aligned}\Big|F1+F2\Big| &:= \Big|\Big(d_t\big((\eta^+)^j-(\eta^-)^j\big)\nabla\psi_C^{j-1}, \tau^j d_t\mathbf{e}^j\Big) + \Big(\big((\eta^+)^{j-1}-(\eta^-)^{j-1}\big)\nabla d_t\psi_C^{j-1}, \tau^j d_t\mathbf{e}^j\Big)\\ &\quad \Big(d_t\big((n^+)^j_C-(n^-)^j_C\big)\nabla\zeta^{j-1}, \tau^j d_t\mathbf{e}^j\Big) + \Big(\big((n^+)^{j-1}_C-(n^-)^{j-1}_C\big)\nabla d_t\zeta^{j-1}, \tau^j d_t\mathbf{e}^j\Big)\Big|\\ &\le C\tau^j\Big\{\|\psi_C^{j-1}\|^2_{H^2}\|d_t\mathbf{e}^j\|^2 + \|\nabla d_t\psi_C^{j-1}\|^2\Big[\|(\eta^+)^{j-1}\|^2_{H^1} + \|(\eta^-)^{j-1}\|^2_{H^1}\Big]\Big\}\\ &\quad + \frac{1}{5}\tau^j\Big\{\|d_t(\eta^+)^j\|^2_{H^1} + \|d_t(\eta^-)^j\|^2_{H^1} + \|d_t\mathbf{e}^j\|^2 + \|\nabla d_t\mathbf{e}^j\|^2\Big\}\\ &\quad + C\tau^j\Big\{\Big[\|d_t(n^+)^j_C\|^2_{H^1} + \|d_t(n^-)^j_C\|^2_{H^2}\Big]\|\zeta^{j-1}\|^2_{H^2}\\ &\quad + \Big[\|(n^+)^{j-1}_C\|^2_{H^2} + \|(n^-)^{j-1}_C\|^2_{H^2}\Big]\|\nabla d_t\zeta^{j-1}\|^2\Big\} + \frac{1}{5}\tau^j\|\nabla d_t\mathbf{e}^j\|^2.\end{aligned}$$

In the same way we control $\tau^j\|d_t(\eta^\pm)^j\|$ and sum up the resulting estimate with (5.56) to obtain with Gronwall's inequality the required pressure bound.

Still we have to establish the *a priori* bounds $k\sum_{j=1}^J\|d_t\xi_i^j\|^2 \le C$, $i\in\{1,3,4\}$. But since the arguments correspond to the verification of (5.56) by omitting time-weights, we skip the elaboration of this argument at this place. □

5.4. **Spatial discretization of Scheme B, Corollary 3.8**

We first reformulate the Scheme B for $(\mathbf{v}^j, q^j, (\phi^\pm)^j, (\psi)^j) \in H^1_0(\Omega,\mathbb{R}^N)\times\big[H^1(\Omega)\cap L^2_0(\Omega)\big]\times[H^1(\Omega)]^2\times H^2(\Omega)$ for the purely temporal discretization in the context of strong solutions as

$$\begin{aligned}&\Big(\mathrm{d}_t\mathbf{u}^j, \mathbf{v}\Big) + \Big(\nabla\mathbf{u}^j, \nabla\mathbf{v}\Big) + \Big((\mathbf{u}^{j-1}\cdot\nabla)\mathbf{u}^j, \mathbf{v}\Big) + \frac{1}{2}\Big((\operatorname{div}\mathbf{u}^{j-1})\mathbf{u}^j, \mathbf{v}\Big) - \Big(p^j, \operatorname{div}\mathbf{v}\Big)\\ &\quad = -\Big(\big((n^+)^j-(n^-)^j\big)\nabla\psi^{j-1}, \mathbf{v}\Big),\\ &\quad \Big(\operatorname{div}\mathbf{u}^j, q\Big) + k\Big(\nabla p^j, \nabla q\Big) = 0,\\ &\quad \Big(\mathrm{d}_t(n^\pm)^j, \phi^\pm\Big) + \Big(\nabla(n^\pm)^j, \nabla\phi^\pm\Big) \pm \Big((n^\pm)^j\nabla\psi^{j-1}, \nabla\phi^\pm\Big) - \Big(\mathbf{u}^{j-1}(n^\pm)^j, \nabla\phi^\pm\Big) = 0,\\ &\quad \Big(\nabla\psi^{j-1}, \nabla\phi\Big) = \Big(n^+)^{j-1} - (n^-)^{j-1}, \phi\Big),\end{aligned}$$

from which we subtract the conforming finite element version of Scheme B rewritten for $(\mathbf{V}, Q, \Phi^{\pm}, \Phi) \in \mathbf{V}_h \times M_h \times [Y_h]^3$,

$$\begin{aligned}
&\Big(\mathrm{d}_t \mathbf{U}^j, \mathbf{V}\Big) + \Big(\nabla \mathbf{U}^j, \nabla \mathbf{V}\Big) + \Big((\mathbf{U}^{j-1} \cdot \nabla)\mathbf{U}^j, \mathbf{V}\Big) + \frac{1}{2}\Big((\operatorname{div} \mathbf{U}^{j-1})\mathbf{U}^j, \mathbf{V}\Big) \\
&\qquad\qquad - \Big(\Pi^j, \operatorname{div} \mathbf{V}\Big) = -\Big(\big((N^+)^j - (N^-)^j\big)\nabla\Psi^{j-1}, \mathbf{V}\Big), \\
&\Big(\operatorname{div} \mathbf{U}^j, Q\Big) + k\Big(\nabla \Pi^j, \nabla Q\Big) = 0, \\
&\Big(d_t (N^{\pm})^j, \Phi^{\pm}\Big) + \Big(\nabla (N^{\pm})^j, \nabla \Phi^{\pm}\Big) \pm \Big((N^{\pm})^j \nabla \Psi^{j-1}, \nabla \Phi^{\pm}\Big) - \Big(\mathbf{U}^{j-1}(N^{\pm})^j, \nabla \Phi^{\pm}\Big) = 0, \\
&\Big(\nabla \Psi^{j-1}, \nabla \Phi\Big) = \Big(\big((N^+)^{j-1} - (N^-)^{j-1}\big), \Phi\Big).
\end{aligned} \tag{5.57}$$

The result then follows from standard error estimates that base on corresponding stability arguments as provided in the proof of Lemma 5.7.

Remark 5.8. The finite element spaces chosen in the above space discretization (5.57) of Scheme B does not have to satisfy the compatibility condition (2.6) as for Scheme A.

6. Computational studies

Section 6.1 studies the convergence behavior of the two Schemes A and B. In the next four sections, we relax step by step the academic assumptions to be able to distinguish the effects originating by the system itself from the external ones which we impose by boundary conditions. These steps allow us to recover the pure influence of the quasi-electrostatic forces as a driving force to the fluid. All the computations are done for uniform triangulations. The computational demand of the fixed point iterations used in the Algorithm A_1 is for Examples 1 and 2 at most three iterations, and for Examples 3 and 4 up to six iterations. The comparison of Algorithm A_1 with Scheme B indicates through all computations that the time-splitting Scheme B requires only half of the computational time than Algorithm A_1. Especially, if small time scales are needed to obtain more accurate results, then the fixed point iterations consume a significant amount of CPU-time. Therefore, it is reasonable to only use Scheme A_1, if physically relevant properties such as non-negativity, discrete maximum principle, energy and entropy characterizations are needed to be preserved.

6.1. L^2-Convergence

The L^2-convergence behavior of iterates belonging to Schemes A and B is studied in this section. Let $\Omega = (0,1)^2$. We consider the exact solutions

$$u_1(x,y,t) = -t\cos(\pi x)\sin(\pi y), \qquad u_2(x,y,t) = t\sin(\pi x)\cos(\pi y) \tag{6.1}$$

$$p(x,y,t) = -\frac{1}{4}\Big(\cos(2\pi x) + \cos(2\pi y)\Big) \tag{6.2}$$

$$\psi(x,y,t) = \frac{t}{\pi^2}\Big(\cos(\pi x) - \sin(\pi y)\Big) \tag{6.3}$$

$$n^+(x,y,t) = t\cos(\pi x), \qquad n^-(x,y,t) = t\sin(\pi y) \tag{6.4}$$

for the system (1.1)–(1.10). The convergence results for the time discretization $k = 0.001$ and the space discretizations $h = 0.25,\ 0.125,\ 0.0625,\ 0.03125,\ 0.0156$ are shown by a double logarithmic plot in Figure 1. The snapshot on the right hand side of Figure 1 indicates that the asymptotic regime is reached for a mesh-size smaller than $h = 0.0312$.

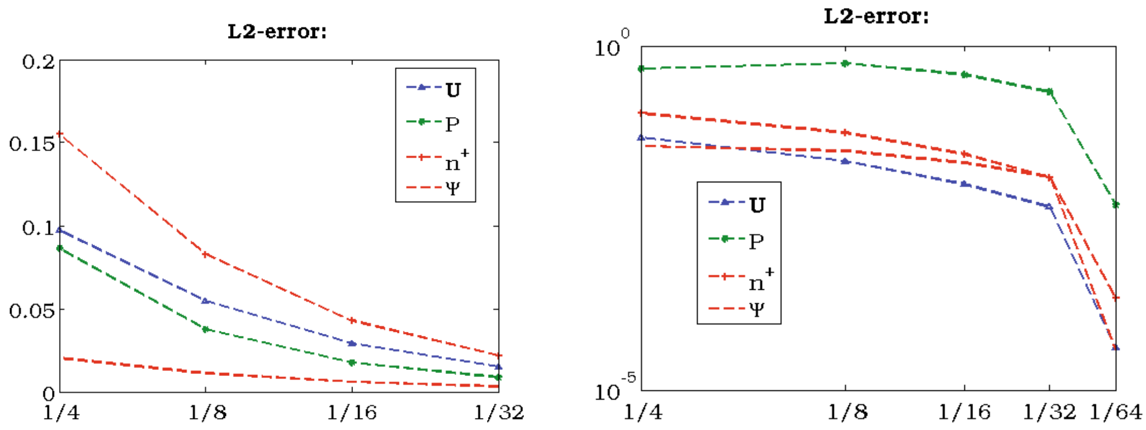

FIGURE 1. L^2**-Convergence:** Scheme A (left) and Scheme B (right) for different mesh sizes.

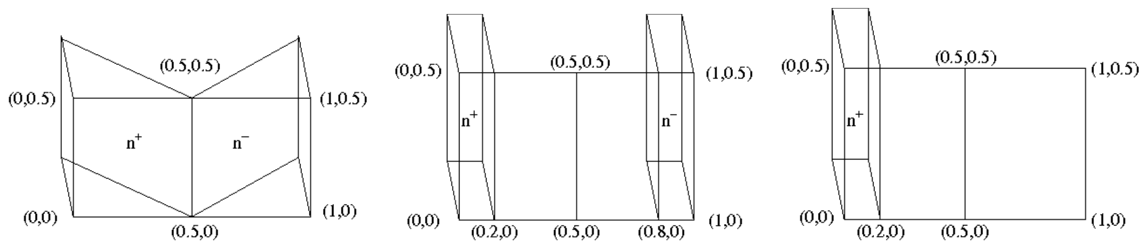

FIGURE 2. **Domain Geometries and Initial Configurations** of the positive and negative charges for the Academic Example 1 (left), Example 2 (middle), and Examples 3 and 4 (right).

6.2. Academic Example 1

The only driving force originates from the initial concentrations of positive n^+ and negative n^- charges for which the initial configuration is depicted in Figure 2. In applications, the concentration differences originate between the interface of the electrolyte and the solid surfaces. The atomic structure of the solid induces counter ions stemming from the electrolyte on the solid surface. This movement of the ions around the solid particle is called electroosmosis. Hence we consider the system (1.1)–(1.10) for the initial data

$$
\begin{aligned}
u_1(x,y,0) &= 0, \qquad u_2(x,y,0) = 0,\\
n^+(x,y,0) &= \begin{cases} 1-2x & \text{if } 0 \le x \le 0.5,\\ 0 & \text{else,} \end{cases} \qquad n^-(x,y,0) = \begin{cases} 0 & \text{if } 0 \le x \le 0.5,\\ 2(x-0.5) & \text{else,} \end{cases}
\end{aligned}
$$

and vanishing Neumann boundary conditions as required in Definition 3.1. Such assumptions for Scheme A and B result in the energy and entropy behavior plotted in Figure 3, where

$$
H_\delta[P,N] := \int_\Omega F_\delta(P) + F_\delta(N)\,\mathrm{d}\mathbf{x},
$$

for $F_\delta(x) := x\ln(x+\delta)$ and $\delta \ge 0$. In Examples 2, 3 and 4, we use $\delta = 0.00001$. The characteristic plots of both, energy and entropy show an asymptotic ($t \to \infty$) exponential decay of almost the same rate. Moreover,

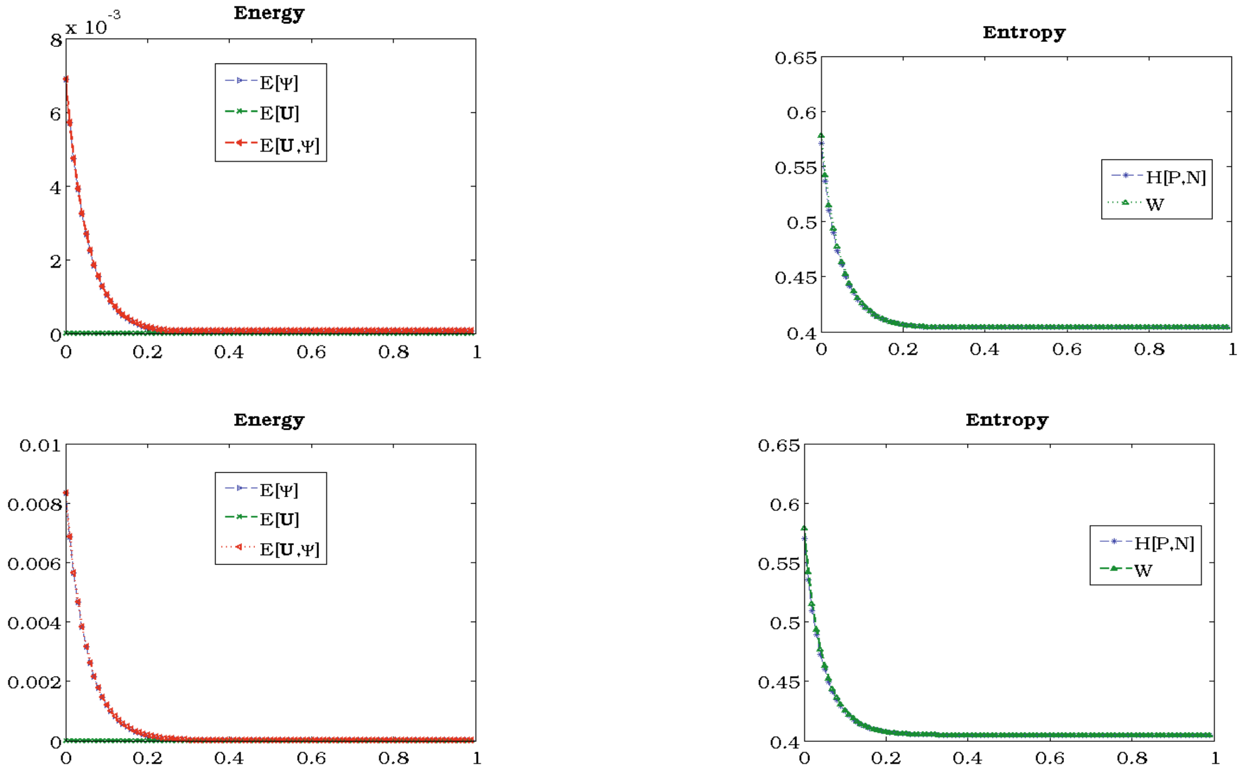

FIGURE 3. **Example 1:** 1st line: energy (left) and entropy (right) for the Scheme A ($h = 0.03125$, $k = 0.01$). 2nd line: corresponding results for Scheme B.

the entropy curve shows that the system is mainly active in the first 0.3 seconds. We choose $h = 0.0312$, $k = 0.0015$ on the time interval $[0, 0.3]$.

6.3. Academic Example 2

We investigate the influence of L^∞-initial data. More precisely, the only difference to Example 1 is that we change initial concentrations presented on the left hand side of Figure 2 to the situation depicted in the middle. One recognizes slightly smaller values of the energy for Schemes A and B in Figure 4. Such a behavior seems to arise because of the smaller mass $M^+ := \|n^+\|_{L^1}$ in Example 2 where $M^+ = 0.1$ compared to $M^+ = 0.375$ in Example 1. Conversely, the entropy is larger for rough initial data for both Schemes A and B. Here, the entropy functional is regularized for $\delta = 0.0001$.

In Figure 7, we provide some snapshots for the most interesting values obtained for the mesh parameters $h = 0.0312$, $k = 0.00015$ on a time interval $[0, 0.3]$.

6.4. Academic Example 3

We neglect the negative concentrations by setting $n_0^- \equiv 0$. In order not to be inconsistent with the given vanishing Neumann boundary conditions for ψ, we change them below. Such a configuration is motivated to recover the pure influence of an external electrical field as driving force. For such a situation, the existence of contrary charged species would unnecessarily disturb our configuration with compensating effects. Hence, in difference to Example 2, we consider the initial concentration of positively charged species as depicted in Figure 2

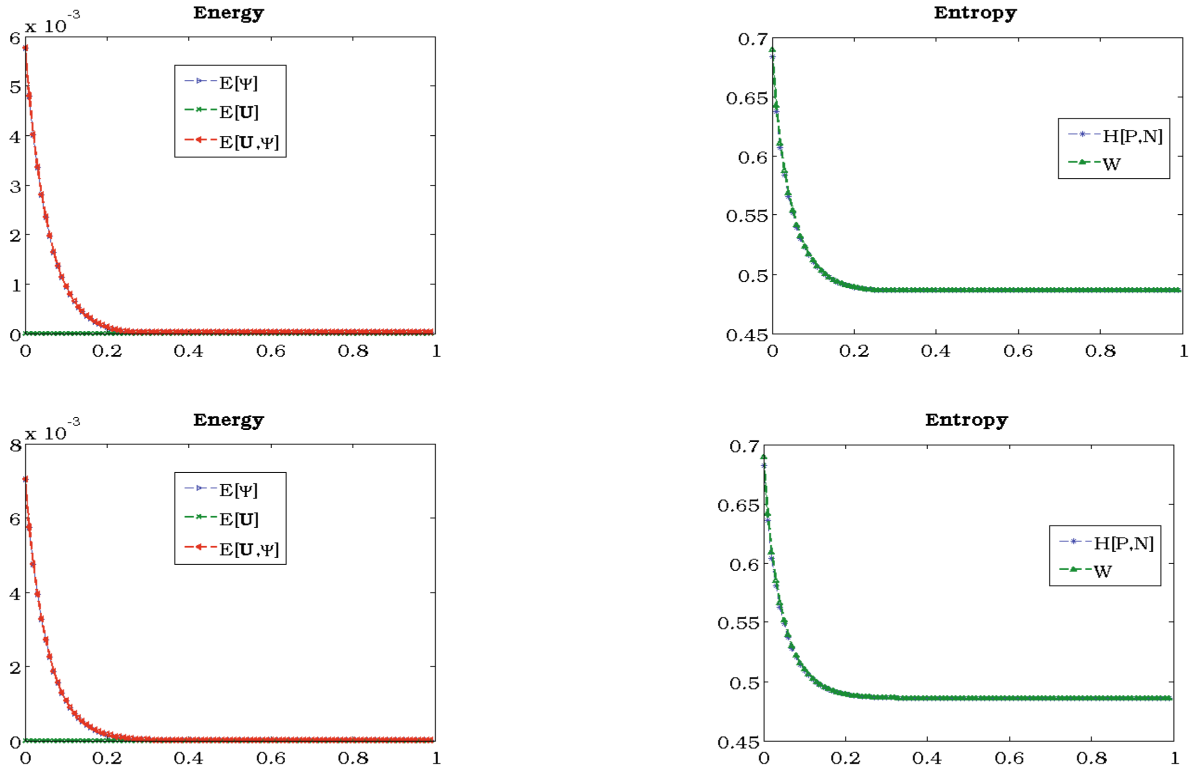

FIGURE 4. **Example 2:** 1st line: energy (left) and entropy (right) for the Scheme A and $h = 0.03125$, $k = 0.01$. 2nd line: represents corresponding results for Scheme B in the same order.

on the right, and we set for the electrical potential the Dirichlet boundary conditions

$$\psi(x,y,t) = \begin{cases} 1 & \text{for } (x,y) \in \{0\} \times [0,0.5] \\ 0 & \text{for } (x,y) \in \{1\} \times [0,0.5], \end{cases} \tag{6.5}$$

and for the remaining part of the boundary we set $\partial_{\mathbf{n}}\psi = 0$.

As in the examples before, we compute the energy and the entropy for the rather coarse mesh parameters $k = 0.01$ and $h = 0.0312$. Again, we regularize the entropy functional by the parameter $\delta = 0.00001$. The resulting screenshots are given in Figure 5. The influence of the new boundary conditions acting as external forces results in non-dissipative energy and entropy values.

6.5. **Academic Example 4**

Conversely to Example 3, the channel is already streamed by a certain fluid. Hence we are interested in how a previously defined amount of positively charged species n^+ with an initial rectangular geometry given by the right picture in Figure 2 evolves starting from the right hand side of the channel. Therefore, the fluid velocity satisfies the initial conditions

$$u_1(x,y,0) \quad = \quad 1 \qquad \text{on } \Omega \setminus \Big\{\{0\} \times [0,1] \cup \{1\} \times [0,1]\Big\}, \tag{6.6}$$

$$u_2(x,y,0) \quad = \quad 0 \qquad \text{on } \Omega, \tag{6.7}$$

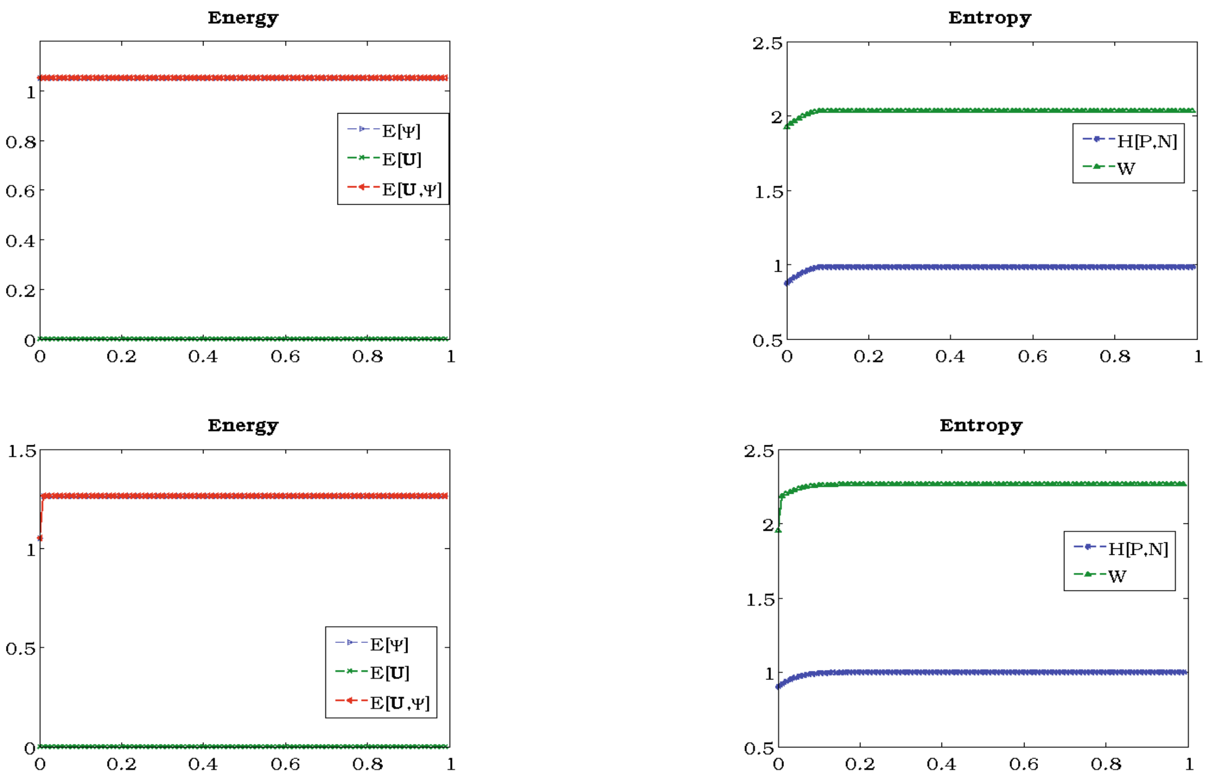

FIGURE 5. **Example 3:** 1st line: energy (left) and entropy (right) computed for $h = 0.03125$, $k = 0.01$, and $\delta = 0.00001$ for Scheme A. 2nd line: corresponding values for Scheme B.

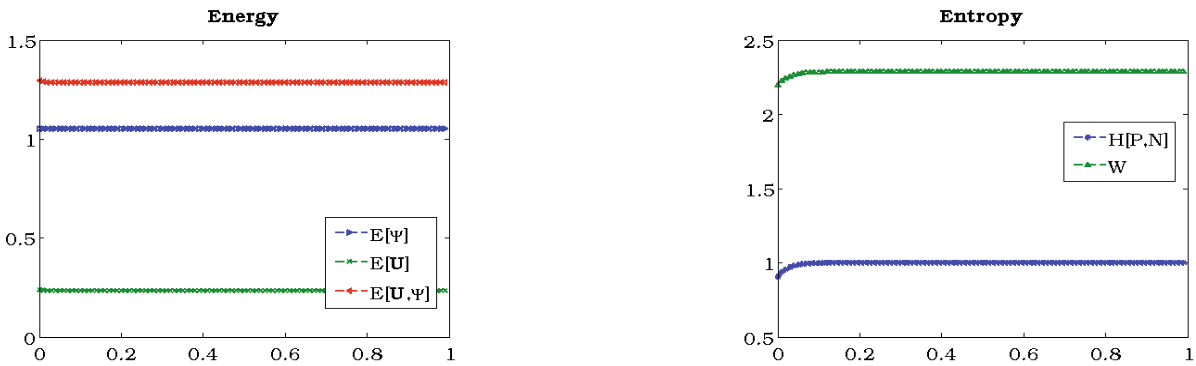

FIGURE 6. **Example 4:** The energy (left) and entropy (right) computed for $h = 0.03125$, $k = 0.01$ and $\delta = 0.00001$. In this example we obtain corresponding results for Scheme A and B.

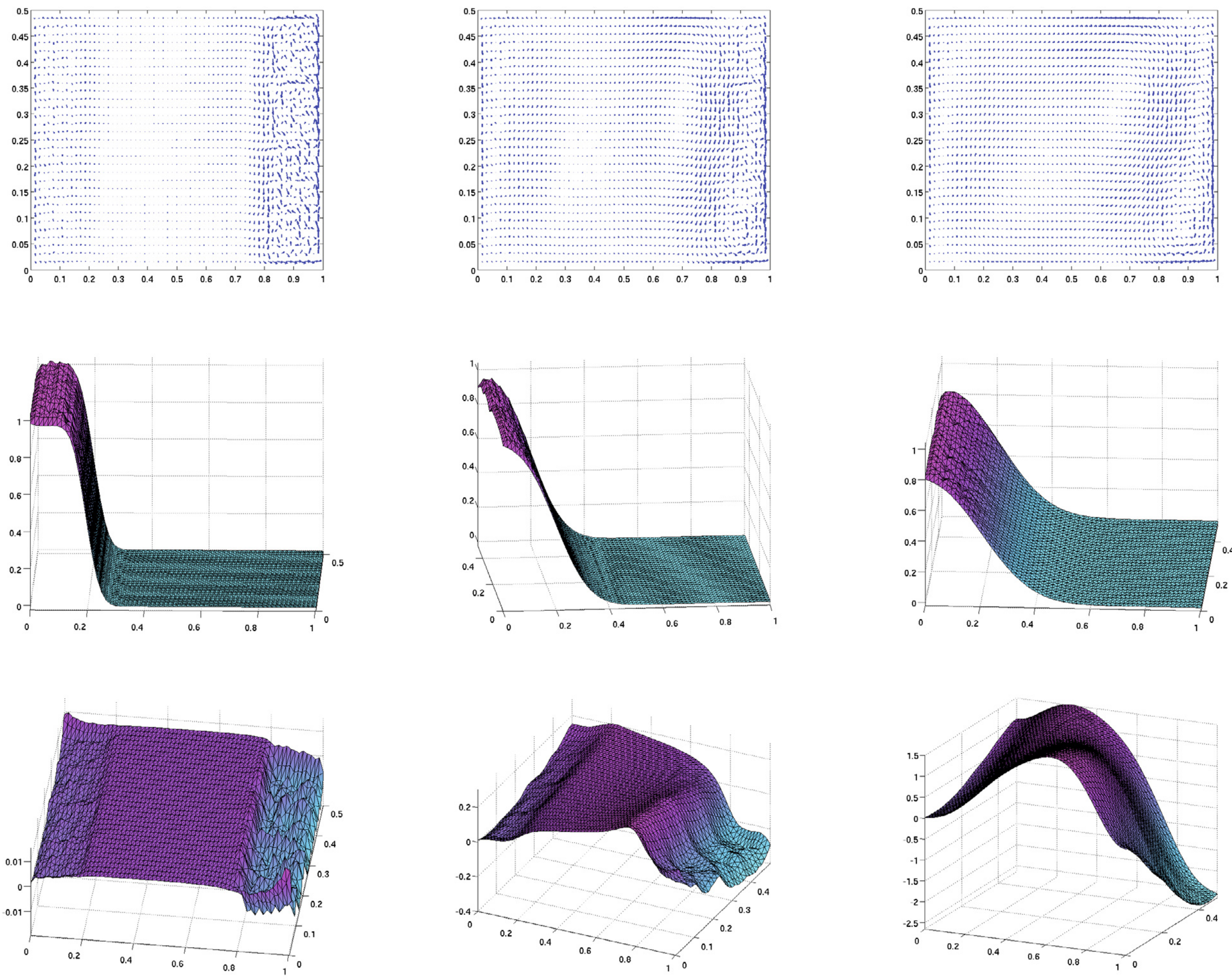

FIGURE 7. **Snapshots for Example 2:** 1st line: fluid velocities $\mathbf{U}^j$ for the times $t_{30} = 0.005$, $t_{200} = 0.03$, $t_{450} = 0.07$. 2nd line: corresponding concentration $(N^+)^j$. 3rd line: pressure Π^j for time steps $t_1 = 0.0002$, $t_{30} = 0.005$, $t_{450} = 0.07$. Moreover, Π^1 shows a shock whose size depends on the temporal discretization k. This shock is a result of the switch-on character of Example 2. The pictures for Schemes A and B are similar. ($h = 0.0312$, $k = 0.00015$.)

and the boundary conditions

$$u_1(x,y,t) = \begin{cases} 1 & \text{for } (x,y) \in \{0\} \times (0,0.5) \cup \{1\} \times (0,0.5), \\ 0 & \text{else,} \end{cases} \tag{6.8}$$

$$u_2(x,y,t) = 0 \qquad \text{on } \partial\Omega. \tag{6.9}$$

Again, we compute the energy and entropy for the rough mesh parameters $k = 0.01$ and $h = 0.0312$. Further we regularize the logarithms in the entropy with $\delta = 0.00001$. The plots are given in Figure 6. If compared to Example 3, the resulting energy and entropy values are higher, as expected due to the strong influence of the constant streaming fluid. Hence the energy $E(\mathbf{U}^j)$ is now remarkably away from zero. An interesting consequence of the streaming fluid is that the energy density of the electric field $E(\Psi^j)$ decreases in such a way that the total energy $E(\mathbf{U}, \Psi)$ remains on the same value as in Example 3.

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