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AN UNCONDITIONALLY STABLE FINITE ELEMENT-FINITE VOLUME PRESSURE CORRECTION SCHEME FOR THE DRIFT-FLUX MODEL

Laura Gastaldo[1], Raphaèle Herbin[2] and Jean-Claude Latché[1]

Abstract. We present in this paper a pressure correction scheme for the drift-flux model combining finite element and finite volume discretizations, which is shown to enjoy essential stability features of the continuous problem: the scheme is conservative, the unknowns are kept within their physical bounds and, in the homogeneous case (*i.e.* when the drift velocity vanishes), the discrete entropy of the system decreases; in addition, when using for the drift velocity a closure law which takes the form of a Darcy-like relation, the drift term becomes dissipative. Finally, the present algorithm preserves a constant pressure and a constant velocity through moving interfaces between phases. To ensure the stability as well as to obtain this latter property, a key ingredient is to couple the mass balance and the transport equation for the dispersed phase in an original pressure correction step. The existence of a solution to each step of the algorithm is proven; in particular, the existence of a solution to the pressure correction step is derived as a consequence of a more general existence result for discrete problems associated to the drift-flux model. Numerical tests show a near-first-order convergence rate for the scheme, both in time and space, and confirm its stability.

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1. Introduction

Dispersed two-phase flows and, in particular, bubbly flows are widely encountered in industrial applications as, for instance, nuclear safety studies, which are the context of the present work. Within the rather large panel of models dealing with such flows, the simplest is the so-called drift-flux model, which consists in balance equations for an equivalent continuum representing both the gaseous and the liquid phase. For isothermal flows, this approach leads to a system of three balance equations, namely the overall mass balance, the gas mass balance and the momentum balance, which reads:

$$\partial_t \rho + \nabla\cdot(\rho\, u) = 0 \tag{1.1a}$$

$$\partial_t(\rho\, y) + \nabla\cdot(\rho\, y\, u) = -\nabla\cdot(\rho\, y\,(1-y)\, u_r) + \nabla\cdot(D\nabla y) \tag{1.1b}$$

$$\partial_t(\rho\, u) + \nabla\cdot(\rho\, u\otimes u) + \nabla p - \nabla\cdot\tau(u) = \pi \tag{1.1c}$$

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[1] Institut de Radioprotection et de Sûreté Nucléaire (IRSN), France. `laura.gastaldo@irsn.fr; jean-claude.latche@irsn.fr`
[2] Université de Provence, France. `herbin@cmi.univ-mrs.fr`

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where t stands for the time, ρ, u and p are the (average) density, velocity and pressure in the flow and y stands for the gas mass fraction. The forcing term π may represent, for instance, the gravity forces. The tensor τ is the viscous part of the stress tensor, given by the following expression:

$$\tau(u) = \mu\,(\nabla u + \nabla^t u) - \frac{2}{3}\,\mu\,(\nabla \cdot u)\; I. \tag{1.2}$$

For a constant (in space) viscosity, this relation yields:

$$\nabla \cdot \tau = \mu \left[\Delta u + \frac{1}{3}\; \nabla \nabla \cdot u \right]. \tag{1.3}$$

Finally, the diffusion coefficient D is supposed to be non-negative. Diffusion terms represent in most applications small scale perturbations of the flow due to the presence of the dispersed phase, sometimes called "diphasic turbulence" and u_r is the relative velocity between the liquid and the gaseous phase (the so-called drift velocity); for both these quantities, a phenomenologic relation must be supplied.

This system must be complemented by an equation of state, which takes the general form:

$$\rho = \varrho^{\,p,\alpha}(p, \alpha_g) = (1 - \alpha_g)\rho_\ell + \alpha_g \varrho_g(p) \tag{1.4}$$

where α_g stands for the void fraction and $\varrho_g(p)$ expresses the gas density as a function of the pressure; in the ideal gas approximation and for an isothermal flow, which we consider here, ϱ_g is simply a linear function:

$$\varrho_g(p) = \frac{p}{a^2} \tag{1.5}$$

where a is a constant characteristic of the gas, equal to the sound velocity in an isothermal (monophasic) flow. The density of the liquid phase ρ_ℓ is assumed to be constant. Introducing the gas mass fraction y in (1.4) by using the relation $\alpha_g\, \varrho_g = \rho\, y$ leads to the following form of the equation of state:

$$\rho = \varrho^{\,p,y}(p,\, y) = \frac{\varrho_g(p)\, \rho_\ell}{\rho_\ell\, y + (1 - y)\;\varrho_g(p)}. \tag{1.6}$$

The problem is supposed to be posed over Ω, an open bounded connected subset of $\mathbb{R}^d$, $d \le 3$, and over a finite time interval $(0, T)$. It must be supplemented by suitable boundary conditions, and initial conditions for ρ, u and y.

To design a numerical scheme for the solution of the system (1.1), one is faced with several difficulties. Firstly, since the fluid density ρ_ℓ is supposed not to depend on the pressure, almost incompressible zones, *i.e.* zones where the void fraction is low, may coexist in the flow with compressible zones, *i.e.* zones where the void fraction remains significant. The problem is thus particularly difficult to solve from a numerical point of view, because the employed numerical scheme has to cope with a wide range of Mach numbers, starting from zero to a fraction of unity for low to moderate speed flows. Secondly, the gas mass fraction y can be expected, both for physical and mathematical reasons, to remain in the $[0, 1]$ interval, and it appears strongly desirable that the numerical scheme reproduces this behaviour at the discrete level. Finally, it appears from numerical experiments that, in order to avoid numerical instabilities, the algorithm should preserve a constant pressure through moving interfaces between phases (*i.e.* contact discontinuities of the underlying hyperbolic system). Indeed, when testing a fractional step method for the same problem (1.1) which did not satisfy this property [19], we observed instabilities when computing flows involving phases of very different densities, the cure to which seems to need a drastic reduction of the time step. To obtain a scheme stable in the low Mach number limit, the solution that we adopt here is to use an algorithm inspired from the incompressible flow numerics, namely from the class of finite element pressure correction methods, and which degenerates to a classical projection scheme when the fluid density is constant. The last two requirements are met thanks to an original pressure correction step in which the mass balance equation is solved simultaneously with a part of the gas mass balance. For technical reasons, the solution of this latter equation is itself split in two steps, with the first step incorporated to the pressure correction and the second one performed independently. The coupling of the equations in the projection step is a major difference with the scheme proposed in [19].

This work takes benefit of ideas developed in a wide literature, so we are only able to quote here some references, the choice of which will unfortunately probably appear somewhat arbitrary. For a description of projection schemes for incompressible flow, see *e.g.* [21,26] and references herein. An extension to barotropic Navier-Stokes equations close to the scheme developed here can be found in [16], together with references to (a large number of) related works (see *e.g.* [23] for the seminal work and [34] for a comprehensive introduction). Extensions of pressure correction algorithms for multi-phase flows are scarcer, and seem to be restricted to iterative algorithms, often similar in spirit to the usual SIMPLE algorithm for incompressible flows [24,28,33]. Note that the drift flux model is also considered in the hyperbolic literature (see *e.g.* [2,3,10,11,14,27,30]), where Riemann solver based algorithms are proposed; this direction is not taken here, essentially because we want to address low (down to zero) low Mach number flows. The gas mass balance equation (1.1b) is a convection-diffusion equation which differs from the usual mass balance for chemical species in compressible multi-component flows studied in [25] by the addition of a non-linear term of the form $\nabla \cdot \rho\, \varphi(y)\, u_r$, where φ is a regular function such that $\varphi(0) = \varphi(1) = 0$ (in the present case, $\varphi(y) = y\,(1-y)$).

Several theoretical issues concerning the proposed scheme are studied in this paper. First, the existence of a solution to the pressure correction step, which consists in an algebraic non-linear system, is obtained by a topological degree argument. Second, we address the stability of the scheme. At the continuous level, the existence of an entropy for the system when the drift velocity vanishes (*i.e.* the homogeneous model) is well-known. It is shown in [22], by a Chapman-Enskog expansion technique, that the two-fluid model can be reduced to the drift-flux model when a strong coupling of both phases is assumed, with a Darcy-like closure relation for the drift velocity, *i.e.* an expression of the form:

$$u_r = \frac{1}{\lambda}\,(1-\alpha_g)\,\alpha_g\,\frac{\varrho_g(p) - \rho_\ell}{\rho}\,\nabla p \tag{1.7}$$

where λ is a positive phenomenological coefficient. The same relation can also be obtained by neglecting in the two-fluid model the difference of acceleration between both phases [32]. With such an expression for u_r, the drift term becomes a second order term, and it is shown in [22] that it is consistent with the entropy of the homogeneous model (*i.e.* that it generates a non-negative dissipation of the entropy). These results are proven here at the discrete level: up to a minor modification of the proposed scheme, which seems useless in practice, the entropy is conserved when u_r is equal to zero, and when the closure relation (1.7) applies and with a specific discretization, the drift term generates a dissipation.

This paper is built as follows. The fractional step algorithm for the solution of the whole problem is first presented in Section 2. The next section is devoted to the analysis of the scheme. We first state the existence of a solution to each step of the algorithm, the fact that the unknowns are kept within their physical bounds (Sect. 3.1) and that the algorithm is able to preserve a constant pressure and a constant velocity through moving interfaces between phases (Sect. 3.2). The proof of the existence of the solution to the pressure correction step is obtained as a consequence of a more general existence theory for some discrete problems associated to the drift-flux model, which is exposed in the appendix. The next two sub-sections are devoted to the stability analysis of the scheme: we first address the case $u_r = 0$ (Sect. 3.3), then the case where u_r is given by the Darcy-like closure relation (1.7) (Sect. 3.4). An inequality for the term corresponding to the work of the pressure forces is obtained as the consequence of a general result proven in appendix, which may be seen as a discrete renormalization identity which applies to a system of transport equations. Finally, numerical tests are reported in Section 4; they include a problem exhibiting an analytical solution which allows to assess convergence properties of the discretization, a sloshing transient in a cavity, and the evolution of a bubble column.

For the sake of simplicity, we suppose for the presentation of the scheme and its analysis that the velocity is prescribed to zero on the whole boundary $\partial\Omega$ of the computational domain, and that the gas mass flux through $\partial\Omega$ also vanishes, so that both the normal component of u_r and ∇y are zero on the boundary. Moreover, the analysis of the scheme assumes that pure liquid zones do not exist in the flow; with the proposed algorithm, this is a consequence of the fact that such zones are not present at the initial time (*i.e.*, at $t = 0$, $y \in (0, 1]$). Indeed, getting rid of this latter limitation for the theoretical study seems to be a difficult task. However, the numerical tests presented in Section 4 are not restricted to these situations. In particular, $y = 0$ in the liquid column in the sloshing problem, up to spurious phases mixing by the numerical diffusion near the free surface; it is also the case at the initial time in the bubble column simulation.

In the presentation of the scheme, the drift velocity is supposed to be known, *i.e.* to be given by a closure relation independent of the unknowns of the problem, and this still holds in numerical experiments. The case where u_r is given by (1.7) is thus only treated from a theoretical point of view in Section 3.4.

2. The numerical algorithm

2.1. Time semi-discrete formulation

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which is supposed uniform for the sake of simplicity. Let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the constant time step. In a time semi-discrete setting, denoting by ρ^{-1} and u^0 initial guesses for the density and velocity, the algorithm proposed in this paper is the following.

0. Initialization:

$$\frac{\rho^0 - \rho^{-1}}{\delta t} + \nabla \cdot (\rho^0 u^0) = 0. \tag{2.1}$$

Then, for $n \geq 0$:

1. Prediction solve for $\tilde{u}^{n+1}$

$$\frac{\rho^n\ \tilde{u}^{n+1} - \rho^{n-1}\ u^n}{\delta t} + \nabla \cdot (\rho^n\ u^n \otimes \tilde{u}^{n+1}) + \nabla p^n - \nabla \cdot \tau(\tilde{u}^{n+1}) = \pi^{n+1}. \tag{2.2}$$

2. Solve for p^{n+1}, u^{n+1}, ρ^{n+1} and z^{n+1}

$$\rho^n\ \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla (p^{n+1} - p^n) = 0 \tag{2.3a}$$

$$\frac{\varrho^{p,z}(p^{n+1},\ z^{n+1}) - \rho^n}{\delta t} + \nabla \cdot (\varrho^{p,z}(p^{n+1},\ z^{n+1})\ u^{n+1}) = 0 \tag{2.3b}$$

$$\frac{z^{n+1} - \rho^n y^n}{\delta t} + \nabla \cdot (z^{n+1}\ u^{n+1}) = 0 \tag{2.3c}$$

$$\rho^{n+1} = \varrho^{p,z}(p^{n+1}, z^{n+1}). \tag{2.3d}$$

3. Solve for y^{n+1}

$$\frac{\rho^{n+1} y^{n+1} - z^{n+1}}{\delta t} + \nabla \cdot (\rho^{n+1}\ y^{n+1}(1 - y^{n+1})\ u_r^{n+1}) = \nabla \cdot (D \nabla y^{n+1}). \tag{2.4}$$

Step 0 is introduced to obtain a compatible discretization of ρ^0 for step 1 at $n = 0$ which is required for the stability of the scheme.

Step 1 consists in a classical semi-implicit solution of the momentum balance equation to obtain a predicted velocity.

Step 2 is an original nonlinear pressure correction step, which couples the total mass balance equation (2.3b) with the transport terms of the gas mass balance equation (2.3c). In this step, we consider as new unknown the partial gas density z given by $z = \rho\, y$, rather than the gas mass fraction y. Thus, the equation of state must be reformulated to express the mixture density as a function of the partial gas density and of the pressure, which, from equation (1.6), yields:

$$\rho = \varrho^{p,z}(p,z) = z\left(1 - \frac{\rho_\ell\, a^2}{p}\right) + \rho_\ell. \tag{2.5}$$

When the liquid and the gas densities are very different, the variations of ρ with respect to z are smoother than those with respect to y in (1.6), especially in the neighbourhood of $y = 0$; this change of variable thus makes the resolution of this step much easier, and the overall algorithm more robust. In counterpart, it leads to split the gas mass balance equation: transport terms are dealt with in the present step, and the gas mass fraction is corrected in a next step (step 3) to take into account the drift terms. The pressure correction step would degenerate in the usual projection step as used in incompressible flows solvers if the density were constant (*i.e.* $z = 0$). Taking the discrete divergence of (2.3a) and using (2.3b) to eliminate the unknown velocity u^{n+1} yields a non-linear elliptic problem for the pressure. Solving simultaneously this elliptic problem and (2.3c) by Newton's algorithm, we obtain the pressure and the gas mass fraction. Once the pressure is computed, (2.3a) yields the updated velocity and (2.3d) gives the end-of-step density.

Finally, in the third step, the remaining terms of the gas mass balance are considered, and the end-of-step gas mass fraction is computed.

This time discretization is designed to keep the mass fraction y in the physical range $[0,1]$, to allow the transport of phases interfaces without generating spurious pressure and velocity variations and to ensure the stability of (*i.e.* the conservation of the entropy by) the scheme. To show how this time splitting algorithm achieves these goals is the aim of the remainder of this paper.

2.2. Mesh and discrete spaces

Let $\mathcal{M}$ be a decomposition of the domain Ω into either convex quadrilaterals ($d = 2$) or hexahedra ($d = 3$) or simplices. By $\mathcal{E}$ and $\mathcal{E}(K)$ we denote the set of all $(d-1)$-edges σ of the mesh and of the element $K \in \mathcal{M}$ respectively. The set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal ones (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$. The decomposition $\mathcal{M}$ is supposed to be regular in the usual sense of the finite element literature (*e.g.* [7]), and, in particular, $\mathcal{M}$ satisfies the following properties: $\bar\Omega = \bigcup_{K\in\mathcal{M}} \bar K$; if $K,\, L \in \mathcal{M}$, then $\bar K \cap \bar L$ is reduced to the empty set, to a vertex or (if $d = 3$) to a segment, or $\bar K \cap \bar L$ is (the closure of) a common $(d-1)$-edge of K and L, which is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, n_{KL} stands for the normal vector to σ, oriented from K to L. By $|K|$ and $|\sigma|$ we denote the measure, respectively, of the control volume K and of the edge σ.

For stability reasons, the spatial discretization must preferably be based on pairs of velocity and pressure approximation spaces satisfying the so-called *inf-sup* or Babuska-Brezzi condition (*e.g.* [5]). We choose here as in [19] the Rannacher and Turek element [29] for quadrilateral or hexahedric meshes, or the Crouzeix-Raviart element (see [8] for simplicial meshes. These are non-conforming approximations with degrees of freedom for the velocity located at the center of the faces which seem to be well suited to a coupling with a finite volume treatment of the total mass balance (2.3b) and gas mass balance (2.4), ensuring that physical bounds on ρ and y are respected. The reference element $\widehat{K}$ for the rotated bilinear element is the unit d-cube (with edges parallel to the coordinate axes); the discrete functional space on $\widehat{K}$ is $\tilde Q_1(\widehat{K})^d$, where $\tilde Q_1(\widehat{K})$ is defined as follows:

$$\tilde Q_1(\widehat{K}) = \operatorname{span}\left\{1,\, (x_i)_{i=1,\ldots,d},\, (x_i^2 - x_{i+1}^2)_{i=1,\ldots,d-1}\right\}.$$

The reference element for the Crouzeix-Raviart is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. For both velocity elements used here, the degrees of freedom are determined by the following set of nodal functionals:

$$\{F_{\sigma,i},\ \sigma \in \mathcal{E}(K),\, i = 1,\ldots,d\} \qquad F_{\sigma,i}(v) = |\sigma|^{-1} \int_\sigma v_i \,\mathrm{d}\gamma. \tag{2.6}$$

The mapping from the reference element to the actual one is, for the Rannacher-Turek element, the standard Q_1 mapping and, for the Crouzeix-Raviart element, the standard affine mapping. Finally, in both cases, the continuity of the average value of discrete velocities (*i.e.*, for a discrete velocity field v, $F_{\sigma,i}(v)$, $1 \le i \le d$) across each face of the mesh is required, thus the discrete space W_h is defined as follows:

$$\begin{aligned} W_h = \ \{ \ & v_h \in L^2(\Omega) \, : \, v_h|_K \in W(K)^d, \forall K \in \mathcal{M}; \ F_{\sigma,i}(v_h) \text{ continuous across each edge } \sigma \in \mathcal{E}_{\text{int}}, \ 1 \le i \le d \, ; \\ & F_{\sigma,i}(v_h) = 0, \ \forall \sigma \in \mathcal{E}_{\text{ext}}, \ 1 \le i \le d \ \} \end{aligned}$$

where $W(K)$ is the space of functions on K generated by the reference element and the mapping described above. For both Rannacher-Turek and Crouzeix-Raviart discretizations, the pressure is approximated by the space L_h of piecewise constant functions:

$$L_h = \left\{ q_h \in L^2(\Omega) \, : \, q_h|_K = \text{ constant}, \forall K \in \mathcal{M} \right\}.$$

From the definition (2.6), each velocity degree of freedom can be univoquely associated to an element edge. Hence, the velocity degrees of freedom may be indexed by the number of the component and the associated edge, and the set of velocity degrees of freedom reads:

$$\{ v_{\sigma,i}, \ \sigma \in \mathcal{E}_{\text{int}}, \ 1 \le i \le d \}.$$

We define $v_\sigma = \sum_{i=1}^d v_{\sigma,i} \, \mathrm{e}^{(i)}$ where $\mathrm{e}^{(i)}$ is the ith vector of the canonical basis of $\mathbb{R}^d$. We denote by $\varphi_\sigma^{(i)}$ the vector shape function associated to $v_{\sigma,i}$, which, by the definition of the considered finite elements, reads:

$$\varphi_\sigma^{(i)} = \varphi_\sigma \, \mathrm{e}^{(i)},$$

where φ_σ is a scalar function.

Each degree of freedom for the pressure is associated to a cell K, and the set of pressure degrees of freedom is denoted by $\{ p_K, \ K \in \mathcal{M} \}$. The density ρ, the gas mass fraction y and the gas partial density z are also approximated by piecewise constant functions over each element, and the associated sets of degrees of freedom are denoted by $\{ \rho_K, \ K \in \mathcal{M} \}$, $\{ y_K, \ K \in \mathcal{M} \}$ and $\{ z_K, \ K \in \mathcal{M} \}$ respectively.

In the definition of the scheme, we also need a dual mesh, which is defined as follows. For any $K \in \mathcal{M}$ and any face $\sigma \in \mathcal{E}(K)$, let $D_{K,\sigma}$ be the cone of basis σ and of opposite vertex the mass center of K. The volume $D_{K,\sigma}$ is referred to as the half-diamond mesh associated to K and σ. For $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we now define the diamond mesh D_σ associated to σ by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$. We denote by $\varepsilon = D_\sigma | D_{\sigma'}$ the face separating two diamond meshes D_σ and $D_{\sigma'}$ (see Fig. 1).

2.3. **The fully discrete scheme**

The mass balance equations (2.1) and (2.3b) are discretized by a finite-volume technique. The fully discrete version of (2.3b) is:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t} \left(\varrho^{p,z}(p_K^{n+1}, z_K^{n+1}) - \rho_K^n \right) + \sum_{\sigma = K|L} F_{\sigma,K}^{n+1} = 0 \tag{2.7}$$

where $F_{\sigma,K}^{n+1}$ is an approximation of the integral over $\sigma = K|L$ of $\varrho^{p,z}(p_K^{n+1}, z_K^{n+1}) \, u^{n+1} \cdot n_{KL}$. To ensure the positivity of the density, we use an upwinding technique for the convection term:

$$F_{\sigma,K}^{n+1} = (\mathrm{v}_{\sigma,K}^+)^{n+1} \, \varrho^{p,z}(p_K^{n+1}, z_K^{n+1}) - (\mathrm{v}_{\sigma,K}^-)^{n+1} \, \varrho^{p,z}(p_L^{n+1}, z_L^{n+1})$$

where $(\mathrm{v}_{\sigma,K}^+)^{n+1} = \max(\mathrm{v}_{\sigma,K}^{n+1}, \ 0)$ and $(\mathrm{v}_{\sigma,K}^-)^{n+1} = -\min(\mathrm{v}_{\sigma,K}^{n+1}, \ 0)$ with $\mathrm{v}_{\sigma,K}^{n+1} = |\sigma| \, u_\sigma^{n+1} \cdot n_{KL}$.

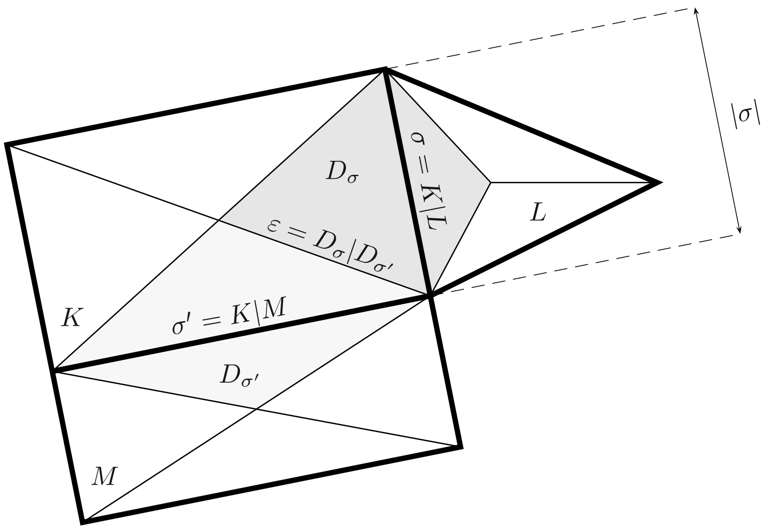

Figure 1. Notations for control volumes and diamond cells.

The velocity prediction equation is approximated by a combination of a dual mesh finite volume technique for the unsteady term and convection term, and a finite element technique for the other terms:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \text{for } 1 \le i \le d, \quad \frac{|D_\sigma|}{\delta t}\,(\rho_\sigma^n\ \tilde u_{\sigma,i}^{n+1} - \rho_\sigma^{n-1}\ u_{\sigma,i}^n) + \sum_{\substack{\varepsilon\in\mathcal{E}(D_\sigma),\\ \varepsilon = D_\sigma|D_{\sigma'}}} \frac{1}{2}\, F_{\varepsilon,\sigma}^n\ (\tilde u_{\sigma,i}^{n+1} + \tilde u_{\sigma',i}^{n+1}) + a_d(\tilde u^{n+1}, \varphi_\sigma^{(i)})$$
$$- \int_{\Omega,h} p^n\ \nabla\cdot\varphi_\sigma^{(i)} = \int_\Omega \pi^{n+1}\cdot\varphi_\sigma^{(i)} \qquad (2.8)$$

where $F_{\varepsilon,\sigma}^n$ is the discrete mass flux through the dual edge ε outward D_σ and the bilinear form a_d represents the viscous term and is defined as follows:

$$\forall v \in W_h,\ \forall w \in W_h,$$
$$a_d(v,w) = \left|\begin{array}{ll} \displaystyle \mu \int_{\Omega,h} \left[\nabla v : \nabla w + \frac{1}{3}\,\nabla\cdot v\ \nabla\cdot w\right]\,{\rm d}x & \text{if (1.3) holds (case of constant viscosity),} \\ \displaystyle \int_{\Omega,h} \tau(v) : \nabla w\ {\rm d}x & \text{with } \tau \text{ given by (1.2) otherwise.} \end{array}\right.$$

The main motivation to implement a finite volume approximation for the first two terms is to obtain a discrete equivalent of the kinetic energy theorem, which reads:

$$\sum_{\sigma\in\mathcal{E}_{\rm int}} \left[\frac{|D_\sigma|}{\delta t}\,(\rho_\sigma^n\ \tilde u_\sigma^{n+1} - \rho_\sigma^{n-1}\ u_\sigma^n) + \sum_{\substack{\varepsilon\in\mathcal{E}(D_\sigma),\\ \varepsilon = D_\sigma|D_{\sigma'}}} \frac{1}{2}\, F_{\varepsilon,\sigma}^n\ (\tilde u_\sigma^{n+1} + \tilde u_{\sigma'}^{n+1})\right]\cdot\ u_\sigma$$
$$\ge \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{\rm int}} \frac{|D_\sigma|}{\delta t}\ \left[\varrho_\sigma^n\ |\tilde u_\sigma^{n+1}|^2 - \varrho_\sigma^{n-1}\ |u_\sigma^n|^2\right]. \qquad (2.9)$$

For this result to be valid, the necessary condition is that the convection operator vanishes for a constant velocity, *i.e.* that the following discrete mass balance over the diamond cells is satisfied [1,19]:

$$\forall \sigma \in \mathcal{E}_{\rm int}, \qquad \frac{|D_\sigma|}{\delta t}\,(\rho^n_\sigma - \rho^{n-1}_\sigma) + \sum_{\substack{\varepsilon\in\mathcal{E}(D_\sigma),\\ \varepsilon=D_\sigma|D_{\sigma'}}} F^n_{\varepsilon,\sigma} = 0.$$

This governs the choice for the definition of the density approximation ρ_σ and the mass fluxes $F_{\varepsilon,\sigma}$. The density ρ_σ is defined by a weighted average:

$$\forall \sigma \in \mathcal{E}_{\rm int}, \qquad |D_\sigma|\;\rho_\sigma = |D_{K,\sigma}|\;\rho_K + |D_{L,\sigma}|\;\rho_L \tag{2.10}$$

and the flux $F_{\varepsilon,\sigma}$ through the dual edge ε of the half diamond cell $D_{K,\sigma}$ is computed as the flux through ε of a constant divergence lifting of the mass fluxes through the edges of the primal cell K, *i.e.* the quantities $(F_{\sigma,K})_{\sigma\in\mathcal{E}(K)}$ appearing in (2.7). For a detailed construction of this approximation, we refer to [1,19].

The discretization of (2.3a) is consistent with that of the momentum balance (2.8), *i.e.* we use a mass lumping technique for the unsteady term and a standard finite element formulation for the gradient of the pressure increment:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \text{for } 1\le i\le d, \qquad \frac{|D_\sigma|}{\delta t}\,\rho^n_\sigma\,(u^{n+1}_{\sigma,i} - \tilde u^{n+1}_{\sigma,i}) - \int_{\Omega,h} (p^{n+1}-p^n)\ \nabla\cdot\varphi^{(i)}_\sigma\ {\rm d}x = 0.$$

Since the pressure is piecewise constant, the transposed of the discrete gradient operator takes the form of the finite volume standard discretization of the divergence based on the finite element mesh, thus the previous relation can be rewritten as follows:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma=K|L, \qquad \frac{|D_\sigma|}{\delta t}\,\rho^n_\sigma\,(u^{n+1}_\sigma - \tilde u^{n+1}_\sigma) + |\sigma|\,\big[(p^{n+1}_L - p^n_L) - (p^{n+1}_K - p^n_K)\big]\,n_{KL} = 0. \tag{2.11}$$

Consistently with the mass balance equation, we use for the discretization of (2.3c), *i.e.* the transport of the gas partial density z, a finite volume method with an upwind technique for the convection term $\nabla\cdot(z\,u)$. This yields the following discrete equation:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\,(z^{n+1}_K - \rho^n_K\,y^n_K) + \sum_{\sigma=K|L} ({\rm v}^+_{\sigma,K})^{n+1}\,z^{n+1}_K - ({\rm v}^-_{\sigma,K})^{n+1}\,z^{n+1}_L = 0. \tag{2.12}$$

For the practical solution of the projection step, equations (2.11) and (2.7) are combined to eliminate the end-of-step velocity and obtain a discrete elliptic problem for the pressure; these algebraic manipulations are detailed in [19]. We are thus left with a system of two-coupled nonlinear equations; this system is solved by a Newton technique, which requires in practice a few iterations, and the linear systems involved in Newton's method are solved by a GMRES algorithm.

Finally, so as to be consistent with the discretization of the first part of the gas mass balance (2.12), the correction step for y is discretized by the finite volume method, and the resulting discrete problem reads:

$$|K|\,\frac{\rho^{n+1}_K\,y^{n+1}_K - z^{n+1}_K}{\delta t} + \sum_{\sigma=K|L} G^{n+1}_{\sigma,K}\ \Phi_\sigma(y^{n+1}_K, y^{n+1}_L) + D\sum_{\sigma=K|L}\frac{|\sigma|}{d_\sigma}(y^{n+1}_K - y^{n+1}_L) = 0 \tag{2.13}$$

where:

– The flux $G^{n+1}_{\sigma,K}$ is given by:

$$G^{n+1}_{\sigma,K} = \rho^{n+1}_{\sigma,{\rm up}}\int_\sigma u^{n+1}_r\cdot n_\sigma,$$

where $\rho_{\sigma,\mathrm{up}}^{n+1}$ stands for $\rho_{\sigma,\mathrm{up}}^{n+1} = \rho_K^{n+1}$ if $u_r^{n+1} \cdot n_\sigma \geq 0$ and $\rho_{\sigma,\mathrm{up}}^{n+1} = \rho_L^{n+1}$ otherwise. Note that this upwind choice with respect to u^{n+1} has no theoretical justification: in fact, the developments of this paper hold with any discretization for this density, and we use here the same discretization as in the mass balance merely for ease of implementation.

- The quantity $\Phi_\sigma(y_K^{n+1}, y_L^{n+1})$ stands for $g(y_K^{n+1}, y_L^{n+1})$ if $G_{\sigma,K}^{n+1} \geq 0$ and for $g(y_L^{n+1}, y_K^{n+1})$ otherwise, g being a monotone numerical flux function [13] for $\varphi(y) = \max[\, y\,(1-y),\ 0\,]$ given by:

$$g(a_1, a_2) = g_1(a_1) + g_2(a_2),$$

  where $g_1(a_1) = a_1$ if $a_1 \in [0,1]$ and $g_1(a_1) = 0$ otherwise, and $g_2(a_2) = -(a_2)^2$ if $a_2 \in [0,1]$ and $g_2(a_2) = 0$ otherwise. Note that this choice does not exactly match the definition, as neither g_1 nor g_2 are continuous at $a_1 = 1$ and $a_2 = 1$ respectively. However, this is unimportant, as one can prove that the solution y remains in the interval $(0,1]$ in any case, as stated in [19], Theorem 2.2.
- For all edge $\sigma = K|L$, d_σ denotes the Euclidean distance between two points x_K and x_L of the adjacent cells K and L, supposed to be such that the segment $[x_K, x_L]$ is perpendicular to $K|L$. These points may be defined as follows: if the control volume K is a rectangle or a cuboid, x_K is the barycenter of K; if the control volume K is a simplex, x_K is the circumcenter of the vertices of K. Note that, in this latter case, the condition $x_K \in K$ implies some geometrical constraints on the cells K, which are of course no longer needed if the diffusion coefficient D is zero.

3. Properties of the scheme

3.1. Well posedness, conservativity and physical bounds

Theorem 3.1. *Let the density of the liquid phase be constant and the gas phase obey the ideal gas law. In addition, we assume that the initial density is positive and the initial gas mass fraction belongs to the interval $(0,1]$. Then there exists a solution $(u^n)_{1\leq n\leq N}$, $(p^n)_{1\leq n\leq N}$, $(\rho^n)_{1\leq n\leq N}$, $(z^n)_{1\leq n\leq N}$ and $(y^n)_{1\leq n\leq N}$ to the scheme which enjoys the following properties, for all $n \leq N$:*

- *the unknowns lie in their physical range:*

$$\forall K \in \mathcal{M}, \qquad \rho_K^n > 0, \qquad z_K^n > 0, \qquad p_K^n > 0, \qquad y_K^n \in (0,1];$$

- *the total mass and the gas mass are conserved:*

$$\sum_{K\in\mathcal{M}} |K|\ \rho_K^n = \sum_{K\in\mathcal{M}} |K|\ \rho_K^0,$$
$$\sum_{K\in\mathcal{M}} |K|\ z_K^n = \sum_{K\in\mathcal{M}} |K|\ \rho_K^n y_K^n = \sum_{K\in\mathcal{M}} |K|\ \rho_K^0 y_K^0.$$

Proof. The fact that the velocity prediction step has a unique solution is a straightforward consequence of the discrete kinetic energy balance relation (2.9); indeed the problem is linear and, by dissipativity of the diffusion term, an estimate can be given for the solution. We prove in Appendix B that, if $\forall K \in \mathcal{M}$, $\rho_K^n > 0$ and $y_K^n \in (0,1]$, the system (2.11)–(2.12) has a solution, and any solution of this step is such that:

$$\forall K \in \mathcal{M}, \qquad \rho_K^{n+1} > 0, \quad p_K^{n+1} > 0, \quad z_K^{n+1} > 0 \quad \text{and} \quad \frac{z_K^{n+1}}{\rho_K^{n+1}} \in (0,1].$$

Finally, using these last bounds, we get $y_K^{n+1} \in (0,1]$ from [19], Theorem 2.2 and Remark 2.1. The proof of the first assertion of the theorem follows by an easy induction. □

Remark 3.2 (uniqueness of the solution)**.** Although it has not been proved, the solution of the system is probably unique. At any rate, the solvers never seem to oscillate between two solutions.

3.2. Transport of interfaces

We now turn to another feature of the scheme, which numerical experiments have shown to be crucial for the robustness of the algorithm. In this section, the drift velocity u_r, the diffusion coefficient D and the forcing term π are set to zero. In addition, we momentarily forget the boundary condition and consider the problem posed over $\mathbb{R}^n$. Then the continuous problem enjoys the following property: if the initial velocity and the initial pressure are constant, $u = u_0$ and $p = p_0$ respectively, then they remain constant in time, while ρ or z are transported by this (constant) velocity; this solution corresponds to the transport of the contact discontinuity of the underlying hyperbolic system, the wave structure of which is quite similar to that of the Euler equations [22]. Let us then prove that the numerical scheme considered in this paper presents the same behaviour: if, at the initial time, $u^0_K = u_0$ and $p^0_K = p_0$ for all $K \in \mathcal{M}$, then $p^{n+1}_K = p_0$ and $u^{n+1}_K = u_0$, for all $K \in \mathcal{M}$ and $n < N$.

Assume that, at time $t = t_n$, the velocity u^n and the pressure p^n take the constant value u_0 and p_0 respectively and let us check that there exists a solution u^{n+1}, p^{n+1}, z^{n+1} and y^{n+1} to the scheme such that $u^{n+1} = u_0$ and $p^{n+1} = p_0$. For $\pi = 0$, the discrete momentum balance equation (2.8) reads:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \text{for } 1 \leq i \leq d,$$
$$\frac{|D_\sigma|}{\delta t}\,(\rho^n_\sigma \tilde u^{n+1}_{\sigma,i} - \rho^{n-1}_\sigma u^n_{\sigma,i}) + \sum_{\substack{\varepsilon\in\mathcal{E}(D_\sigma),\\ \varepsilon=D_\sigma|D_{\sigma'}}} \frac{1}{2}\,F^n_{\varepsilon,\sigma}\,(\tilde u^{n+1}_{\sigma,i} + \tilde u^{n+1}_{\sigma',i}) - \int_{\Omega,h} p^n\ \nabla\cdot\varphi^{(i)}_\sigma\,{\rm d}x = 0.$$

Replacing u^n and p^n by u_0 and p_0 respectively and taking $\tilde u^{n+1}_\sigma = u_0$ for all $\sigma \in \mathcal{E}_{\rm int}$, this system becomes:

$$\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \qquad u_0 \left[\frac{|D_\sigma|}{\delta t}\,(\rho^n_\sigma - \rho^{n-1}_\sigma) + \sum_{\varepsilon\in\mathcal{E}(D_\sigma)} F^n_{\varepsilon,\sigma}\right] = 0,$$

which indeed holds thanks to the equivalence between mass balances over primal and dual meshes [1]. We now turn to the pressure correction step, which we recall:

$$\left|\begin{array}{ll}
\dfrac{|D_\sigma|}{\delta t}\,\rho^n_\sigma\,(u^{n+1}_\sigma - \tilde u^{n+1}_\sigma) + |\sigma|\ \left[(p^{n+1}_L - p^n_L) - (p^{n+1}_K - p^n_K)\right]\ n_{KL} = 0, & \forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L \\
\dfrac{|K|}{\delta t}\left[\varrho^{p,z}(p^{n+1}_K, z^{n+1}_K) - \rho^n_K\right] & \\
\qquad + \displaystyle\sum_{\sigma\in\mathcal{E}(K)} \left[({\rm v}^+_{\sigma,K})^{n+1}\,\varrho^{p,z}(p^{n+1}_K, z^{n+1}_K) - ({\rm v}^-_{\sigma,K})^{n+1}\,\varrho^{p,z}(p^{n+1}_L, z^{n+1}_L)\right] = 0, & \forall K \in \mathcal{M} \\
\dfrac{|K|}{\delta t}(z^{n+1}_K - z^n_K) + \displaystyle\sum_{\sigma\in\mathcal{E}(K)} \left[({\rm v}^+_{\sigma,K})^{n+1}\,z^{n+1}_K - ({\rm v}^-_{\sigma,K})^{n+1}\,z^{n+1}_L\right] = 0, & \forall K \in \mathcal{M}.
\end{array}\right.$$

Taking $u^{n+1}_\sigma = u_0$ for all $\sigma \in \mathcal{E}_{\rm int}$ and $p^{n+1}_K = p_0$ for all $K \in \mathcal{M}$, the left-hand-side of the first equation of this system vanishes. We then proceed as in [15]; remarking that for a fixed pressure, the equation of state giving the density ρ as a function of z becomes an affine function:

$$\rho = \varrho^{p,z}(p_0, z) = z\left(1 - \frac{\rho_\ell\, a^2}{p_0}\right) + \rho_\ell,$$

we get from the mass balance equation (2.7):

$$\frac{|K|}{\delta t}\left[\left(z_K^{n+1}-z_K^n\right)\left(1-\frac{\rho_\ell\, a^2}{p_0}\right)\right]+\sum_{\sigma\in\mathcal{E}(K)}(\mathrm{v}_{\sigma,K}^{+})^{n+1}\left[z_K^{n+1}\left(1-\frac{\rho_\ell\, a^2}{p_0}\right)+\rho_\ell\right] \\ -(\mathrm{v}_{\sigma,K}^{-})^{n+1}\left[z_L^{n+1}\left(1-\frac{\rho_\ell\, a^2}{p_0}\right)+\rho_\ell\right]=0,$$

or equivalently:

$$\left(1-\frac{\rho_\ell\, a^2}{p_0}\right)\left[\frac{|K|}{\delta t}(z_K^{n+1}-z_K^n)+\sum_{\sigma\in\mathcal{E}(K)}\left[(\mathrm{v}_{\sigma,K}^{+})^{n+1}\, z_K^{n+1}-(\mathrm{v}_{\sigma,K}^{-})^{n+1}\, z_L^{n+1}\right]\right]+\rho_\ell\sum_{\sigma\in\mathcal{E}(K)}\mathrm{v}_{\sigma,K}^{n+1}=0.$$

The last term vanishes for $u^{n+1}=u_0$ of the left-hand-side of this equality and thus we get exactly the same equation as that the partial gas mass balance (2.12). Thus, $u^{n+1}=u_0$, $p^{n+1}=p_0$, z^{n+1} given by this latter equation and y^{n+1} satisfying the correction step (which, for $u_r=0$ and $D=0$ becomes $\rho^{n+1}y^{n+1}=z^{n+1}$) is a solution to the scheme. Consequently, provided that the solution is unique, the algorithm indeed preserves constant pressure and velocity through moving interfaces between phases, and transports this interface with this constant velocity.

Remark 3.3 (boundary conditions)**.** The same property holds with a bounded computational domain when prescribing on the boundary either $u=u_0$ or a Neumann condition compatible with $u=u_0$ and $p=p_0$; this fact has been confirmed by numerical experiments, although we omit its proof here, to avoid the technicalities of the description of these latter discrete boundary conditions.

3.3. **Stability analysis: the homogeneous model case**

The aim of this section is to provide some results concerning the stability of (*i.e.* the conservation of the entropy by) the scheme considered in this paper, in the case where both the drift velocity u_r and the diffusion coefficient for the mass fraction of the dispersed phase D vanish (*i.e.* for the homogeneous model). Precisely speaking, we prove that the usual entropy associated to the homogeneous model is conserved by the scheme, if we add to the algorithm a pressure renormalization step. Note however, that this step should not be implemented in practice, since its beneficial effects are not clear in numerical tests.

We begin by introducing the volumetric free energy of the system. In the two-phase mixture considered here, ρ_ℓ is constant and ρ_g is linearly increasing with the pressure:

$$\rho_g=\frac{1}{a^2}\,p$$

where a is a positive real number (from a physical point of view, it is the sound velocity in a pure gaseous isothermal flow). For any positive ρ and z such that $z-\rho+\rho_\ell>0$, the relation (1.6) giving the mixture density as a function of the gas mass fraction and the phase densities may be recast through (2.5) under the following form:

$$\frac{1}{a^2}\,p=\varrho_g^{\rho,z}(\rho,z)=\frac{z\,\rho_\ell}{z+\rho_\ell-\rho}\cdot \tag{3.1}$$

The volumetric free energy of the mixture is given by:

$$\mathcal{F}(\rho,z)=a^2\,z\,\ln(\varrho_g^{\rho,z}(\rho,z)). \tag{3.2}$$

Lemma 3.4. *The function $\mathcal{F}$ defined by (3.2) enjoys the following properties:*

- *the function $\mathcal{F}$ is continuously differentiable over the convex subset of $\mathbb{R}^2$:*

$$\mathcal{C} = \{(\rho,\, z) \in \mathbb{R}^2 \ s.t.\ \ \rho > 0,\ \ z > 0,\ \ z - \rho + \rho_\ell > 0\}; \tag{3.3}$$

- *we have the following identity:*

$$\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F} - \mathcal{F} = p; \tag{3.4}$$

- *the function $\mathcal{F}$ is convex over $\mathcal{C}$.*

Proof. To prove relation (3.4), let us remark that $\mathcal{F}$ can be written as:

$$\mathcal{F}(\rho, z) = z\ \mathcal{F}_g(\varrho_g^{\rho,z}(\rho, z)) \quad \text{with} \quad \mathcal{F}_g(s) = a^2 \ln(s) \quad \text{so that} \quad \mathcal{F}_g'(s) = \frac{\wp(s)}{s^2}$$

where $\wp : s \mapsto a^2 s$ is the function giving the pressure as a function of the gas density (thus, in particular, $\mathcal{F}_g'(\rho_g) = p/\rho_g^2$). Such a function $\mathcal{F}_g$ is usually referred to as the specific free energy of the gaseous phase. Developing the derivatives and using the definition of $\mathcal{F}_g$, we get:

$$\begin{aligned}\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F} - \mathcal{F} \quad &= \rho\, z\, \mathcal{F}_g'(\rho_g)\partial_\rho \varrho_g^{\rho,z} + z^2\, \mathcal{F}_g'(\rho_g)\partial_z \varrho_g^{\rho,z} + z\mathcal{F}_g(\rho_g) - z\mathcal{F}_g(\rho_g) \\ &= z\, \frac{p}{\rho_g^2}\big[\rho\, \partial_\rho \varrho_g^{\rho,z} + z\partial_z \varrho_g^{\rho,z}\big].\end{aligned} \tag{3.5}$$

From the expression (3.1), we have:

$$\partial_\rho \varrho_g^{\rho,z} = \frac{\rho_g^2}{\rho_\ell\ z} \qquad \text{and} \qquad \partial_z \varrho_g^{\rho,z} = \frac{\rho_g^2(\rho_\ell - \rho)}{\rho_\ell\ z^2}\cdot \tag{3.6}$$

Substituting in (3.5) leads to:

$$\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F} - \mathcal{F} = p.$$

The convexity of $\mathcal{F}$ is obtained from its explicit form:

$$\mathcal{F}(\rho, z) = a^2\ z\ \ \ln\left(\frac{z\ \rho_\ell}{z + \rho_\ell - \rho}\right)\cdot$$

Differentiating twice this expression, we get:

$$\partial_\rho^2 \mathcal{F} = a^2\, \frac{z}{(z + \rho_\ell - \rho)^2}, \qquad \partial_z^2 \mathcal{F} = a^2\, \frac{(\rho_\ell - \rho)^2}{z\,(z + \rho_\ell - \rho)^2}, \qquad \partial_{\rho z}^2 \mathcal{F} = \partial_{z\rho}^2 \mathcal{F} = a^2\, \frac{\rho_\ell - \rho}{(z + \rho_\ell - \rho)^2}\cdot$$

It is thus easy to check that the determinant of the Hessian matrix A of $\mathcal{F}$ is zero while its trace is positive. One eigenvalue of A is thus zero and the second one is positive, and $\mathcal{F}$ is convex. □

In this section, we use the following discrete norm and semi-norm:

$$\begin{aligned}&\forall v \in W_h, \qquad && \|v\|_{h,\rho}^2 = \sum_{\sigma \in \mathcal{E}_{\rm int}} |D_\sigma|\ \rho_\sigma\, |v_\sigma|^2 \\ &\forall q \in L_h, \qquad && |q|_{h,\rho}^2 = \sum_{\sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L} \frac{1}{\rho_\sigma}\ \frac{|\sigma|^2}{|D_\sigma|}\,(q_K - q_L)^2\end{aligned} \tag{3.7}$$

where $\rho = (\rho_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ is a family of positive real numbers. The function $\|\cdot\|^2_{h,\rho}$ defines a norm over W_h, and $|\cdot|_{h,\rho}$ can be seen as a weighted version of the discrete H^1 semi-norm classical which is in the finite volume context [13].

With a zero drift velocity and a zero diffusion coefficient, the numerical scheme at hand reads, leaving it for short in the time semi-discrete setting:

1. solve for $\tilde u^{n+1}$

$$\frac{\rho^n\ \tilde u^{n+1} - \rho^{n-1}\ u^n}{\delta t} + \nabla\cdot(\rho^n\ u^n \otimes \tilde u^{n+1}) + \nabla p^n - \nabla\cdot\tau(\tilde u^{n+1}) = \pi^{n+1}; \tag{3.8}$$

2. solve for p^{n+1}, u^{n+1}, ρ^{n+1} and z^{n+1}

$$\left|\begin{aligned} &\rho^n\ \frac{u^{n+1}-\tilde u^{n+1}}{\delta t} + \nabla(p^{n+1}-p^n) = 0\\ &\frac{\varrho^{p,z}(p^{n+1},\ z^{n+1}) - \rho^n}{\delta t} + \nabla\cdot(\varrho^{p,z}(p^{n+1},\ z^{n+1})\ u^{n+1}) = 0\\ &\frac{z^{n+1}-\rho^n y^n}{\delta t} + \nabla\cdot(z^{n+1}\ u^n) = 0\\ &\rho^{n+1} = \varrho^{p,z}(p^{n+1}, z^{n+1});\end{aligned}\right. \tag{3.9}$$

3. solve for y^{n+1}

$$\rho^{n+1} y^{n+1} = z^{n+1}. \tag{3.10}$$

Lemma 3.5. *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law and let $\mathcal{F}(\rho,z)$ be the corresponding volumetric free energy of the mixture, defined by (3.2). We assume that the density ρ^n is positive and the gas mass fraction y^n belongs to the interval $(0,1]$. Let u^{n+1}, p^{n+1}, z^{n+1} and ρ^{n+1} be a solution to the discrete equations associated to (3.9),* i.e. *(2.11), (2.7) and (2.12). Then the following inequality holds:*

$$-\int_{\Omega,h} p^{n+1}\ \nabla\cdot u^{n+1}\,\mathrm{d}x \geq \int_\Omega \mathcal{F}(\rho^{n+1},z^{n+1})\,\mathrm{d}x - \int_\Omega \mathcal{F}(\rho^n,z^n)\,\mathrm{d}x.$$

Proof. By assumption, for any $K\in\mathcal{M}$, (ρ^n_K, z^n_K) belongs to the convex subset $\mathcal{C}$ of $\mathbb{R}^2$ defined in Lemma 3.4. By Theorem B.4, $(\rho^{n+1}_K, z^{n+1}_K)$ exists and also belongs to $\mathcal{C}$. Thanks to Lemma 3.4, we may thus invoke Theorem A.1, which yields the result. □

Proposition 3.6 (a partial stability result)**.** *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law and let $\mathcal{F}(\rho,z)$ be the corresponding volumetric free energy of the mixture, defined by (3.2). We assume that the density ρ^n is positive and the gas mass fraction y^n belongs to the interval $(0,1]$. Let $\tilde u^{n+1}$, u^{n+1}, p^{n+1}, z^{n+1} and ρ^{n+1} be a solution to the discrete equations associated to (3.8)–(3.9),* i.e. *(2.8), (2.11), (2.7) and (2.12), with a zero forcing term. Then the following bound holds:*

$$\begin{aligned}\frac12\,\|u^{n+1}\|^2_{h,\rho^n} + \int_\Omega \mathcal{F}(\rho^{n+1},z^{n+1})\,\mathrm{d}x + \delta t\, a_d(\tilde u^{n+1},\tilde u^{n+1}) + \frac{\delta t^2}{2}|p^{n+1}|^2_{h,\rho^n}\\ \leq \frac12\,\|u^n\|^2_{h,\rho^{n-1}} + \int_\Omega \mathcal{F}(\rho^n,\rho^n y^n)\,\mathrm{d}x + \frac{\delta t^2}{2}|p^n|^2_{h,\rho^n}.\end{aligned} \tag{3.11}$$

Proof. Multiplying each component of (2.8) by the corresponding unknown $\tilde{u}_{\sigma,i}^{n+1}$ and summing over the edges and the components yields, by virtue of the kinetic energy identity (2.9):

$$\frac{1}{2\,\delta t}\,\|\tilde{u}^{n+1}\|^2_{h,\rho^n} \;-\frac{1}{2\,\delta t}\,\|u^n\|^2_{h,\rho^{n-1}} \;+a_d(\tilde{u}^{n+1},\tilde{u}^{n+1})-\int_{\Omega,h} p^n \nabla\cdot\tilde{u}^{n+1}\,\mathrm{d}x\le 0. \tag{3.12}$$

On the other hand, the first relation of the projection step (2.11) reads, for any $\sigma=K|L\in\mathcal{E}_{\rm int}$:

$$\left[\frac{|D_\sigma|\;\rho^n_\sigma}{\delta t}\right]^{1/2} u^{n+1}_\sigma+\left[\frac{|D_\sigma|\;\rho^n_\sigma}{\delta t}\right]^{-1/2}|\sigma|\;(p^{n+1}_L-p^{n+1}_K)\;n_{KL}$$
$$=\left[\frac{|D_\sigma|\;\rho^n_\sigma}{\delta t}\right]^{1/2} \tilde{u}^{n+1}_\sigma+\left[\frac{|D_\sigma|\;\rho^n_\sigma}{\delta t}\right]^{-1/2}|\sigma|\;(p^{n}_L-p^{n}_K)\;n_{KL}.$$

Squaring this relation gives $(T_1)_\sigma=(T_2)_\sigma$, where $(T_1)_\sigma$ and $(T_2)_\sigma$ are the square of the norm of the left and right-hand-side respectively. We get, by the definition (3.7) of the velocity norm and pressure semi-norm:

$$(T_1)_\sigma=\frac{|D_\sigma|}{\delta t}\rho^n_\sigma\;|u^{n+1}_\sigma|^2+2|\sigma|\;(p^{n+1}_L-p^{n+1}_K)\;u^{n+1}_\sigma\cdot n_{KL}+\delta t\frac{|\sigma|^2}{|D_\sigma|}\;\frac{1}{\rho^n_\sigma}\;(p^{n+1}_L-p^{n+1}_K)^2.$$

Summing over the internal edges and reordering the sums, we get:

$$\sum_{\sigma\in\mathcal{E}_{\rm int}}(T_1)_\sigma=\frac{1}{\delta t}\,\|u^{n+1}\|^2_{h,\rho^n}\;-2\int_{\Omega,h}p^{n+1}\nabla\cdot u^{n+1}\,\mathrm{d}x+\delta t\;|p^{n+1}|^2_{h,\rho^n}.$$

By the same computation for the left-hand-side, we get:

$$\frac{1}{2\delta t}\,\|u^{n+1}\|^2_{h,\rho^n}\;-\int_{\Omega,h}p^{n+1}\nabla\cdot u^{n+1}\,\mathrm{d}x+\frac{\delta t}{2}\;|p^{n+1}|^2_{h,\rho^n}=\frac{1}{2\delta t}\,\|\tilde{u}^{n+1}\|^2_{h,\rho^n}\;-\int_{\Omega,h}p^{n}\nabla\cdot \tilde{u}^{n+1}\,\mathrm{d}x+\frac{\delta t}{2}\;|p^{n}|^2_{h,\rho^n}.$$

Summing this last relation with (3.12) yields:

$$\frac{1}{2\delta t}\,\|u^{n+1}\|^2_{h,\rho^n}\;-\frac{1}{2\,\delta t}\,\|u^n\|^2_{h,\rho^{n-1}}\;+a_d(\tilde{u}^{n+1},\tilde{u}^{n+1})$$
$$+\frac{\delta t}{2}\,|p^{n+1}|^2_{h,\rho^n}-\frac{\delta t}{2}\,|p^{n}|^2_{h,\rho^n}-\int_{\Omega,h}p^{n+1}\;\nabla\cdot u^{n+1}\,\mathrm{d}x\le 0.$$

We thus conclude the proof by invoking Lemma 3.5. □

The following theorem yields an estimate on the discrete entropy of the system, which is defined as the sum of the kinetic energy and the free energy; its expression at time t_{n+1} is:

$$\frac{1}{2}\,\|u^{n+1}\|^2_{h,\rho^n}\;+\int_\Omega\mathcal{F}(\rho^{n+1},z^{n+1}).$$

Theorem 3.7 (stability of the scheme, case $u_r=D=0$)**.** *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law and let $\mathcal{F}(\rho,z)$ be the corresponding volumetric free energy of the mixture, defined by* (3.2)*. We suppose that the initial density is positive and the initial gas mass fraction belongs to the interval* $(0,1]$.

We now add to the scheme (3.8)–(3.10) the following renormalization step of the pressure, to be performed at the very beginning of the time step, before the velocity prediction step (3.8):

$$\text{Solve for } \tilde{p}^{n+1}: \qquad -\nabla\cdot\left(\frac{1}{\rho^n}\nabla\tilde{p}^{n+1}\right) = -\nabla\cdot\left(\frac{1}{\sqrt{\rho^n\,\rho^{n-1}}}\nabla p^n\right)$$

or, in the fully discrete setting:

$$\forall K\in\mathcal{M},\qquad \sum_{\sigma=K|L}\frac{|\sigma|^2}{|D_\sigma|}\,\frac{1}{\rho^n_\sigma}\,(\tilde{p}^{n+1}_K-\tilde{p}^{n+1}_L) = \sum_{\sigma=K|L}\frac{|\sigma|^2}{|D_\sigma|}\,\frac{1}{(\rho^n_\sigma\,\rho^{n-1}_\sigma)^{1/2}}\,(p^n_K-p^n_L). \tag{3.13}$$

Accordingly, the pressure used in the velocity prediction step must be changed to $\tilde{p}^{n+1}$.

Let $(\tilde{u}^n)_{0\le n\le N}$, $(u^n)_{0\le n\le N}$, $(p^n)_{0\le n\le N}$, $(z^n)_{0\le n\le N}$ and $(\rho^n)_{0\le n\le N}$ be the solution to this modified scheme, i.e., *for the discrete equations (3.13), (2.8), (2.11), (2.7) and (2.12), with a zero forcing term. Then the following entropy conservation result holds for $0\le n<N$:*

$$\frac{1}{2}\,\|u^{n+1}\|^2_{h,\rho^n} + \int_\Omega \mathcal{F}(\rho^{n+1},z^{n+1})\,\mathrm{d}x + \delta t\sum_{k=1}^{n+1} a_d(\tilde{u}^k,\tilde{u}^k) + \frac{\delta t^2}{2}|p^{n+1}|^2_{h,\rho^n}$$
$$\le \frac{1}{2}\,\|u^0\|^2_{h,\rho^0} + \int_\Omega \mathcal{F}(\rho^0,z^0)\,\mathrm{d}x + \frac{\delta t^2}{2}|p^0|^2_{h,\rho^0}. \tag{3.14}$$

Proof. By the same proof as for the scheme without the pressure renormalization step, we get:

$$\frac{1}{2\delta t}\,\|u^{n+1}\|^2_{h,\rho^n} + a_d(\tilde{u}^{n+1},\tilde{u}^{n+1}) + \frac{\delta t}{2}\,|p^{n+1}|^2_{h,\rho^n} + \frac{1}{\delta t}\int_\Omega \mathcal{F}(\rho^{n+1},z^{n+1})\,\mathrm{d}x$$
$$\le \frac{1}{2\,\delta t}\,\|u^n\|^2_{h,\rho^{n-1}} + \frac{\delta t}{2}\,|\tilde{p}^{n+1}|^2_{h,\rho^n} + \frac{1}{\delta t}\int_\Omega \mathcal{F}(\rho^n,\rho^n y^n)\,\mathrm{d}x$$

and the conclusion follows by summing over the time steps, remarking that $z^{n+1}=\rho^{n+1}y^{n+1}$ and, that thanks to the renormalization step (see [16] for a detailed computation):

$$|\tilde{p}^{n+1}|^2_{h,\rho^n} \le |p^n|^2_{h,\rho^{n-1}}.$$

□

Note that a similar pressure renormalization step has already been introduced for variable density incompressible flows [20].

3.4. Stability analysis: dissipativity of the drift term

We address in this section the case where the drift velocity is given by the Darcy-like closure relation (1.7):

$$u_r = \frac{1}{\lambda}\,(1-\alpha_g)\,\alpha_g\,\frac{\varrho_g(p)-\rho_\ell}{\rho}\,\nabla p.$$

In this relation, λ is a positive phenomenological coefficient and α_g is the void fraction, which can be expressed as a function of the unknowns used in the scheme as $\alpha_g = z/\varrho_g(p)$. With this expression, a natural discretization for the mass flux associated to u_r reads:

$$G_{\sigma,K} = |\sigma|\;\rho_{\sigma,\mathrm{up}}\left[\frac{\alpha_g\,(1-\alpha_g)}{\lambda\,\rho}\,(\rho_g-\rho_\ell)\right]_\sigma (p_L-p_K) \tag{3.15}$$

where $\rho_{\sigma,\mathrm{up}}$ is a density on σ, which, for practical implementation reasons, we choose to be an upwind approximation with respect to the mean velocity u. The goal of this section is to show that it is possible to approximate:

$$\left[\frac{\alpha_g\,(1-\alpha_g)}{\lambda\,\rho}\,(\rho_g-\rho_\ell)\right]_\sigma$$

in such a way that the drift term is dissipative with respect to the entropy of the system.

We begin this section by stating a consequence of the equation of state for the mixture which is central to the present development.

Lemma 3.8. *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law, let $\mathcal{F}(\rho,z)$ be the corresponding volumetric free energy of the mixture, defined by (3.2), and let $h(\rho,z)=\partial_z\mathcal{F}(\rho,z)$. Then the following results hold:*

(1) *h only depends on the pressure,* i.e. *there exists a function h_p such that, for ρ and z in the convex set $\mathcal{C}$ defined by (3.3), $h(\rho,z)=h_p(\wp(\rho,z))$, where $\wp$ is the function giving the pressure as a function of ρ and z:*

$$p=\wp(\rho,z)=a^2\,\varrho_g^{\rho,z}(\rho,z)=a^2\,\frac{z\,\rho_\ell}{z+\rho_\ell-\rho};$$

(2) *the derivative of h_p is given by:*

$$h_p'(p)=\frac{\rho_\ell-\rho_g(p)}{\rho_\ell\,\rho_g(p)};$$

(3) *for any positive real numbers p_1 and p_2 such that $p_1<p_2$, there exists $p_{1,2}\in[p_1,p_2]$ such that:*

$$h_p'(p_{1,2})\,\frac{h_p(p_1)-h_p(p_2)}{p_1-p_2}\geq 0.$$

Proof. Let $(\rho,z)\in\mathcal{C}$, then the pressure or, equivalently, the gas density ρ_g can be expressed as a function of (ρ,z) by $\rho_g=\varrho_g^{\rho,z}(\rho,z)$. By the definition of $\mathcal{F}$, we thus have, using the notations of the proof of Lemma 3.4:

$$h(\rho,z)=\partial_z\mathcal{F}(\rho,z)=\mathcal{F}_g(\rho_g(\rho,z))+z\,\partial_z\mathcal{F}_g(\rho,z)=a^2\,\ln\left(\frac{p}{a^2}\right)+z\,\mathcal{F}_g'(\rho_g)\,\partial_z\varrho_g^{\rho,z}(\rho,z).$$

Then using the expression (3.6) of the derivative of $\varrho_g^{\rho,z}$ with respect to the second variable, we get:

$$h(\rho,z)=a^2\,\ln\left(\frac{p}{a^2}\right)+p\,\frac{\rho_\ell-\rho}{\rho_\ell\,z}\cdot$$

Using the fact that $\rho=(1-\alpha_g)\rho_\ell+\alpha_g\,\rho_g$ and thus $\rho_\ell-\rho=\alpha_g\,(\rho_\ell-\rho_g)=\dfrac{z}{\rho_g}\,(\rho_\ell-\rho_g)$, we have:

$$h(\rho,z)=a^2\,\ln\left(\frac{p}{a^2}\right)+p\,\frac{\rho_\ell-\rho_g}{\rho_\ell\,\rho_g}\cdot$$

By definition of ρ_g, *i.e.* $\rho_g=p/a^2$, we thus get:

$$h(\rho,z)=a^2\left[\ln\left(\frac{p}{a^2}\right)+\frac{\rho_\ell-p/a^2}{\rho_\ell}\right]=h_p(p).$$

Taking the derivative of this relation yields the desired expression for h_p' and, since h_p is continuously differentiable in $[p_1,p_2]$, the existence of $p_{1,2}$ follows by Lagrange's theorem. □

Proposition 3.9. *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law and let $\mathcal{F}(\rho, z)$ be the corresponding volumetric free energy of the mixture, defined by (3.2). Let $\mathcal{C}$ be the convex set defined by (3.3) and let $(\rho_K)_{K\in\mathcal{M}}$, $(y_K)_{K\in\mathcal{M}}$ and $(z_K)_{K\in\mathcal{M}}$ be such that, $\forall K \in \mathcal{M}$, $(\rho_K, \rho_K y_K) \in \mathcal{C}$, $(\rho_K, z_K) \in \mathcal{C}$, and the following relation is satisfied:*

$$\frac{|K|}{\delta t}\ (\rho_K\, y_K - z_K) + \sum_{\sigma=K|L} G^+_{\sigma,K}\, g(y_K, y_L) - G^-_{\sigma,K}\, g(y_L, y_K) = 0 \tag{3.16}$$

where g corresponds to an approximation of $\varphi(y) = \max[\, y\,(1-y),\ 0\,]$ by a monotone numerical flux function, $G^+_{\sigma,K} = \max(G_{\sigma,K}, 0)$, $G^-_{\sigma,K} = -\min(G_{\sigma,K}, 0)$ and $G_{\sigma,K}$ is given by the relation (3.15). Then, if $g(y_K, y_L) \geq 0$ for all $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, there exists a discretization for the term:

$$\left[\frac{\alpha_g\,(1-\alpha_g)}{\lambda\,\rho}\,(\rho_g - \rho_\ell)\right]_\sigma$$

in (3.15) such that the following stability estimate holds:

$$\frac{1}{\delta t} \sum_{K\in\mathcal{M}} |K|\ \ [\mathcal{F}(\rho_K, \rho_K\, y_K) - \mathcal{F}(\rho_K, z_K)] \leq 0$$

which means that the drift term is dissipative with respect to the entropy of the system.

Proof. We multiply equation (3.16) by $\partial_z \mathcal{F}(\rho_K, \rho_K\, y_K)$ and sum up over the control volumes of the mesh:

$$\sum_{K\in\mathcal{M}} h(\rho_K, \rho_K\, y_K) \left[\frac{|K|}{\delta t}\,(\rho_K\, y_K - z_K) + \sum_{\sigma=K|L} [G^+_{\sigma,K}\, g(y_K, y_L) - G^-_{\sigma,K}\, g(y_L, y_K)]\right] = T_1 + T_2 = 0$$

where T_1 and T_2 read:

$$\begin{aligned} T_1 &= \sum_{K\in\mathcal{M}} \frac{|K|}{\delta t}\ h(\rho_K, \rho_K\, y_K)\ [\rho_K\, y_K - z_K] \\ T_2 &= \sum_{K\in\mathcal{M}} h(\rho_K, \rho_K\, y_K) \left[\sum_{\sigma=K|L} [G^+_{\sigma,K}\, g(y_K, y_L) - G^-_{\sigma,K}\, g(y_L, y_K)]\right]. \end{aligned}$$

Since the function $\mathcal{F}$ is convex, we have:

$$T_1 \geq \frac{1}{\delta t} \sum_{K\in\mathcal{M}} |K|\ \ [\mathcal{F}(\rho_K, \rho_K\, y_K) - \mathcal{F}(\rho_K, z_K)]. \tag{3.17}$$

Let us turn to T_2. Reordering the sum, we get:

$$T_2 = \sum_{\sigma\in\mathcal{E}_{\rm int}} |\sigma|\ \rho_{\sigma,{\rm up}}\, g_{\rm up}(y_K, y_L, u_r) \left[\frac{\alpha_g\,(1-\alpha_g)}{\lambda\,\rho}\,(\rho_g - \rho_\ell)\right]_\sigma (p_L - p_K)\,[h(\rho_K, \rho_K\, y_K) - h(\rho_L, \rho_L\, y_L)]$$

where $g_{\rm up}(y_K, y_L, u_r) = g(y_K, y_L)$ if $u_r \cdot n_{KL} \geq 0$ and $g_{\rm up}(y_K, y_L, u_r) = g(y_L, y_K)$ otherwise; in any case, we have, by assumption, $g_{\rm up}(y_K, y_L, u_r) \geq 0$. We now choose, for the approximation of the quantity defined on σ in the preceding relation, an expression of the form:

$$\left[\frac{\alpha_g\,(1-\alpha_g)}{\lambda\,\rho}\,(\rho_g - \rho_\ell)\right]_\sigma \equiv \frac{(\alpha_g)_\sigma\ [1-(\alpha_g)_\sigma]}{\lambda\,\rho_\sigma}\,(\varrho_g(p_\sigma) - \rho_\ell)$$

where $(\alpha_g)_\sigma$ and ρ_σ stand for approximations of the void fraction and the density on σ, respectively, which are only assumed to be non-negative. Applying Lemma 3.8, T_2 reads:

$$T_2 = \sum_{\sigma\in\mathcal{E}_{\rm int}} |\sigma|\ \rho_{\sigma,{\rm up}}\ g_{\rm up}(y_K, y_L, u_r)\ \frac{(\alpha_g)_\sigma\,(1-\alpha_g)_\sigma}{\lambda\,\rho_\sigma}\ \rho_\ell\ \varrho_g(p_\sigma)\ h'_p(p_\sigma)\ (p_K - p_L)\ [h_p(p_K) - h_p(p_L)].$$

If $p_K = p_L$, the term associated to $K|L$ in this sum vanishes. Otherwise, from the third assertion of Lemma 3.8, there exists $p_\sigma \in [\min(p_K, p_L), \max(p_K, p_L)]$ such that the product $h'_p(p_\sigma)\ (p_K - p_L)\ [h_p(p_K) - h_p(p_L)]$ is positive. Since we choose $(\alpha_g)_\sigma$ such that $(\alpha_g)_\sigma \geq 0$, all the other quantities are positive, and this concludes the proof. □

The following proposition extends the stability result of the preceding section to the case $u_r \neq 0$.

Proposition 3.10 (stability of the scheme, Darcy case)**.** *Let the density of the liquid phase be constant, let the gas phase obey the ideal gas law and let $\mathcal{F}(\rho, z)$ be the corresponding volumetric free energy of the mixture, defined by (3.2). We suppose that the density ρ^n is positive and the gas mass fraction y^n belongs to the interval $(0, 1]$. Let $\tilde u^{n+1}$, u^{n+1}, p^{n+1}, z^{n+1}, ρ^{n+1} and y^{n+1} be a solution to the equations of one time step of the scheme,* i.e. *(2.8), (2.11), (2.7), (2.12) and (2.13), with a zero forcing term. We suppose that the drift velocity is given by a the Darcy-like relation (1.7) and that the discretization of the correction step for the gas mass fraction y^{n+1} is such that the stability result of Proposition 3.9 applies. Then the following inequality holds:*

$$\frac{1}{2}\,\|u^{n+1}\|^2_{h,\rho^n}\ + \int_\Omega \mathcal{F}(\rho^{n+1}, \rho^{n+1} y^{n+1})\,{\rm d}x + \delta t\, a_d(\tilde u^{n+1}, \tilde u^{n+1}) + \frac{\delta t^2}{2} |p^{n+1}|^2_{h,\rho^n}$$
$$\leq \frac{1}{2}\,\|u^n\|^2_{h,\rho^{n-1}}\ + \int_\Omega \mathcal{F}(\rho^n, \rho^n y^n)\,{\rm d}x + \frac{\delta t^2}{2} |p^n|^2_{h,\rho^n}.$$

Proof. Proposition 3.9 yields:

$$\frac{1}{\delta t} \sum_{K\in\mathcal{M}} |K|\ \left[\mathcal{F}(\rho_K^{n+1}, \rho_K^{n+1}\, y_K^{n+1}) - \mathcal{F}(\rho_K^{n+1}, z_K^{n+1})\right] \leq 0.$$

The conclusion thus follows by summing this relation with the estimate of Proposition 3.6. □

Finally, note that, as in the preceding section, this partial stability result yields the same entropy decrease estimate for the whole scheme as in the preceding section if a renormalization step for the pressure is added to the scheme.

Remark 3.11 (on the choice of the monotone numerical flux function)**.** As stated in Section 2, we have adopted for the numerical tests presented hereafter the following flux-splitting formula:

$$g(a_1, a_2) = g_1(a_1) + g_2(a_2)$$

where $g_1(a_1) = a_1$ if $a_1 \in [0, 1]$ and zero otherwise and and $g_2(a_2) = -(a_2)^2$ if $a_2 \in [0, 1]$ and zero otherwise. This numerical monotone flux does not satisfy the hypothesis of Proposition 3.10, as it is not always non-negative. However, several other choices are possible for the numerical flux function g (see *e.g.* [13]), and some of them solve this problem. Thanks to the fact that $\varphi(s) = s\,(1-s)$ is positive $\forall s \in [0, 1]$, it is the case, for example, for the flux obtained with a one-dimensional Godunov scheme for each interface:

$$g(a_1, a_2) = \left|\begin{array}{ll} \max\{\varphi(s),\ a_2 \leq s \leq a_1\} & \text{if } a_2 \leq a_1 \\ \min\{\varphi(s),\ a_1 \leq s \leq a_2\} & \text{if } a_1 \leq a_2. \end{array}\right.$$

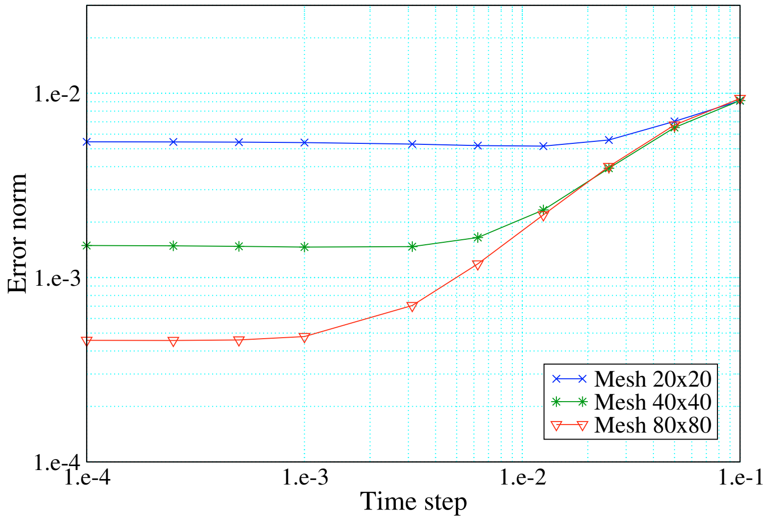

FIGURE 2. Error for the velocity at $t = 0.5$, as a function of the time step (L^2 norm).

4. Numerical results

This section is devoted to numerical tests of the proposed scheme. We first address a problem built in such a way that it admits an analytical solution, to assess the convergence properties of the scheme. Then several additional tests are performed, to check the stability of the algorithm and the quality of the results.

4.1. Assessing the convergence against an analytic solution

We address here a problem built by the so-called technique of manufactured solutions: the computational domain and the solution are chosen *a priori* and the initial conditions, the boundary conditions and the forcing terms are adjusted consequently. Let thus the computational domain be $\Omega = (0, 1) \times (-1/2, 1/2)$, and the density and the momentum take the following expressions:

$$\rho = 1 + \frac{1}{4} \sin(\pi t) \, [\cos(\pi x_1) - \sin(\pi x_2)] \qquad \rho\, u = -\frac{1}{4} \cos(\pi t) \begin{bmatrix} \sin(\pi x_1) \\ \cos(\pi x_2) \end{bmatrix}.$$

The pressure and the partial gas density are linked to the density by the equation of state (2.5), where the liquid density ρ_ℓ is set at $\rho_\ell = 5$ and the quantity a^2 in the equation of state of the gas (1.5) is given by $a^2 = 1$ (so $\rho_g = p$). We choose the following expression for the unknowns y and z:

$$y = \frac{2.5 - 0.5\, \rho}{4.5\, \rho} \qquad z = \rho\, y = \frac{2.5 - 0.5\, \rho}{4.5}.$$

The relative velocity is constant and given by $u_r = (0, 1)^t$ and the diffusive coefficient D is set to $D = 0.1$. The analytical expression for the pressure is obtained from the equation of state (*i.e.* relation (3.1)). These functions satisfy the mass balance equation; for the gas mass fraction and momentum balance, we add the corresponding right-hand side. In this latter equation, we suppose that the divergence of the stress tensor is given by:

$$\nabla \cdot \tau(u) = \mu\, \Delta u + \frac{\mu}{3} \nabla \nabla \cdot u, \qquad \mu = 10^{-2}$$

and we use for the viscous term the corresponding form for the bilinear form a_d (see Sect. 2.3).

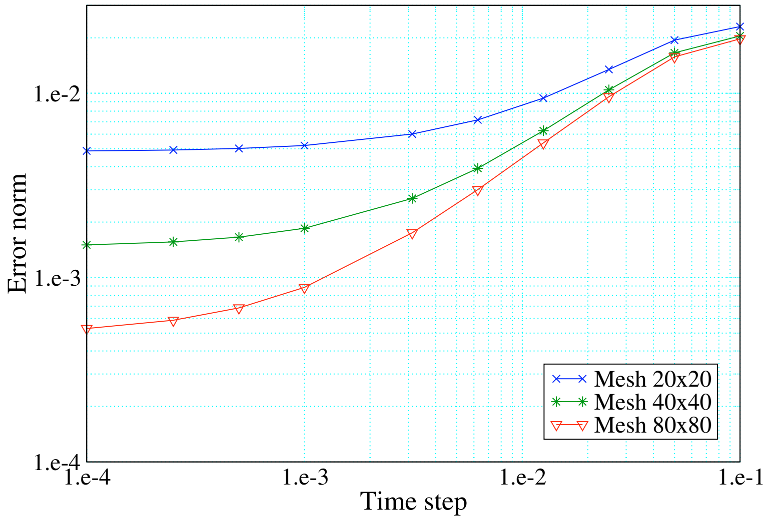

FIGURE 3. Error for the pressure at $t = 0.5$, as a function of the time step (discrete L^2 norm).

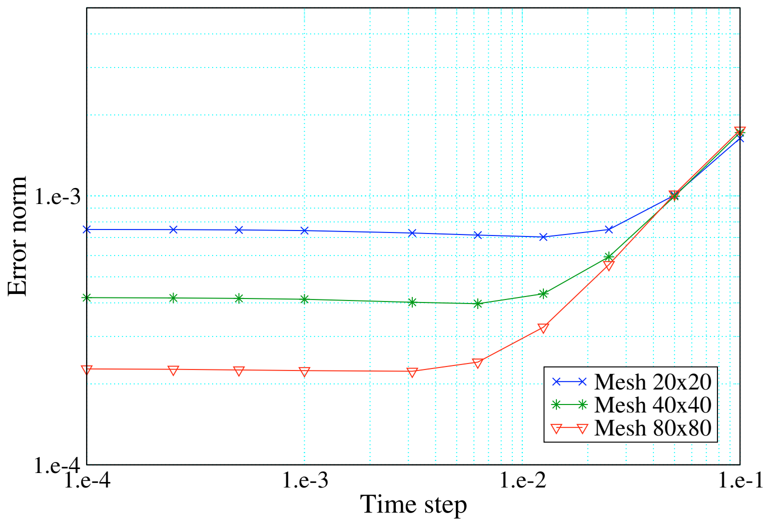

FIGURE 4. Error for the gas mass fraction at $t = 0.5$, as a function of the time step (discrete L^2 norm).

Errors for the velocity, pressure and gas mass fraction obtained at $t = 0.5$, as a function of the time step and for various meshes, are drawn in Figures 2, 3 and 4, respectively. These errors are evaluated in the L^2 norm for the velocity and in the discrete L^2 norms for the pressure and the gas mass fraction. Computations are made with 20×20, 40×40 and 80×80 uniform meshes (so with square cells and the Rannacher-Turek element). For large time steps, these curves show a decrease corresponding to approximately a first order convergence in time, until a plateau is reached, due to the fact that errors are bounded by below by the residual spatial discretization error. The value of the errors on this plateau then show a spatial convergence order close to one, which is consistent with the choice of an upwind discretization for the advection terms in the mass and gas mass fraction balance equations.

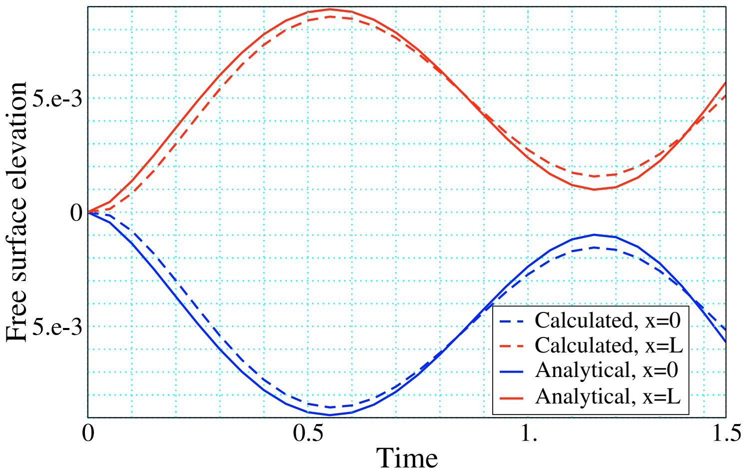

FIGURE 5. Sloshing in cavity: analytical solution and numerical solution with $\mu = \rho/100$.

4.2. **Two-dimensional sloshing in cavity**

Two layers of non-miscible fluids (air and water) are superimposed with the lighter one on top of the heavier one. The gravity (with $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$) is acting in the vertical downward direction. The length of the rectangular cavity is $L = 1$ m, the height of each layer is respectively $h_\ell = 1$ m and $h_g = 1.25$ m, so the total height of the box is 2.25 m. The water and air densities are respectively $\rho_\ell = 1000\,\mathrm{kg{\cdot}m^{-3}}$ and $\rho_g = p/a^2$ where a^2 is such that $\rho_g = 1.2\ \mathrm{kg{\cdot}m^{-3}}$ at $p = 10^5$ Pa. The diffusion coefficient D and the drift velocity are set to zero. A perfect slip condition is imposed on the whole boundary. At initial time, both fluids are at rest, then the cavity is submitted to an horizontal acceleration given by $a_0 = 0.1\,\mathrm{m{\cdot}s^{-2}}$.

In the case where both fluids are supposed incompressible and the convection and diffusion terms may be neglected, an analytical solution for the flow in a rectangular cavity is provided in [6]. In particular, the shape of the interface is given by the following relation:

$$\xi = \frac{a_0}{g}\left[x - \frac{L}{2} + \sum_{n\geq 0} \frac{4}{L\,k_{2n+1}^2}\ \cos(\omega_{2n+1}\,t)\ \cos(k_{2n+1}\,t)\right]$$

where the wave number k_n is defined by:

$$k_n = \frac{2\,\pi\,n}{L}$$

and ω_n is given by:

$$\omega_n^2 = \frac{g\,k_n\,(\rho_\ell - \rho_g)}{\rho_g\ \coth(k_n\,h_g) + \rho_\ell\ \coth(k_n\,h_\ell)}.$$

In practice, to compute this analytical solution, we perform the summation up to $n = 200$.

So as to remain in the domain of validity of the solution, the amplitude of the fluid oscillations must be very small; hence a very fine mesh is necessary near the free surface, to capture its motion. The mesh is thus made of about 41 000 rectangular cells (with the Rannacher-Turek element) and, in the vertical direction, the space step is adapted in such a way that it is smaller near the interface between the two phases and equal to $\delta x_2 = 0.0005$ m, and increases when moving away the free surface, up to $\delta x_2 = 0.05$ m at the top and bottom sections. In the horizontal direction, the mesh is uniform with step size $\delta x_1 = 1/70$ m. Calculations with different viscosities have been performed, these latter being supposed to vary with the mixture density: $\mu = \rho/100$, $\mu = \rho/1000$, $\mu = \rho/10\,000$.

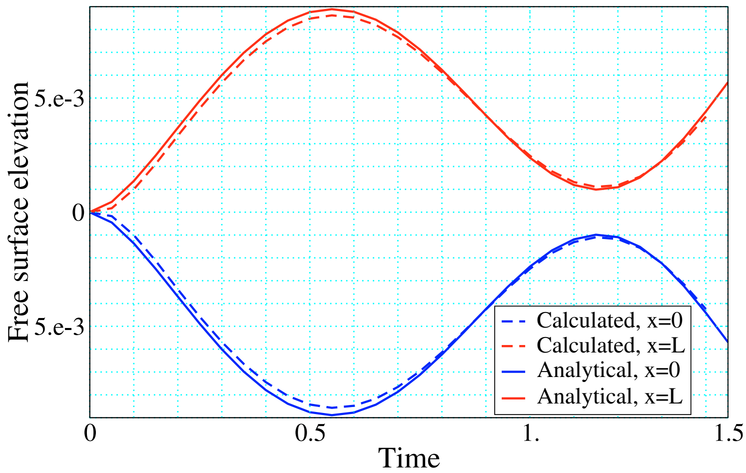

FIGURE 6. Sloshing in cavity: analytical solution and numerical solution with $\mu = \rho/1000$.

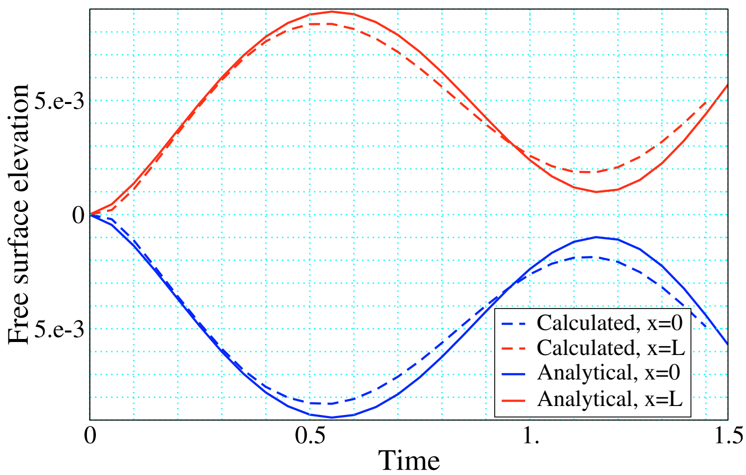

FIGURE 7. Sloshing in cavity: analytical solution and numerical solution with $\mu = \rho/10\,000$.

The numerical results are reported on Figure 5 ($\mu = \rho/100$), Figure 6 ($\mu = \rho/1000$), and Figure 7 ($\mu = \rho/10\,000$) respectively. Comparing the obtained shape for the interface with the analytical solution, we observe that the numerical solution is closer to the analytical one with $\mu = \rho/1000$ than with $\mu = \rho/100$, certainly because the fluid is too viscous in this latter case. More surprisingly, when reducing the viscosity to $\mu = \rho/10\,000$, the numerical solution also becomes less accurate. Our explanation is that, to obtain a good solution, it is necessary to respect a balance between approaching the physical problem (which, in this case, would suggest $\mu = 0$) and keeping sufficient coercivity to ensure a reasonable convergence of the numerical approximation (which, on the contrary, requires a high value for the viscosity). With a more refined mesh, viscosity thus probably could be decreased, and the solution be closer to the analytical one. However, with this mesh already, results seem to be rather more accurate than those available in the literature [6].

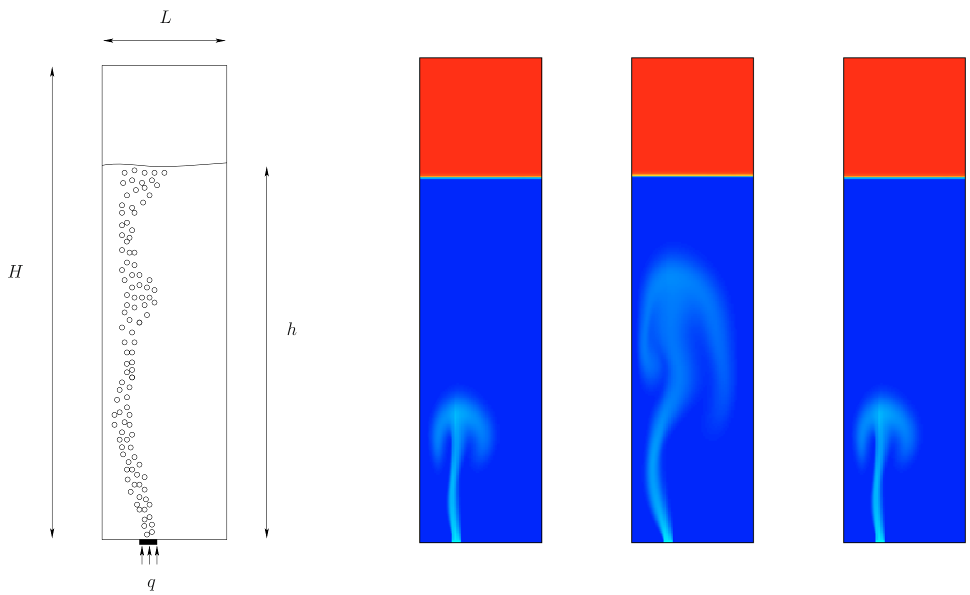

FIGURE 8. Bubble column: geometry of the problem and void fraction at times 2 s, 4 s and 40 s.

4.3. **Bubble column**

We address in this section a classical benchmark for diphasic flow solvers, namely the flow in a pseudo two dimensional bubble column investigated experimentally by Becker *et al.* [4]. The apparatus has a rectangular cross section with the following dimensions: its width is $L = 50\,\mathrm{cm}$, its depth is 8 cm and it is $H = 200$ cm high (see Fig. 8). It is filled with water up to the height $h = 150$ cm. A gas sparger, positioned 15 cm from the left wall, is used to introduce an air flow of $q = 8\ \mathrm{L{\cdot}min^{-1}}$ into the system. The circular sparger has a diameter of 40 mm and a pore size of 40 μm. Several liquid circulation cells can be observed in the column, the location and size of which continuously change. The bubble swarm is influenced by these vortices and therefore rises in a meander-like way. The direction of its lower part is stable and directed towards the nearest sidewall; its upper part changes its shape and location in a quasiperiodic way, according to transient liquid circulations [31].

To simulate this experiment, we choose the following data. The boundary conditions are defined at the inlet as follows:

$$u_{\mathrm{imp}} = \frac{q}{S\,\alpha_{g,\mathrm{imp}}}$$

where S is the gas inlet area and $\alpha_{g,\mathrm{imp}} = 1$ is the void fraction imposed at the inlet. Along the walls and at the outlet of the column, homogeneous Dirichlet conditions are used for the velocity. Initial conditions are set to $u = 0\ \mathrm{m{\cdot}s^{-1}}$ and $p = p_0$ where $p_0 = 10^5\,\mathrm{Pa}$ is the ambient pressure. The density of the liquid is $\rho_\ell = 1000\,\mathrm{kg{\cdot}m^{-3}}$; the gas obeys an ideal gas equation of state $\rho_g = p/a^2$, where a^2 is such that $\rho_g = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ at $p = 10^5\,\mathrm{Pa}$. The diffusion coefficient D is set to zero, the drift velocity is constant and given by $u_r = (0,\, 0.2)^t\ \mathrm{m{\cdot}s^{-1}}$.

For this test case, we use a regular mesh composed of rectangular cells (with the Rannacher-Turek element) with 76 cells in the horizontal direction, out of which 4 are for the gas inlet, and 300 in the vertical one. Calculations with time steps up to $\delta t = 10^{-1}$ s have been performed, observing that smaller time steps yield a thinner free surface.

The viscosity is a parameter which is difficult to adjust, since, in this simulation which is based on the system of equations governing a laminar flow, it must represent in some way the turbulent diffusion, *i.e.* the effects of fluctuations of the flow at microscopic scales, which may originate from the usual turbulence phenomena (sometimes termed "monophasic turbulence") and from the perturbation of the velocity field due to the motion of the bubbles (sometimes termed "diphasic turbulence"). Calculations with a viscosity ranging from $\mu = 10^{-3}$ Pa·s to $\mu = 10^2$ Pa·s have been performed. With smaller viscosities, we observe more oscillations of the free surface, the bubble swarm reaches the free surface faster and is farther from the sidewall.

Finally, the numerical results obtained with $\delta t = 10^{-2}$ s and a viscosity of $\mu = 1$ Pa·s are reported on Figure 8. With this value of the viscosity and these mesh and time steps, numerical convergence seems to be reached, at least visually. One can observe the stability and the thinness of the free surface. Results qualitatively reproduce the expected behaviour, which is the best we can hope with the rather crude modelling of turbulence which we adopted.

5. Conclusion

In this paper, we address the drift-flux model, which, for isothermal flows, consists in a system of three balance equations, namely the overall mass balance, the gas mass balance and the momentum balance, complemented by an equation of state and a phenomenologic relation for the drift velocity.

For this problem, we develop a pressure correction scheme with a finite element – finite volume space discretization. The existence of a solution to each step of the algorithm is proven. Essential stability properties of the continuous problem still hold at the discrete level: the unknowns are kept within their physical bounds (in particular, the gas mass fraction remains in the $[0, 1]$ interval); in the homogeneous case (*i.e.* when the drift velocity vanishes), the discrete entropy of the system decreases; in addition, when using for the drift velocity the Darcy-like relation suggested in [22], the drift term is dissipative. Since, when the density is constant, this fractional step algorithm degenerates to a usual incremental projection method based on an *inf-sup* stable approximation, stability can be expected in the zero Mach number limit. Finally, the present algorithm preserves a constant pressure and a constant velocity through moving interfaces between phases (*i.e.* contact discontinuities of the underlying hyperbolic system). To achieve this latter goal, the key ingredient is to couple the mass balance and the transport terms of the gas mass balance in an original pressure correction step.

We chose in this paper to only consider the case of a constant density liquid phase and of a gaseous phase obeying the ideal gas law. Dealing with a more general barotropic gas phase is certainly the simplest generalization, and the present theory also extends to the case of a compressible liquid with minor modifications [18]: for the stability study, essentially, the expression for the volumetric free energy of the mixture has to be replaced by the usual expression applying when both phases are compressible, see for instance [22]; the existence theory is simpler, since an upper bound for the density would provides an estimate for the pressure. Returning to the case of an incompressible fluid, extending the present theory to deal with pure liquid zones appears on the contrary to be a difficult task, since the role played by the pressure in such a system seems to deserve some clarifications. Ongoing research also concerns the extension of the present analysis to homogeneous models including an energy balance equation.

Numerical tests show a near-first-order convergence in space and time, consistent with the implemented discretization: first order backward Euler method in time and standard upwinding of the convection terms in the mass and gas mass fraction balance equations. With respect to this latter point, using more accurate space discretization (typically, MUSCL-like techniques) should certainly be desirable.

To assess the robustness of this algorithm, various numerical tests have been performed. They show in particular that free surface flows are computed without any instability, keeping a rather sharp interface throughout the computation. In addition, pure monophasic liquid zones are supported, although, as already mentioned, this case remains beyond the scope of the theory developed here. Tests are currently being performed on the Sod test case, and first results seem to show that shock solutions may be correctly approximated. This scheme is now implemented in the TOPASE code developed at IRSN and daily used for industrial applications.

A. Appendix: A discrete renormalization inequality for a system of transport equations

Let ρ and z satisfy:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \tag{A.1}$$

$$\partial_t z + \nabla \cdot (z u) = 0 \tag{A.2}$$

where u is a given velocity supposed to vanish at the boundary. These equations are discretized by an upwind finite volume scheme, and the aim of this section is to prove a discrete analog of the formal identity, for a regular function $\mathcal{F}$:

$$-\int_\Omega \big[\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F} - \mathcal{F}\big]\ \nabla \cdot u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathcal{F}(\rho, z)\,\mathrm{d}x. \tag{A.3}$$

This result is used to prove the existence of a solution to the scheme, together with its stability (see Lem. 3.5). The formal computation which allows to derive this relation in the continuous setting is the following. Multiplying equation (A.1) by $\partial_\rho \mathcal{F}$, equation (A.2) by $\partial_z \mathcal{F}$ and finally summing these relations, we obtain:

$$\partial_\rho \mathcal{F}\ \ [\partial_t \rho + \nabla \cdot (\rho u)] + \partial_z \mathcal{F}\ \ [\partial_t z + \nabla \cdot (z u)] = 0$$

which yields:

$$\partial_t \mathcal{F}(\rho(x,t), z(x,t)) + \partial_\rho \mathcal{F}\ \nabla \cdot (\rho u) + \partial_z \mathcal{F}\ \nabla \cdot (z u) = 0.$$

Developing the divergence terms, we get:

$$\partial_t \mathcal{F}(\rho(x,t), z(x,t)) + u \cdot [\partial_\rho \mathcal{F}\ \nabla \rho + \partial_z \mathcal{F}\ \nabla z] + \nabla \cdot u\,[\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F}] = 0. \tag{A.4}$$

The second term of this relation is equal to $u \cdot \nabla \mathcal{F}(\rho(x,t), z(x,t))$. Adding and subtracting $\mathcal{F}\ \nabla \cdot u$, we thus have:

$$\partial_t \mathcal{F}(\rho(x,t), z(x,t)) + \nabla \cdot (\mathcal{F}(\rho(x,t), z(x,t))\, u) + \nabla \cdot u\,[\rho\ \partial_\rho \mathcal{F} + z\ \partial_z \mathcal{F} - \mathcal{F}] = 0. \tag{A.5}$$

Since the integral of $\nabla \cdot (\mathcal{F}(\rho(x,t), z(x,t))\, u)$ over the computational domain is zero thanks to the fact that the velocity is supposed to vanish at the boundary, this equation is the relation we are seeking.

We are going now to reproduce this computation at the discrete level.

Theorem A.1. *Let $\mathcal{C}$ be an open convex subset of $\mathbb{R}^2$ and $\mathcal{F}$ be a convex continuously differentiable function from $\mathcal{C}$ to $\mathbb{R}$. We suppose that $(\rho_K)_{K\in\mathcal{M}}$, $(\rho^*_K)_{K\in\mathcal{M}}$, $(z_K)_{K\in\mathcal{M}}$ and $(z^*_K)_{K\in\mathcal{M}}$ are four families of real numbers such that, $\forall K \in \mathcal{M}$, $(\rho_K, z_K) \in \mathcal{C}$, $(\rho^*_K, z^*_K) \in \mathcal{C}$ and the following relations hold:*

$$\left|\begin{array}{l} \dfrac{|K|}{\delta t}\,(\rho_K - \rho^*_K) + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}\, \rho_\sigma = 0 \\ \dfrac{|K|}{\delta t}\,(z_K - z^*_K) + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}\, z_\sigma = 0 \end{array}\right. \tag{A.6}$$

where ρ_σ and z_σ are given by $\rho_\sigma = \rho_K$ and $z_\sigma = z_K$ if $\mathrm{v}_{\sigma,K} \geq 0$, $\rho_\sigma = \rho_L$ and $z_\sigma = z_L$ otherwise. Then the following estimate holds:

$$\sum_{K\in\mathcal{M}} -p_K \left[\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}\right] \geq \sum_{K\in\mathcal{M}} |K|\ \frac{\mathcal{F}(\rho_K, z_K) - \mathcal{F}(\rho^*_K, z^*_K)}{\delta t},$$

where the family of real numbers $(p_K)_{K\in\mathcal{M}}$ is given by:

$$\forall K \in \mathcal{M}, \qquad p_K = \rho_K\ \partial_\rho \mathcal{F}(\rho_K, z_K) + z_K\ \partial_z \mathcal{F}(\rho_K, z_K) - \mathcal{F}(\rho_K, z_K).$$

Proof. Let us multiply the first relation of (A.6) by $\partial_\rho \mathcal{F}$, the second one by $\partial_z \mathcal{F}$, both being evaluated at $(\rho_K,\, z_K)$, and sum:

$$\underbrace{\frac{|K|}{\delta t}\,[(\rho_K-\rho_K^*)\partial_\rho\mathcal{F}(\rho_K,\, z_K)+(z_K-z_K^*)\partial_z\mathcal{F}(\rho_K,\, z_K)]}_{T_{\partial/\partial t}} \\ \underbrace{+\,\partial_\rho\mathcal{F}(\rho_K,\, z_K)\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,\rho_\sigma+\partial_z\mathcal{F}(\rho_K,\, z_K)\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,z_\sigma}_{T_{\mathrm{div},K}}=0. \tag{A.7}$$

The second term of the previous relation, $T_{\mathrm{div},K}$, can be recast as:

$$\begin{aligned} T_{\mathrm{div},K}=&\,\partial_\rho\mathcal{F}(\rho_K,\, z_K)\left[\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,(\rho_\sigma-\rho_K)+\rho_K\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\right] \\ &+\partial_z\mathcal{F}(\rho_K,\, z_K)\left[\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,(z_\sigma-z_K)+z_K\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\right]. \end{aligned} \tag{A.8}$$

This relation is the discrete equivalent to equation (A.4): up to the multiplication by $1/|K|$, the first summations in the first term and the second term at the right-hand-side are the analogues of $u\cdot\nabla\rho$ and $u\cdot\nabla z$ respectively, while the second summations are the analogues of $\rho\nabla\cdot u$ and $z\,\nabla\cdot u$ respectively. Adding and subtracting $\mathcal{F}(\rho_K,\, z_K)$, we obtain a discrete equivalent of relation (A.5):

$$\begin{aligned} T_{\mathrm{div},K}=&\,\partial_\rho\mathcal{F}(\rho_K,\, z_K)\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,(\rho_\sigma-\rho_K) \\ &+\partial_z\mathcal{F}(\rho_K,\, z_K)\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\,(z_\sigma-z_K)+\mathcal{F}(\rho_K,\, z_K)\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K} \\ &+[\rho_K\;\partial_\rho\mathcal{F}(\rho_K,\, z_K)+z_K\partial_z\mathcal{F}(\rho_K,\, z_K)-\mathcal{F}(\rho_K,\, z_K)]\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}. \end{aligned}$$

In the last term, we recognize, as in the continuous setting, $p_K\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}$. The process will be completed if we put the first three terms of the right-hand-side in the divergence form. To this end, let us sum up the term $T_{\mathrm{div},K}$ over $K\in\mathcal{M}$ and reorder the summation:

$$\sum_{K\in\mathcal{M}}T_{\mathrm{div},K}=\sum_{K\in\mathcal{M}}p_K\left[\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}\right]+\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}T_{\mathrm{div},\sigma} \tag{A.9}$$

where, if $\sigma=K|L$:

$$\begin{aligned} T_{\mathrm{div},\sigma}=&\,\mathrm{v}_{\sigma,K}\big[\partial_\rho\mathcal{F}(\rho_K,\, z_K)(\rho_\sigma-\rho_K)+\partial_z\mathcal{F}(\rho_K,\, z_K)(z_\sigma-z_K)+\mathcal{F}(\rho_K,\, z_K) \\ &-\partial_\rho\mathcal{F}(\rho_L,\, z_L)(\rho_\sigma-\rho_L)-\partial_z\mathcal{F}(\rho_L,\, z_L)(z_\sigma-z_L)-\mathcal{F}(\rho_L,\, z_L)\big]. \end{aligned}$$

In this relation, there are two possible choices for the orientation of σ, *i.e.* $K|L$ or $L|K$; we choose this orientation in order to have $\mathrm{v}_{\sigma,K}\geq 0$. The function $(\rho,\, z)\mapsto\mathcal{F}(\rho,\, z)$ is by assumption continuously differentiable and convex on the convex set $\mathcal{C}$ containing both $(\rho_K,\, z_K)$ and $(\rho_L,\, z_L)$, so the technical Lemma A.2 hereafter applies and there exists $(\bar\rho_\sigma,\,\bar z_\sigma)$ in the segment $[(\rho_K,\,z_K),\ (\rho_L,\,z_L)]$ (itself included in $\mathcal{C}$) such that:

$$\left|\begin{array}{l} \text{if } (\rho_K,\,z_K)\neq(\rho_L,\,z_L):\\ \qquad \begin{aligned}&\partial_\rho\mathcal{F}(\rho_K,\,z_K)(\bar\rho_\sigma-\rho_K)+\partial_z\mathcal{F}(\rho_K,\,z_K)(\bar z_\sigma-z_K)+\mathcal{F}(\rho_K,\,z_K)\\ &\qquad =\partial_\rho\mathcal{F}(\rho_L,\,z_L)(\bar\rho_\sigma-\rho_L)+\partial_z\mathcal{F}(\rho_L,\,z_L)(\bar z_\sigma-z_L)+\mathcal{F}(\rho_L,z_L)\end{aligned}\\ \text{otherwise:}\quad (\bar\rho_\sigma,\,\bar z_\sigma)=(\rho_K,\,z_K)=(\rho_L,\,z_L).\end{array}\right. \tag{A.10}$$

By definition, the choice $(\rho_\sigma,z_\sigma)=(\bar\rho_\sigma,\bar z_\sigma)$ is such that the term $T_{\mathrm{div},\sigma}$ vanishes, which means that the first three terms at the right-hand-side of equation (A.8) are a conservative approximation of the quantity $\nabla\cdot(fu)$ appearing in equation (A.5), with the following expression for the flux:

$$F_{\sigma,K}=\mathcal{F}_\sigma\,\mathrm{v}_{\sigma,K},\quad \text{with:}$$
$$\begin{aligned}\mathcal{F}_\sigma&=\partial_\rho\mathcal{F}(\rho_K,\,z_K)(\bar\rho_\sigma-\rho_K)+\partial_z\mathcal{F}(\rho_K,\,z_K)(\bar z_\sigma-z_K)+\mathcal{F}(\rho_K,z_K)\\ &=\partial_\rho\mathcal{F}(\rho_L,\,z_L)(\bar\rho_\sigma-\rho_L)+\partial_z\mathcal{F}(\rho_L,\,z_L)(\bar z_\sigma-z_L)+\mathcal{F}(\rho_L,z_L).\end{aligned}$$

Then the term $T_{\mathrm{div},\sigma}$ can be rewritten as:

$$T_{\mathrm{div},\sigma}=\mathrm{v}_{\sigma,K}\ (\rho_\sigma-\bar\rho_\sigma)\ \big[\partial_\rho\mathcal{F}(\rho_K,\,z_K)-\partial_\rho\mathcal{F}(\rho_L,\,z_L)\big]+\mathrm{v}_{\sigma,K}\ (z_\sigma-\bar z_\sigma)\ \big[\partial_z\mathcal{F}(\rho_K,\,z_K)-\partial_z\mathcal{F}(\rho_L,\,z_L)\big].$$

With the orientation taken for σ, an upwind choice yields:

$$T_{\mathrm{div},\sigma}=\mathrm{v}_{\sigma,K}\ (\rho_K-\bar\rho_\sigma)\ \big[\partial_\rho\mathcal{F}(\rho_K,\,z_K)-\partial_\rho\mathcal{F}(\rho_L,\,z_L)\big]+\mathrm{v}_{\sigma,K}\ (z_K-\bar z_\sigma)\ \big[\partial_z\mathcal{F}(\rho_K,\,z_K)-\partial_z\mathcal{F}(\rho_L,\,z_L)\big]$$

and, by the inequality of Lemma A.2 hereafter, $T_{\mathrm{div},\sigma}$ can be seen to be non-negative. Let us now turn to $T_{\partial/\partial t}$. As the function $(\rho,\,z)\mapsto\mathcal{F}(\rho,\,z)$ is convex on the convex set $\mathcal{C}$ and both $(\rho_K,\,z_K)$ and $(\rho^*_K,\,z^*_K)$ belong to $\mathcal{C}$, we have:

$$T_{\partial/\partial t}\geq|K|\ \frac{\mathcal{F}(\rho_K,\,z_K)-\mathcal{F}(\rho^*_K,z^*_K)}{\delta t}. \tag{A.11}$$

Then, summing for $K\in\mathcal{M}$ and using relations (A.7), (A.9) and (A.11) concludes the proof. □

In the course of the preceding proof, we used the following technical lemma.

Lemma A.2. *Let $\mathcal{C}$ be an open convex subset of $\mathbb{R}^2$, $\mathcal{F}$ be a convex continuously differentiable function from $\mathcal{C}$ to $\mathbb{R}$ and $(\rho_1,\,z_1)$ and $(\rho_2,\,z_2)$ be two distinct elements of $\mathcal{C}$. Then there exists $\zeta\in[0,1]$ such that $(\bar\rho,\,\bar z)=(1-\zeta)\ (\rho_1,\,z_1)+\zeta\ (\rho_2,\,z_2)$ satisfies the following relation:*

$$\begin{aligned}\mathcal{F}(\rho_1,z_1)+\partial_\rho\mathcal{F}(\rho_1,\,z_1)(\bar\rho-\rho_1)+\partial_z\mathcal{F}(\rho_1,\,z_1)(\bar z-z_1)=&\,\mathcal{F}(\rho_2,z_2)+\partial_\rho\mathcal{F}(\rho_2,\,z_2)(\bar\rho-\rho_2)\\ &+\partial_z\mathcal{F}(\rho_2,\,z_2)(\bar z-z_2).\end{aligned} \tag{A.12}$$

In addition, the following inequality holds:

$$T=(\rho_1-\bar\rho)\big[\partial_\rho\mathcal{F}(\rho_1,\,z_1)-\partial_\rho\mathcal{F}(\rho_2,\,z_2)\big]+(z_1-\bar z)\big[\partial_z\mathcal{F}(\rho_1,\,z_1)-\partial_z\mathcal{F}(\rho_2,\,z_2)\big]\geq 0.$$

Proof. Let us consider the function g defined by:

$$\zeta\mapsto\mathcal{F}((1-\zeta)\ (\rho_1,\,z_1)+\zeta\ (\rho_2,\,z_2)).$$

By assumption, the function g is defined over $[0,1]$, convex and continuously differentiable. Moreover, it may be checked that equation (A.2) equivalently reads:

$$g(0) + g'(0)\ \zeta = g(1) + g'(1)\ (\zeta - 1)$$

or, reordering terms:

$$[g'(1) - g'(0)]\ \ \zeta = g(0) - (g(1) - g'(1)).$$

Since g is convex, if $g'(1) = g'(0)$, the function g is affine and $g(0) - (g(1) - g'(1))$ vanishes, so the preceding relation is satisfied with any value of ζ. Otherwise, the preceding relation allows to compute ζ and, still by convexity of g, both $g'(1) - g'(0)$ and $g(0) - (g(1) - g'(1))$ are positive, and so is ζ. Still in this second case, this relation equivalently reads:

$$[g'(1) - g'(0)]\ \ (\zeta - 1) = g(0) + g'(0) - g(1)$$

which, as $g(0)+g'(0)-g(1)$ is negative, shows that $\zeta \le 1$. Finally, the quantity T simply reads $\zeta\,[g'(1) - g'(0)]$, and is thus non-negative. □

Remark A.3. From the above computation, it appears that the choice of $\bar\rho_\sigma$ and $\bar z_\sigma$ defined by equation (A.10), for the convective terms equations (A.1) and (A.2), is convenient to obtain an exact (*i.e.* without any dissipation) discrete counterpart of the continuous identity (A.3).

B. Appendix: Existence of a solution to a class of discrete diphasic problems

We address in this section the following abstract discrete problem:

$$\left|\begin{array}{ll}
a(u,\varphi_\sigma^{(i)}) - \displaystyle\int_{\Omega,h} p\ \nabla\cdot\varphi_\sigma^{(i)}\mathrm{d}x = \int_\Omega \pi\cdot\varphi_\sigma^{(i)}\mathrm{d}x, & \forall\sigma\in\mathcal{E}_{\rm int},\ 1\le i\le d \\
\dfrac{|K|}{\delta t}\,[\varrho^{p,z}(p_K,\,z_K) - \varrho^{p,z}(p_K^*,\,z_K^*)] + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}^+\,\varrho^{p,z}(p_K,z_K) - \mathrm{v}_{\sigma,K}^-\,\varrho^{p,z}(p_L,z_L) = 0, & \forall K\in\mathcal{M} \\
\dfrac{|K|}{\delta t}(z_K - z_K^*) + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}^+\,z_K - \mathrm{v}_{\sigma,K}^-\ z_L = 0, & \forall K\in\mathcal{M}.
\end{array}\right. \tag{B.1}$$

This problem is supposed to be obtained from (part of) a continuous problem by a space discretization combining Rannacher-Turek or Crouzeix-Raviart finite elements and finite volumes; notations related to discrete quantities are given in Section 2.2 and are not recalled here. The bilinear form a is only assumed to be such that $\|u\|_{\rm a} = [a(u,u)]^{1/2}$ defines a norm over the discrete space W_h. The quantities $(\mathrm{v}_{\sigma,K}^+)^{n+1}$ and $(\mathrm{v}_{\sigma,K}^-)^{n+1}$ stands respectively for $\max(\mathrm{v}_{\sigma,K}^{n+1},\,0)$ and $-\min(\mathrm{v}_{\sigma,K}^{n+1},\,0)$ with $\mathrm{v}_{\sigma,K}^{n+1} = |\sigma|\,u_\sigma^{n+1}\cdot n_{KL}$. Note that, in the last two equations, the flux summation excludes the external edges, which implicitly expresses the fact that the velocity is supposed to vanish on the boundary.

This system must be completed by three equations of state. The first two ones give the liquid density ρ_ℓ and the gas density ρ_g as a function of the pressure: we suppose here that the density of the liquid is constant and that the gas obeys the equation of state of ideal gases, which, for the sake of conciseness, we suppose here to be simply $\rho_g = p$. The last equation relates the mixture density ρ with the gas mass fraction y or the gas partial density $z = \rho\,y$ and the phases density, and may take the three following forms:

$$p = \wp(\rho,z) = \frac{z\ \rho_\ell}{z+\rho_\ell-\rho} \qquad \rho = \varrho^{p,z}(p,z) = \rho_\ell + z\ \left(1 - \frac{\rho_\ell}{p}\right) \qquad \rho = \varrho^{p,y}(p,y) = \frac{1}{\dfrac{1-y}{\rho_\ell} + \dfrac{y}{p}}\cdot \tag{B.2}$$

These three relations are equivalent as soon as the following assumptions for the unknowns of this system are satisfied:

$$\rho > 0, \qquad p > 0, \qquad z > 0 \qquad \text{and} \quad 0 < y \leq 1. \tag{B.3}$$

These assumptions are natural, except for the hypothesis that y or z does not vanish, which excludes the existence of purely liquid zones. This latter assumption is assumed to hold for the initial quantities, *i.e.* we suppose that:

$$\forall K \in \mathcal{M}, \qquad y^*_K = \frac{z^*_K}{\rho^*_K} \in (0,1] \tag{B.4}$$

where $\rho^*_K = \varrho^{p,z}(p^*_K,\, z^*_K)$.

Our aim in this section is to prove that there exists a solution to system (B.1) complemented with one of the relations of (B.2), under the assumption (B.4), and that any such solution satisfies the inequalities (B.3).

We begin this section with two preliminary lemmas.

Lemma B.1. *Let $(x^*_K)_{K\in\mathcal{M}}$ and $(x_K)_{K\in\mathcal{M}}$ be two families of real numbers satisfying the following set of equations:*

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\,(x_K - x^*_K) + \sum_{\sigma=K|L} \left[\mathrm{v}^+_{\sigma,K}\, x_K - \mathrm{v}^-_{\sigma,K}\, x_L\right] = 0.$$

We suppose that, $\forall K \in \mathcal{M},\ x^*_K > 0$. Let $\|\nabla_h \cdot u\|_\infty$ be defined by:

$$\|\nabla_h \cdot u\|_\infty \;=\; \max_{K\in\mathcal{M}} \left[0, \frac{1}{|K|} \sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}\right].$$

Then, $\forall K \in \mathcal{M}$, x_K satisfies:

$$\frac{\min\limits_{K\in\mathcal{M}} x^*_K}{1 + \delta t\, \|\nabla_h \cdot u\|_\infty} \leq x_K \leq \frac{1}{\min\limits_{K\in\mathcal{M}} |K|} \sum_{K\in\mathcal{M}} |K|\; x^*_K.$$

Proof. The first inequality follows from an application of the discrete maximum principle lemma which can be found in [16] (Lem. 2.5, Sect. 2.3). The second one then follows from the fact that, by conservativity, $\sum_{K\in\mathcal{M}} x_K = \sum_{K\in\mathcal{M}} x^*_K$, remarking that, by the preceding relation, the values x_K, for $K \in \mathcal{M}$, are all positive. □

The proof of the following result can be found in [25].

Lemma B.2. *Let $(\rho^*_K)_{K\in\mathcal{M}}$, $(x^*_K)_{K\in\mathcal{M}}$, $(\rho_K)_{K\in\mathcal{M}}$ and $(x_K)_{K\in\mathcal{M}}$ be four families of real numbers satisfying the following set of equations:*

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\,(\rho_K\, x_K - \rho^*_K\, x^*_K) + \sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\, \rho_K\, x_K - \mathrm{v}^-_{\sigma,K}\, \rho_L\, x_L = 0.$$

We suppose that, $\forall K \in \mathcal{M},\ \rho^*_K > 0$, $\rho_K > 0$ and:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}\,(\rho_K - \rho^*_K) + \sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\, \rho_K - \mathrm{v}^-_{\sigma,K}\, \rho_L = 0.$$

Then the following discrete maximum principle holds:

$$\forall K \in \mathcal{M}, \qquad \min_{L\in\mathcal{M}} x^*_L \leq x_K \leq \max_{L\in\mathcal{M}} x^*_L.$$

We now state the abstract theorem which will be used hereafter; this result follows from standard arguments of the topological degree theory (see [9] for an overview of the theory and *e.g.* [12,16] for other uses in the same objective as here, namely the proof of existence of a solution to a numerical scheme).

Theorem B.3 (a result from the topological degree theory)**.** *Let N and M be two positive integers and V be defined as follows:*

$$V = \{(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M \text{ such that } y > 0 \text{ and } z > 0\}$$

where, for any real number c and vector y, the notation $y > c$ means that each component of y is greater than c. Let $b \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M$ and f and F be two continuous functions respectively from V and $V \times [0,1]$ to $\mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M$ satisfying:

(i) *$F(\cdot, 1) = f(\cdot)$;*
(ii) *$\forall \theta \in [0,1]$, if an element v of $\bar{\mathcal{O}}$ (the closure of $\mathcal{O}$) is such that $F(v, \theta) = b$, then $v \in \mathcal{O}$, where $\mathcal{O}$ is defined as follows:*

$$\mathcal{O} = \{(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M \text{ s.t. } \|x\| < M \text{ and } \epsilon < y < M \text{ and } \epsilon < z < M\}$$

with M and ϵ two positive constants and $\|\cdot\|$ a norm defined over $\mathbb{R}^N$;
(iii) *the topological degree of $F(\cdot, 0)$ with respect to b and $\mathcal{O}$ is equal to $d_0 \neq 0$.*

Then the topological degree of $F(\cdot, 1)$ with respect to b and $\mathcal{O}$ is also equal to $d_0 \neq 0$; consequently, there exists at least a solution $v \in \mathcal{O}$ such that $f(v) = b$.

We are now in position to prove the existence of a solution to the considered discrete system.

Theorem B.4 (existence of a solution)**.** *Under the assumption* (B.4)*, the nonlinear system* (B.1) *complemented with the relation* (B.2) *admits at least one solution, and any possible solution is such that:*

$$\forall K \in \mathcal{M}, \qquad \rho_K > 0, \qquad z_K > 0, \qquad 0 < y_K = \frac{z_K}{\rho_K} \leq 1, \qquad p_K > 0$$

Proof. This proof makes use of Theorem B.3 twice, by linking the initial problem (B.1) to a linear one through two successive homotopies. Let $N = d\ \mathrm{card}(\mathcal{E}_{\rm int})$ and $M = \mathrm{card}(\mathcal{M})$; we identify the finite element velocity space with $\mathbb{R}^N$ and the finite volume space of pressure and partial density with $\mathbb{R}^M$. Let V be defined by $V = \{(u, p, z) \in \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M$ such that $p > 0$ and $z > 0\}$.

Step 1. First homotopy

We consider the function $F:\ V \times [0,1] \to \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M$ given by:

$$F(u,p,z,\theta) = \left|\begin{array}{ll} v_{\sigma,i} = a(u, \varphi_\sigma^{(i)}) - \displaystyle\int_{\Omega,h} p\ \nabla \cdot \varphi_\sigma^{(i)}\, \mathrm{d}x - \int_\Omega \pi \cdot \varphi_\sigma^{(i)}\, \mathrm{d}x, & \sigma \in \mathcal{E}_{\rm int},\ 1 \leq i \leq d \\ q_K = \dfrac{|K|}{\delta t} \left[\varrho_\theta^{p,z}(p_K, z_K) - \varrho_\theta^{p,z}(p_K^*, z_K^*)\right] & \\ \qquad + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}^+\, \varrho_\theta^{p,z}(p_K, z_K) - \mathrm{v}_{\sigma,K}^-\, \varrho_\theta^{p,z}(p_L, z_L), & K \in \mathcal{M} \\ s_K = \dfrac{|K|}{\delta t} \left[z_K - \varrho_\theta^{p,z}(p_K^*, z_K^*)\, y_K^*\right] + \displaystyle\sum_{\sigma=K|L} \mathrm{v}_{\sigma,K}^+\, z_K - \mathrm{v}_{\sigma,K}^-\, z_L, & K \in \mathcal{M} \end{array}\right. \tag{B.5}$$

where the function $\varrho_\theta^{p,z}$ is implicitly defined by the following relation:

$$\varrho_\theta^{p,z}(p,z)=\varrho_\theta^{p,y}(p,y)=\frac{1}{\dfrac{1-y}{\varrho_{\ell,\theta}(p)}+\dfrac{y}{p}} \qquad \text{with} \quad \varrho_{\ell,\theta}(p)=\frac{1}{\dfrac{\theta}{\rho_\ell}+\dfrac{1-\theta}{p}} \qquad \text{and } z=\rho\, y.$$

Note that this definition makes sense (*i.e.* using $z=\rho y$, the function $\varrho^{p,z}$ can be explicitly computed from the expression of $\varrho^{p,y}$) as soon as $p>0$, and thus for any $(u,p,z)\in V$.

Solving the problem $F(u,p,z,1)=0$ is exactly the same as solving the system (B.1).

Let ϵ and M be two positive real numbers, and $\mathcal{O}$ be defined by:

$$\mathcal{O}=\{(u,p,z)\in\mathbb{R}^N\times\mathbb{R}^M\times\mathbb{R}^M \text{ s.t. } \|u\|_{\rm a}<M,\ \epsilon<p<M \text{ and } \epsilon<z<M\}.$$

We now suppose that $(u,p,z)\in\bar{\mathcal{O}}$ (and thus, in particular, $p\geq\epsilon$) and that $F(u,p,z,\theta)=0$ and provide estimates for (u,p,z).

We begin by the following elementary bound, which is useful throughout the proof. From the definition of $\varrho_{\ell,\theta}(p)$, we observe that $\min(\rho_\ell,p)\leq\varrho_{\ell,\theta}(p)\leq\max(\rho_\ell,p)$. In the same way, provided that $y\in[0,1]$, $\min(\varrho_{\ell,\theta}(p),p)\leq\varrho_\theta^{p,z}(p,z)\leq\max(\varrho_{\ell,\theta}(p),p)$. Hence, $\min(\rho_\ell,p)\leq\varrho_\theta^{p,z}(p,z)\leq\max(\rho_\ell,p)$ and, thanks to assumption (B.4):

$$\forall\theta\in[0,1],\ \forall K\in\mathcal{M},\qquad \varrho_\theta^{p,z}(p_K^*,z_K^*)\leq\bar\rho^* \qquad\text{with}\quad \bar\rho^*=\max\left[\left(\max_{K\in\mathcal{M}}p_K^*\right),\ \rho_\ell\right].$$

Step 1.1. $\|\cdot\|_{\rm a}$ estimate for the velocity

Let us first recast the equation of state of the mixture under a more convenient form. Substituting its definition for $\varrho_{\ell,\theta}(p)$ in $\varrho_\theta^{p,y}(p,y)$, we get:

$$\rho=\frac{1}{\dfrac{\theta\,(1-y)}{\varrho_\ell}+\dfrac{y+(1-\theta)(1-y)}{p}}=\frac{1}{\dfrac{1-y'}{\varrho_\ell}+\dfrac{y'}{p}} \tag{B.6}$$

with $y'(y,\theta)=y+(1-\theta)(1-y)$. Then, taking $y=z/\varrho_\theta^{p,z}(p,z)$ as unknown in the third equation of $F(u,p,z,\theta)=0$, we get, for any $K\in\mathcal{M}$:

$$\frac{|K|}{\delta t}\,[\varrho_\theta^{p,z}(p_K,z_K)\,y_K-\varrho_\theta^{p,z}(p_K^*,z_K^*)\,y_K^*]+\sum_{\sigma=K|L}{\rm v}_{\sigma,K}^+\,\varrho_\theta^{p,z}(p_K,z_K)\,y_K-{\rm v}_{\sigma,K}^-\,\varrho_\theta^{p,z}(p_L,z_L)\,y_L=0.$$

As, by the second equation of $F(u,p,z,\theta)=0$, this relation vanishes for the constant function $y_K=1$, $\forall K\in\mathcal{M}$, we also obtain:

$$\frac{|K|}{\delta t}\,[\varrho_\theta^{p,z}(p_K,z_K)\,y'_K-\varrho_\theta^{p,z}(p_K^*,z_K^*)\,(y')_K^*]+\sum_{\sigma=K|L}{\rm v}_{\sigma,K}^+\,\varrho_\theta^{p,z}(p_K,z_K)\,y'_K-{\rm v}_{\sigma,K}^-\,\varrho_\theta^{p,z}(p_L,z_L)\,y'_L=0 \tag{B.7}$$

where, by assumption (B.4), $\forall K\in\mathcal{M}$, $(y')_K^*=y_K^*+(1-\theta)(1-y_K^*)\in(1-\theta+\theta\underline{y}^*,1]$, with $\underline{y}^*=\min_{K\in\mathcal{M}}y_K^*$. We thus obtain a new problem, which keeps the structure of system (B.1), with the same equation of state (*i.e.* relation (B.6)) and just a modified initial value for z (*i.e.* z_K^* changed to $\varrho_\theta^{p,z}(p_K^*,z_K^*)\,(y')_K^*$). The unknown p is still an unknown of this new problem, and we thus have by assumption $p\geq0$. In addition, by Lemma B.2, any solution of this new problem is such that the gas mass fraction verifies $1-\theta+\theta\underline{y}^*<y\leq1$, and thus the density and the gas partial density are positive. The unknowns thus belong to the domain where

the free energy is correctly defined, and Theorem A.1 applies. Multiplying the first equation of $F(u,p,z,\theta)=0$ by $u_{\sigma,i}$, summing over $\sigma\in\mathcal{E}_{\rm int},\ 1\leq i\leq d$ and using Young's inequality thus yields:

$$\frac{1}{2}\,\|u\|_{\rm a}^2+\frac{1}{\delta t}\sum_{K\in\mathcal{M}}|K|\ \varrho_\theta^{p,z}(p_K,z_K)\,y'_K\ \ln(p_K)\leq\frac{1}{\delta t}\sum_{K\in\mathcal{M}}|K|\ \varrho_\theta^{p,z}(p^*_K,z^*_K)\,(y')^*_K\ \ln(p^*_K)+\frac{1}{2}\,\|\pi\|_{-\rm a}^2$$

where $\|\cdot\|_{-\rm a}$ stands for the dual norm of $\|\cdot\|_{\rm a}$ with respect to the L^2 inner product. The summation at the right-hand-side of this relation is bounded by $(1/\delta t)\ |\Omega|\ \bar\rho^*\ln(\bar p^*)$ where $\bar p^*=\max_{K\in\mathcal{M}}p^*_K$. By conservativity of equation (B.7), $\sum_{K\in\mathcal{M}}|K|\ \varrho_\theta(p^*_K,z^*_K)\,(y')^*_K\leq|\Omega|\ \bar\rho^*$. Since, by assumption, $p\geq\epsilon$, we thus get:

$$\|u\|_{\rm a}^2\leq\frac{2}{\delta t}\ |\Omega|\ \bar\rho^*\ |\ln(\epsilon)|+\frac{2}{\delta t}\ |\Omega|\ \bar\rho^*\ln(\bar p^*)+\ \|\pi\|_{-\rm a}^2\,.$$

For $\epsilon>0$ small enough, we thus have:

$$\|u\|_{\rm a}\leq c_1\ |\ln(\epsilon)|^{1/2} \tag{B.8}$$

where, in this relation and throughout the proof, we denote by c_i a real number only depending on the data of the problem, *i.e.* Ω, $\bar\rho^*$, $\bar p^*$, π, a, δt and the mesh, and the expression "ϵ small enough" stands for $\epsilon<c'_1$ where c'_1 is a positive real number itself only depending on the data.

Step 1.2. L^∞ estimates for z

By equivalence of the norms over finite dimensional spaces, inequality (B.8) also yields a bound for u in the L^∞ norm and, finally, for $\|\nabla_h\cdot u\|_\infty$:

$$\|\nabla_h\cdot u\|_\infty\leq c_2\ |\ln(\epsilon)|^{1/2}.$$

By Lemma B.1, we thus get from the third relation of the system $F(u,p,z,\theta)=0$, still for ϵ small enough:

$$z\geq c_3\ |\ln(\epsilon)|^{-1/2}. \tag{B.9}$$

On the other hand, we get from the same relation by conservativity:

$$z\leq c_4. \tag{B.10}$$

Step 1.3. L^∞ estimates for p

From the first relation of (B.2), using the bounds for z, we get:

$$p\geq c_5\ |\ln(\epsilon)|^{-1/2}. \tag{B.11}$$

To obtain an upper bound for p, we first remark that, as the considered spatial discretization satisfies a discrete *inf-sup* condition, a bound for u provides a bound for $p-m(p)$ where $m(p)$ stands for the mean value of p. By equivalence of norms on finite dimensional spaces, we can choose to express this bound in the seminorm defined by $\forall q\in L_h,\ |q|_{1,1,h}=\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)}|q_K-q_L|$. With this semi-norm, the mean value of p disappears, and we get for ϵ small enough:

$$\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)}|p_K-p_L|\leq c_6\ |\ln(\epsilon)|^{1/2}. \tag{B.12}$$

An upper bound for p in one cell of the mesh, say K_0, would then provide an upper bound for p, since, for any $K\in\mathcal{M}$, it is possible to build a path from K_0 to K crossing each internal edge at most once. To obtain such an estimate, we follow the following idea. If the pressure is somewhere lower than ρ_ℓ, we are done; otherwise, with the chosen equation of state $\varrho_\theta^{p,z}$, when θ varies, the liquid is everywhere denser than for $\theta=1$ and we are going to show that, even if its total mass also increases, the volume that it occupies is lower than for $\theta=1$. Hence, the remaining volume for the gas is bounded away from zero, and, by conservation of the gas mass, the pressure cannot blow up everywhere. First, we need to introduce the phase volumetric fractions. The equation of state (B.2) can be written as:

$$\frac{\rho}{p} + \frac{\rho - z}{\rho_\ell} = 1$$

and, as $\rho - z = \rho\ (1-y)$ and $y \leq 1$, both fractions at the left-hand-side of this relation are non-negative. We may thus define $\alpha_g \in [0,1]$ and $\alpha_\ell \in [0,1]$, referred to as the gas and liquid volume fraction respectively, by:

$$\alpha_g = \frac{\rho}{p} \qquad\qquad \alpha_\ell = \frac{\rho - z}{\rho_\ell}\cdot$$

Note that $\alpha_g + \alpha_\ell = 1$. Combining the second and the third relation of the system $F(u,p,z,\theta) = 0$, summing over the control volumes of the mesh and remarking that the fluxes cancel by conservativity, we get:

$$\sum_{K\in\mathcal{M}} |K|\ (\alpha_\ell)_K = \sum_{K\in\mathcal{M}} |K|\ \frac{(1-y_K^*)\ \varrho_\theta^{p,z}(p_K^*, z_K^*)}{\rho_\ell}\cdot \tag{B.13}$$

Let us denote by $(\alpha_\ell^*)_{K,1}$ the liquid void fraction with the equation of state corresponding to $\theta = 1$:

$$(\alpha_\ell^*)_{K,1} = \frac{1 - y_K^*}{\rho_\ell \left[\dfrac{y_K^*}{p_K^*} + \dfrac{1-y_K^*}{\rho_\ell}\right]}\cdot$$

Exploiting the form (B.6) of the equation of state for $\theta \neq 0$, we obtain from relation (B.13):

$$\sum_{K\in\mathcal{M}} |K|\ (\alpha_\ell)_K = \sum_{K\in\mathcal{M}} |K| (\alpha_\ell^*)_{K,1}\ \frac{\dfrac{y_K^*}{p_K^*} + \dfrac{1-y_K^*}{\rho_\ell}}{\dfrac{(y')_K^*}{p_K^*} + \dfrac{1-(y')_K^*}{\rho_\ell}} \qquad \text{with} \quad (y')_K^* = y_K^* + \theta(1-y_K^*).$$

If we suppose that $p_K \geq \rho_\ell$, the fraction in the above equation is bounded by 1: indeed, both the numerator and the denominator are harmonic averages of p_K^* and ρ_ℓ, the weight associated to p_K^* being larger in the denominator, since $(y')_K^*$ is closer to 1 than y_K^*. We thus get:

$$\sum_{K\in\mathcal{M}} |K|\ (\alpha_g)_K \geq c_7 = |\Omega| - \sum_{K\in\mathcal{M}} |K| (\alpha_\ell)_{K,1}$$

where c_7 is positive by assumption, since $\forall K \in \mathcal{M},\ y_K^* > 0$. Thus there exists $K_0 \in \mathcal{M}$ such that $(\alpha_g)_{K_0} \geq c_8 = c_7/|\Omega|$. On the other hand, we have, still by conservativity:

$$\sum_{K\in\mathcal{M}} |K|\ (\alpha_g)_K\ p_K = \sum_{K\in\mathcal{M}} |K|\ z_K = \sum_{K\in\mathcal{M}} |K|\ z_K^* = \sum_{K\in\mathcal{M}} |K|\ \varrho_\theta^{p,z}(p_K^*, z_K^*)\, y_K^* \leq |\Omega|\ \bar\rho^*.$$

We thus get, since all the $(\alpha_g)_K$ and p_k are non-negative:

$$(\alpha_g)_{K_0}\ p_{K_0} \leq \frac{|\Omega|}{|K_0|}\ \bar\rho^*$$

and thus, as $(\alpha_g)_{K_0}$ is bounded by below, the pressure is bounded by a quantity only depending on the data. As a consequence, for ϵ small enough:

$$p \leq c_9\ |\ln(\epsilon)|^{1/2}. \tag{B.14}$$

Step 2. Second homotopy

We consider the function $F:\ V\times[0,1]\rightarrow\mathbb{R}^N\times\mathbb{R}^M\times\mathbb{R}^M$ given by:

$$F(u,p,z,\theta)=\left|\begin{array}{ll} v_{\sigma,i}=a(u,\varphi_\sigma^{(i)})-\theta\displaystyle\int_{\Omega,h}p\ \nabla\cdot\varphi_\sigma^{(i)}\,\mathrm{d}x-\int_\Omega\pi\cdot\varphi_\sigma^{(i)}\,\mathrm{d}x, & \sigma\in\mathcal{E}_{\rm int},\ 1\leq i\leq d\\ q_K=\dfrac{|K|}{\delta t}(p_K-p_K^*)+\theta\displaystyle\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}^+\,p_K-\mathrm{v}_{\sigma,K}^-\,p_L, & K\in\mathcal{M}\\ s_K=\dfrac{|K|}{\delta t}\left(z_K-\dfrac{p_K^*}{\rho_K^*}\,z_K^*\right)+\theta\displaystyle\sum_{\sigma=K|L}\mathrm{v}_{\sigma,K}^+\,z_K-\mathrm{v}_{\sigma,K}^-\,z_L, & K\in\mathcal{M}.\end{array}\right. \tag{B.15}$$

The system $F(u,p,z,1)=0$ is the same as the system obtained at the end of the preceding homotopy for $\theta=0$, and the system $F(u,p,z,0)=0$ is linear and clearly regular (by stability of the bilinear form a).

In addition, the third equation is now decoupled from the first two ones, and these latter have the structure of a monophasic compressible problem as studied in [16]. From this theory, an estimate similar to the first one in the preceding step is available and reads:

$$\frac{1}{2}\,\|u\|_{\rm a}^2+\frac{1}{\delta t}\sum_{K\in\mathcal{M}}|K|\ p_K\ \ln(p_K)\leq\frac{1}{\delta t}\sum_{K\in\mathcal{M}}|K|\ p_K^*\ \ln(p_K^*)+\frac{1}{2}\,\|\pi\|_{-\rm a}^2\,.$$

Since the function $s\mapsto s\ln(s)$ is bounded by below on $(0,+\infty)$, this latter relation yields:

$$\|u\|_{\rm a}\ \leq c_{10}. \tag{B.16}$$

By Lemma B.1, we thus directly get:

$$c_{11}\leq p\leq c_{12},\qquad c_{13}\leq z\leq c_{14}. \tag{B.17}$$

Conclusion

We choose ϵ small enough for the relations (B.8), (B.9), (B.12) and (B.14) to hold, $\epsilon<\min(c_{11},\,c_{13})$ and, in addition:

$$\epsilon<\max(c_3,\,c_5)\ |\ln(\epsilon)|^{-1/2}$$

which is possible because the function $s\mapsto s\ln s$ tends to zero when s tends to zero. Let now M be such that:

$$M>\max\Big[\max(c_1,c_9)\ |\ln(\epsilon)|^{1/2},\ c_4,\,c_{10},\,c_{12},\,c_{14}\Big].$$

Then, from inequalities (B.8), (B.9), (B.10), (B.11), (B.14), (B.16) and (B.17), we get that throughout both homotopies, the unknown (u,p,z) remains in $\mathcal{O}$. As the last linear system is regular and admits a solution in $\mathcal{O}$, the topological degree of $F(\cdot,\theta)$ with respect to $\mathcal{O}$ and zero remains different of zero all along both homotopies, which proves the existence of a solution in $\mathcal{O}$.

We now turn to the proof of the *a priori* estimates $\rho>0$, $z>0$, $0<y^*\leq 1$ and $p>0$. The fact that, if $\rho^*>0$ and $z^*>0$, then $\rho>0$ and $z>0$ is a direct consequence of Lemma B.1 applied to the second and third relation of problem (B.1). In addition, as both $\rho^*>0$ and $\rho>0$, Lemma B.2 applies and thus, as $0<y^*\leq 1$, we have $0<y\leq 1$. If $p\geq\rho_\ell$, the fact that $p>0$ is evident. In the other case, by the equation of state written as a function of p and y (third form of (B.2)), we get first that:

$$p\leq\rho<\rho_\ell \tag{B.18}$$

and, second, that, since $\rho > 0$, the pressure does not vanish. Thus the second form of this same relation (B.2) can be written:

$$\rho = \frac{z}{p}\, p + \left(1 - \frac{z}{p}\right) \rho_\ell = \alpha_g\, p + (1 - \alpha_g)\ \rho_\ell.$$

As $p \neq \rho_\ell$, the void fraction α_g thus reads:

$$\alpha_g = \frac{\rho - \rho_\ell}{\rho_\ell - p}$$

which, by inequalities (B.18), yields $\alpha_g > 0$ and, finally, since $z > 0$, $p > 0$. □

This existence result applies directly to the pressure correction step used in the algorithm presented in this paper, with a particular expression for the bilinear form a, which reads, dropping for short the time exponents:

$$a(u, v) = \sum_{\sigma \in \mathcal{E}_{\rm int}} \frac{|D_\sigma|}{\delta t}\ \rho_\sigma\ u_\sigma \cdot v_\sigma.$$

Note that the analysis is performed here with a very simple equation of state for the gas ($p = \rho$), but would be readily extended to general barotropic laws $p = \wp(\rho)$, under the mild assumptions that the corresponding free energy exists and is convex and the function $\wp$ is increasing and one to one from $(0, +\infty)$ to $(0, +\infty)$.

Let us now turn to the discretization of a stationary diphasic problem. As happens in the monophasic case [17], it is likely that, in the case where the velocity is prescribed on the whole boundary, the problem is well posed if the data of the total mixture mass (say M_m) and of the total gas mass (say M_g) present in the computational domain are prescribed. A natural way to impose these two conditions is to add to this problem two regularizing terms in the mass balance and the gas mass balance:

$$\left|\begin{array}{ll} c(h)\ |K|\ \left[\varrho^{p,z}(p_K, z_K) - \dfrac{M_m}{|\Omega|}\right] + \displaystyle\sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\ \varrho^{p,z}(p_K, z_K) - \mathrm{v}^-_{\sigma,K}\ \varrho^{p,z}(p_L, z_L) = 0 & \forall K \in \mathcal{M} \\ c(h)\ |K|\ \left[z_K - \dfrac{M_g}{|\Omega|}\right] + \displaystyle\sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\ z_K - \mathrm{v}^-_{\sigma,K}\ z_L = 0 & \forall K \in \mathcal{M} \end{array}\right.$$

where $c(h)$ is a regularization parameter tending to zero with the size of the mesh. In this case, the present existence theory directly applies, provided that the momentum balance equation remains linear with respect to the velocity. Of course, under the same restriction, this is also true for an implicit discretization of a time-dependent problem.

In view of the stability results provided for the advection operator, adding such a term to the first relation of the problem (*i.e.* the momentum balance) should lead to a rather straightforward extension of the present existence result; the advection term would be multiplied by the homotopy parameter and the stability (*i.e.* an analogue to estimate (B.8)) would stem from the diffusion term. Note that, in this case, to keep the stability of the advection term, a regularization term consistent with the mass balance one should also be introduced in the momentum balance equation.

We have shown in this paper that, with Darcy's law for the drift velocity and a particular discretization for this term, the drift term is dissipative. Hence this term does not seem to prevent the obtention of stability estimates such as (B.8); this suggests that the existence theory developed here might be extended to the complete drift flux model.

Finally, we have not dealt in this study with the case where liquid monophasic zones ($z = 0$) exist in the flow. In such zones, the pressure changes of mathematical nature: it is no more a parameter entering the equation of state and determined by the local density, but a Lagrange multiplier for the incompressibility constraint. Note that this fact is already underlying in the present study: indeed, the incompressibility of the liquid prevents to derive L^∞ estimates for the pressure from L^∞ estimates for the density (which are readily obtained using a conservation argument), and we must invoke to this purpose the stability of the discrete gradient (*i.e.* the discrete *inf-sup* condition), that is typically the argument allowing to control the pressure in incompressible flow problems. However, obtaining *a priori* estimates when z may vanish in the flow seems a difficult task, which should deserve more efforts. On the contrary, obtaining existence results for two barotropic phases seems to be rather simpler than the analysis performed here [18].

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