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ON SUBSTITUTION INVARIANT STURMIAN WORDS: AN APPLICATION OF RAUZY FRACTALS

Valérie Berthé[1], Hiromi Ei[2], Shunji Ito[3] and Hui Rao[4]

Abstract. Sturmian words are infinite words that have exactly $n+1$ factors of length n for every positive integer n. A Sturmian word $s_{\alpha,\rho}$ is also defined as a coding over a two-letter alphabet of the orbit of point ρ under the action of the irrational rotation $R_\alpha :\ x \mapsto x+\alpha \pmod 1$. A substitution fixes a Sturmian word if and only if it is invertible. The main object of the present paper is to investigate Rauzy fractals associated with two-letter invertible substitutions. As an application, we give an alternative geometric proof of Yasutomi's characterization of all pairs (α,ρ) such that $s_{\alpha,\rho}$ is a fixed point of some non-trivial substitution.

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1. Introduction

1.1. Sturmian words and substitution invariance

Sturmian words are infinite words over a binary alphabet, say, $\{1,2\}$, that have exactly $n+1$ factors of length n for every positive integer n. Sturmian words can also be defined in a constructive way as follows. Let $0<\alpha<1$. Let $\mathbb{T}^1=\mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x+\alpha$. For a given real number α, we introduce the following

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[1] LIRMM 161 rue Ada F-34392 Montpellier cedex 5, France; berthe@lirmm.fr

[2] Department of Information and System Engineering, Faculty of Science Engineering, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8851, Japan

[3] Department of Information and System Engineering, Kanazawa University, Kanazawa, Japan

[4] Department of Mathematics, Tsinghua University, Beijing, China

two partitions of $\mathbb{T}^1$:

$$\underline{I}_1 = [0, 1-\alpha), \quad \underline{I}_2 = [1-\alpha, 1); \quad \overline{I}_1 = (0, 1-\alpha], \quad \overline{I}_2 = (1-\alpha, 1].$$

Tracing the orbit of $R_\alpha^n(\rho)$, we define two infinite words for $\rho \in \mathbb{T}^1$:

$$\underline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \overline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \overline{I}_2. \end{cases}$$

It is well known [13, 25] that an infinite word is a Sturmian word if and only if it is equal either to $\overline{s}_{\alpha,\rho}$ or to $\underline{s}_{\alpha,\rho}$ for some irrational number α. The word $\underline{s}_{\alpha,\rho}$ is called *lower Sturmian word* whereas the word $\overline{s}_{\alpha,\rho}$ is called *upper Sturmian word. The notation $s_{\alpha,\rho}$ stands in all that follows indifferently for $\overline{s}_{\alpha,\rho}$ or for $\underline{s}_{\alpha,\rho}$ when there is no need to distinguish between the two.* A detailed description of Sturmian words can be found in Chapter 2 of [23], see also [28].

Let $\{1,2\}^*$ be the free monoid over $\{1,2\}$ endowed with the concatenation operation. A non-erasing homomorphism σ of the free monoid $\{1,2\}^*$ is called a *substitution.* An infinite word $s \in \{1,2\}^{\mathbb{N}}$ is a *fixed point* of the substitution σ if $\sigma(s) = s$.

It is well known that the famous Fibonacci word, *i.e.*, the fixed point of the Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$, is a Sturmian word. It is thus natural to ask when a Sturmian word is a fixed point of some non-trivial substitution. More precisely, we want to know:

Question 1. For which α and ρ is the Sturmian word $\underline{s}_{\alpha,\rho}$ (resp. $\overline{s}_{\alpha,\rho}$) a fixed point of some non-trivial substitution?

By non-trivial substitution, we mean here a substitution that is distinct from the identity. In all that follows, we say that a Sturmian word is *substitution invariant* if it is a fixed point of a non-trivial substitution.

There is a substantial literature devoted to Question 1. The first step has been made in [14] (Th. 1.1 below). When $\rho = \alpha$, we have $\underline{s}_{\alpha,\alpha} = \overline{s}_{\alpha,\alpha}$ since α is an irrational number. We thus denote this word by $s_{\alpha,\alpha}$. It is usually called the *characteristic word* of α. For a number x in a quadratic field, we denote by x' the conjugate of x in this field.

Theorem 1.1 (Crisp *et al.* [14])**.** *Let $0 < \alpha < 1$ be an irrational number. Then the following two conditions are equivalent:*

(i) *the characteristic word $s_{\alpha,\alpha}$ is substitution invariant;*
(ii) *α is a quadratic irrational with $\alpha' \notin [0,1]$.*

A quadratic number α with $0 < \alpha < 1$ and $\alpha' \notin [0,1]$ is called a *Sturm number* according to [2]. Let us note that the simplification of Condition (ii) in Theorem 1.1 to its present form is due to [2]. Furthermore, the expression of substitutions which fix $s_{\alpha,\alpha}$ can be explicitly obtained from the continued fraction expansion of α (see [14]).

For more results on the homogeneous case (*i.e.*, the case $\rho = \{n\alpha\}$ for $n \in \mathbb{Z}$, where $\{x\}$ stands for the fractional part of x), see for instance [7, 8, 11, 16, 21, 23]; for results in the non-homogeneous case, see [6,22,26]. Some variants of Question 1 are also considered in [10, 27].

Yasutomi has given a complete answer to Question 1 in [35]. Its characterization involves the conjugate of the quadratic real number x and can be compared to Galois' theorem for simple continued fractions describing numbers having a purely periodic continued fraction expansion.

Theorem 1.2 (Yasutomi [35])**.** *Let* $0 < \alpha < 1$ *and* $0 \le \rho \le 1$. *Then* $s_{\alpha,\rho}$ *is substitution invariant if and only if the following two conditions are satisfied:*

(i) α *is an irrational quadratic number and* $\rho \in \mathbb{Q}(\alpha)$*;*
(ii) $\alpha' > 1,\ 1-\alpha' \le \rho' \le \alpha'$ *or* $\alpha' < 0,\ \alpha' \le \rho' \le 1-\alpha'$.

Remark 1.3. Let us note the symmetry between both cases in Assertion (ii) of Theorem 1.2. Indeed, let $E : 1 \mapsto 2, 2 \mapsto 1$ be the substitution exchanging letters; then $\underline{s}_{\alpha,\rho}$ (resp. $\overline{s}_{\alpha,\rho}$) is substitution invariant if and only if $\underline{s}_{1-\alpha,1-\rho}$ (resp. $\overline{s}_{1-\alpha,1-\rho}$) which is equal to $E(\underline{s}_{\alpha,\rho})$ (resp. $E(\overline{s}_{1-\alpha,1-\rho})$); furthermore, (α,ρ) satisfies $\alpha' > 1,\ 1-\alpha' \le \rho' \le \alpha'$ if and only if $(1-\alpha, 1-\rho)$ satisfies $1-\alpha' < 0,\ \alpha' \le 1-\rho' \le 1-\alpha'$.

As a corollary of Theorem 1.2, we easily obtain:

Corollary 1.4. *Let* α *be a Sturm number. Then*

(i) for any $\rho \in \mathbb{Q} \cap (0,1)$, $\underline{s}_{\alpha,\rho} = \overline{s}_{\alpha,\rho}$ *is substitution invariant;*
(ii) let $\rho \in [0,1)$. *The Sturmian word* $\underline{s}_{\alpha,\{n\alpha\}}$ *(resp.* $\overline{s}_{\alpha,\{n\alpha\}}$*) is substitution invariant if and only if* $n = -1, 0, 1$. *In total we obtain exactly five substitution invariant Sturmian words*

$$\{21s_{\alpha,\alpha}, 12s_{\alpha,\alpha}, 2s_{\alpha,\alpha}, 1s_{\alpha,\alpha}, s_{\alpha,\alpha}\}$$

in the homogeneous case.

Note that (ii) is also proven in [35] and in [16].

Proof.

(i) Since ρ is a rational number, we have $\rho' = \rho$. Hence condition (ii) of Theorem 1.2 is fulfilled if $\alpha' > 1$ or $\alpha' < 0$;
(ii) let us first assume that $\alpha' > 1$. Let $n, p \in \mathbb{Z}$ such that $\rho = \{n\alpha\} = n\alpha - p$. One has $p = [n\alpha]$.

For $n = -1, 0, 1$, we have $\rho = 1-\alpha, 0, \alpha$, respectively, so that $\rho' = 1-\alpha', 0, \alpha'$. Hence $\rho' \in [1-\alpha', \alpha']$. Therefore $\overline{s}_{\alpha,\rho}$ and $\underline{s}_{\alpha,\rho}$ are substitution invariant.

For $n \ge 2$, $\rho' = n\alpha' - p > \alpha'$ since $p = [n\alpha] \le n-1$; for $n \le -2$, one has $p = [n\alpha] > n\alpha - 1 \ge n-1$. Hence $p+1 \ge n+1$ and $\rho' = n\alpha' - p < 1-\alpha'$. Therefore, $\overline{s}_{\alpha,\rho}$ and $\underline{s}_{\alpha,\rho}$ are not substitution invariant.

We deduce the case $\alpha' < 0$ by applying Remark 1.3. $\square$

1.2. Invertible substitutions

Let σ be a substitution over $\{1,2\}$ and let $M_\sigma = (m_{ij})$ be its *incidence matrix*, where m_{ij} counts the number of occurrences of the letter i in $\sigma(j)$. We assume that $\det M_\sigma = \pm 1$ (the substitution is said to be *unimodular*) and M_σ is *primitive* (M_σ^n has only positive entries for some non-negative integer n).

A substitution is said to be *invertible* if it is an automorphism of the free group generated by the alphabet $\{1,2\}$. Note that if σ is an invertible substitution, then its incidence matrix is unimodular.

Theorem 1.5 (Wen and Wen [34])**.** *Every invertible substitution over* $\{1,2\}$ *is a composition of the following three invertible substitutions:*

$$1 \mapsto 2, 2 \mapsto 1;\ 1 \mapsto 12, 2 \mapsto 1;\ 1 \mapsto 21, 2 \mapsto 1. \tag{1}$$

Question 1 is related to invertible substitutions according to the following well-known result (see for instance [23]).

Theorem 1.6. *A word is a Sturmian substitution invariant word if and only if it is a fixed point of some primitive and invertible substitution.*

Let us illustrate the main idea of the proof of Theorem 1.2 in [35]. According to the three substitutions in Theorem 1.5, Ito and Yasutomi [21] define three transformations from $[0,1]^2$ to $[0,1]^2$, namely:

$$T_1(\alpha,\rho) = \left(\frac{\alpha}{1+\rho}, \frac{\rho}{1+\alpha}\right), \quad T_2(\alpha,\rho) = \left(\frac{1}{2-\alpha}, \frac{\rho}{2-\alpha}\right),$$

$$T_3(\alpha,\rho) = (1-\alpha, 1-\rho).$$

Then it is proven that a Sturmian word $s_{\alpha,\rho}$ is substitution invariant if and only if there exists a sequence $S_1,\ldots,S_n$ with $S_i \in \{T_1,T_2,T_3\}$ such that $(\alpha,\rho) = S_1 \circ \cdots \circ S_n(\alpha,\rho)$. Since there are three transformations, the task of determining such (α,ρ) is tedious. Yasutomi's original proof of Theorem 1.2 in [35] is somewhat technical and lengthy.

Since Theorem 1.2 is a key elementary result, it is worth giving a proof that is more transparent and accessible. Let us note that a geometric proof based on the use of cut-and-project schemes has also been given in [4]. The proof we present here is based on Rauzy fractals.

1.3. Rauzy fractals

Rauzy fractals (first introduced in [30] in the Tribonacci case) are compact attractors of a graph-directed iterated function system associated with primitive substitutions with some prescribed algebraic properties. For more details, see for instance Chapter 7 in [28]. Rauzy fractals have numerous applications in number theory, ergodic theory, dynamical systems, fractal geometry and tiling theory (see for instance [3, 18–20, 30, 32], and Chap. 7 in [28]). The main purpose of the present paper is to describe a new application of Rauzy fractals to Sturmian words and more precisely, to study Rauzy fractals associated with invertibe two-letter substitutions according to [15].

Let us first describe an intuitive approach to Rauzy fractals for two-letter substitutions. We give a more formal definition in Section 2. Let σ be a primitive and unimodular substitution over $\{1,2\}$. If σ does not admit a fixed point, that is, if the image of 1 (resp. 2) begins with 2 (resp. 1), then σ^2 admits a fixed point. Otherwise, a fixed point of σ is still a fixed point of σ^2. Let $s = s_0s_1s_2\dots$ be a fixed point of σ^2. Let $(1-\alpha,\alpha)$ be the eigenvector with positive entries of M_σ corresponding to the Perron-Frobenius eigenvalue. We shall call α the *characteristic length* of the matrix M_σ or of the substitution σ, according to the context.

We define an oriented walk on the real line as follows. We start from the origin; in the nth step, if $s_{n-1} = 1$, we move to the right side by α; if $s_{n-1} = 2$, we move to the left side by $1-\alpha$. Taking the closure of the orbit of the origin under this transformation, we obtain

$$X = \text{closure } \{|s_0s_1\dots s_{k-1}|_1 \cdot \alpha + |s_0s_1\dots s_{k-1}|_2 \cdot (\alpha-1);\ k \geq 0\},$$

where $|s_0s_1\dots s_{n-1}|_j$ stands for the number of occurrences of the letter j in the word $s_0s_1\dots s_{n-1}$. Furthermore, we define

$$\begin{array}{rcl} X_1 & = & \text{closure } \{|s_0s_1\dots s_{k-1}|_1 \cdot \alpha + |s_0s_1\dots s_{k-1}|_2 \cdot (\alpha-1); \\ & & k \geq 0, s_k = 1\}, \\ X_2 & = & \text{closure } \{|s_0s_1\dots s_{k-1}|_1 \cdot \alpha + |s_0s_1\dots s_{k-1}|_2 \cdot (\alpha-1); \\ & & k \geq 0, s_k = 2\}. \end{array} \tag{2}$$

The Rauzy fractals of σ are defined as the set $X = X_1 \cup X_2, X_1, X_2$ in (2). (To be more precise, we shall see in Sect. 2 that X, X_1, X_2 are an affine image of the Rauzy fractals.)

A central property for our study is that the fixed points of an invertible substitution are Sturmian (see Th. 1.6), and hence the associated Rauzy fractals are intervals.

Theorem 1.7 [12]**.** *Let σ be a primitive unimodular substitution over $\{1,2\}$. Then the Rauzy fractals X_1, X_2 and $X_1 \cup X_2$ are intervals if and only if σ is invertible.*

A simple proof of this result is given in Section 2.4. Let us note that we only use here in the present paper the following easy implication: the Rauzy fractals of an invertible substitution are intervals.

Let us give a sketch of our proof of Theorem 1.2. By Theorems 1.6 and 1.7, if a Sturmian word is substitution invariant, then it is a fixed point of some primitive substitution with connected Rauzy fractals.

Let σ be an invertible substitution with characteristic length α. Then α is a Sturm number, and the Rauzy fractals X_1, X_2 are intervals with length $1-\alpha$ and α, respectively. Suppose $s = \overline{s}_{\alpha,\rho}$ or $s = \underline{s}_{\alpha,\rho}$ is a fixed point of σ^2. (According to Prop. 2.3 below, we can indifferently consider any of these two words.) One checks that $\rho = 1-\alpha-h$, where $\{h\} = X_1 \cap X_2$.

Let V' be the line $y = \frac{1-\alpha'}{\alpha'}x$, where α' is the algebraic conjugate of α. A broken line in $\mathbb{R}^2$, the so-called stepped surface, is associated with line V', defined as a discretization of V' (see Fig. 3).

The sets X_1,X_2 have a self-similar structure: indeed they satisfy a set equation which is controlled by the stepped surface of V' (see Lem. 3.1 and Th. 3.2). Hence, by connectedness and self-similarity of Rauzy fractals, we express the intersection $X_1 \cap X_2$ in terms of the stepped surface (see Th. 4.2).

Then we show that the stepped surface is associated with the rotation R_γ with $\gamma = \frac{\alpha'-1}{2\alpha'-1}$, which may be considered as the *dual rotation* of R_α. An arithmetic characterization of the stepped surface is obtained (see Th. 5.1). This allows us to get an algebraic description of the intersection set $X_1 \cap X_2$ for an invertible substitution σ, which yields a proof of Theorem 1.2.

This paper is organized as follows. We first review in Section 2 some basic facts on Rauzy fractals. We then discuss in Section 2.4 the connectedness of Rauzy fractals for a two-letter alphabet. Theorem 1.7 is proven in this section. In Section 3, we study set equations of Rauzy fractals, especially in the invertible case. The intersection set $X_1 \cap X_2$ for invertible substitutions is determined in Section 4. In Section 5, an algebraic characterization of the stepped surface is given. A proof of Theorem 1.2 is given in Section 6.

2. Rauzy fractals

In this section we review some basic facts on Rauzy fractals. We present here all definitions that apply to a two-letter alphabet, which is sufficient for our purpose. Note that the notation, which is adapted from [19], is slightly different from [3].

2.1. Sturm numbers

Let σ be a primitive unimodular substitution over $\{1, 2\}$. Let β be the maximal eigenvalue of its incidence matrix M_σ. Its algebraic conjugate β' is also an eigenvalue of M_σ. By the Perron-Frobenius' theorem, we have $\beta > 1$. Now $\beta\beta' = \det M_\sigma = \pm 1$ implies $|\beta'| < 1$. Therefore β is a Pisot number and the substitution σ is said to be of *Pisot type*.

It is well-known that the densities of letters exist in fixed points of primitive substitutions (see [29]). Furthermore, the vector of densities of the letters 1 and 2 denoted by $(1-\alpha, \alpha)$, with $0 \leq \alpha \leq 1$, is easily proven to be an *expanding eigenvector*, *i.e.*, an eigenvector associated with the expanding eigenvalue β. Let us recall that α is called the *characteristic length* of M_σ. The characteristic length α is (irrational) quadratic; the vector $(1-\alpha', \alpha')$ is an eigenvector associated with the eigenvalue β'. Still by Perron-Frobenius' theorem, coordinates $1-\alpha'$, α' cannot both be positive, hence $\alpha'(1-\alpha') \leq 0$, which implies that $\alpha' \notin]0, 1[$. Hence α is a Sturm number.

Conversely, any Sturm number is the characteristic length of a primitive unimodular matrix M of size 2×2. Indeed, if α is a Sturm number, then $s_{\alpha,\alpha}$ is a fixed point of an invertible primitive substitution σ following Theorem 1.1, and hence α is the characteristic length of M_σ. We thus have proven the lemma below.

Lemma 2.1. *A number $\alpha \in (0,1)$ is a Sturm number if and only if there exists a 2×2 primitive unimodular matrix M with non-negative integral entries such that $(1-\alpha, \alpha)$ is an expanding eigenvector of M. Consequently, if the Sturmian word $s_{\alpha,\rho}$ is substitution invariant, then this implies that α is a Sturm number.*

Example 1. Let σ be the substitution $1 \mapsto 121$, $2 \mapsto 12$, *i.e.*, the square of the Fibonacci substitution. This substitution admits as a unique fixed point the Fibonacci word $s_{\alpha,\alpha}$, with $\alpha = \frac{3-\sqrt{5}}{2}$, whose first terms are

$$121121211211212112121$$

One has $M_\sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\beta = \frac{3+\sqrt{5}}{2}$, and $\beta' = \frac{3-\sqrt{5}}{2} = \alpha = \frac{1}{\beta} > 0$.

We will also need the following lemma.

Lemma 2.2 ([8, 24, 34])**.** *Let σ be a non-trivial substitution over $\{1,2\}$. The following three conditions are equivalent:*

(i) σ is primitive invertible;
(ii) for any Sturmian word s, $\sigma(s)$ is still a Sturmian word;
(iii) there exists a Sturmian word s such that $\sigma(s)$ is a Sturmian word.

The equivalence between (i) and (ii) is due to [24] and [34], the equivalence with (iii) is proven in [8]. For more details, see [23].

2.2. Upper and lower Sturmian sequences

In this subsection, we show that $\underline{s}_{\alpha,\rho}$ is substitution invariant if and only if $\overline{s}_{\alpha,\rho}$ is also substitution invariant.

Proposition 2.3. *Let $0 < \alpha < 1$ be an irrational number and $0 \le \rho \le 1$. Then $\underline{s}_{\alpha,\rho}$ is substitution invariant if and only $\overline{s}_{\alpha,\rho}$ is also substitution invariant.*

Proof. Suppose $\underline{s}_{\alpha,\rho} = s_0 s_1 s_2 \dots$ is a fixed point of the non-trivial substitution σ. According to Lemma 2.2, σ is primitive invertible. By primitivity, one has

$$|\sigma^2(1)| \ge 2 \text{ and } |\sigma^2(2)| \ge 2. \tag{3}$$

Note that $\underline{s}_{\alpha,\rho} = s_0 s_1 s_2 \dots$ is also a fixed point of the non-trivial substitution σ^2. Let us prove that $\overline{s}_{\alpha,\rho}$ is a fixed point of σ^2.

Let us assume that $\underline{s}_{\alpha,\rho} \neq \overline{s}_{\alpha,\rho}$ (otherwise, there is nothing to prove). One has either

$$\begin{aligned} \underline{s}_{\alpha,\rho} &= s_0 \dots s_{n-1} 21 s_{n+2} \dots = s_0 \dots s_{n-1} 21 s_{\alpha,\alpha}, \\ \overline{s}_{\alpha,\rho} &= s_0 \dots s_{n-1} 12 s_{n+2} \dots = s_0 \dots s_{n-1} 12 s_{\alpha,\alpha} \end{aligned} \tag{4}$$

or

$$\begin{aligned} \underline{s}_{\alpha,\rho} &= 1 s_{\alpha,\alpha}, \\ \overline{s}_{\alpha,\rho} &= 2 s_{\alpha,\alpha}. \end{aligned} \tag{5}$$

Let $s = \underline{s}_{\alpha,\rho}$ and $s' = \overline{s}_{\alpha,\rho}$. We assume that we are in case (4); case (5) can be handled in the same way.

It is shown in [33] (as a consequence of Th. 1.5) that if τ is an invertible substitution over a two-letter alphabet, then there exist two words u and v such that either $\tau(12) = u12v$, $\tau(21) = u21v$, or $\tau(12) = u21v$, $\tau(21) = u12v$. By applying twice this result, one deduces that there exist a finite word w and an infinite word t, such that

$$\sigma^2(s) = w21t, \quad \sigma^2(s') = w12t. \tag{6}$$

One first deduces that $t = s_{\alpha,\alpha}$. Indeed, $12t$ and $21t$ are two Sturmian words with the same angle α. Second, we deduce from $\sigma^2(s_0 s_1 \dots s_{n-1}) = s_0 s_1 \dots s_{n-1} u = w$ and (3) that w and $s_0 s_1 \dots s_{n-1}$ are equal to the empty word. Again by (4) and (6), we have

$$\sigma^2(s) = 21t, \quad \sigma^2(s') = 12t = s'.$$

Hence s' is a fixed point of σ^2. □

2.3. Definition of Rauzy fractals

Let $\vec{e}_1, \vec{e}_2$ be the canonical basis of $\mathbb{R}^2$. Let $f : \{1,2\}^* \to \mathbb{Z}^2$ be the *Parikh map*, also called *abelianization homomorphism*, defined by $f(w) = |w|_1 \vec{e}_1 + |w|_2 \vec{e}_2$, where $|w|_i$ denotes the number of occurrences of the letter i in w.

Let V be the expanding eigenspace of the matrix M_σ corresponding to the eigenvalue β, and V' the contracting eigenspace corresponding to β'. The expanding subspace is generated by the vector $\vec{v} = (1-\alpha, \alpha)$, therefore the contracting subspace is generated by the vector $\vec{v'} = (1-\alpha', \alpha')$. Then $V \oplus V' = \mathbb{R}^2$ is a direct sum decomposition of $\mathbb{R}^2$. According to this direct sum, two natural projections are defined:

$$\pi : \mathbb{R}^2 \to V' \quad \text{and} \quad \pi' : \mathbb{R}^2 \to V.$$

We define the *Rauzy fractal* associated with σ as the closure of the projection according to π of the vertices of the broken line (illustrated in Fig. 1) obtained by applying map f to the prefixes of a given fixed point of σ^2. (We recall that σ^2 always admits a fixed point since we work on a two-letter alphabet.)

More precisely, let $s = (s_k)_{k \geq 0}$ be a fixed point of σ^2. We first define

$$Y = \{f(s_0 \dots s_{k-1});\ k \geq 0\},$$

where the notation $s_0 \dots s_{k-1}$ stands for the empty word when $k = 0$. We then divide Y into two parts:

$$Y_1 = \{f(s_0 \dots s_{k-1});\ s_k = 1\},\ Y_2 = \{f(s_0 \dots s_{k-1});\ s_k = 2\}.$$

Projecting Y_1, Y_2 onto the contracting eigenspace V' and taking the closures, we get

$$\vec{X}_1 = \overline{\pi(Y_1)},\ \vec{X}_2 = \overline{\pi(Y_2)}.$$

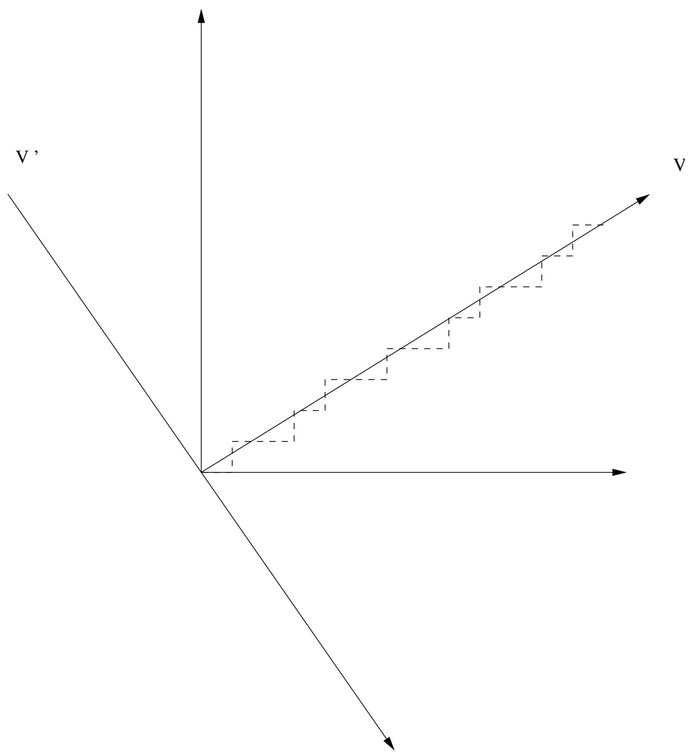

FIGURE 1. The broken line.

We call $\vec{X}_1$ and $\vec{X}_2$ the *Rauzy fractals* of the substitution σ. It is shown in [19] that the Rauzy fractals are independent of the choice of the fixed point in the definition.

Clearly, Rauzy fractals $\vec{X}_1$ and $\vec{X}_2$ are one-dimensional objects. One has

$$\vec{e}_1 = -\frac{\alpha'}{\alpha-\alpha'}\vec{v} + \frac{\alpha}{\alpha-\alpha'}\vec{v'} \text{ and } \vec{e}_2 = \frac{1-\alpha'}{\alpha-\alpha'}\vec{v} + \frac{\alpha-1}{\alpha-\alpha'}\vec{v'}. \tag{7}$$

Hence an easy computation shows that

$$X_1 = \phi(\vec{X}_1), \quad X_2 = \phi(\vec{X}_2),$$

where X_1, X_2 are defined in (2) and ϕ is the linear map defined by

$$\phi: \ V' \to \mathbb{R}, \ \phi(\frac{x\vec{v'}}{\alpha-\alpha'}) = x. \tag{8}$$

By abuse of language, we also call X, X_1 and X_2 the *Rauzy fractals* of the substitution σ.

Barge and Diamond showed in [5] that every Pisot substitution over a two-letter alphabet satisfies a certain combinatorial condition, called the *strong coincidence condition.* Thanks to this, one can show that

Lemma 2.4 ([19]). *Let σ be a primitive Pisot substitution over two letters. Then*

$$\mu(X_1) = 1 - \alpha, \quad \mu(X_2) = \alpha,$$

where μ is the Lebesgue measure and α is the characteristic length of M_σ.

2.4. Connectedness of Rauzy fractals

It is generally hard to decide whether Rauzy fractals are connected (see for instance [1,12]). However, in the two-letter case we have a complete characterization given by Theorem 1.7. We provide an elementary proof of this folklore result.

Proof of Theorem 1.7. Let σ be a primitive invertible substitution. Let s be a fixed point of σ^2. By Theorem 1.6, s is a Sturmian word. Indeed, if s' is any Sturmian word with the same initial letter as s, then the sequence of Sturmian words (according to Lem. 2.2) $(\sigma^{2n}(s'))_{n\geq 1}$ converges to s. Hence s has at most $n+1$ factors of length n. Since σ is both unimodular and primitive, we infer that the density of the letter 1 in s is irrational, which implies that s is aperiodic and thus, a Sturmian word.

Let α, ρ such that $s = s_{\alpha,\rho}$ (which means indifferently either $\underline{s}_{\alpha,\rho}$ or $\overline{s}_{\alpha,\rho}$). Let us first prove that the points $f(s_0 \cdots s_{k-1})$, for $k \in \mathbb{N}$, stay at a bounded distance of the line V; more precisely, they stay between the lines $y = \frac{\alpha}{1-\alpha}x + \frac{\rho-1}{1-\alpha}$ and $y = \frac{\alpha}{1-\alpha}x + \frac{\rho}{1-\alpha}$, which directly implies that $\mu(X_1 \cup X_2) \leq 1$. Indeed, the broken line defined by the vertices $f(s_0 ... s_{k-1})$, for $k \in \mathbb{N}$, is a cutting sequence (see for instance [23]), that is, it corresponds to the approximation of the line $y = \frac{\alpha}{1-\alpha}x + \frac{\rho}{1-\alpha} - 1$ by the broken line with integer vertices obtained by progressing by unit segments, either up or to the right, always going in the direction of the line, and starting from the origin point $(0,0)$: one first notes that $s_0 = 1$ if and only if $\frac{\rho}{1-\alpha} - 1 < 0$; furthermore, if $\alpha < 1/2$ (resp. $\alpha > 1/2$), the vertex $f(s_0 \cdots s_{k-1})$ is below (resp. above) the line $y = \frac{\alpha}{1-\alpha}x + \frac{\rho}{1-\alpha} - 1$ if and only if $s_k = 2$ (resp. $s_k = 1$).

Moreover, by (7), $\phi \circ \pi \circ f(1) = \alpha$, and $\phi \circ \pi \circ f(2) = \alpha - 1$. This implies that

$$\begin{cases} \phi \circ \pi \circ f(s_0 \cdots s_k) = \phi \circ \pi \circ f(s_0 \cdots s_{k-1}) + \alpha & \text{when } s_k = 1 \\ \phi \circ \pi \circ f(s_0 \cdots s_k) = \phi \circ \pi \circ f(s_0 \cdots s_{k-1}) + \alpha - 1 & \text{when } s_k = 2. \end{cases} \tag{9}$$

Hence

$$\forall k \in \mathbb{N},\ \phi \circ \pi \circ f(s_0 \cdots s_{k-1}) \equiv k\alpha \bmod 1. \tag{10}$$

By irrationality of α, we deduce that Rauzy fractals are intervals.

Conversely, let σ be a primitive unimodular substitution over $\{1,2\}$. We first assume that the Rauzy fractals of σ, namely X_1, X_2, and $X = X_1 \cup X_2$, are intervals. Let $s = (s_k)_{k\geq 0}$ be a fixed point of σ^2 which defines X_1 and X_2. Let α stand for the characteristic length of M_σ. Equations (9) and (10) still hold.

According to Lemma 2.4, $\mu(X_1 \cap X_2) = 0$, $\mu(X_1) = 1 - \alpha$ and $\mu(X_2) = \alpha$. Furthermore, $X_1 + \alpha \subset X = X_1 \cup X_2$, by (9). Hence there exists $h \in \mathbb{R}$ such that $X_1 = [-1 + \alpha + h, h]$ and $X_2 = [h, h + \alpha]$.

If the sequence $(\phi\circ\pi\circ f(s_0 \cdots s_{k-1}))_{k\geq 1}$ never takes as value one of the endpoints of X_1 and X_2, then one has according to (9)

$$\begin{cases} \forall k \in \mathbb{N},\ s_k = 1 \text{ if and only if } \phi \circ \pi \circ f(s_0 \cdots s_{k-1}) \in (-1 + \alpha + h, h) \\ \forall k \in \mathbb{N},\ s_k = 2 \text{ if and only if } \phi \circ \pi \circ f(s_0 \cdots s_{k-1}) \in (h, h + \alpha). \end{cases}$$

We deduce from (10) that $s = \overline{s}_{\alpha,1-\alpha-h} = \underline{s}_{\alpha,1-\alpha-h}$.

If there exists $k \geq 1$ such that $\phi \circ \pi \circ f(s_0 \cdots s_{k-1}) = -1 + \alpha + h$, then $s = \underline{s}_{\alpha,1-\alpha-h}$. Similarly, if there exists $k \geq 1$ such that $\phi\circ\pi\circ f(s_0 \cdots s_{k-1}) = h+\alpha$, then $s = \overline{s}_{\alpha,1-\alpha-h}$.

We assume now that there exists $k \geq 1$ such that $\phi \circ \pi \circ f(s_0 \cdots s_{k-1}) = h$. If $s_k = 1$, then $\phi \circ \pi \circ f(s_0 \cdots s_k) = h + \alpha$, and $s = \overline{s}_{\alpha,1-\alpha-h}$. If $s_k = 2$, then $\phi \circ \pi \circ f(s_0 \cdots s_k) = h + \alpha - 1$, and $s = \underline{s}_{\alpha,1-\alpha-h}$.

We thus have proved that s is a Sturmian word. According to Lemma 2.2, this implies that σ^2, and thus σ, are invertible. □

We deduce from the previous proof the following:

Corollary 2.5. *Let σ be a primitive invertible substitution. Then there exists $h \in \mathbb{Z}$ such that the Rauzy fractals satisfy*

$$X_1 = [-1 + \alpha + h, h], \quad X_2 = [h, \alpha + h],$$

where α is the characteristic length of σ. Furthermore, if $\overline{s}_{\alpha,\rho}$ or $\underline{s}_{\alpha,\rho}$ is a fixed point point of σ^2, then $\rho = 1 - \alpha - h$.

Example 2. Let us continue Example 1. One has $X_1 = [-\alpha, 1 - 2\alpha]$, $X_2 = [1 - 2\alpha, 1 - \alpha]$, $h = 1 - 2\alpha$.

3. Self-similarity of Rauzy fractals

In this section, we discuss the self-similar structure of Rauzy fractals X_1 and X_2, while paying special attention to the case σ invertible. The stepped surface is shown to play an important role.

3.1. Set equations of Rauzy fractals

Let σ be a primitive substitution over $\{1, 2\}$ and let β be the Perron-Frobenius eigenvalue of M_σ.

It is well-known [3, 19, 32]) that $\vec{X}_1$ and $\vec{X}_2$, and thus X_1 and X_2, have a self-similar structure, *i.e.*, both $\frac{1}{\beta'}X_1$ and $\frac{1}{\beta'}X_2$ are unions of translated copies of X_1

and X_2. (We recall that $|\beta'| < 1$.) In order to describe the corresponding set equations, we introduce the following notation: let D_1 (resp. D_2) be the set of these $(a, i) \in \mathbb{R} \times \{1, 2\}$ such that $X_i + a \subset \frac{1}{\beta'} X_1$ (resp. $X_i + a \subset \frac{1}{\beta'} X_2$), that is,

$$\frac{1}{\beta'} X_1 = \bigcup_{(a,i)\in D_1} X_i + a, \quad \frac{1}{\beta'} X_2 = \bigcup_{(b,i)\in D_2} X_i + b.$$

For the explicit form of D_1, D_2, we refer to [3] for the general case, and to Section 3.4, in the present case. To give an intuitive flavour of the explicit form, let us just note that any vertex $f(s_0 \cdots s_{k-1})$ of the broken line has form $f(\sigma(s_0 \cdots s_{q-1})) + f(p)$, for a prefix p of $\sigma(s_q)$. Its projection yields the multiplication by $1/\beta'$, and thus belongs to X_{s_k}/β'. The first part $f(\sigma(s_0 \cdots s_{q-1}))$ contributes by projection to an interval X_{s_q} and $f(p)$ induces a translation of this interval.

Example 3. We continue Example 2. One checks that

$$\frac{X_1}{\beta'} = [-1, 1/\alpha - 2] = [-1, 1 - \alpha] = (X_1 + \alpha - 1) + X_1 + X_2,$$
$$\frac{X_2}{\beta'} = [1/\alpha - 2, 1/\alpha - 1] = [1 - \alpha, 2 - \alpha] = (X_1 + 1) + (X_2 + 1).$$

One has $D_1 = \{(\alpha - 1, 1), (0, 1), (0, 2)\}$ and $D_2 = \{(1, 1), (1, 2)\}$.

3.2. The stepped surface

Recall that V' is the contracting eigenline of M_σ. We denote the upper closed half-plane delimited by V' as $(V')^+$, and the lower open half-plane delimited by V' as $(V')^-$. We define

$$S = \{[z, i^*]; \quad z \in \mathbb{Z}^2, z \in (V')^+ \text{ and } z - \vec{e}_i \in (V')^-\},$$

where the notation $[z, i^*]$, for $z \in \mathbb{Z}^2$ and $i^* \in \{1^*, 2^*\}$, endows the point z in $\mathbb{Z}^2$ with color $i^* = 1^*, 2^*$. Intuitively S consists of the collection of these colored points $[z, i^*]$ which are close to the contracting eigenline V'.

We now define $\overline{[z, 1^*]}$ (resp. $\overline{[z, 2^*]}$) as the closed line segment from z to $z + \vec{e}_2$ (resp. to $z + \vec{e}_1$) (see Fig. 2). Then the *stepped surface* $\overline{S}$ of V' is defined as the broken line consisting of the following segments

$$\overline{S} = \bigcup_{[z,i^*]\in S} \overline{[z, i^*]}.$$

It is easily seen to be connected. A piece of a stepped surface is depicted in Figure 3 for the example of Example 1. By abuse of language, the formal set S will also be called the stepped surface of V'.

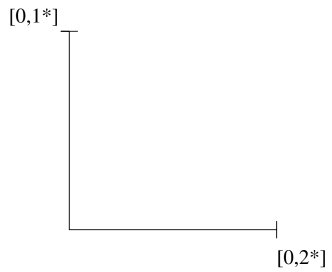

FIGURE 2. The segments $\overline{[0,1^*]}$ and $\overline{[0,2^*]}$.

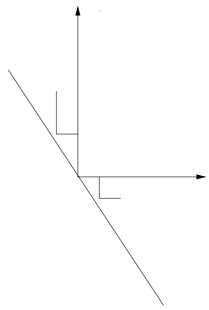

FIGURE 3. A piece of the stepped surface for $1 \mapsto 121$, $2 \mapsto 12$.

It turns out that the set equations of the Rauzy fractals are controlled by the stepped surface. An explicit expression of sets D_1 and D_2 is given in [3], from which one immediately deduces the following facts:

Lemma 3.1 [3,19]**.** *Using the notation above:*

(i) for any $(a,i) \in D_1 \cup D_2$, there exists an element $[z,i^*] \in S$ such that $\phi \circ \pi(z) = a$;
(ii) $(0,1),(0,2) \in D_1 \cup D_2$;
(iii) $(n_{ij})_{1 \le i,j \le 2} = {}^tM_\sigma$, *where n_{ij} counts the number of elements (a,i) in the set D_j.*

3.3. Tiling associated with the stepped surface

Projecting the stepped surface $\overline{S}$ onto V', we first obtain a tiling $\mathcal{J}'$ of V':

$$\mathcal{J}' = \{\pi(\overline{[z,i^*]});\ [z,i^*] \in S\}.$$

Applying the linear transformation ϕ (see (8)), we then get a tiling $\mathcal{J}$ of the real line:

$$\mathcal{J} = \{\phi \circ \pi(\overline{[z,i^*]});\ [z,i^*] \in S\}.$$

Tiling $\mathcal{J}$ is a tiling with two prototiles. Indeed

$$\mathcal{J} = \{\phi \circ \pi(z) + J_i;\ [z, i^*] \in S\},$$

where

$$J_1 = \phi \circ \pi\overline{[0, 1^*]} = [-1 + \alpha, 0],\ J_2 = \phi \circ \pi\overline{[0, 2^*]} = [0, \alpha].$$

We label the tiles of $\mathcal{J}$ on the right side of the origin by the sequence $T_0, T_1, T_2, \dots$, where T_{n+1} is the rightside neighbour of T_n. Likewise we label the tiles of $\mathcal{J}$ on the left side of the origin by $T_{-1}, T_{-2}, \dots$ One has $\mathcal{J} = \{T_k;\ k \in \mathbb{Z}\}$. We furthermore define the two-sided sequence $(g_k)_{k\in\mathbb{Z}}$ as the sequence of left endpoints of tiles T_k (one has $g_0 = 0$). An arithmetic description of the sequence $(g_k)_{k\in\mathbb{Z}}$ is given in Section 5.

Example 4. We continue Example 3. One has $g_{-2} = 2(\alpha - 1)$, $g_{-1} = \alpha - 1$, $g_0 = 0$, $g_1 = \alpha$, $g_2 = 1$.

3.4. Set equations of connected Rauzy fractals

According to Corollary 2.5, if σ is a primitive invertible substitution, then there exists a real number h such that $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$, that is,

$$X_1 = J_1 + h, \quad X_2 = J_2 + h,$$

where $J_1 = [-1 + \alpha, 0]$ and $J_2 = [0, \alpha]$ are the two prototiles of tiling $\mathcal{J}$.

Let $(a, i) \in D_1$. There exists an element $[z, i^*] \in S$ such that $\phi \circ \pi(z) = a$ by Lemma 3.1. Let $k \in \mathbb{Z}$ such that $\phi \circ \pi\overline{[z, i^*]} = T_k$; then

$$X_i + a = J_i + h + a = T_k + h.$$

We thus can introduce two subsets $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathcal{J}$ such that

$$\frac{X_1}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h, \quad \frac{X_2}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_2} T\right) + h.$$

On the one hand, the tiles in $\mathcal{D}_1 \cup \mathcal{D}_2$ do not overlap according to [5] and [3]. On the other hand, these tiles must form a connected patch of $\mathcal{J}$ since $X_1, X_2, X_1 \cup X_2$ are intervals according to Theorem 1.7. Hence we have proven that

Theorem 3.2. *Let $X_1 = [-1 + \alpha + h, h], X_2 = [h, h + \alpha]$ be the Rauzy fractals of the primitive invertible substitution σ. Then*

$$\frac{X_1}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h, \quad \frac{X_2}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_2} T\right) + h,$$

where $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ are connected patches of the tiling $\mathcal{J}$.

Example 5. We continue Example 4. One has $\mathcal{D}_1 = \{T_{-2}, T_{-1}, T_0\}$, $\mathcal{D}_2 = \{T_1, T_2\}$, $\frac{X_1}{\beta'} = h + T_{-2} + T_{-1} + T_0$, $\frac{X_2}{\beta'} = h + T_1 + T_2$.

4. Invertible substitutions with a given incidence matrix

In this section, we give a more detailed description of the Rauzy fractals of invertible substitutions with a given incidence matrix.

4.1. A list of invertible substitutions with a given incidence matrix

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix. A very interesting result on invertible substitutions is given in [31]:

Theorem 4.1 (Séébold [31])**.** *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a primitive unimodular matrix with non-negative entries. The number of invertible substitutions with incidence matrix* M *is equal to* $a+b+c+d-1$.

Let σ be an invertible substitution with incidence matrix $M_\sigma = M$. According to Lemma 3.1, $(0,1), (0,2) \in D_1 \cup D_2$, hence we have

$$T_{-1}, T_0 \in \mathcal{D}_1 \cup \mathcal{D}_2. \tag{11}$$

By Lemma 3.1iii, we have

$$\text{Card } \mathcal{D}_1 = \text{Card } D_1 = a+b, \quad \text{Card } \mathcal{D}_2 = \text{card } D_2 = c+d. \tag{12}$$

Let us assume that the determinant of M is equal to 1. (We will not need to subsequently consider the case $\det(M) = -1$, but a similar study can be conducted.) In this case, $1/\beta' = \beta > 0$ so that $\frac{X_1}{\beta'}$ is on the left side of $\frac{X_2}{\beta'}$. Hence by Theorem 3.2, the patch $\mathcal{D}_1$ is on the left side of $\mathcal{D}_2$. By Theorem 3.2, (11), and (12), we infer that there exists k with $1 \le k \le a+b+c+d-1$ such that

$$\begin{aligned} \mathcal{D}_1 &= \{T_{-k}, T_{-k+1}, \ldots, T_{-k+a+b-1}\}, \\ \mathcal{D}_2 &= \{T_{-k+a+b}, T_{-k+a+b+1}, \ldots, T_{-k+a+c+b+d-1}\}. \end{aligned} \tag{13}$$

Hence there are at most $a+b+c+d-1$ invertible substitutions with incidence matrix M, and their set equations are deduced from (13). On the other hand, Theorem 4.1 asserts that there are exactly $a+b+c+d-1$ such substitutions. Since the set equations for different substitutions are distinct, we conclude that there is a one-to-one correspondence between the invertible substitutions with incidence matrix M and the set equations determined by (13). We denote these substitutions by σ_k, $1 \le k \le a+b+c+d-1$.

4.2. Intersection point of Rauzy fractals

For each of the substitutions σ_k defined in the previous section, there exists ρ_k such that s_{α,ρ_k} (which means indifferently either $\overline{s}_{\alpha,\rho_k}$ or $\underline{s}_{\alpha,\rho_k}$) is a fixed point of σ_k^2 according to the proof of Proposition 2.3.

Let $1 \leq k \leq a+b+c+d-1$. Let $X_1 = [-1+\alpha+h_k, h_k]$, $X_2 = [h_k, \alpha+h_k]$ be the Rauzy fractals of σ_k. One has $\rho_k = 1-\alpha-h_k$ according to Corollary 2.5. Below we use the connectedness and the self-similarity of Rauzy fractals to determine h_k and thus ρ_k. Let us recall that $(g_k)_{k\in\mathbb{Z}}$ stands for the sequence of left endpoints of tiles T_k in $\mathcal{J}$.

Theorem 4.2. *Let M be a 2×2 primitive matrix with non-negative entries such that $\det M = 1$. Let σ_k, $1 \leq k \leq a+b+c+d-1$, be the invertible substitutions with incidence matrix M, and let $X_1 = [-1+\alpha+h_k, h_k]$, $X_2 = [h_k, \alpha+h_k]$ be the Rauzy fractals of σ_k^2. Let β be the maximal eigenvalue of M. Then*

$$h_k = \frac{g_{-k+a+b}}{\beta-1}\cdot$$

Proof. On the one hand, $\frac{X_1}{\beta'} \cap \frac{X_2}{\beta'} = \{(\beta')^{-1}h_k\} = \{\beta h_k\}$. On the other hand, this intersection point is the left endpoint of the interval $\cup\{T+h_k;\ T \in \mathcal{D}_2\}$, *i.e.*, the left endpoint of $T_{-k+a+b}+h_k$. So we get $g_{-k+a+b}+h_k = \beta h_k$, and $h_k = \frac{g_{-k+a+b}}{\beta-1}$. □

Theorem 4.3. *Let M be a 2×2 primitive matrix with non-negative entries such that $\det M = 1$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{a+b+c+d-1}$ be the invertible substitutions with incidence matrix M. Let $G := \{g_k;\ k \in \mathbb{Z}\}$. Then the Sturmian word $s_{\alpha,\rho}$ is a fixed point of the substitution σ_k^2 if and only if*

$$0 \leq \rho \leq 1 \text{ and } (\rho+\alpha-1) \in \frac{G}{1-\beta}\cdot$$

Proof. By Theorem 4.2, one has $h_{a+b+c+d-1} < \cdots < h_2 < h_1$. Hence a real number h belongs to the set $\{h_1, h_2, \ldots, h_{a+b+c+d-1}\}$ if and only if

$$h \in \frac{G}{\beta-1} \text{ and } h_{a+b+c+d-1} \leq h \leq h_1. \tag{14}$$

The values h_1 and $h_{a+b+c+d-1}$ remain to be determined. For the substitution σ_1, the set $\mathcal{D}_1$ is equal to $\{T_{-1}, T_0, \ldots, T_{a+b-2}\}$. By Lemma 3.1iii, the numbers of tiles in $\mathcal{D}_1$ of length $1-\alpha$ and α are a and b, respectively. Since $|T_{-1}| = 1-\alpha$, we have

$$g_{a+b-1} = (a-1)(1-\alpha)+b\alpha = (\beta-1)(1-\alpha).$$

Here we use the equality $a(1-\alpha)+b\alpha = \beta(1-\alpha)$, which follows from the fact that $(1-\alpha,\alpha)$ is an expanding eigenvector of M. Therefore $h_1 = 1-\alpha$. A similar argument shows that $h_{a+b+c+d-1} = -\alpha$.

Remember now that $\rho_k = 1-\alpha-h_k$. The theorem follows from (14). □

5. The stepped surface

In this section, we give an arithmetic description of the stepped surface $\overline{S}$. We first define the two-sided word $(t_n)_{n\in\mathbb{Z}}$ as:

$$\forall n \in \mathbb{Z},\ t_n = \begin{cases} 1, & \text{if } |T_n| = 1-\alpha \\ 2, & \text{if } |T_n| = \alpha. \end{cases}$$

It is well known that Sturmian words can also be described as cutting sequences (see for instance [23]). One checks according to [3] that $(t_n)_{n\in\mathbb{Z}}$ is the upper two-sided cutting sequence of the line $V' : y = \frac{1-\alpha'}{\alpha'}x$. Hence

$$t_{-1}t_{-2}t_{-3}\cdots = 1s_{\gamma,\gamma}, \quad t_0t_1t_2\cdots = 2s_{\gamma,\gamma}, \tag{15}$$

where

$$\gamma = \frac{\alpha'-1}{2\alpha'-1}. \tag{16}$$

Let $R_\gamma : x \mapsto x+\gamma$ be the rotation of angle γ of the torus $\mathbb{T}^1$. We deduce from (15) that for all positive k

$$|t_{-1}t_{-2}\ldots t_{-k}|_1 \cdot \gamma + |t_{-1}t_{-2}\ldots t_{-k}|_2 \cdot (\gamma-1) = R_\gamma^k(0)$$

$$|t_0t_1\ldots t_{k-1}|_1 \cdot \gamma + |t_0t_1\ldots t_{k-1}|_2 \cdot (\gamma-1) = -R_\gamma^{-k}(0).$$

By definition of $(g_k)_{k\in\mathbb{Z}}$, one has for every nonnegative k

$$\begin{aligned} g_{-k} &= |t_{-1}t_{-2}\ldots t_{-k}|_1 \cdot (\alpha-1) + |t_{-1}t_{-2}\ldots t_{-k}|_2 \cdot (-\alpha), \\ g_k &= |t_0t_1\ldots t_{k-1}|_1 \cdot (1-\alpha) + |t_0t_1\ldots t_{k-1}|_2 \cdot \alpha. \end{aligned}$$

Hence

$$\forall k \in \mathbb{Z},\ \frac{g'_k}{2\alpha'-1} = R_\gamma^{-k}(0), \tag{17}$$

where g'_k denotes the conjugate of g_k. This thus provides an arithmetic description of the stepped surface.

Theorem 5.1. *One has*

$$\begin{aligned} G &= \{g \in \mathbb{Z}[\alpha];\ 0 \le g' < 2\alpha'-1\} \text{ when } \alpha' > 1, \\ G &= \{g \in \mathbb{Z}[\alpha];\ 2\alpha'-1 < g' \le 0\} \text{ when } \alpha' < 0. \end{aligned}$$

Proof. We assume that $\alpha' > 1$. The case $\alpha' < 0$ can be handled similarly.
Note that

$$\{R_\gamma^k(0);\ k \in \mathbb{Z}\} = \{m\gamma + n;\ 0 \le m\gamma + n < 1\}.$$

This together with (17) imply that

$$\begin{aligned} G &= \{g;\ g' = m(\alpha'-1) + n(2\alpha'-1);\ m,n \in \mathbb{Z},\ 0 \le g' < 2\alpha'-1\} \\ &= \{g;\ g = m(\alpha-1) + n(2\alpha-1);\ m,n \in \mathbb{Z},\ 0 \le g' < 2\alpha'-1\} \\ &= \{g \in \mathbb{Z}[\alpha];\ 0 \le g' < 2\alpha'-1\}. \end{aligned}$$

□

Remark 5.2. For a Sturm number α, it is easy to check that $\gamma = \frac{\alpha'-1}{2\alpha'-1}$ is also a Sturm number. We say that γ is the *dual* of α. One checks that γ and α are duals of each other. In some sense, rotation R_γ is the dual rotation of R_α.

6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Theorem 1.2 (Yasutomi [35])**.** *Let $0 < \alpha < 1$ and $0 \leq \rho \leq 1$. Then $s_{\alpha,\rho}$ is substitution invariant if and only if the following two conditions are satisfied:*

(i) α is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
(ii) $\alpha' > 1,\ 1-\alpha' \leq \rho' \leq \alpha'$ or $\alpha' < 0,\ \alpha' \leq \rho' \leq 1-\alpha'$.

6.1. An algebraic lemma

We first need a preliminary lemma.

Lemma 6.1. *Let β be a quadratic algebraic unit, and α be an irrational number in $\mathbb{Q}(\beta)$. Then for any $\rho \in \mathbb{Q}(\beta)$, there exists an arbitrary large even number n such that $\rho(\beta^n - 1) \in \mathbb{Z}[\alpha]$.*

Proof. Let $\mathcal{A}$ stand for the ring of algebraic integers in $\mathbb{Q}(\beta)$. First we claim that for any $\rho \in \mathbb{Q}(\beta)$, there exists an arbitrary large number n such that $\rho(\beta^n - 1) \in \mathcal{A}$. Indeed, let $\delta \in \mathcal{A}$ such that $\delta\rho \in \mathcal{A}$. Then at least two terms in the sequence $(\delta\rho\beta^n)_{n\geq 0}$ belong to the same residue class modulo the principal ideal of $\mathcal{A}$ generated by δ. Hence $\delta\rho(\beta^{n_1} - \beta^{n_2})$ is divisible by δ in $\mathcal{A}$ for some $n_1 > n_2$. Since β is an algebraic unit, $\delta\rho(\beta^n - 1)$ is also divisible by δ for $n = n_1 - n_2$, and $\rho(\beta^n - 1) \in \mathcal{A}$. This proves our claim. Note furthermore that obviously we can decide that n is an even number. We thus have proven that for every $N > 0$,

$$\mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2n} - 1}.$$

We then prove that there is a rational number K such that $\mathcal{A} \subset K\mathbb{Z}[\alpha]$. Indeed, let d be the square-free integer such that $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d})$. Then there are integers a, b and $c \neq 0$ such that $\alpha = \frac{a+b\sqrt{d}}{c}$. Note that $b \neq 0$ since α is irrational. It is well known that any element in $\mathcal{A}$ must have the form $(m\sqrt{d} + n)/2$. Since

$$(m\sqrt{d} + n)/2 = \frac{mc\alpha - ma + nb}{2b}$$

is an element of $\frac{\mathbb{Z}[\alpha]}{2b}$, our assertion is true by taking $K = \frac{1}{2b}$.

Therefore for any $N > 0$, we have

$$\mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2n} - 1} \subset K \bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2n} - 1} \subseteq \mathbb{Q}(\beta).$$

Multiplying every term of the above formula by K^{-1}, we obtain

$$\mathbb{Q}(\beta) = \bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2n} - 1}\cdot \qquad \square$$

6.2. Proof of Theorem 1.2

Now we are in a position to prove Theorem 1.2.
Necessity. Let us suppose that $s_{\alpha,\rho}$ is a fixed point of the non-trivial primitive invertible substitution σ. Let β be the maximal eigenvalue of M_σ. We may assume that $\det M = 1$, for otherwise we consider σ^2 instead of σ.

By Lemma 2.1, α must be a Sturm number. From Theorem 4.3, we deduce

$$1 - \alpha - \rho = h \in \frac{G}{\beta - 1} \subseteq \frac{\mathbb{Z}[\alpha]}{\beta - 1} \subseteq \mathbb{Q}(\beta).$$

Hence $\rho \in \mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$, so condition (i) is necessary.

Concerning (ii), we need only to consider the case $\alpha' > 1$ according to Remark 1.3. Note that $s_{\alpha,\rho}$ is also a fixed point of σ^n, for any $n \geq 1$, and in particular for any even number n; furthermore, substitutions σ^n share the same stepped surface. Hence

$$\rho + \alpha - 1 \in \frac{G}{1 - \beta^n},$$

$$\rho' + \alpha' - 1 \in \frac{\{g';\ g \in G\}}{1 - (\beta')^n}\cdot$$

By Theorem 5.1, we have

$$0 \leq \rho' + \alpha' - 1 < \frac{2\alpha' - 1}{1 - (\beta')^n}\cdot$$

Note that the above formula holds for every even number n. By letting n tend to infinity, $(\beta')^n$ vanishes, and we conclude that $1 - \alpha' \leq \rho' \leq \alpha'$.

Sufficiency. Suppose that (α, ρ) satisfies (i) and (ii). According to Remark 1.3, we may assume here again that $\alpha' > 1$, so $\rho' + \alpha' - 1 \in [0, 2\alpha' - 1]$. Since α is a Sturm number, there exists a primitive substitution σ such that $s_{\alpha,\alpha}$ is a fixed point of σ (Th. 1.1). Let β be the maximal eigenvalue of the incidence matrix M_σ. We may assume that $\det M_\sigma = 1$, otherwise we consider σ^2 instead of σ.

Obviously $\alpha \in \mathbb{Q}(\beta)$. Condition (i) implies that $\rho \in \mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Hence $\rho + \alpha - 1 \in \mathbb{Q}(\beta)$, so by Lemma 6.1, there exist an even number n and $g \in \mathbb{Z}[\alpha]$ such that

$$\rho + \alpha - 1 = \frac{g}{1 - \beta^n}\cdot$$

Let us prove that g is actually an element of G. Assumptions $\alpha' > 1$ and $1 - \alpha' \leq \rho' \leq \alpha'$ imply that $0 \leq \rho' + \alpha' - 1 \leq 2\alpha' - 1$. Now $0 < 1 - (\beta')^n < 1$ since n is even. Hence

$$g' = (\rho' + \alpha' - 1)(1 - (\beta')^n) \in [0, 2\alpha' - 1)$$

so $g \in G$ by Theorem 5.1. We thus have proven that $\rho + \alpha - 1 \in \frac{G}{1-\beta^n}$. This together with $0 \leq \rho \leq 1$ implies that $s_{\alpha,\rho}$ is substitution invariant (by Th. 4.3). $\square$

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