RAIRO-Theor. Inf. Appl. **42** (2008) 121–135
DOI: 10.1051/ita:2007047

Available online at:
www.rairo-ita.org

DECIDING WHETHER A RELATION DEFINED IN PRESBURGER LOGIC CAN BE DEFINED IN WEAKER LOGICS

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Abstract. We consider logics on $\mathbb{Z}$ and $\mathbb{N}$ which are weaker than Presburger arithmetic and we settle the following decision problem: given a k-ary relation on $\mathbb{Z}$ and $\mathbb{N}$ which are first order definable in Presburger arithmetic, are they definable in these weaker logics? These logics, intuitively, are obtained by considering modulo and threshold counting predicates for differences of two variables.

Mathematics Subject Classification. 03B10, 68Q70.

1. Introduction

Cast in general terms, the problem we consider is as follows. Consider two structures $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same domain such that each basic predicate of $\mathcal{M}_2$ is definable in $\mathcal{M}_1$. Is it decidable whether or not a given a relation which is first order definable in $\mathcal{M}_1$ is first order definable in $\mathcal{M}_2$? A striking example considering structures close to those investigated here is Muchnik's well-celebrated result [8], stating that given a relation defined in the structure $\langle \mathbb{Z}; <, +, V_k\rangle$ where $V_k(x)$ denotes the highest power of k dividing the integer x, it is decidable whether or not it is definable in the structure $\langle \mathbb{Z}; <, +\rangle$. In that case we say that $\mathcal{M}_2$ is *decidable* in $\mathcal{M}_1$.

The purpose of this short note is to investigate logics on $\mathbb{Z}$ and $\mathbb{N}$ for which we solve the above general question. Concerning $\mathbb{N}$, these logics have been used in different contexts in the field of theoretical computer science, *e.g.*, model checking with timed automata in [2] and varieties of recognizable languages in [9]. Independently of these bibliographic references, they seem to be natural enough as to deserve some investigation. The first order theory of the structure $\mathcal{Z}_p = \langle \mathbb{Z}, <, +, 0, 1\rangle$

Keywords and phrases. Presburger arithmetic, first order logic, decidability.

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(resp. $\mathcal{N}_p = \langle \mathbb{N}, <, +, 0, 1 \rangle$) is the famous Presburger arithmetic of the integers (resp. of the nonnegative integers) with addition. The validity of a sentence was proved to be decidable by using quantifier elimination in the extension including all predicates $x = b \mod a$ with $a \in \mathbb{N}$ and $0 \leq b < a$, see *e.g.*, [13] Chap. III.4. Here we restrict the predicates of $\mathcal{Z}_p$ (resp. $\mathcal{N}_p$) in the following way. Based on the characterization of relations definable in Presburger arithmetic and recalled at the beginning of the next section, it is easily seen that the structure $\langle \mathbb{Z}; (x \geq c)_{c \in \mathbb{Z}}, (x \geq y + c)_{c \in \mathbb{Z}}, x = y + z \rangle$ which has no functions, is equivalent to Presburger arithmetic. One structure studied here is obtained by substituting the predicates $x = b \mod a$, $0 \leq b < a$, for the predicate $x = y + z$.

The next table shows the different structures studied in this paper. The integers a, b, c satisfy the conditions $0 \leq b < a$ and $c \in \mathbb{Z}$ (resp. $c \in \mathbb{N}$). The subscripts used should suggest the general idea of *modulo* and *threshold* counting. The subscript "s" stands for *simple* as the predicates involve a unique variable.

structure	domain	signature
$\mathcal{Z}_{\text{thresh,mod}}$	$\mathbb{Z}$	$(x \geq c)_{c \in \mathbb{Z}}, (x - y \geq c)_{c \in \mathbb{Z}}, (x = b \mod a)_{0 \leq b < a}$
$\mathcal{N}_{\text{thresh,mod}}$	$\mathbb{N}$	$(x \geq c)_{c \in \mathbb{N}}, (x - y \geq c)_{c \in \mathbb{N}}, (x = b \mod a)_{0 \leq b < a}$
$\mathcal{Z}_{\text{mod}}$	$\mathbb{Z}$	$x - y \geq 0, (x = b \mod a)_{0 \leq b < a}$
$\mathcal{N}_{\text{mod}}$	$\mathbb{N}$	$x - y \geq 0, (x = b \mod a)_{0 \leq b < a}$
$\mathcal{Z}_{\text{thresh}}$	$\mathbb{Z}$	$(x \geq c)_{c \in \mathbb{Z}}, (x - y \geq c)_{c \in \mathbb{Z}}$
$\mathcal{N}_{\text{thresh}}$	$\mathbb{N}$	$(x \geq c)_{c \in \mathbb{N}}, (x - y \geq c)_{c \in \mathbb{N}}$
$\mathcal{Z}_{\text{s}}$	$\mathbb{Z}$	$(x \geq c)_{c \in \mathbb{Z}}, (x = b \mod a)_{0 \leq b < a}$
$\mathcal{N}_{\text{s}}$	$\mathbb{N}$	$(x \geq c)_{c \in \mathbb{N}}, (x = b \mod a)_{0 \leq b < a}$

Our main contribution is the following result:

Theorem. *Given a relation which is first order definable in Presburger arithmetic of $\mathbb{Z}$ (resp. $\mathbb{N}$) it is recursively decidable whether or not it is first order definable in the structure $\mathcal{Z}_{\text{thresh,mod}}$ (resp. $\mathcal{N}_{\text{thresh,mod}}$).* The same result holds when $\mathcal{Z}_{\text{thresh,mod}}$ is replaced by $\mathcal{Z}_{\text{mod}}$ or $\mathcal{Z}_{\text{thresh}}$ ($\mathcal{N}_{\text{thresh,mod}}$ is replaced by $\mathcal{N}_{\text{mod}}$ or $\mathcal{N}_{\text{thresh}}$).

Observe that if a theory T_1 is undecidable, then any strict subtheory T_2 is undecidable in T_1. Indeed, let ψ be a closed formula in T_1 and let $\theta(x_1, \ldots, x_k)$ be a formula in $T_1 \setminus T_2$. If ψ holds then $\psi \wedge \theta$ is equivalent to θ which is not T_2-definable. If ψ does not hold, then $\psi \wedge \theta$ defines the empty set which is T_2-definable. *E.g.*, Presburger arithmetic is not decidable in ordinary arithmetic with the sum and the product.

We now discuss the relationships of our result with previous works in the literature. Decidability of $\mathcal{N}_{\text{s}}$ in Presburger arithmetic is implicitly solved in [5], p. 1048. Indeed, it is an easy exercise, using Elgot and Mezei's characterization recalled in paragraph 3.2, to see that the relations defined on $\mathcal{N}_{\text{s}}$ are the so-called *recognizable* relations of direct products of free commutative monoids. Ginsburg and Spanier proved in the early sixties, that given a relation definable in Presburger arithmetic, it is recursively decidable whether or not it is recognizable.

Decidability of $\mathcal{Z}_{\rm s}$ in Presburger arithmetic is also relatively straightforward: for the sake of completeness we prove it in paragraph 2.2. It is interesting to note that Ginsburg and Spanier proof consists of expressing in Presburger arithmetic the property of a relation to be recognizable. Though resorting to more elaborate techniques, we proceed in a similar way concerning the first six structures of the above table.

The structure $\mathcal{N}_{\rm thresh,mod}$ is a particular case of structures studied by Läuchli and Savioz. Indeed, they introduced the so-called "special relations" on free monoids of which they gave different logical and algebraic characterizations [7]. The relations definable in the structure $\mathcal{N}_{\rm thresh,mod}$ correspond to the case where the free monoid has a unique generator.

A last reference is the paper of Koubarakis tackling the complexity problem of quantifier's elimination in the structures $\langle \mathbb{K}; (x \geq c)_{c\in\mathbb{K}}, (x-y \geq c)_{c\in\mathbb{K}}\rangle$ where $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$ [6].

2. Preliminaries

We assume the reader familiar with the basic notions of rational and recognizable families of subsets in an arbitrary monoid. In particular, we recall that these families coincide when the monoid is free which covers the case of the additive monoid $\mathbb{N}$ (but not the case of the additive group $\mathbb{Z}$).

2.1. Semilinear sets

The main result on first order definable subsets in Presburger arithmetic is captured in the next theorem, see [4, 5, 11], which, in passing, defines the notion of semilinear sets.

Theorem 2.1. *Given a subset $X \subseteq \mathbb{Z}^k$ (resp. $X \subseteq \mathbb{N}^k$) the following conditions are equivalent:*

(1) *X is a semilinear set which is a finite union of linear subsets, i.e, of subsets of the form*

$$v_0 + \mathbb{N}v_1 + \cdots + \mathbb{N}v_n$$

for some $n \geq 0$ and some $v_0, v_1, \ldots v_n \in \mathbb{Z}^k$ (resp. $\mathbb{N}^k$).

(2) *X is first order definable in the structure $\langle \mathbb{Z}, =, <, +, 0, 1\rangle$ (resp. $\langle \mathbb{N}, =, <, +, 0, 1\rangle$).*

Furthermore, there exists a procedure which converts one form into another.

Example 2.2. *The following relations are definable in Presburger arithmetic with domain $\mathbb{N}$ or $\mathbb{Z}$ accordingly, and their representation as semilinear subsets will serve as illustration throughout this paper.*

$$\begin{array}{ll} X_1 & : \mathbb{N}(1,2) = \{(x,y) \in \mathbb{N}^2 \mid y = 2x\}, \\ X_2 & : (1,2) + \mathbb{N}(2,2) + \mathbb{N}(0,2) \\ & \qquad = \{(x,y) \in \mathbb{N}^2 \mid x \leq y, x = 1 \mod 2, y = 0 \mod 2\}, \\ X_3 & : (1,1) + \mathbb{N}(2,0) + \mathbb{N}(0,2) = \{(x,y) \in \mathbb{N}^2 \mid x, y = 1 \mod 2\}, \\ X_4 & : \mathbb{N}(1,-1) \cup \mathbb{N}(-1,1) = \{(x,y) \in \mathbb{Z}^2 \mid x + y = 0\}. \end{array}$$

2.2. The logics with unary predicates only

We observed in the introduction that Ginsburg and Spanier proved a result which can be interpreted as asserting that the structure $\mathcal{N}_{\mathrm{s}}$ is decidable in Presburger arithmetic. Here we show that a similar result also holds for the structure $\mathcal{Z}_{\mathrm{s}}$. This is achieved by reducing the latter case to the former case.

Indeed, we leave it to the reader to verify that a relation $R \subseteq \mathbb{Z}^k$ is definable in $\langle \mathbb{Z}; (x \geq c)_{c\in\mathbb{Z}}, (x = b \bmod a)_{0\leq b<a}\rangle$ if and only if it is a finite union of the form $X_1 \times \cdots \times X_k$ where X_i is a rational subset of $\mathbb{Z}$. We claim that this is equivalent to asserting that each intersection $R \cap \mathbb{D}_1 \times \cdots \times \mathbb{D}_k$ with $\mathbb{D}_i = \mathbb{N}$ or $-\mathbb{N}$, is recognizable in the monoid $\mathbb{D}_1 \times \cdots \times \mathbb{D}_k$. Indeed, the condition is clearly sufficient. In order to verify that it is necessary, observe that an intersection $R \cap \mathbb{D}_1 \times \cdots \times \mathbb{D}_k$ is a finite union of direct products $X_1 \times \cdots \times X_k$ where each X_i is rational in $\mathbb{D}_i$, thus recognizable in $\mathbb{D}_i$ by Kleene theorem on the equality of the two families of rational and recognizable sets in free monoids. Now it should be clear how we proceed. Given a Presburger formula $\varphi(x_1, \ldots, x_k)$, for all subsets $I \subset \{1, \ldots, k\}$, define the predicate $\sigma_i(x_i)$ which is equivalent to $x \geq 0$ if $i \in I$ and to $x \leq 0$ otherwise. Then we use Ginsburg and Spanier's algorithm to verify whether or not the following formula defines a recognizable subset

$$\sigma_1(x_1) \wedge \cdots \wedge \sigma_k(x_k) \wedge \varphi(x_1, \ldots, x_k).$$

2.3. Quantifier elimination

We sketch a proof that the theories considered in the table of the introduction admit quantifier elimination. Actually, without the modulo predicates, this result is known as Fourier-Motzkin elimination, see *e.g.*, [12], p. 155. It suffices to verify that a formula of the form $\phi(x_1, \ldots, x_n) \wedge \exists y \psi(x_1, \ldots, x_n, y)$ admits quantifier elimination where ψ is as follows:

$$\bigwedge_{i\in I}(x_i - y \geq a_i) \wedge \bigwedge_{j\in J}(x_j - y \leq a_j) \wedge (c \leq y \leq d) \wedge (y \equiv b \bmod a). \tag{1}$$

Set $m = \max\{\max\{x_j - a_j \mid j \in J\}, c\}$ and $M = \min\{\min\{x_i - a_i \mid i \in I\}, d\}$. Then the expression is equivalent to a disjunction of the form

$$(M - m \geq a) \vee ((0 \leq M - m < a) \wedge \chi(x_1, \ldots, x_n)$$

where $\chi(x_1, \ldots, x_n)$ asserts that for some $0 \leq \alpha \leq b \leq \beta \leq a$, m is equal to α and M is equal to β modulo a or m is equal to β and M is equal to α modulo a.

3. Recognizable relations

We recall in this section the definition and some elementary properties of a proper subfamily of the semilinear subsets. A relation $X \subseteq \mathbb{N}^k$ is *recognizable* if there exists a morphism $h : \mathbb{N}^k \to M$ where M is a finite monoid, such that $X = h^{-1}h(X)$ holds. It is *group recognizable* if M is a group and *aperiodic recognizable* if M does not contain a subset which is isomorphic to some group.

3.1. Folklore on $\mathbb{N}$

Let us linger on the very simple case where $k = 1$. The next lines are folklore and may be skipped by most readers. A finite monoid M generated by one element z consists of the set $1 = z^0, z, z^2, \ldots, z^n$ along with the operation defined by the following equalities for some $m \leq n$

$$z \cdot z^i = \begin{cases} z^{i+1} & \text{if } i < n, \\ z \cdot z^n = z^m & \text{otherwise.} \end{cases}$$

The monoid M is a group if $m = 0$ and it does not contain a nontrivial group if $m = n$. Consequently, a subset $X \subseteq \mathbb{N}$ is recognizable if and only if for some integer a and for some finite sets A and B we have $X = B \cup (A + \mathbb{N}a)$. It is group recognizable if there exists an integer a and a subset A of integers less than a such that $X = A + \mathbb{N}a$ and it is aperiodic recognizable if it is finite or if there there exists an integer a and a subset A of integers less than a such that $X = A \cup (a + \mathbb{N})$. It is clear that a recognizable, resp. group recognizable, resp. aperiodic recognizable subset of $\mathbb{N}$ is definable in $\mathcal{N}_{\text{thresh,mod}}$, resp. $\mathcal{N}_{\text{mod}}$, resp. $\mathcal{N}_{\text{thresh}}$.

Given two nonnegative integers x and y, consider the partial operation $x \dot{-} y = z$ on $\mathbb{N}$ defined by $x - y = z$ if such a nonnegative integer z exists. Extend this notation to subsets $X, Y \subseteq \mathbb{N}$: $X \dot{-} Y = \{z \in \mathbb{N} \mid \exists x \in X, \exists y \in Y : x = y + z\}$. It is well-known, actually a very special case in the theory of semivarieties of finite monoids, *cf.* [10] (but easy to verify directly) that if X and Y are recognizable, resp. group recognizable, resp. aperiodic recognizable, then so is $X \dot{-} Y$. All these claims will be used in numerous places without explicit references. The following elementary result will also be used.

Lemma 3.1. *Consider two recognizable subsets $X, Y \subseteq \mathbb{N}$. Then there exists a finite disjoint decomposition $X = X_1 \cup \cdots \cup X_r$ such that for $i = 1, \ldots, r$ and $x, x' \in X_i$, the equality $Y \dot{-} x = Y \dot{-} x'$ holds.*

Proof. Let $f : \mathbb{N} \to F$ and $h : \mathbb{N} \to H$ be two morphisms into finite commutative monoids such that $X = f^{-1}f(X)$ and $Y = h^{-1}h(Y)$ holds. Consider two elements $x, x' \in \mathbb{N}$ satisfying $f(x), f(x') \in f(X)$ and $h(x) = h(x')$. Then for all $z \in \mathbb{N}$

we have

$$x + z \in Y \Leftrightarrow h(x+z) = h(x'+z) \in h(Y) \Leftrightarrow x' + z \in Y.$$

The claimed decomposition of X is thus $X = \bigcup_{a \in h(Y)} X \cap h^{-1}(a)$. □

3.2. Elgot and Mezei characterization

Turning to the case where k is an arbitrary integer, there exists a very useful characterization of the recognizable relations of a direct product in terms of the direct product of recognizable sets of each component. Eilenberg attributes this result to Elgot and Mezei [3] p. 265. We state it in the particular case of a direct product of copies of $\mathbb{N}$ but it holds under more general conditions.

Proposition 3.2. *A subset $X \subseteq \mathbb{N}^k$ is recognizable (resp. group recognizable, resp. aperiodic recognizable) if it is a finite union of direct products such as*

$$Z_1 \times \cdots \times Z_k$$

where for $i = 1, \ldots, k$, Z_i is recognizable, resp. group recognizable, resp. aperiodic recognizable in $\mathbb{N}$.

With this charaterization, one sees that in example 2.2, X_3 is recognizable but that X_1 and X_2 are not. Standard constructions show that the family of recognizable, resp. group recognizable, resp. aperiodic recognizable subsets is closed under the Boolean operations, projection onto an arbitrary collection of components and direct product. It should not require much effort to the interested reader to directly work these constructions out by himself.

4. Subfamilies of semilinear sets

The purpose of this section is to introduce three new families of subsets of $\mathbb{Z}^k$ which are contained in the family of semilinear sets. Their definition is based on the following notion on vectors of the $\mathbb{Z}^k$. The *support* of $u \in \mathbb{Z}^k$ is the set of integers $1 \leq i \leq k$ such that its i-th component is nonzero: $\mathtt{Supp}(u) = \{1 \leq i \leq k \mid u_i \neq 0\}$.

We set

$$\begin{array}{l} P = \{e \neq 0 \mid e_i = 0, 1, \quad \text{for all } 1 \leq i \leq k\} \\ N = \{e \neq 0 \mid e_i = 0, -1, \quad \text{for all } 1 \leq i \leq k\} \end{array} \tag{2}$$

and we define a partial order on the set $P \cup N$ via the following rules.

- If $e \in N$ and $f \in P$ have disjoint support, then $e < f$.
- If the support of $e \in P$ is strictly included in the support of $f \in P$ then $f < e$.
- If the support of $e \in N$ is strictly included in the support of $f \in N$ then $e < f$.

E.g., here is a strictly increasing sequence of vectors in $P \cup N$

$$(-1, 0, 0, 0, 0) < (-1, 0, 0, -1, 0) < (0, 0, 1, 0, 1) < (0, 0, 0, 0, 1).$$

We define $\mathcal{E}$ as the collection of all strictly increasing sequences of vectors in $N \cup P$ and by convention we assume the empty sequence belongs to $\mathcal{E}$. We now design an algorithm which expresses each vector of $u \in \mathbb{Z}^k$ as a linear combination of these vectors all of whose scalars are positive nonzero integers, in a unique way. Assume first all components of $u \neq 0$ are non negative. Then there exists a unique vector $e \in P$ such that by subtraction, all zero components of u remain unchanged and the absolute value of all nonzero components decreases by one. Repeating this process until obtaining the nullvector yields $u = \lambda_1 e_1 + \cdots + \lambda_r e_r$ for some integers $r, \lambda_1, \cdots \lambda_r \in \mathbb{N}$ and some increasing sequence $e_1 < \cdots < e_r$ of vectors in P. *E.g.*, $(0,3,1,6) = (0,1,1,1) + 2(0,1,0,1) + 3(0,0,0,1)$. Similarly, if all components of $u \neq 0$ are non positive, a direct adaptation of the above procedure leads to an expression of the form $u = \mu_1 f_1 + \cdots + \mu_s f_s$ for some integers $s, \mu_1, \ldots \mu_r \in \mathbb{N}$ and some decreasing sequence $f_1 > \cdots > f_s$ of vectors in N. Consider now an arbitrary vector $u \neq 0$ with positive and negative components. It can be written as $u = v + w$ where v and w have all nonpositive and nonnegative components respectively. Apply the previous procedure to v and to w

$$v = \mu_1 f_1 + \cdots + \mu_s f_s$$
$$w = \lambda_1 e_1 + \cdots + \lambda_r e_r.$$

Then we get

$$u = \mu_s f_s + \cdots + \mu_1 f_1 + \lambda_1 e_1 + \cdots + \lambda_r e_r$$

where $f_s < \cdots < f_1 < e_1 < \cdots < e_r$ is in $\mathcal{E}$ and $\mu_s, \ldots, \mu_1, \lambda_1, \ldots, \lambda_r \in \mathbb{N}$. *E.g.*, we have $(-4,2,-1,3) = 3(-1,0,0,0) + (-1,0,-1,0) + 2(0,1,0,1) + (0,0,0,1)$. Consequently we have the disjoint union

$$\mathbb{Z}^k = \bigcup_{E \in \mathcal{E}} \left(\sum_{e \in E} (\mathbb{N} - 0) e \right). \tag{3}$$

We adopt the convention that $\sum_{e \in E} (\mathbb{N} - 0)e$ represents the zero vector when E is the empty sequence.

We are now in condition to define the new families of subsets of $\mathbb{Z}^k$. We denote by $\mathcal{F}_{\text{rec}}$ (resp. $\mathcal{F}_{\text{mod}}$, resp. $\mathcal{F}_{\text{ap}}$) the family of subsets $X \subseteq \mathbb{Z}^k$ such that for each strictly increasing sequence of vectors $e_1 < \cdots < e_p$, the subset

$$\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid x_1 e_1 + \ldots + x_p e_p \in X\} \tag{4}$$

is recognizable (resp. group recognizable, resp. aperiodic recognizable) in $\mathbb{N}^p$. The family $\mathcal{F}_{\text{rec}}$ (resp. $\mathcal{F}_{\text{mod}}$, resp. $\mathcal{F}_{\text{ap}}$) is the union of the families when k takes on arbitrary integer values.

Example 4.1. (Example 2.2 cont'd). *With the sequence* $(1,1) < (0,1)$ *we have for the first three subsets, respectively* $\{n(1,1) + n(0,1) \mid n \in \mathbb{N}\}$*,* $(1+2\mathbb{N})(1,1) + (1+2\mathbb{N})(0,1)$ *and* $(1+2\mathbb{N})(1,1) + 2\mathbb{N}(0,1)$ *and with the sequence* $(1,1) < (1,0)$ *we have* $\{(0,0)\}$*,* $\emptyset$ *and* $(1+2\mathbb{N})(1,1) + 2\mathbb{N}(1,0)$*. More generally, we have* $X_1 \notin \mathcal{F}_{\text{rec}}$ *and*

$X_2, X_3 \in \mathcal{F}_{\text{mod}}$. *Concerning the relation* X_4*, with the sequence* $(0,-1) < (1,0)$ *we have* $\{n(0,-1) + n(1,0) \mid n \in \mathbb{N}$ *which is not recognizable in* $\mathbb{N}^2$*, consequently,* $X_4 \notin \mathcal{F}_{\text{rec}}$.

4.1. Closure properties

Before proving some nice closure properties of the subfamilies introduced in the previous paragraph, we wish to give different equivalent definitions of these families. We extend to strictly increasing subsequences the standard notations for subsets. *E.g.*, if E and F are two strictly increasing sequences of vectors, $E \cap F$ denotes their maximal common increasing subsequence. Also, given a vector e, we write $e \in E$ to express the fact that it belongs to the sequence E.

The following technical lemma is a consequence of the fact that each vector of $\mathbb{Z}^k$ has a unique decomposition as a linear combination of strictly increasing vectors. The proof is rather straightforward and we omit it.

Lemma 4.2. *Let* E *and* F *be strictly increasing sequences of vectors. Consider* $\sum_{e \in E} X_e e$ *and* $\sum_{f \in F} Y_f f$ *where* $X_e, Y_f \subseteq \mathbb{N}$ *for* $e \in E$ *and* $f \in F$*. The intersection*

$$\left(\sum_{e \in E} X_e e\right) \cap \left(\sum_{f \in F} Y_f f\right)$$

is nonempty if and only if the following conditions hold:

1) for all $e \in E \setminus F$ *we have* $0 \in X_e$ *and for all* $f \in F \setminus E$ *we have* $0 \in Y_f$*,*

2) for all $g \in E \cap F$ *we have* $X_g \cap Y_g \neq \emptyset$*.*

Furthermore, when the intersection is nonempty then it is equal to

$$\sum_{g \in E \cap F} (X_g \cap Y_g) g.$$

The following gives an alternative definition of the subfamilies of semilinear subsets.

Proposition 4.3. *Given a subset* $X \subseteq \mathbb{Z}^k$*, the following conditions are equivalent.*

1) For each strictly increasing sequence of vectors, the subset

$$\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid x_1 e_1 + \ldots + x_p e_p \in X\} \tag{5}$$

is recognizable (resp. group recognizable, resp. aperiodic recognizable) in $\mathbb{N}^p$.

2) X *is a finite union of subsets of the form*

$$X_1 e_1 + \cdots + X_p e_p \tag{6}$$

where for some $0 < p \leq k$*, the sequence* $e_1, \ldots, e_p$ *is strictly increasing and where the* X_i*'s are recognizable (resp. group recognizable, resp. aperiodic recognizable) in* $\mathbb{N}$.

Proof. The previous lemma shows that 1 implies 2. Let us verify the other implication. It suffices to treat the particular case where X is of the form $\sum_{e\in E} X_e e$. Consider the intersection $(\sum_{e\in E} X_e e) \cap (\sum_{f\in F} \mathbb{N}f)$ where F is a strictly increasing sequence of vectors. Because of Lemma 4.2, this intersection is of the form $\sum_{g\in E\cap F} X_g g$. Conversely, condition 1 implies condition 2. Indeed, from equality 3 we get $\mathbb{Z}^k = \bigcup_E \left(\sum_{e\in E} \mathbb{N}e\right)$. Consequently, X is the union of all intersections $X \cap (\sum_{e\in E} \mathbb{N}e)$ where E ranges over all strictly increasing sequences of vectors. Since, by hypothesis, each of these subsets is of the form $\sum_{e\in E} X_e e$ where the X_e's are recognizable (resp. group recognizable, resp. aperiodic recognizable) the implication follows. $\square$

The previous result clears the way to the closure properties of these families relative to the main elementary set operations.

Theorem 4.4. *Let* $\mathcal{F}$ *be any one of the three families* $\mathcal{F}_{\mathrm{rec}}$, $\mathcal{F}_{\mathrm{mod}}$ *or* $\mathcal{F}_{\mathrm{ap}}$*. Then* $\mathcal{F}$ *is closed under the Boolean operations, the projection over an arbitrary subset of components and direct product, i.e.,*

(i) $X, Y \in \mathcal{F}^{(k)} \Rightarrow X \cup Y \in \mathcal{F}^{(k)}$,

(ii) $X \in \mathcal{F}^{(k)} \Rightarrow \overline{X} \in \mathcal{F}^{(k)}$,

(iii) $X \in \mathcal{F}^{(k)} \Rightarrow \pi(X) \in \mathcal{F}^{(k-1)}$ *where* $(x_1, \ldots, x_k) \mapsto \pi(x_1, \ldots, x_k) = (x_2, \ldots, x_k)$ *maps* $\mathbb{N}^k$ *onto its* $k-1$ *last components* $\mathbb{N}^{k-1}$,

(iv) $X \in \mathcal{F}^{(k)} \wedge Y \in \mathcal{F}^{(\ell)} \Rightarrow X \times Y \in \mathcal{F}^{(k+\ell)}$.

Proof. In order to avoid cumbersome repetitions, we prove the result for $\mathcal{F}_{\mathrm{rec}}$, but the arguments can be easily adapted to the other two families. We shall furthermore drop the subscript and write $\mathcal{F}$. In the proof we use whatever equivalent definition of $\mathcal{F}$ given by Proposition 4.3 is more convenient.

Concerning the Boolean operations, the claims are consequences of the closure properties of the family of recognizable relations and of the following elementary equalities, where E is an arbitrary strictly increasing sequence of vectors

$$(X \cup Y) \cap \sum_{e\in E} \mathbb{N}e = \left(X \cap \sum_{e\in E} \mathbb{N}e\right) \cup \left(Y \cap \sum_{e\in E} \mathbb{N}e\right)$$

$$\overline{X} \cap \sum_{e\in E} \mathbb{N}e = \sum_{e\in E} \mathbb{N}e \setminus \left(X \cap \sum_{e\in E} \mathbb{N}e\right).$$

Indeed, the vectors in E are linearly independent and therefore the sum $\sum_{e\in E}\mathbb{N}e$ is isomorphic to the direct product $\mathbb{N}^{|E|}$ allowing us to apply Proposition 3.2.

We now proceed to the claim on the direct product. Consider $X \in \mathcal{F}^{(k)}$ and $Y \in \mathcal{F}^{(\ell)}$ for some $k, \ell \in \mathbb{N}$. By Proposition 4.3, item 2, it suffices to investigate the case where X and Y are of the form

$$X = \sum_{-q\leq i\leq p, i\neq 0} X_i e_i, \quad Y = \sum_{-s\leq j\leq r, j\neq 0} Y_j f_j,$$

where the sequence of the vectors e_i's (resp. f_j's) is strictly increasing in $\mathbb{Z}^k$ (resp. $\mathbb{Z}^\ell$) and where the X_i's (resp. Y_j's) are recognizable, resp. group recognizable, resp. aperiodic in $\mathbb{N}$.

We first observe that without loss of generality, by possibly renumbering the indices, there exists an integer $1 \leq k' \leq k$ such that $\mathtt{Supp}(e_1)$ (resp. $\mathtt{Supp}(e_{-1})$) is included in the first k' (resp. last $k - k'$) components. Denote by $e'_1, \ldots, e'_p$ the restrictions of the vectors $e_1, \ldots, e_p$ to their first k' components and by $e'_{-1}, \ldots, e'_{-q}$ the restrictions of the vectors $e_{-1}, \ldots, e_{-q}$ to their last $k - k'$ components. Then X is equal to the direct product

$$(X_1 e'_1 + \cdots + X_p e'_p) \times (X_{-q} e'_{-q} + \cdots + X_{-1} e'_{-1}).$$

Similarly, assuming that there exists an integer $1 \leq \ell' \leq \ell$, such that $\mathtt{Supp}(f_1)$ (resp. $\mathtt{Supp}(f_{-1})$) is included in the first ℓ' (resp. last $\ell - \ell'$) components, without loss of generality Y is the direct product

$$(Y_1 f'_1 + \cdots + Y_r f'_r) \times (Y_{-s} f'_{-s} + \cdots + Y_{-1} f'_{-1})$$

and therefore $X \times Y$, up to a renumbering of the indices, is equal to $A \times B$ where

$$\begin{aligned} A &= \sum_{1\leq i\leq p} X_i e'_i \times \sum_{1\leq j\leq r} Y_j f'_j \\ B &= \sum_{1\leq i\leq q} X_{-i} e'_{-i} \times \sum_{1\leq j\leq s} Y_{-j} f'_{-j}. \end{aligned} \tag{7}$$

We claim that A is a finite union of sets of the form $\sum_{1\leq h\leq u} Z_h g_h$ where $g_1 < \cdots < g_u$ is a strictly increasing sequence of vectors in $\mathbb{Z}^{k'+\ell'}$ with nonnegative components and the Z_h's are recognizable, resp. group recognizable, resp. aperiodic. Indeed, by Lemma 3.1 the recognizable subsets X_1 and Y_1 have finite decompositions $X_1 = \bigcup_\alpha X_1^{(\alpha)}$ and $Y_1 = \bigcup_\beta Y_1^{(\beta)}$ where the $X_1^{(\alpha)}$ and the $Y_1^{(\beta)}$ are recognizable, resp. group recognizable, resp. aperiodic, such that for all α and all $x, x' \in X_1^{(\alpha)}$ we have $Y_1 \dot{-}\, x = Y_1 \dot{-}\, x'$ and for all β and all $y, y' \in Y_1^{(\beta)}$ we have $X_1 \dot{-}\, y = X_1 \dot{-}\, y'$.

Then we obtain

$$A = \bigcup_{\alpha} [X_1^{\alpha}(e'_1, f'_1) + U] \cup \bigcup_{\beta} \left[Y_1^{\beta}(e'_1, f'_1) + V\right]$$

where

$$\begin{aligned} U &= (X_2 e'_2 + \cdots + X_p e'_p) \times ((Y_1 \dot{-} X_1^{\alpha}) f'_1 + \cdots + Y_r f'_r) \\ V &= \Big((X_1 \dot{-} Y_1^{\beta}) e'_1 + \cdots + X_p e'_p\Big) \times (Y_2 f'_2 + \cdots + Y_r f'_r). \end{aligned}$$

The claim follows by induction on $p + r$ applied to the subsets U and V, *i.e.*, A is a finite union of subsets of the form

$$Z_1 g_1 + \cdots + Z_u g_u$$

where $g_1 < \cdots < g_u$ is a strictly increasing sequence of vectors with nonnegative components. Similarly, B is a finite union of subsets of the form

$$Z_{-v} g_{-v} + \cdots + Z_{-1} g_{-1}$$

where $g_{-v} < \cdots < g_{-1}$ is a strictly increasing sequence of vectors with nonpositive components. Observe that the common support of the vectors $g_1, \ldots, g_u$ is disjoint of the common support of the vectors $g_{-v}, \ldots, g_{-1}$ and the union of these two supports is the set $\{1, \ldots, k + \ell\}$. The conclusion follows from the equality

$$\begin{aligned} &(Z_1 g_1 + \cdots + Z_u g_u) \times (Z_{-v} g_{-v} + \cdots + Z_{-1} g_{-1}) = \\ &Z_{-v}(0, g_{-v}) + \cdots + Z_{-1}(0, g_{-1}) + Z_1(g_1, 0) + \cdots + Z_u(g_u, 0) \end{aligned}$$

where *e.g.*, $(0, g_{-v})$ is the $k + \ell$-vector with $k' + \ell'$ leading zeros.

Finally, let π be the projection onto the $k - 1$ last components

$$\pi(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k)$$

and let F be a strictly increasing sequence of vectors in $\mathbb{N}^{k-1}$. Then $\pi(X) \cap \sum_{f \in F} \mathbb{N} f$ is equal to the union over all strictly increasing sequences E of $\mathbb{N}^k$ which map onto F by π, of the sets $\pi(X \cap \sum_{e \in E} \mathbb{N} e)$. $\square$

5. The decision procedure

The decision procedure is based on the following characterization which shows that definable relations are precisely the relations belonging to the subfamilies of semilinear sets.

Theorem 5.1. *A subset $X \subseteq \mathbb{Z}^k$ is first order definable in $\mathcal{Z}_{\text{thresh,mod}}$ (resp. $\mathcal{Z}_{\text{mod}}$, resp. $\mathcal{Z}_{\text{thresh}}$) if and only if it belongs to the family $\mathcal{F}_{\text{rec}}$, (resp. $\mathcal{F}_{\text{mod}}$, resp. $\mathcal{F}_{\text{ap}}$).*

Proof. The following notation is standard. Given a formula $\phi(x_1, \dots, x_k)$ with free variables $x_1, \dots, x_k$, $[\![\phi]\!]$ denotes the *interpretation* of ϕ, *i.e.*, the set of k-tuples satisfying the formula.

We now prove that the condition is necessary. We first show that the relations defined by the atomic predicates belong to $\mathcal{F}_{\mathrm{rec}}$, (resp. $\mathcal{F}_{\mathrm{mod}}$, resp. $\mathcal{F}_{\mathrm{ap}}$). We start with the predicates involving a unique variable:

$$\begin{aligned}&[\![x \geq a]\!] = a + \mathbb{N} \cdot 1,\\ &[\![x = b \mod a]\!] = (b + \mathbb{N}a)1 \cup (\mathbb{N}a \dot{-} b)(-1).\end{aligned}$$

We now consider the predicate $x - y \geq a$ and we proceed by case study.

Case 1: $x, y \geq 0$. If $a \geq 0$, the relation is $\mathbb{N}(1,1) + (a + \mathbb{N})(1,0)$ otherwise it is $[\mathbb{N}(1,1) + \{0 \leq b \leq -a\}(0,1)] \cup [\mathbb{N}(1,1) + \mathbb{N}(1,0)]$.

Case 2: $x \geq 0, y \leq 0$. If $a \geq 0$, the relation is equal to

$$(a + \mathbb{N})(0,-1) + \mathbb{N}(1,0) \cup \bigcup_{0 \leq b \leq a} b(0,-1) + ((a-b) + \mathbb{N})(1,0)$$

otherwise the relation is $\mathbb{N} \times (-\mathbb{N}) = \mathbb{N}(0,-1) + \mathbb{N}(1,0)$.

Case 3: $x \leq 0, y \geq 0$. The is equivalent to $y - x \leq -a$, *i.e.*, it is the negation of the condition $y - x \geq -a + 1$. Using the previous case, the interpretation is the complement of the relation

$$(-a + 1 + \mathbb{N})(-1,0) + \mathbb{N}(0,1) \cup \bigcup_{0 \leq b \leq -a+1} b(-1,0) + ((-a + 1 - b) + \mathbb{N})(0,1)$$

if $-a + 1 \geq 0$ and of $-\mathbb{N} \times \mathbb{N}$ otherwise and we may conclude by Theorem 4.4.

Case 4: $x, y \leq 0$. In this last case, the condition is equivalent to $(-y) - (-x) \geq a$. Case 1 yields the relation $\mathbb{N}(-1,-1) + (a + \mathbb{N})(0,-1)$ if $a \geq 0$ otherwise it is $[\mathbb{N}(-1,-1) + \{0 \leq b \leq -a\}(-1,0)] \cup [\mathbb{N}(-1,-1) + \mathbb{N}(0,-1)]$.

We now proceed by structural induction on the formula. Assume $[\![\phi(x_1, \dots, x_k)]\!]$ is in $\mathcal{F}_{\mathrm{rec}}$, resp. $\mathcal{F}_{\mathrm{mod}}$, resp. $\mathcal{F}_{\mathrm{ap}}$. Then the equalities $[\![\neg\phi(x_1, \dots, x_k)]\!] = \mathbb{N}^k \setminus [\![\phi(x_1, \dots, x_k)]\!]$ and $[\![\exists x_1 : \phi(x_1, \dots, x_k)]\!] = \pi([\![\phi(x_1, \dots, x_k)]\!])$ show that $[\![\neg\phi]\!]$ and $[\![\exists x_1 : \phi]\!]$ are in $\mathcal{F}_{\mathrm{rec}}$, resp. $\mathcal{F}_{\mathrm{ap}}$, resp. $\mathcal{F}_{\mathrm{mod}}$. Concerning the conjunction, assume without loss of generality that by renaming the variables, we are given two formulas of the form $\phi_1(x_1, x_2, \dots, x_k)$ and $\phi_2(x_\ell, x_{\ell+1}, \dots, x_r)$ where $1 \leq \ell \leq k+1$. Then the relation

$$[\![\phi_1(x_1, x_2, \dots, x_k) \wedge \phi_2(x_\ell, x_{\ell+1}, \dots, x_r)]\!]$$

is equal to

$$\left([\![\phi_1(x_1, x_2, \dots, x_k)]\!] \times \mathbb{Z}^{r-k}\right) \cap \left(\mathbb{Z}^{\ell-1} \times [\![\phi_2(x_\ell, x_{\ell+1}, \dots, x_r)]\!]\right).$$

Observing that $\mathbb{Z}^{r-k}$ and $\mathbb{Z}^{\ell-1}$ are in $\mathcal{F}_{\mathrm{rec}}$, resp. $\mathcal{F}_{\mathrm{ap}}$, resp. $\mathcal{F}_{\mathrm{mod}}$ and using the closure properties of Theorem 4.4, completes the verification of this direction.

Let us now prove that the condition is sufficient. A relation in $\mathcal{F}$ is a finite union of subsets of the form

$$X_{-r}e_{-r} + \cdots + X_{-1}e_{-1} + X_1e_1 \cdots + X_se_s,$$

where $e_{-r}, \cdots, e_{-1}, e_1, \cdots, e_s$ is a strictly increasing sequence of vectors and the X_i's are recognizable, group recognizable, aperiodic recognizable. The vectors e_{-i} for $i = 1, \ldots, r$ have only negative or null components and the vectors e_i for $i = 1, \ldots, s$ have only positive components. We adopt the convention that the expression reduces to $X_1e_1 \cdots + X_se_s$ (resp. $X_{-r}e_{-r} + \cdots + X_{-1}e_{-1}$) if $r = 0$ (resp. $s = 0$). Let ϕ_i be the formula defining X_i (see the paragraph 3.1). Set $I_i = \mathtt{Supp}(e_i)$ and define $I_{-r-1} = I_{s+1} = \emptyset$. Set $J_{-i} = I_{-i} \setminus I_{-i-1}$ if $1 \leq i \leq r$ and $J_i = I_i \setminus I_{i+1}$ if $1 \leq i \leq s$. We choose an arbitrary element α_i in J_i for $i = -r \cdots -1, 1 \cdots s$. Then this relation is defined by the formula

$$\begin{aligned}
&\left(y_{\alpha_{-1}} \leq 0 \wedge \bigwedge_{2\leq i\leq r} y_{\alpha_{-i}} - y_{\alpha_{-i+1}} \leq 0\right) \wedge \left(y_{\alpha_1} \geq 0 \wedge \bigwedge_{2\leq i\leq s} y_{\alpha_i} - y_{\alpha_{i-1}} \geq 0\right)\\
&\wedge \left(\phi_{-1}(-y_{\alpha_{-1}}) \wedge \bigwedge_{j\in J_{-1}} (y_j - y_{\alpha_{-1}} = 0)\right) \wedge \bigwedge_{i=2}^{r} \Big(\phi_{-i}(-(y_{\alpha_{-i}} - y_{\alpha_{-i+1}}))\\
&\wedge \bigwedge_{j\in J_{-i}} (y_j - y_{\alpha_i} = 0)\Big) \wedge \left(\phi_1(y_{\alpha_1}) \wedge \bigwedge_{j\in J_1} (y_j - y_{\alpha_1} = 0)\right)\\
&\wedge \bigwedge_{i=2}^{s} \left(\phi_i(y_{\alpha_i} - y_{\alpha_{i-1}}) \wedge \bigwedge_{j\in J_i} (y_j - y_{\alpha_i} = 0)\right). \qquad \square
\end{aligned}$$

In [1] Corollary 4.5, it is proved to be recursively decidable, given a semilinear subset of $\mathbb{N}^k$ whether or not it is belongs to $\mathcal{F}_{\text{rec}}$ (where the family is called the family of *synchronous* relations). Here we go one step further.

Theorem 5.2. *Given a relation in $\mathbb{Z}^k$ (resp. in $\mathbb{N}^k$) which is first order definable in the structure $\mathcal{Z}_p$ (resp. $\mathcal{N}_p$), it is recursively decidable whether or not it is first order definable in the structure $\mathcal{Z}_{\text{thresh,mod}}$ (resp. $\mathcal{N}_{\text{thresh,mod}}$). The same result holds when $\mathcal{Z}_{\text{thresh,mod}}$ is replaced by $\mathcal{Z}_{\text{mod}}$ or $\mathcal{Z}_{\text{thresh}}$ ($\mathcal{N}_{\text{thresh,mod}}$ is replaced by $\mathcal{N}_{\text{mod}}$ or $\mathcal{N}_{\text{thresh}}$.*

Proof. Let $\phi(x_1, \ldots, x_k)$ be a formula of the Presburger arithmetic on the integers with addition, defining a relation $X \subseteq \mathbb{Z}^k$. It is definable in the structure $\mathcal{Z}_{\text{thresh,mod}}$ resp. $\mathcal{Z}_{\text{mod}}$, $\mathcal{Z}_{\text{thresh}}$ if and only if for all strictly increasing sequences $e_{-r}, \cdots, e_{-1}, e_1, \cdots, e_s$ of vectors in $\mathcal{E}$ where the e_{-i} for $i = 1, \ldots, r$ have only negative or null components and the e_i's for $i = 1, \ldots, s$ have only positive components, the intersection $X \cap \mathbb{N}e_{-r} + \cdots + \mathbb{N}e_s$, considered as embedded in $\mathbb{N}^{r+s}$, is a recognizable, resp. group recognizable, aperiodic recognizable subset of $\mathbb{N}^{r+s}$. However, this intersection can be defined in Presburger arithmetic.

Indeed, let I_i be the support of e_i and define $J_{-r-1} = J_{s+1} = \emptyset$ and more generally $J_{-i} = I_{-i} \setminus I_{-i-1}$ if $1 \leq i \leq r$ and $J_i = I_i \setminus I_{i+1}$ if $1 \leq i \leq s$. For all $1 \leq j \leq k$ let $\sigma(x_j)$ be the predicate equal to $x_j < 0$ if $j \in I_{-1}$, to $x_j > 0$ if $j \in I_1$ and to $x_j = 0$ otherwise. Then the relation $X \cap (\mathbb{N}e_{-r} + \cdots + \mathbb{N}e_{-1} + \mathbb{N}e_1 + \cdots + \mathbb{N}e_s)$ is defined by the formula

$$\begin{aligned}
&\psi(y_{-r}, \ldots, y_{-1}, y_1, \ldots, y_s) \equiv y_{-r} > 0, \cdots y_{-1} > 0, y_1 > 0, \cdots, y_s > 0 \\
&\exists x_1 \cdots \exists x_k : \sigma(x_1) \wedge \cdots \wedge \sigma(x_k) \wedge \phi(x_1, \ldots, x_k) \wedge \\
&\bigwedge_{j \in J_{-1}} (y_{-1} = -x_j) \wedge \bigwedge_{1<i\leq r} \left(\bigwedge_{j \in J_{-i}, \ell \in J_{-i+1}} (y_{-i} = x_\ell - x_j) \right) \\
&\wedge \bigwedge_{j \in J_1} (y_1 = x_j) \wedge \bigwedge_{1<i\leq s} \left(\bigwedge_{\ell \in J_i, j \in J_{i-1}} (y_i = x_\ell - x_j) \right).
\end{aligned}$$

We now proceed as follows. Assume that for some increasing sequence of vectors the previous formula does not define a recognizable set, which can be decided by [5] page 1048, then ϕ is not definable in the structure $\mathcal{Z}_{\text{thresh,mod}}$. Otherwise, it is definable in $\mathcal{Z}_{\text{thresh,mod}}$ and it remains to check whether or not it is definable in one of the two substructures $\mathcal{Z}_{\text{mod}}$ and $\mathcal{Z}_{\text{thresh}}$. At this point we know that there exists a morphism f of $\mathbb{N}^p$ onto a finite commutative monoid M and a subset $F \subseteq M$ such that : $[\![\psi(y_1, \ldots, y_p)]\!] = f^{-1}(F)$ holds. We may assume further that M is minimal in the sense that for all $u, v \in M$ the following holds:

$$(\forall x \in M : \quad u + x \in F \Leftrightarrow v + x \in F) \Rightarrow u = v. \tag{8}$$

Then M is a group, resp. an aperiodic monoid if and only if ϕ is definable in the structure $\mathcal{Z}_{\text{mod}}$, resp. $\mathcal{Z}_{\text{thresh}}$. It thus suffices to list all possible triples (f, M, F) satisfying (8). Consider a triple for which equality $[\![\psi(y_1, \ldots, y_p)]\!] = f^{-1}(F)$ holds. If M is a group then ϕ is definable in $\mathcal{Z}_{\text{mod}}$, if it is an aperiodic monoid then ϕ is definable in $\mathcal{Z}_{\text{thresh}}$, otherwise ϕ is definable in $\mathcal{Z}_{\text{thresh,mod}}$ but neither in $\mathcal{Z}_{\text{thresh}}$ nor in $\mathcal{Z}_{\text{mod}}$.

Turning now to the domain $\mathbb{N}$, assume $\phi(x_1, \ldots, x_k)$ is a first order formula in the structure $\mathcal{N}_p$. It can be viewed as a formula in $\mathcal{Z}_p$ by transforming it into $(x_1 \geq 0) \wedge \cdots \wedge (x_k \geq 0) \wedge \phi'(x_1, \ldots, x_k)$ where ϕ' is obtained by substituting $\exists y \geq 0$ (resp. $\forall y \geq 0$) for each occurrence of $\exists y$ (resp. $\forall y$) in ϕ. If $\phi(x_1, \ldots, x_k)$ is first order definable in $\mathcal{N}_p$, then the transformed formula is first order definable in $\mathcal{Z}_p$. Furthermore if it is definable in $\mathcal{N}_{\text{thresh,mod}}$ (resp. $\mathcal{N}_{\text{mod}}$, resp. $\mathcal{N}_{\text{thresh}}$) it is also definable in $\mathcal{Z}_{\text{thresh,mod}}$ (resp. $\mathcal{Z}_{\text{mod}}$, resp. $\mathcal{Z}_{\text{thresh}}$). Conversely, if the relation is definable in $\mathcal{Z}_{\text{thresh,mod}}$ (resp. $\mathcal{Z}_{\text{mod}}$, resp. $\mathcal{Z}_{\text{thresh}}$), under the hypothesis that the variables x and y are nonnegative the predicate $x - y \geq a$ is equivalent to a predicate in $\mathcal{N}_{\text{thresh}}$. Indeed, the predicate $x - y \geq a$ with $a \leq 0$ is equivalent to $y - x \geq -a$ which completes the proof. $\square$

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