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ON THE HIERARCHIES OF Δ_2^0-REAL NUMBERS *

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Abstract. A real number x is called Δ_2^0 if its binary expansion corresponds to a Δ_2^0-set of natural numbers. Such reals are just the limits of computable sequences of rational numbers and hence also called computably approximable. Depending on how fast the sequences converge, Δ_2^0-reals have different levels of effectiveness. This leads to various hierarchies of Δ_2^0 reals. In this survey paper we summarize several recent developments related to such kind of hierarchies shown by the author and his collaborators.

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1. Introduction

We consider only the reals of the unit interval $[0, 1]$ in this paper except explicitly stated otherwise. Thus, each real x corresponds naturally to a set A of natural numbers such that $x = x_A := \sum_{n\in A} 2^{-(n+1)}$. In this paper we identify a set with its characteristic sequence, *i.e.*, $n \in A \iff A(n) = 1$ and $n \notin A \iff A(n) = 0$ for all n. Thus the real x_A of binary expansion A can also be denoted by $0.A$. In this way, the computability of subsets of natural numbers can be transferred straightforwardly to reals. For example, according to Turing [25], a real x is *computable* if there is an effective procedure to write down its binary expansion one bit after another[2]. In other words, x has a computable binary expansion in the sense that $x = x_A$ for a computable $A \subseteq \mathbb{N}$. This definition is robust because Robinson [18] and others (see [6, 10]) have shown that, if we define the

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[2] Turing's original definition in [25] uses the decimal expansion instead of binary expansion and he suggested also the Cauchy sequence representation in the addendum [26]. But they are obviously equivalent.

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computability of reals by means of Dedekind cuts or Cauchy representations, we achieve the same notion. More precisely, x is computable iff it has a computable Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$; and iff there is a computable sequence (x_s) of rational numbers which converges to x *effectively* in the sense that

$$(\forall n \in \mathbb{N})(|x_n - x_{n+1}| \leq 2^{-n}). \tag{1}$$

The class of computable reals is denoted by **EC** (for **E**ffctively **C**omputable). Actually, a reasonable effectivization of any representation of reals leads to the same computability notion. This means that the computability of reals is independent of their representations. By a simple relativization we can easily show that the notion of Turing reducibility of reals is independent of their representations too (see, *e.g.*, [5]). Here the Turing reducibility of reals is defined as follows: x_A is *Turing reducible to* x_B (denoted by $x_A \leq_T x_B$) iff $A \leq_T B$. Two reals x, y are *Turing equivalent* (denoted by $x \equiv_T y$) if $x \leq_T y$ and $y \leq_T x$. The Turing degree $\deg_T(x)$ of a real x is defined as the class of all reals which are Turing equivalent to x, *i.e.*, $\deg_T(x) := \{y \in \mathbb{R} : y \equiv_T x\}$. Because of the correspondence between reals and subsets of natural numbers, we can identify the Turing degree $\deg_T(x_A)$ of a real x_A and the Turing degree $\deg_T(A) := \{B \subseteq \mathbb{N} : A \equiv_T B\}$ of the set $A \subseteq \mathbb{N}$. Thus, we can say that a degree of real is c.e. if it contains at least a c.e. set.

Unfortunately, the nice story of independence has to stop here. For stronger computability[3] the representation does play a critical role. Specker shows for example in [24] that, the primitive recursiveness of reals based on Dedekind cuts is strictly stronger than that based on binary expansion which is again stronger than one defined based on the representation of Cauchy sequences. For polynomial time computability, Ko [8] shows that, binary expansions and Dedekind cuts lead to the same notion of polynomial time computability of reals, but they are strictly stronger than that of Cauchy representation. Moreover, only the class of polynomial time computable reals defined based on Cauchy sequence representation is closed under arithmetical operations.

The situation for weak computability of reals is quite similar. For instance, we can consider the following three versions of "computably enumerable" reals: $\mathbf{C}_1$ consists of all reals x_A for c.e. sets A; $\mathbf{C}_2$ contains all reals x of c.e. left Dedekind cuts L_x; and $\mathbf{C}_3$ is the class of limits of computable sequences of rational numbers (which form, of course, c.e. sets of rational numbers). These notions are not equivalent and we have actually $\mathbf{C}_1 \subsetneq \mathbf{C}_2 \subsetneq \mathbf{C}_3$. As a result, both strong and weak computability of reals depend on their representations and in general the notion corresponding to Cauchy sequence representation has also nice analytical properties. In the preceding example, only the class $\mathbf{C}_3$ is an algebraic field. The elements of $\mathbf{C}_3$ are naturally called *computably approximable* (c.a., for short) and the class $\mathbf{C}_3$ is hereafter denoted by **CA**. Computably approximable reals is a very

[3]For two properties α and β, we say that α is stronger than β if the set of objects of the property α is a proper subset of that of β. A notion which is stronger (weaker) than normal computability is called stronger (weaker) computability.

interesting class which shares a lot of properties with the class of computable reals. Ho [7] shows that a real is computably approximable iff it is $\mathbf{0}'$-computable, *i.e.*, there exists a $\mathbf{0}'$-computable sequence (x_s) of rational numbers which converges to x effectively in the sense of (1). Thus, a computably approximable real has a Δ_2^0 binary expansion and hence it is also called a Δ_2^0-real. Specker [24] was the first to show that **CA** is different from **EC**. Since then, a lot of classes of reals between **EC** and **CA** have been introduced and investigated. They are closely related to various hierarchies of Δ_2^0-sets (*cf.* [12,13,22]). Especially, since c.a. reals are limits of computable sequences of rational numbers, we can introduce various subclasses of **CA** by add some extra conditions on the convergence. This leads to a lot of hierarchies of Δ_2^0-reals which classify computability levels of reals. In the following, we summarize some of such kind of hierarchies which are recently introduced by the author and his collaborators.

2. A finite hierarchy

Let's begin with a finite hierarchy of Δ_2^0-reals in this section. As mentioned, the class **EC** of computable reals is the first and the smallest subclass of Δ_2^0-real which we are really interested in. If we consider the increasing and decreasing monotone instead of effectively converging computable sequences of rational numbers, we obtain the classes of c.e. and co-c.e., respectively, reals which both extend the class **EC**. As arithmetical closure of c.e. reals we obtain the class of d-c.e. reals which can be further extended to its closure (the class of dbc reals) under computable total real functions. All these classes are defined in purely analytical way and they form a finite hierarchy of **CA**.

2.1. Computably enumerable reals

In computability theory, the recursive enumerability (r.e.) or, recently more popular after Soare [19], computable enumerability (c.e.) of sets is one of the most important notion besides computability. For real numbers, there are two straightforward ways to define their "computable enumerability": A real x is called *binary c.e.* or *Dedekind c.e.* if $x = x_A$ for a c.e. set A or if its Dedekind cut L_x is a c.e. set of rational numbers[4], respectively. Obviously, a real x is Dedekind c.e. iff there is an increasing computable sequence (x_s) of rational numbers which converges to x. In addition, it is easy to see that any binary c.e. real is also Dedekind c.e. But the converse does not hold in general as observed by C. G. Jockusch (see [20]). For example, if A is a non-computable c.e. set and $B := A \oplus \overline{A}$, then the real x_B is Dedekind c.e. but not binary c.e., where $A \oplus B := \{2n : n \in A\} \cup \{2n+1 : n \in B\}$ is the *join* of sets A and B. Actually, if (A_s) is a computable enumeration of A, *i.e.*, (A_s) is a computable sequence of finite sets such that $A_0 = \emptyset$, $A_s \subseteq A_{s+1}$ for all s and $\lim_s A_s = A$, then $(x_{A_s \oplus \overline{A}_s})$ is an increasing computable sequence of

[4]We use rational numbers $\mathbb{Q}$ and dyadic rational numbers $\mathbb{D}$ exchangeably. They are equivalent for the computability consideration.

rational numbers which converges to $x_{A\oplus\overline{A}}$. These and other observations (see *e.g.* [20, 21]) show that the computable enumerability of reals according to Dedekind cuts is more proper than that according to binary expansion and hence we have the following definition.

Definition 2.1. A real x is c.e. (co-c.e.) if there is an increasing (decreasing) computable sequence (x_s) of rational numbers which converges to x. The classes of c.e. and co-c.e. reals are denoted by **CE** and co-**CE**.

C.e. and co-c.e. reals are also called *left* and *right computable* because they can be approximated from the left and right side in the real axis, respectively. Left and right computable reals together are called *semi-computable* and the class of all semi-computable reals is denoted by **SC**. A real x is semi-computable iff there is a computable sequence (x_s) of rational numbers which converges to x *1-monotonically* in the sense that $|x - x_t| \leq |x - x_s|$ for all $t > s$ (see [1, 27]).

The computable enumerability of a real is different from the computable enumerability of its binary expansion. It is, however, equivalent to the strongly ω-computable enumerability of its binary expansion as shown by Calude, Hertling, Khoussainov and Wang [2]. Let's explain the notion of strongly ω-c.e. now. According to Ershov [9], a set $A \subseteq \mathbb{N}$ is *h-c.e.* for a function $h : \mathbb{N} \to \mathbb{N}$ if there is a computable sequence (A_s) of finite sets which converges to A such that $A_0 = \emptyset$ and $|\{s \in \mathbb{N} : A_s(n) \neq A_{s+1}(n)\}| \leq h(n)$ for all $n \in \mathbb{N}$. Ershov shows that (so called Ershov's hierarchy theorem), if $f(n) < h(n)$ hold for infinitely many n, then there is an h-c.e. set which is not f-c.e. For any constant $k \in \mathbb{N}$, A is k-c.e. if it is h-c.e. for the constant function $h := \lambda n.k$, and A is ω-c.e. if it is h-c.e. for a computable function h. Obviously, 1-c.e. sets are just c.e. sets and the 2-c.e. sets are usually called *d-c.e.* (standing for difference of c.e. sets) because for any 2-c.e. set A there exist c.e. sets B, C such that $A = B \setminus C$. Thus, an ω-c.e. set can be constructed in such a way that, for any $n \in \mathbb{N}$, the mind-changes for n (*i.e.*, the change of the value $A_s(n)$) is bounded by a computable function. As a variant of ω-c.e., A is called *strongly ω-c.e.* if there is a computable sequence (A_s) of finite sets which converges to A such that

$$(\forall n \forall s)(n \in A_s - A_{s+1} \Longrightarrow (\exists m < n)(m \in A_{s+1} - A_s)). \tag{2}$$

That is, whenever some number n leaves A, some smaller number m has to enter A at the same time according to the enumeration (A_s). Thus, any strongly ω-c.e. set is h-c.e. for $h := \lambda n.2^n$, and, by Ershov's hierarchy theorem, not every ω-c.e. set is strongly ω-c.e. One of the most important property of strongly ω-c.e. sets is the binary characterization of c.e. reals.

Theorem 2.2 (Calude, Hertling, Khoussainov and Wang [2]). *A real x is c.e. if and only if $x = x_A$ for a strongly ω-c.e. set A.*

The binary expansion leads naturally to an infinite hierarchy of c.e. reals. Soare [20] called a c.e. real x *stably c.e.*[5] if its binary expansion is d-c.e. For example, the real of Jockusch's example is a stably c.e. Since there exists a strongly ω-c.e. set which is not d-c.e., the class of all stably c.e. reals is strictly between the classes of binary c.e. and c.e. reals. There is no reason to stop here. In general, a c.e. real is called *h-stably c.e.* for a function h if its binary expansion is an h-c.e. set. Thus, the k-stably c.e., for constant $k \in \mathbb{N}$, and the ω-stably c.e. reals can be defined accordingly [27]. By Theorem 2.2, the classes of h-stably c.e. reals collapse to the level $\lambda n.2^n$-stably c.e. for $h(n) \geq 2^n$ for all n. For lower levels, however, we have a proper hierarchy.

Theorem 2.3 (Weihrauch and Zheng [27])**.** *For any constant k, there is a $(k+1)$-stably c.e. real which is not k-stably c.e. and there exists an ω-stably c.e. real which is not k-stably c.e. for any $k \in \mathbb{N}$.*

Additionally, Downey [4] calls a real *strongly c.e.* if its binary expansion is c.e. Moreover, Wu [28] calls a real *k-strongly c.e.* if it is the sum of up to k strongly c.e. reals. All k-strongly c.e. reals are called *regular.* Wu shows that, for any $k \in \mathbb{N}$, a k-strongly c.e. real is $2k$-stably c.e. and there is a $(k+1)$-strongly c.e. real which is not k-stably c.e. This, together with Theorem 2.3, implies that, for any $k \in \mathbb{N}$, there is a $(k+1)$-strongly c.e. real which is not k-strongly c.e. and there exists a c.e. real which is not regular.

Besides $\lambda n.2^n$-stably computable enumerability, there is another very useful necessary condition of semi-computability as follows.

Theorem 2.4 (Ambos-Spies, Weihrauch and Zheng [1])**.** *If $A, B \subseteq \mathbb{N}$ are Turing incomparable c.e. sets, then the real $x_{A\oplus\overline{B}}$ is not semi-computable.*

In particular, the previous theorem implies that any non-computable c.e. degree contains a non-semi-computable real and the class of c.e. reals is not closed under subtraction.

2.2. Difference of c.e. reals

The classes of c.e. and semi-computable reals are introduced naturally by the monotonicity of sequences and have a lot of nice computability-theoretical properties. However, neither of them have nice analytical property. For example, they are not closed under subtraction. This motivates us to explore their arithmetical closure and leads to the following definition.

Definition 2.5. A real x is called *d-c.e.* (difference of c.e.) if there are c.e. reals y, z such that $x = y - z$. The class of all d-c.e. reals is denoted by **DCE**.

By Theorem 2.4, the class **DCE** is a proper superset of **CE** and **SC**, because $x_{A\oplus\overline{B}} = x_{2A} - x_{2B+1}$ is d-c.e. but not semi-computable if A and B are Turing

[5] Of course, Soare [20] uses the term "stably r.e." In this paper we always use "computable" instead of "recursive" after the suggestion of Soare in [19].

incomparable c.e. sets. But the difference hierarchy collapses since **DCE** is obviously closed under addition and subtraction. Moreover, the class **DCE** is also closed under multiplication and division and hence is a filed. This follows from another nice characterization of d-c.e. reals as follows.

Theorem 2.6 (Ambos-Spies, Weihrauch and Zheng [1]). *A real x is d-c.e. iff there is a computable sequence (x_s) of rational numbers which converges to x* weakly effectively *in the sense that the sum $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}|$ is finite.*

Because of Theorem 2.6, d-c.e. reals are also called *weakly computable* in literatures [1,27,29]. Now it is easy to see that **DCE** is closed under arithmetical operations $+, -, \times, \div$ and hence it is the arithmetical closure of **CE** (and of co-**CE**). Recently, Raichev [14] and, independently, Ng [11] show that **DCE** is actually a real closed field.

There is another very interesting characterization of d-c.e. reals related to the Solovay reduction which classifies the relative randomness of the real numbers. A real x is Solovay reducible to y if there exist two computable sequences (x_s) and (y_s) of rational numbers converging to x and y, respectively, such that $|x - x_s| \leq c(|y - y_s| + 2^{-s})$ for some constant c and all s (see [23, 33]). Solovay [23] shows that any c.e. real is Solovay reducible to a c.e. random real. For d-c.e. reals we have the following result.

Theorem 2.7 (Rettinger and Zheng [16]). *A real number is d-c.e. if and only if it is Solovay reducible to a c.e. random real.*

The binary expansion of a c.e real can only be up to $\lambda n.2^n$-c.e. in Ershov's hierarchy. However, the d-c.e. real can have much more complicated binary expansion even beyond the ω-c.e. sets as the next result showed.

Theorem 2.8 (Zheng [31]). *There are two (strongly) c.e. reals y and z such that the difference $x := y - z$ does not have an ω-c.e. Turing degree. That is, there exists a d-c.e. real which has a non-ω-c.e. Turing degree.*

More recently, the following results about the Turing degrees of d-c.e. reals are shown.

Theorem 2.9 (Downey, Wu and Zheng [3]).

(1) *Every ω-c.e. Turing degree contains a d-c.e. real; and*
(2) *there exists a Δ^0_2-Turing degree which does not contain any d-c.e. reals.*

Thus, the class of Turing degrees of d-c.e. reals contains all ω-c.e. degrees and some (but not all) non-ω-c.e. degrees. Theorem 2.9 is proved by an interesting finite priority construction using double witnesses technique. The main idea will be explained in the proof of Theorem 2.13 which extends Theorem 2.9 to a larger class.

We close our discussion about d-c.e. reals with an interesting necessary condition shown by Ambos-Spies, Weihrauch and Zheng [1] as follows.

Theorem 2.10 (Ambos-Spies, Weihrauch and Zheng [1]). *For any set A, if x_{2A} is a d-c.e. real, then A is h-c.e. for $h = \lambda n.2^{3n}$.*

By Ershov's hierarchy theorem, there exists a Δ^0_2-set A which is not $\lambda n.2^{3n}$-c.e. and hence x_{2A} is not a d-c.e. real. This implies immediately that $\mathbf{DCE} \subsetneq \mathbf{CA}$.

2.3. Divergence bounded computable reals

The class **DCE** is an arithmetical closure of **CE** and has nice arithmetical properties. However, it is not closed under total computable real functions (see [35]). This leads to another class of reals which extends **DCE** properly. To understand better the computability contents of these reals, we introduce first a new class as follows.

Definition 2.11 (Rettinger *et al.* [17])**.** A real x is called dbc (*divergence bounded computable*) if there is a computable total function h and a computable sequence (x_s) of rational numbers which converges to x *h-bounded effectively* in the sense that there are at most $h(n)$ non-overlapping index-pairs (i, j) with $|x_i - x_j| \geq 2^{-n}$.

The pair (i, j) with $|x_i - x_j| \geq 2^{-n}$ is called a 2^{-n}-jump. For any 2^{-n}-jump (i, j), the binary expansions of x_i and x_j differ at their first n-positions. As a result, any real number of an ω-c.e. binary expansion is dbc. The class of all dbc reals is denoted by **DBC**. Surprisingly, the class **DBC** is just the closure of c.e. reals (and hence of d-c.e. reals) under total computable real functions.

Theorem 2.12 (Rettinger *et al.* [17])**.** *A real x is dbc iff there is a d-c.e. real y and a total computable real function f such that $x = f(y)$.*

Notice that, if A is an ω-c.e. but not $\lambda n.2^{3n}$-c.e. set, then x_{2A} is divergence bounded computable (because $2A$ is ω-c.e. too) but not d-c.e. by Theorem 2.10. This implies that **DBC** properly extends the class **DCE**. On the other hand, by a diagonalization argument we can show that there is a Δ^0_2-real which is not dbc. The next theorem shows that, even the Turing degrees of dbc reals do not exhaust all Δ^0_2-Turing degrees and this extends Theorem 2.9.

Theorem 2.13 (Zheng and Rettinger [32])**.** *There is a Δ^0_2-Turing degree which does not contain any divergence bounded computable reals.*

Proof. We construct a computable sequence (A_s) of finite subsets of natural numbers which converges to A such that A is not Turing equivalent to any divergence bounded computable real. To this end, the set A has to satisfy all the following requirements.

$$R_e: \quad \left.\begin{array}{l} b_e \text{ and } h_e \text{ are total functions and } (b_e(s))_s \\ \text{converges } h_e\text{-bounded effectively to } x_{B_e} \end{array}\right\} \Longrightarrow A \neq \Phi_e^{B_e} \vee B_e \neq \Psi^A,$$

where $(b_e, h_e, \Phi_e, \Psi_e)$ is an effective enumeration of all tuples of computable partial functions $b_e :\subseteq \mathbb{N} \to \mathbb{D}$, $h_e :\subseteq \mathbb{N} \to \mathbb{N}$, and computable functionals Φ_e, Ψ_e. In this proof, we use the corresponding lower case letters φ_e and ψ_e to denote the use functions of Φ_e and Ψ_e, which record the largest oracle queries during the computations Φ_e^B and Ψ_e^A, respectively. For any $e, s \in \mathbb{N}$, if $b_e(s)$ is defined, then let $B_{e,s}$ be the finite set of natural numbers such that $b_e(s) = x_{B_{e,s}}$. We define the length function l as follows:

$$l(e,s) \quad := \quad \max\{x : A_s \upharpoonright x = \Phi_{e,s}^{B_{e,s}} \upharpoonright x \ \& \ B_{e,s} \upharpoonright \varphi_{e,s}(x) = \Psi_{e,s}^{A_s} \upharpoonright \varphi_{e,s}(x)\}.$$

Thus, to satisfy R_e, it suffices to guarantee that $l(e,s)$ is bounded above, if the premisses of R_e hold.

To satisfy R_e, we choose a witness n_e large enough and let $A(n_e-1)A(n_e) = 00$ at the beginning. Then wait for a stage s such that $l(e,s) > n_e$. If this does not happen, then we are done. Otherwise, let s_1 be the first stage such that $l(e,s_1) > n_e$ and let $m_e := \psi_{e,s_1}(\varphi_{e,s_1}(n_e))$. Assume w.l.o.g. that $n_e < m_e$. If $h_{e,s_1}(m_e)$ is also defined, then we put $n_e - 1$ into A to destroy the agreement and wait for a new stage $s_2 > s_1$ such that $l(e,s_2) > n_e$ holds again. If no such stage exists, then we are done again. Otherwise, we put n_e into A too. If there exists another stage $s_3 > s_2$ such that $l(e,s_3) > n_e$, then we delete both $n_e - 1$ and n_e from A. In this case, the set A_{s_3+1} is recovered to that of stage s_1, *i.e.*, $A_{s_3+1} = A_{s_1}$. This closes a circle in which the values $A(n_e-1)A(n_2)$ changes in the order $00 \to 10 \to 11 \to 00$. This process will continue as long as the number of 2^{-m_e}-jumps of the sequence $(x_{B_{e,s}})$ does not exceed $h_e(m_e)$.

In this way, we achieve a temporary disagreement between A and $\Phi_e^{B_e}$ by change the values $A(n_e-1)A(n_e)$ whenever the length of agreement goes beyond the witness n_e. After that, if the agreement becomes bigger than n_e again, then the corresponding value $\Phi_e^{B_e}(n_e-1)\Phi_e^{B_e}(n_e)$ has to be changed too and this forces the initial segment $B_e \upharpoonright \varphi_e(n_e)$ to be changed, say, $B_{e,s} \upharpoonright \varphi_{e,s}(n_e) \neq B_{e,t} \upharpoonright \varphi_{e,t}(n_e)$. If $|x_{B_{e,s}} - x_{B_{e,t}}| \geq 2^{-m_e}$, then (s,t) is a 2^{-m_e}-jump and this can happen at most $h_e(m_e)$ times if the sequence $(b_e(s))$ converges h_e-bounded effectively.

On the other hand, if $|x_{B_{e,s}} - x_{B_{e,t}}| = 2^{-m} < 2^{-m_e}$ for a natural number $m > m_e$. Then there exists a (least) natural number $n < m_e$ such that $B_{e,s}(n) \neq B_{e,t}(n)$ because $B_{e,s} \upharpoonright m_e \neq B_{e,t} \upharpoonright m_e$ (remember that $m_e \geq \varphi_e(n_e)$). This implies that, $B_{e,s} = 0.w10^kv$ or $B_{e,s} = 0.w01^kv$ for some $w, v \in \{0,1\}^*$ and $k := m - n$. Correspondingly, the combination $\Phi_{e,s}^{B_{e,s}}(n_e-1)\Phi_{e,s}^{B_{e,s}}(n_e)$ can have at most two possibilities too. However, in every circle described above, $A(n_e-1)A(n_e)$ takes three different forms, *i.e.*, $00, 10$ and 11. In other words, we can always achieve a disagreement $A \neq \Phi_e^B$ at some stages and hence the requirement R_e is satisfied eventually.

To satisfy all requirements simultaneously, we apply a finite injury priority construction. In this case, only elements larger than $\psi_e\varphi_e(n_e)$ are allowed to be appointed as witnesses of R_i for $i > e$ to preserve R_e from disturbance by lower priority R_i. □

The relationship among the classes discussed cab be summarized by the following diagram

$$\mathbf{EC} \subsetneq \begin{array}{c}\mathbf{CE}\\ \mathbf{co\text{-}CE}\end{array} \subsetneq \mathbf{SC} \subsetneq \mathbf{DCE} \subsetneq \mathbf{DBC} \subsetneq \mathbf{CA}.$$

They form a proper finite hierarchy of Δ^0_2-reals. The classes $\mathbf{EC}, \mathbf{DCE}, \mathbf{DBC}$ and $\mathbf{CA}$ are closed fields.

3. Ershov's hierarchy

Ershov's hierarchy [9] of Δ^0_2-subsets of natural numbers can be transferred to reals straightforwardly if any real number is appointed to a set. Thus, by means of binary expansion and Dedekind cut representations, we can introduce two hierarchies of reals of Ershov's type.

3.1. Binary computability

Since any real x corresponds naturally to its binary expansion set A in the sense that $x = x_A$, the Ershov's hierarchy on sets can be transferred to the Δ^0_2-reals straightforwardly as follows.

Definition 3.1 (Zheng and Rettinger [34]). Let h be a function. A real x is *h-binary computable* if $x = x_A$ for an h-c.e. set A.

The k-binary computable for any constant k and ω-binary computable reals are defined accordingly. Let k-b$\mathbf{EC}$ (for $k \in \mathbb{N}$), ω-b$\mathbf{EC}$ and h-b$\mathbf{EC}$ denote the classes of all k-, ω- and h-binary computable reals, respectively. In addition, the class $\bigcup_{k\in\mathbb{N}} k$-b$\mathbf{EC}$ is denoted by $*$-b$\mathbf{EC}$. By Ershov's hierarchy theorem, we have an infinite hierarchy k-b$\mathbf{EC} \subsetneq (k+1)$-b$\mathbf{EC} \subsetneq *$-b$\mathbf{EC} \subsetneq \omega$-b$\mathbf{EC} \subsetneq \mathbf{CA}$ for all constant k. Obviously, 1-b$\mathbf{EC}$ is the class of strongly c.e. reals and hence 1-b$\mathbf{EC} \subsetneq \mathbf{CE}$. Furthermore, we have

Theorem 3.2 (Zheng and Rettinger [34]).

(1) *k-b$\mathbf{EC} \nsubseteq \mathbf{SC}$ for $k \geq 2$;*
(2) *$\mathbf{CE} \nsubseteq *$-b$\mathbf{EC}$ and $*$-b$\mathbf{EC} \subsetneq \mathbf{DCE}$;*
(3) *ω-b$\mathbf{EC}$ is incomparable with $\mathbf{DCE}$.*

Proof.

1. Let A, B be two Turing incomparable c.e. sets. Then the join $A \oplus \overline{B}$ is obviously a 2-c.e. set and hence the real $x_{A\oplus\overline{B}}$ is 2-binary computable. But it is not semi-computable because of Theorem 2.4. This means that 2-b$\mathbf{EC} \nsubseteq \mathbf{SC}$ and hence k-b$\mathbf{EC} \nsubseteq \mathbf{SC}$ for all $k \geq 2$.

2. By Theorem 2.3, there exists an ω-stably c.e. real x which is not k-stably c.e. for any $k \in \mathbb{N}$. Thus, x is c.e. but not k-binary computable and hence $\mathbf{CE} \nsubseteq *$-b$\mathbf{EC}$.

For the second part of item 2, it suffices to prove the inclusion part. Assume by induction hypothesis that k-b$\mathbf{EC} \subseteq \mathbf{DCE}$ for some $k \in \mathbb{N}$. Let $A \subseteq \mathbb{N}$ be a $(k+1)$-c.e. set. Then there exist a c.e. set B and a k-c.e. set C such that $A = B \setminus C$. Obviously, the set $B \cup C$ is k-c.e. too. Then, both $x_{B\cup C}$ and x_C are k-binary computable and hence are d-c.e. by induction hypothesis, *i.e.*, $x_{B\cup C}, x_C \in \mathbf{DCE}$. Since the class $\mathbf{DCE}$ is closed under subtraction and $x_A = x_{B\setminus C} = x_{(B\cup C)\setminus C} = x_{B\cup C} - x_C$, x_A is d-c.e. too. Therefore $(k+1)$-b$\mathbf{EC} \subseteq \mathbf{DCE}$.

3. **DCE** $\not\subseteq$ ω-b**EC** follows from Theorem 2.8. To prove ω-b**EC** $\not\subseteq$ **DCE**, we can choose by Ershov's hierarchy theorem an ω-c.e. set A which is not $\lambda n.2^{3n}$-c.e. Then the set $2A$ is obviously also an ω-c.e. set and hence x_{2A} is ω-binary computable. However x_{2A} is not d-c.e. by Theorem 2.10. □

Thus, any class k-b**EC**, for $k \geq 2$, is incomparable with the classes of **CE** and **SC**. Moreover, since ω-b**EC** contains **CE** but not its arithmetical closure **DCE**, ω-b**EC** is not closed under addition and subtraction. It is also not difficult to see that, all classes k-b**EC** for $k > 0$ are not closed under addition and subtraction too.

3.2. Dedekind computability

The Ershov's hierarchy of Δ_2^0-subsets of natural numbers can be easily extended to the subsets of dyadic rational numbers. Of course, we have to consider functions $h : \mathbb{D} \to \mathbb{N}$ for the h-c.e. sets of dyadic rational numbers. Thus, the Ershov's hierarchy can be transferred directly to reals by means of Dedekind cuts as follows.

Definition 3.3 (Zheng and Rettinger [34])**.** Let $h : \mathbb{D} \to \mathbb{N}$ be a function. A real x is called *h-Dedekind computable* if its Dedekind cut L_x is h-c.e.

The k- (for $k \in \mathbb{N}$), $*$- and ω-Dedekind computability are defined accordingly. The classes of h-, k- $*$- and ω-Dedekind computable reals are denote by h-d**EC**, k-d**EC**, $*$-d**EC** and ω-d**EC**, respectively. By definition, the class 1-d**EC** is equal to **CE** and any semi-computable real is obviously 2-Dedekind computable. However, other classes k-d**EC** for any $k \geq 2$ collapse to the second level 2-d**EC**.

Theorem 3.4 (Zheng and Rettinger [34])**.**

(1) k-d**EC** = **SC** *for* $k \geq 2$*;*
(2) ω-b**EC** = ω-d**EC**.

Proof.

1. It suffices to prove that $*$-d**EC** $\subseteq$ **SC**. Let $x \in *$-d**EC** and $k := \min\{n : x \in n\text{-d}\mathbf{EC}\}$. If $k < 2$, then we are done. Suppose now that $k \geq 2$. By choice of k, the Dedekind cut $L_x := \{r \in \mathbb{D} : r < x\}$ is k-c.e. but not $(k-1)$-c.e. Let (A_s) be a computable k-enumeration of L_x. Then there are infinitely many $r \in \mathbb{D}$ such that $|\{s \in \mathbb{N} : r \in A_{s+1} \Delta A_s\}| = k$ where Δ is the symmetric set difference defined by $A \Delta B := (A \setminus B) \cup (B \setminus A)$. Then the set $O_k := \{r \in \mathbb{D} : |\{s \in \mathbb{N} : r \in A_{s+1} \Delta A_s\}| = k\}$ is an infinite c.e. set. If k is even, then $r \notin L_x$ for any $r \in O_k$ (remember $A_0 = \emptyset$) and hence $x \leq r$. Then we can show that $\inf O_k = x$ holds and hence x is co-c.e. Similarly, if k is odd, then x is c.e. Therefore, $*$-d**EC** $\subseteq$ **SC**.

2. "ω-b**EC** $\subseteq$ ω-d**EC**": Let $x_A \in \omega$-b**EC**. There is a computable function h and a computable h-enumeration (A_s) of A. We are going to show that $x_A \in \omega$-d**EC**. To this end, let $E_s := \{r \in \mathbb{D}_s : r \leq x_{A_s}\}$ for all s, where $\mathbb{D}_s := \{n \cdot 2^{-s} : n \in \mathbb{Z}\}$ is the set of all dyadic rational numbers of precision s. We identify a dyadic rational number r with a binary word in the sense that $r = \sum_{i<l(r)} r(i) \cdot 2^{-(i+1)}$ where $l(r)$ is the length of the word r and identify a set A with its characteristic

sequence. Then $r < x_A$, iff $r <_L A$, iff $(\forall^\infty s)(r \leq_L A_s)$, iff $(\forall^\infty s)(r \in E_s)$, iff $r \in E := \lim_{s\to\infty} E_s$, where $<_L$ is the length-lexicographical ordering of binary words and sequences. Thus, the limit $E := \lim_s E_s$ is the left Dedekind cut of the real x_A. On the other hand, for any $r \in \mathbb{D}$ and any $s, t \in \mathbb{N}$, if $A_s \upharpoonright l(r) = A_t \upharpoonright l(r)$, then $r \leq_L x_{A_s} \iff r \leq_L x_{A_t}$ and hence $r \in E_s \iff r \in E_t$. That is, if the membership of r to E_s is changed, there must be a numbers $n < l(r)$ such that $A_s(n)$ changes. Since (A_s) is a computable h-enumeration of A, the sequence (E_s) is a computable g-enumeration of E, where $g : \mathbb{D} \to \mathbb{N}$ is a computable function defined by $g(r) := \sum_{i\leq l(r)} h(i)$. Thus, x is a g-Dedekind computable and hence an ω-Dedekind computable real.

"ω-d**EC** $\subseteq$ ω-b**EC**": Suppose that $x := x_A$ is ω-Dedekind computable. That is, there is a computable function h and a computable sequence (E_s) of finite sets of dyadic rational numbers such that (E_s) is a computable h-enumeration of the left Dedekind cut L_x of x_A. Suppose w.l.o.g. that, for any s, if $\sigma \in E_s$, then $\tau \in E_s$ for any τ such that $l(\tau) \leq l(\sigma)$ and $\tau \leq_L \sigma$. Let r_s be the maximal element of E_s and $A_s := \{n : r_s(n) = 1\}$, *i.e.*, $r_s = x_{A_s}$ for any s. Then, we have $\lim_{s\to\infty} x_{A_s} = \lim_{s\to\infty} r_s = x_A$ and hence $\lim_{s\to\infty} A_s = A$. Since (A_s) is obviously a computable sequence of finite subsets of natural numbers, it suffices to show that there is a computable function g such that (A_s) is a g-enumeration of A. Now we define the computable function $g : \mathbb{N} \to \mathbb{N}$ inductively as follows.

For any $n \in \mathbb{N}$. suppose that $g(m)$ is defined for any $m < n$. To define $g(n)$, let's estimate first how many times $A_s(n)$ can be changed for different s at all. Let σ be a binary word of the length n. If there are $s < t$ such that $A_s \upharpoonright n = A_t \upharpoonright n = \sigma$ and $n \notin A_s$ & $n \in A_t$. Then, we have $r_s = x_{A_s} < \sigma 1$ and $\sigma 1 \leq x_{A_t} = r_t$. Here we regard the binary word $\sigma 1$ as a dyadic rational number. This implies that $\sigma 1 \notin E_s$ and $\sigma 1 \in E_t$ by the choice of the sequence (E_s). Similarly, for the case $n \in A_s$ & $n \notin A_t$, we have $\sigma 1 \in E_s$ and $\sigma 1 \notin E_t$. Since (E_s) is an h-enumeration of L_x, there are at most $h(\sigma 1)$ non-overlapping pairs (s, t). This means that, for any $s < t$, if $n \in A_s \Delta A_t$ and $A_s \upharpoonright n = A_t \upharpoonright n = \sigma$, then $\sigma 1 \in E_s \Delta E_t$. Therefore, $A_s(n)$ can be changed at most $g(n)$ times where $g(n)$ is defined by $g(n) := \sum\{g(m) : m < n\} + \sum\{h(\sigma 1) : \sigma \in \{0,1\}^n\}$. Thus, (A_s) is a computable g-enumeration of A and hence A is an ω-c.e. set, because g is obviously a computable function. That is, x_A is ω-binary computable. □

In summary, the k-binary computability and the k-Dedekind computability are incomparable, for any $k \geq 2$ or $k := *$. But the ω-binary and Dedekind computability are equivalent.

4. Hierarchies based on divergence bounding

In the definitions of binary and Dedekind computability, we looked at the possible changes of bits in binary expansion and the changes of memberships of a dyadic rational numbers to the Dedekind cut, respectively. These reflect somehow the different levels of effectivity of reals. If we consider Cauchy sequence representation, the effectivity levels of reals can be classified according to how fast a real be approximated. A possible way to measure the convergence speed of a sequence is to count its big jump. Depending on two different ways how the big jumps are defined, we introduce in this section two hierarchies. To distinguish them clearly, they are called (without special reason) h-Cauchy computability and h-bounded computability, respectively.

4.1. Cauchy computability

If a sequence (x_s) converges effectively, then $|x_i - x_j| \leq 2^{-n}$ for any $i, j \geq n$. Thus, the pairs (i, j) with $i, j \geq n$ such that $|x_i - x_j| > 2^{-n}$ are the annoying factor which possibly destroy the computability of its limit. Such pairs are called big jumps. It is natural to anticipate, that the less big jumps a sequence possesses, the better computability its limit could have. Since 2^{-n} converges to 0 (for $n \to \infty$), there is no constant upper bound of numbers of jumps (i, j) such that $|x_i - x_j| \geq 2^{-n}$ for all n if the sequence (x_s) is not trivial. However, such kind constant bound seems very important for a hierarchy of Ershov's type. Therefore, we consider only the jumps of sizes between 2^{-n} and 2^{-n+1} firstly.

Definition 4.1 (Zheng and Rettinger [34])**.** Let $i, j, n \in \mathbb{N}$ and let (x_s) be a sequence of reals converging to x and $h : \mathbb{N} \to \mathbb{N}$ be a function.

(1) A pair (i, j) is an *n-jump* of (x_s) if $2^{-n} < |x_i - x_j| \leq 2^{-n+1}$ and $i, j \geq n$;
(2) the sequence (x_s) converges to x *h-effectively* if, for any $n \in \mathbb{N}$, the number of non-overlapping n-jumps of (x_s) is bounded by $h(n)$;
(3) the real x is *h-Cauchy computable* if there is a computable sequence (x_s) of rational numbers which converges to x h-effectively.

Other types of Cauchy computability can be defined accordingly and the classes of h-, k- ($k \in \mathbb{N}$), $*$- and ω-Cauchy computable reals are denoted, respectively, by h-c**EC**, k-c**EC**, $*$-c**EC** and ω-c**EC**. By definition, it is easy to see that 0-, and ω-Cauchy computable reals are just the computable and divergence bounded computable reals, respectively, *i.e.*, 0-c**EC** = **EC** and ω-c**EC** = **DBC**. In addition, for any rational number a and any function h, we have $x \in h$-c**EC** iff $a - x \in h$-c**EC**.

Now we show a general hierarchy theorem for Cauchy computability.

Theorem 4.2 (Zheng and Rettinger [34])**.** *If $f, g : \mathbb{N} \to \mathbb{N}$ are total computable functions such that $f(n) < g(n)$ for infinitely many n, then g-c**EC** $\nsubseteq f$-c**EC**.*

Proof. We construct a computable sequence (x_s) of rational numbers which converges g-effectively to a non-f-Cauchy computable real x. Thus, x satisfies all requirements R_e: "*if $(\varphi_e(s))_s$ converges f-effectively to y_e, then $x \neq y_e$*", where (φ_e) is an effective enumeration of all computable partial functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$.

The strategy to satisfy a single requirement R_e is quite straightforward. Let $I_e := [a, b]$ be a rational interval with length 2^{-n_e+2} for some $n_e \in \mathbb{N}$ such that $f(n_e) < g(n_e)$. Divide it equally into four subintervals $I_i := [a_i, a_{i+1}]$, for $i < 4$, of length 2^{-n_e}. Let $b_1 := a_0 + 3 \cdot 2^{-(n_e+2)}$ and $b_2 := a_2 + 2^{-(n_e+2)}$. Notice that, $2^{-n_e} < 2^{-n_e} + 2^{-(n_e+1)} = |b_1 - b_2| < 2^{-n_e+1}$. Define $x_s := b_1$ as long as the sequence $(\varphi_e(s))_s$ does not enter the interval I_1. Otherwise, if $\varphi_e(s)$ enters into the interval I_1 for some $s \geq n_e$, then let $x_s := b_2$. Later on, if $\varphi_e(t)$ enters I_3 for some $t > s$, then let $x_t := b_1$ again, and so on. If $(\varphi_e(s))_s$ converges f-effectively, then (x_s) can be changed at most $f(n_e) + 1 \leq g(n_e)$ times. This guarantees that the sequence (x_s) converges g-effectively and $\lim x_s \neq \lim_s \varphi_e(s)$. To satisfy all the requirements simultaneously, we use a finite injury priority construction □

Thus, we have an Ershov-type hierarchy that k-c**EC** $\subsetneq$ $(k+1)$-c**EC** $\subsetneq$ $*$-c**EC** $\subsetneq$ ω-c**EC** for any $k \in \mathbb{N}$. In addition, any k-Cauchy computable real (for $k \in \mathbb{N}$) is d-c.e., because any k-effectively convergent sequence converges weakly effectively too. On the other hand, all classes k-c**EC** $(k > 0)$ are not comparable with **CE** and **SC** according to the following theorem.

Theorem 4.3 (Zheng and Rettinger [34]). *The class k-c**EC** is incomparable with the classes **CE** and **SC** for any $k > 0$.*

Proof. It suffices to show that 1-c**EC** $\nsubseteq$ **SC** and **CE** $\nsubseteq$ $*$-c**EC**.

Let A and B be Turing incomparable c.e. sets and let (A_s) and (B_s) be their computable enumerations, respectively. Suppose w.o.l.g. that $A_0 = B_0 = \emptyset$, $A_{2s} = A_{2s+1}$ & $|B_{2s+1} \setminus B_{2s}| = 1$ and $|A_{2s+2} \setminus A_{2s+1}| = 1$& $B_{2s+1} = B_{2s+2}$ for all s. Let $x_s := x_{A_s \oplus \overline{B}_s}$ for any $s \in \mathbb{N}$. Then (x_s) is a computable sequence of rational numbers converging 1-effectively to $x_{A \oplus \overline{B}}$ which is not semi-computable by Theorem 2.4. That is, $x_{A \oplus \overline{B}} \in$ 1-c**EC** $\setminus$ **SC**.

For **CE** $\nsubseteq$ $*$-c**EC**, we construct an increasing computable sequence (x_s) of rational numbers whose limit $x := \lim_s x_s$ is not k-Cauchy computable for any $k \in \mathbb{N}$. That is, x satisfies, for all $i, j \in \mathbb{N}$, the requirements $R_{\langle i,j \rangle}$: "*if $(\varphi_i(s))_s$ converges j-effectively y_e, then $x \neq y_e$*", where (φ_i) is an effective enumeration of all computable partial functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$.

To satisfy a single requirement R_e for $e = \langle i, j \rangle$, we choose a rational interval $[a; b]$. Let n be the minimal natural number such that $2^n \geq 3(j+1)/(b-a)$. Define $a_i := a + i \cdot 2^{-n}$ for $i \leq 3(j+1)$ and $a_{3(j+1)+1} = b$. Then the intervals $I_i := [a_i; a_{i+1}]$ have length 2^{-n} for any $i < 3(j+1)$. We define x_0 as the middle point of the interval I_1. If $\varphi_i(s)$ enters I_1 for some $s \geq n$, then define x_s as the middle point of the interval I_4. If there is a $t > s$ such that $\varphi_i(t) \in I_4$, then define x_t as the middle point of the interval I_7, and so on. In general, if $x_{s_1} \in I_{3k+1}$ and $\varphi_i(s_2) \in I_{3k+1}$ for some $s_2 > s_1$, then redefine x_{s_2} as the middle point of I_{3k+4}. If $(\varphi_i(s))_s$ converges j-effectively, then we can always find a correct x which differs from the limit $\lim_s \varphi_i(s)$, because $\varphi_i(s_1) \in I_{3k+1}$ and $\varphi_i(s_2) \in I_{3k+4}$ implies that $2^{-n+1} \leq |\varphi_i(s_1) - \varphi_i(s_2)| \leq 2^{-n+2}$. To satisfy all requirements, it succeeds to apply the above strategy to an interval tree and use the finite injury priority construction. We omit the details here. □

We have seen that both classes 0-c**EC** and ω-c**EC** are fields. However, the next theorem shows that all other classes of Cauchy computable reals are not closed under addition and subtraction.

Theorem 4.4 (Zheng and Rettinger [34]). *There are $x, y \in$ 1-c**EC** such that $x - y \notin *$-c**EC**. Therefore, k-c**EC** and $*$-c**EC** are not closed under addition and subtraction for any $k > 0$.*

Proof. We construct two computable increasing sequences (x_s) and (y_s) of rational numbers which converge 1-effectively to x and y, respectively, while their difference $z := x - y$ is not $*$-cEC. That is, z satisfies all requirements $R_{\langle i,j\rangle}$: "*if* $(\varphi_i(s))_s$ *converges* j*-effectively to* y_i*, then* $y_i \neq z$", where (φ_i) is an effective enumeration of all partial computable functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$.

To satisfy a single requirement R_e for $e := \langle i, j\rangle$, we choose two natural numbers n_e and m_e large enough such that $n_e < m_e$ and $m_e - n_e \geq j + 3$. Let's begin with $x_s = y_s = z_s = 0$. Whenever $|z_s - \varphi_i(t)| < 2^{-(m_e+2)}$ (for $z_s = x_s - y_s$) holds for some $t \geq m_2$, we increase (x_s, y_s) by $(2^{-(n_e+1)} + 3 \cdot 2^{-(m_e+1)}, 2^{-(n_e+1)})$ or $(2^{-(n_e+1)}, 2^{-(n_e+1)} + 3 \cdot 2^{-(m_e+1)})$ alternatively so that $|z_s - z_{s+1}| = 3/2 \cdot 2^{-m_e}$. We need at most $j + 1$ times of such action to satisfy R_e, if $(\varphi_i(s))$ converges j-effectively. □

Similar to the binary computability, Cauchy computability leads also to an infinite hierarchy by Theorem 4.2. However, they are not equivalent in almost all levels.

Theorem 4.5 (Zheng and Rettinger [34])**.**

(1) ω-b**EC** $\subsetneq$ ω-c**EC**; *and*
(2) k-b**EC** $\subsetneq$ k-c**EC** *for any* $k \geq 1$ *or* $k = *$.

Proof.
1. Since ω-c**EC** is closed under arithmetical operations but ω-b**EC** not, it suffices to prove the inclusion ω-b**EC** $\subseteq$ ω-c**EC**. For any $x_A \in \omega$-b**EC**, there is a computable function h and a computable h-enumeration (A_s) of A. Notice that, if $A_s \upharpoonright n = A_t \upharpoonright n$, then $|x_{A_s} - x_{A_t}| \leq 2^{-n}$ for any s, t and n. This means that, the computable sequence (x_s) defined by $x_s := x_{A_s}$ for all s converges to x_A g-effectively, where g is a computable function defined by $g(n) := \sum_{i \leq n} h(i)$. Thus, $x_A \in \omega$-c**EC**.

2. We prove only the inequality and it suffices to prove 1-c**EC** $\nsubseteq$ $*$-b**EC**. We will construct a computable sequence (x_s) of rational numbers which converges 1-effectively to a non-$*$-cEC real x_A, *i.e.*, A is not k-c.e. for any constant k and hence satisfies all requirements $R_{\langle i,j\rangle}$: "*if* $(W_{i,s})_{s\in\mathbb{N}}$ *is a* j*-enumeration, then* $\lim_{s\to\infty} W_{i,s} \neq A$", where (W_e) is a computable enumeration of all c.e. subsets of $\mathbb{N}$ and $(W_{e,s})$ is its uniformly computable approximation. The strategy to satisfy a single requirement R_e for $e = \langle i, j\rangle$ is as follows. We choose an interval $I_e = [n_e, m_e]$ of natural numbers such that $m_e - n_e > 2j$. This interval is preserved exclusively for the requirement R_e. At the beginning, let $x_0 := 2^{-n_e}$ (n_e is put into A). If at some stage s_0, n_e enters W_{i,s_0}, then define $x_{s_0+1} := x_{s_0} - 2^{-m_e}$ (n_e leaves A) and let $m_e := m_e - 1$. If at a later stage $s_1 > s_0$, n_e leaves W_i, then define $x_{s_1+1} := x_{s_1} + 2^{-m_e}$ (n_e enters A) and let $m_e := m_e - 1$, and so on. We take this action at most j times. Thus, if $(W_{i,s})_{s\in\mathbb{N}}$ is a j-enumeration, then R_e will be satisfied eventually. The sequence (x_s) defined in this way converges obviously 1-effectively. □

4.2. h-BOUNDED COMPUTABILITY

The Cauchy computability discussed in Subsection 4.1 leads to an Ershov-type hierarchy. However, the definition of n-jumps in Definition 4.1 seems quite artificial. There is no reason to consider only the number of jumps between 2^{-n} and 2^{-n+1}. More naturally, we can count all the jumps larger than, say 2^{-n}. This leads to another Cauchy computability in this subsection. In this case, we can also show a general hierarchy theorem and a lot of nice analytical properties. Nevertheless, the Ershov-type hierarchy does not hold any more.

Definition 4.6 (Zheng [30])**.** Let $h : \mathbb{N} \to \mathbb{N}$ be a function, C be a class of functions, and (x_s) a sequence of rational converging to x.

(1) (x_s) converges to x *h-bounded effectively* if there are at most $h(n)$ non-overlapping index-pairs (i, j) such that $|x_i - x_j| > 2^{-n}$ for all n;
(2) x is *h-bounded computable* (h-bc) if there is a computable sequence of rational numbers which converges to x h-bounded effectively;
(3) x is *C-bounded computable* (C-bc) if it is h-bc for some $h \in C$.

The classes of h-bc and C-bc reals are denoted by h-**BC** and C-**BC**, respectively. Obviously, if C is the class of all computable functions, then C-**BC** = **DBC**. Thus, divergence bounded computable reals are also called ω-bounded computable, or simply ω-bc (ω-**BC** for the class). On the other hand, if $\liminf h(n) < \infty$, then only rational numbers can be h-bc, *i.e.*, h-**BC** = $\mathbb{Q}$. Therefore, we cannot anticipate an Ershov-type hierarchy in this case. Actually, even the class **EC** cannot be characterized as C-**BC** for any class C of functions. Notice that, although any computable real is id-bc, *i.e.*, **EC** $\subseteq$ id-**BC** for the identity function id, there is an h-bc real which is not computable for any unbounded nondecreasing computable function h.

Moreover, if there is a constant c such that $|f(n) - g(n)| \leq c$ for all n, then f-**BC** = g-**BC**. This means that a general hierarchy theorem like Theorem 4.2 does not hold neither. Nevertheless, we have another version of hierarchy theorem as follows which implies, *e.g.*, f-**BC** $\subsetneq$ g-**BC** if the computable functions f, g satisfy $f \in o(g)$.

Theorem 4.7 (Zheng [30])**.** *If $f, g : \mathbb{N} \to \mathbb{N}$ are computable and satisfy the condition $(\forall c \in \mathbb{N})(\exists m \in \mathbb{N})(c + f(m) < g(m))$, then g-**BC** $\nsubseteq$ f-**BC**.*

Proof. We construct a computable sequence (x_s) of rational numbers converging g-bounded effectively to some real x which satisfies all requirements R_e: "*if $(\varphi_e(s))$ converges f-bounded effectively to y_e, then $y_e \neq x$*", where (φ_e) is an effective enumeration of the partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. That is, x is not f-bc. The idea to satisfy a single requirement R_e is as follows. We choose an interval I and a natural number m such that $f(m) < g(m)$. Choose further two subintervals $I_e, J_e \subset I$ which have at least a distant 2^{-m}. Then we can find a real x either from I_e or J_e to avoid the limit y_e of the sequence $(\varphi_e(s))$ if it converges f-bounded effectively. To satisfy all the requirements simultaneously, we use a finite injury priority construction. □

Almost all classes of Cauchy computability hierarchy discussed in Section 4.1 are not closed under addition and subtraction. However, by the next theorem a lot of classes C-**BC** are closed under the arithmetic operations.

Theorem 4.8 (Zheng [30]). *Let C be a class of functions $f : \mathbb{N} \to \mathbb{N}$. If, for any $f, g \in C$ and $c \in \mathbb{N}$, there is an $h \in C$ such that $f(n+c) + g(n+c) \leq h(n)$ for all n, then C-**BC** is a field.*

Proof. Let $f, g \in C$. If (x_s) and (y_s) are computable sequences of rational numbers which converge to x and y f- and g-bounded effectively, respectively. By triangle inequations the computable sequences $(x_s + y_s)$ and $(x_s - y_s)$ converge h_1-bounded effectively to $x + y$ and $x - y$, respectively, for the function h_1 defined by $h_1(n) := f(n+1) + g(n+1)$.

For the multiplication, choose a natural number N such that $|x_n|, |y_n| \leq 2^N$ and define $h_2(n) := f(N+n+1) + g(N+n+1)$ for any $n \in \mathbb{N}$. If $|x_i - x_j| \leq 2^{-n}$ and $|y_i - y_j| \leq 2^{-n}$, then $|x_i y_i - x_j y_j| \leq |x_i||y_i - y_j| + |y_j||x_i - x_j| \leq 2^N \cdot 2^{-n+1} = 2^{-(n-N-1)}$. This means that $(x_s y_s)$ converges h_2-bounded effectively to xy.

Now suppose that $y \neq 0$ and w.l.o.g. that $y_s \neq 0$ for all s. Let N be a natural number such that $|x_s|, |y_s| \leq 2^N$ and $|y_s| \geq 2^{-N}$ for all $s \in \mathbb{N}$. If $|x_i - x_j| \leq 2^{-n}$ and $|y_i - y_j| \leq 2^{-n}$, then $|x_i/y_i - x_j/y_j| = |(x_i y_j - x_j y_i)/(y_i y_j)| \leq (|x_i||y_i - y_j| + |y_j||x_i - x_j|)/(|y_i y_j|) \leq 2^{3N} \cdot 2^{-n+1} = 2^{-(n-3N-1)}$. That is, the sequence (x_s/y_s) converges h_3-bounded effectively to (x/y) for the function $h_3 : \mathbb{N} \to \mathbb{N}$ defined by $h_3(n) := f(3N+n+1) + g(3N+n+1)$. Since the functions h_1, h_2, h_3 are bounded by some functions of C, all $x + y, x + y, xy$ and x/y are C-bc and hence the class C-**BC** is closed under arithmetical operations $+, -, \times$ and $\div$. □

As examples of class C which satisfy the condition of Theorem 4.8, we have $Lin := \{\lambda n.(c \cdot n + d) : c, d \in \mathbb{N}\}$; $Log^{(k)} := \{\lambda n.(c \log^k(n) + d) : c, d \in \mathbb{N}\}$; $Poly := \{\lambda n.(c \cdot n^d) : c, d \in \mathbb{N}\}$; $Exp_1 := \{\lambda n.(c \cdot 2^n) : c \in \mathbb{N}\}$, etc. Notice that, the classes Lin-**BC**, Log^k-**BC** and $Poly$-**BC** are all fields which do not even contain all c.e. reals. On the other hand, Exp_1-**BC** is a field which contains **CE** and hence **DCE**. The next theorem shows that there is a smaller class $o(2^n)$-**BC** which contains **DCE** properly. If $o_e(2^n)$ denotes the class of all computable function $f \in o(2^n)$, then $o_e(2^n)$-**BC** does not contain **CE** any more.

Theorem 4.9 (Zheng [30]). **SC** $\nsubseteq o_e(2^n)$-**BC** *and* **DCE** $\subsetneq o(2^n)$-**BC**.

Proof. To prove **SC** $\nsubseteq o_e(2^n)$-**BC**, we construct an increasing computable sequence (x_s) of rational numbers converging to x which satisfies all requirements R_e: "*if φ_i and ψ_j are total functions and $\psi_j \in o(2^n)$ and $(\varphi_i(s))$ converges ψ_j-bounded effectively to y_i then $x \neq y_i$*", where (φ_e) and (ψ_e) are effective enumerations of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$ and $\psi_e :\subseteq \mathbb{N} \to \mathbb{N}$, respectively.

To satisfy a single requirement R_e ($e = \langle i, j \rangle$), we choose a rational interval I_{e-1} of length $2^{-m_{e-1}}$ for some natural number m_{e-1}. Then we look for a witness interval $I_e \subseteq I_{e-1}$ of R_e such that each element of I_e satisfies R_e. At the beginning, the interval I_{e-1} is divided into four equidistant subintervals J_e^t for $t < 4$ and let $I_e := J_e^1$ as the (default) candidate of witness interval of R_e. If the function ψ_j is not a total function such that $\psi_j \in o(2^n)$, then we are done. Otherwise, there exists a natural number $m_e > m_{e-1} + 2$ such that $2(\psi_j(m_e) + 2) \cdot 2^{-m_e} \leq 2^{-(m_{e-1}+2)}$. In this case, we divide the interval J_e^3 (which is of length $2^{-(m_{e-1}+2)}$) into subintervals I_e^t of length 2^{-m_e} for $t < 2^{m_e-(m_{e-1}+2)}$ and let $I_e := I_e^1$ as a new candidate of witness interval of R_e. If the sequence $(\varphi_i(s))$ does not enter the interval I_e^1 at all, then it is a correct witness interval. Otherwise, suppose that $\varphi_i(s_0) \in I_e^1$ for some $s_0 \in \mathbb{N}$. Then we change the witness interval to be I_e^3. If $\varphi_i(s_1) \in I_e^3$ for some $s_1 > s_0$, then let $I_e := I_e^5$, and so on. This can happen at most $\psi_j(m_e)$ times if the sequence $(\varphi_i(s))$ converges ψ_j-bounded effectively. To satisfy all the requirements R_e simultaneously, a finite injury priority construction suffices.

The inclusion **DCE** $\subseteq o(2^n)$-**BC** follows immediately from the fact that any bounded increasing sequence converges h-bounded effectively for some $h \in o(2^n)$. To show that **DCE** $\neq o(2^n)$-**BC**, we construct a computable sequence (x_s) of rational numbers and a (non-computable) function $h \in o(2^n)$ such that the sequence (x_s) converges h-bounded effectively to a real x which satisfies all requirements R_e: "*if* φ_e *is a total function, and* $\sum_{s\in\mathbb{N}} |\varphi_e(s) - \varphi_e(s+1)| \leq 1$*, then* $\lim_{s\to\infty} \varphi_e(s) \neq x$", where (φ_e) is an effective enumeration of all partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. The strategy to satisfy a single requirement R_e is quite simple. Namely, we choose two rational intervals I_e and J_e such that they have a distance 2^{-m_e} for some natural number m_e. Then we choose the middle point of I_e as x whenever the sequence $(\varphi_e(s))$ does not enter the interval I_e. Otherwise, we choose the middle of J_e which can be changed later if the sequence $(\varphi_e(s))$ enters the interval J_e, and so on. Because of the condition $\sum_{s\in\mathbb{N}} |\varphi_e(s) - \varphi_e(s+1)| \leq 1$, we need at most 2^{m_e} changes. By a finite injury priority construction, this works for all requirements simultaneously. However, the real x constructed in this way is only a 2^n-bounded computable real. To guarantee the $o(2^n)$-bounded computability of x, we need several m_e's instead of just one. That is, we choose at first a natural number $m_e > e$, two rational intervals I_e and J_e and implement the above strategy, but at most 2^{m_e-e} times. Then we look for a new $m'_e > m_e$ and apply the same procedure up to $2^{m'_e-e}$ times, and so on. This means that, in worst case, we need 2^e different m_e's to satisfy a single requirement R_e. We can see that the finite injury priority technique can still be applied. □

By Theorem 4.9, we have **DCE** $\not\subseteq o_e(2^n)$-**BC**, However, it is not clear yet, whether $o_e(2^n) \subseteq$ **DCE** hold.

5. Monotone computability hierarchy

The convergence speed of a sequence can also be measured by comparing the absolute error-estimation at different stages. This leads to another hierarchy of Δ_2^0-reals.

Definition 5.1 (Rettinger and Zheng [15])**.** Let $h : \mathbb{N} \to \mathbb{R}$ be a function. A real x is called *h-monotonically computable* (h-mc) if there is a computable sequence

(x_s) of rational numbers which converges to x h-monotonically in the sense that

$$(\forall n, m \in \mathbb{N})(n < m \Longrightarrow h(n)|x - x_n| \geq |x - x_m|). \tag{3}$$

If $h = \lambda n.c$ for a constant $c \in \mathbb{R}$, then any h-mc reals are called c-mc. We call a real *monotonically computable* (mc) if it is c-mc for some constant c and ω-*monotonically computable* (ω-mc) if it is h-mc for a computable function $h : \mathbb{N} \to \mathbb{N}$. The classes of h-mc, c-mc, mc and ω-mc reals are denoted by h-**MC**, c-**MC**, **MC** and ω-**MC**, respectively.

5.1. c-MC and dense hierarchy

By definition, we have obviously 0-**MC** $= \mathbb{Q}$, 1-**MC** $=$ **SC** and c-**MC** $=$ **EC** for $0 < c < 1$. Moreover, c_1-**MC** $\subseteq c_2$-**MC** hold for any constants $c_2 \geq c_1$. The next theorem shows that this inclusion is proper if $c_2 > c_1 \geq 1$.

Theorem 5.2 (Rettinger and Zheng [15])**.** *For any real constants $c_2 > c_1 \geq 1$, c_1-**MC** $\subsetneq c_2$-**MC**.*

Proof. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, it suffices to consider the rational numbers $c_2 > c_1 > 1$. We construct a computable sequence (x_s) of rational numbers which converges c_2-monotonically to a non-c_1-mc real x. That is, x satisfies all requirements R_e: "*if $(\varphi_e(s))_s$ converges to y_e c_1-monotonically, then $x \neq y_e$*", where (φ_e) is a computable enumeration of computable functions $\varphi_i : \mathbb{N} \to \mathbb{Q}$. To satisfy a single requirement R_e, we choose a rational interval I and divide it into seven subintervals I_i for $i < 7$ such that

$$x \in I_1 \ \& \ x_1 \in I_3 \ \& \ x_2 \in I_5 \quad \Longrightarrow \quad c_1|x - x_1| < |x - x_2| \tag{4}$$

and, for any $(i, j, k) \in \{(1, 3, 5), (1, 3, 3), (1, 5, 5), (5, 3, 3)\}$,

$$x \in I_i \ \& \ x_1 \in I_j \ \& \ x_3 \in I_k \quad \Longrightarrow \quad c_2|x - x_1| \geq |x - x_2|. \tag{5}$$

As long as the sequence $(\varphi_e(s))_s$ does not enter the interval I_3, we choose an x_s from I_3. If $\varphi_e(s_1) \in I_3$ for some s_1, then we choose an x_{s_1} from I_5. Moreover, if there is another $s_2 > s_1$ such that $\varphi_e(s_2) \in I_5$, then we choose an x_{s_2} from I_1. Thus, the sequence (x_s) changes at most two times and converges to an limit x c_2-monotonically because of condition (4). If x is in I_3 or I_5, then the sequence $(\varphi_e(s))_s$ does not converge to x at all. For the case $x \in I_1$, the sequence $(\varphi_e(s))_s$ can not converge c_1-monotonically to x because of condition (5). In any case, the limit x satisfies the requirement R_e. To satisfy all requirements simultaneously, a finite injury priority construction suffices. □

Theorem 5.2 implies immediately that **SC** $\subsetneq$ **MC**. Furthermore, it is shown in [15] that, every c-mc real is d-c.e. and there exists a d-c.e. real which is not c-mc for any constant c. That is we have

Theorem 5.3 (Rettinger and Zheng [15])**.** **SC** $\subsetneq$ **MC** $\subsetneq$ **DCE**.

5.2. ω-MONOTONE COMPUTABILITY

The hierarchy theorem for c-mc reals cannot be extended to h-mc reals in general because all unbounded monotone function h corresponds to the same class of h-mc reals.

Theorem 5.4 (Zheng, Rettinger and Barmpalias [36]). *If h is a monotone and unbounded computable function, then h-$\mathbf{MC} = \omega$-$\mathbf{MC}$.*

Proof. Let g be a computable function and let (x_s) be a computable sequence of rational numbers which converges g-monotonically to x. Since h is unbounded, there is an increasing computable sequence (n_s) such that $h(n_s) \geq g(s)$ for all s. Let $n(s)$ be the maximal i such that $n_i \leq s$ and $y_s := x_{n(s)}$. For any $s < t$, if $n(s) = n(t)$, *i.e.*, there is an i such that $n_i \leq s < t < n_{i+1}$ and $n(s) = i$, then $h(s)|x-y_s| = h(s)|x-x_i| \geq |x-x_i| = |x-y_t|$. Otherwise, if $n(s) < n(t)$, then there are $i < j$ such that $n_i \leq s < n_j \leq t$ and $n(s) = i < j = n(t)$. Since h is increasing, this implies that $h(s)|x-y_s| \geq h(n_i)|x-x_i| \geq g(i)|x-x_i| \geq |x-x_j| = |x-y_t|$. Therefore, the computable sequence (y_s) converges to x h-monotonically and hence x is h-mc. □

Theorem 5.5 (Zheng, Rettinger and Barmpalias [36]). *The class of ω-mc reals is incomparable with* **DCE** *and* **DBC**.

Proof. $\mathbf{DCE} \nsubseteq \omega$-$\mathbf{MC}$: it suffices to construct a computable sequence (x_s) of rational numbers which converges weakly effectively to a non-id-mc real x. That is, $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 2$ and x satisfies all requirements R_e that "*if $(\varphi_e(s))_s$ converges id-monotonically to y_e, then $x \neq y_e$*", where (φ_e) is a computable enumeration of computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. To satisfy a single requirement R_e, we fix a rational interval (a, b). At any stage s, let x_s be the middle point of (a, b) as long as there are no $t_1 < t_2$ such that $\varphi_{e,s}(t_1), \varphi_{e,s}(t_1) \in (a, b)$. Otherwise, let $(a_1, b_1) := (\varphi_e(i_1)-\delta, \varphi_e(t_1)+\delta)$, where $\delta := \min\{|\varphi_e(t_1)-\varphi_e(t_2)|/(t_1+1), (\varphi_e(t_1)-a)/2, (b-\varphi_e(t_1))/2, 2^{-4e}\}$. Then we define x_s as the middle poind of the interval (a_1, b_1). Since the sequence $(\varphi_e(s))_s$ does not converge id-monotonically to any element of (a_1, b_1), this x_s satisfies R_e. By priority technique, all requirements can be satisfied simultaneously. Because the new interval (a_1, b_1) has a length less than 2^{-4e}, the constructed sequence converges weakly effectively.

ω-$\mathbf{MC} \nsubseteq \mathbf{DBC}$: we construct a computable function $h : \mathbb{N} \to \mathbb{N}$ and a computable sequence (x_s) of rational numbers converging h-monotonically to x which satisfies all requirements $R_{\langle i,j\rangle}$: "*If $(\varphi_i(s))_s$ converges α_j-bounded effectively to y, then $x \neq y$*", where (φ_e) and (α_e) are computable enumerations of computable functions $\varphi_e : \mathbb{N} \to \mathbb{Q}$ and $\alpha_e : \mathbb{N} \to \mathbb{N}$, respectively. □

5.3. COMPUTABILITY AND SEMI-COMPUTABILITY OF h-MC REALS

Now let's look at h-mc reals for computable functions $h : \mathbb{N} \to (0, 1]_{\mathbb{Q}}$. Obviously, if there is a constant $c < 1$ such that $h(n) \leq c$ for all n, then h-$\mathbf{MC} = \mathbf{EC}$. Therefore, we consider only the computable function h such that $\lim_{n\to\infty} h(n) = 1$.

For any such h, we have h-**MC** $\subseteq$ **SC**. Is it possible that h-**MC** is equal to **EC** or **SC**? The next theorem gives the exact criteria when these can happen. Roughly speaking, if $h(n)$ converges to 1 "very slowly", then h-**MC** = **EC** and if $h(n)$ converges 1 "very fast", then h-**MC** = **SC**. Otherwise, h-**MC** is strictly between **EC** and **SC**.

Theorem 5.6 (Zheng, Rettinger and Barmpalias [36]). *For any computable function $h : \mathbb{N} \to (0, 1]_{\mathbb{Q}}$ we have*

(1) *If $\sum_{n\in\mathbb{N}}(1 - h(n)) = \infty$, then h-**MC** = **EC**;*
(2) *If $\sum_{n\in\mathbb{N}}(1 - h(n))$ is computable, then h-**MC** = **SC**;*
(3) *If $\sum_{n\in\mathbb{N}}(1 - h(n))$ is non-computable, then **EC** $\subsetneq$ h-**MC** $\subsetneq$ **SC**.*

Proof.

1. Suppose that $\sum_{n\in\mathbb{N}}(1 - h(n)) = \infty$ and hence $\prod_{n\in\mathbb{N}} h(n) = 0$. First, since $h(n) \neq 0$ for all n, for any computable sequence (x_s) of rational number converging effectively to x, we can easily construct a computable subsequence which converges to x h-monotonically. That is, **EC** $\subseteq$ h-**MC**. On the other hand, let (x_s) be a computable sequence of rational numbers which converges h-monotonically to x. That is, $h(n)|x - x_n| \geq |x - x_m|$ for any $m > n$. This implies that $|x - x_n| \leq \prod_{i\leq n} h(i)|x - x_0|$. Since $\lim_{n\to\infty} \prod_{i\leq n} h(i) = \prod_{n\in\mathbb{N}} h(n) = 0$, we can choose a computable subsequence of (x_s) which converges to x effectively. That is h-**MC** $\subseteq$ **EC**.

2. Suppose that $\sum_{n\in\mathbb{N}}(1 - h(n)) = u$ and u is a computable real. We show now that **SC** $\subseteq$ h-**MC**. Since u is also co-c.e., there is a decreasing computable sequence (u_s) of rational number which converges to u. If x is a c.e. real and (x_s) is an increasing computable sequence of rational numbers which converges to x. Let (y_s) be a computable sequence defined by $y_s := x_s - u_s + \sum_{i\leq s}(1 - h(i))$ for all s. It is not difficult to see that (y_s) is also increasing and satisfies $y_{s+1} - y_s \geq (x - y_s)(1 - h(s))$ and hence (y_s) converges to x h-monotonically. That is, **SC** $\subseteq$ h-**MC**.

3. Suppose that $\sum_{n\in\mathbb{N}}(1 - h(n)) = u$ and u is not a computable real. We first show that **SC** $\nsubseteq$ h-**MC**. Notice that, for any h-mc c.e. real x, there exists an increasing computable sequence of rational numbers which converges to x h-monotonically. Now we are going to construct an increasing computable sequence (x_s) of rational numbers which converges to a non-h-mc real x. Namely x satisfies all requirements R_e: "*if φ_e is an increasing total function and $(\varphi_e(s))$ converges h-monotonically to y_e, the $y_e \neq x$*", where (φ_e) is an effective enumeration of computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$.

Let $\Sigma_5 := \{0, 1, 2, 3, 4\}$ and $\mathbb{I}$ be set of all rational subintervals of the interval $[0, 1]$. We define at first an interval tree $I : \Sigma_5^* \to \mathbb{I}$ inductively by $I(\lambda) := [0, 1]$ and $I(wi) := I_i$ for all $w \in \Sigma_5^*$ and $i < 5$, where $(I_i)_{i<5}$ is an equidistant subdivision of the interval $I(w)$. The interval $I(w)$ is denoted by $[a_w, b_w]$ for any $w \in \Sigma_5^*$. Then we have $a_w = \sum_{i<|w|} 5^{-(i+1)} \cdot w[i]$ and $b_w = a_w + 5^{-|w|}$.

To satisfy a single requirement R_e we will try to find a witness interval $J \subseteq [0, 1]$ such that any point of J except endpoints satisfies the requirement R_e. Suppose that φ_e is increasing. At the beginning, let $J := I(11)$. If the sequence $(\varphi_e(s))$

does not enter $I(11)$, then we are done. Otherwise, suppose that $\varphi_e(s_0) \in I(11)$ for some s_0. If there exists a $t < s_0$ such that $h(t)(a_3 - \varphi_e(t)) < a_3 - \varphi_e(t+1)$ holds, then we change the witness interval to be $I(3)$. In this case, for any $x \in I(3)$, we have $h(t)(x - \varphi_e(t)) < x - \varphi_e(t+1)$ and hence $I(3)$ is a correct witness interval of R_e. Otherwise, if the inequality $h(t)(a_3-\varphi_e(t)) \geq a_3-\varphi_e(t+1)$ holds for all $t < s_0$, then we choose $I(131)$ to be a new candidate of witness interval. Analogously, if there exists an $s_1 > s_0$ such that $\varphi_e(s_1) \in I(131)$, then we choose either $I(3)$ or $I(1331)$ to be the new candidate of witness interval of R_e, depending on whether there exists a $t < s_1$ such that $h(t)(a_3 - \varphi_e(t)) < a_3 - \varphi_e(t+1)$ holds, and so on. Let's look at the possible outcome of our construction. If the interval $I(3)$ is chosen as a witness interval at some stage, then we are done because the sequence $(\varphi_e(s))$ does not converge to any element of $I(3)$.

Otherwise, if $I(3)$ has never been chosen as the witness interval of R_e, then there are two possibilities. Either there is a $k \in \mathbb{N}$ such that $I(13^k1)$ is chosen as a candidate of witness interval and there does not exist s such that $\varphi_e(s) \in I(13^k1)$ and hence $I(13^k1)$ is a correct witness interval of R_e or each of the following intervals $I(11), I(131), I(1331), \cdots$ is chosen to be a candidate of witness interval at some stage. In the latter case, there exists a computable increasing sequence (s_n) of natural numbers such that $\varphi_e(s_n) \in I(13^n1)$ for all n. This implies that the limit $y_e := \lim_{s\to\infty} \varphi_e(s) = 5^{-1}+3\cdot\sum_{i=2}^{\infty} 5^{-i}$ is a computable real number. In addition, by construction we have the inequality $h(n)(a_3 - \varphi_e(n)) \geq a_3 - \varphi_e(n+1)$ holds for all $n \in \mathbb{N}$. This is equivalent to $(1-h(n))(a_3 - \varphi_e(n)) \leq \varphi_e(n+1) - \varphi_e(n)$. Notice that $a_3 - \varphi_e(n) > 5^{-1}$ for all n. Now we define a computable function $g : \mathbb{N} \to (0,1)_{\mathbb{Q}}$ by $g(n) := 1-5\cdot(\varphi_e(n+1)-\varphi_e(n))$ for all n which satisfies $g(n) \leq h(n)$ for all n. On the other hand, we have $\sum_{n\in\mathbb{N}}(1-g(n)) = 5\sum_{n\in\mathbb{N}}(\varphi_e(n+1)-\varphi_e(n)) = 5(y_e - \varphi_e(0))$. That is, the sum $\sum_{n\in\mathbb{N}}(1-g(n))$ is a computable real number and hence $\sum_{n\in\mathbb{N}}(1-h(n))$ is computable too. This contradicts the hypothesis on h. This contradiction implies that only finitely many intervals can be chosen to be the candidate of a witness interval of R_e and the last one is a correct one. Thus, a finite injury priority construction applies.

At the last, we prove that $\mathbf{EC} \subsetneq h$-$\mathbf{MC}$. Since $\prod_{n\in\mathbb{N}} h(n) = c > 0$ we can choose a rational number q such that $0 < q < c$. We are going to construct an increasing computable sequence (x_s) of rational numbers from $[0,1]$ which h-monotonically converges to a non-computable real x. Let (φ_e) be an effective enumeration of partial computable functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{N}$ and $\delta_e := q \cdot 2^{-(e+2)}$. The sequence (x_s) is construct in stages. At the beginning, $x_0 := 0$. At any stage $s+1$, if there is a "non-used" (minimal) $e \leq s$ such that

$$(\exists t \leq s)\left(2^{-t} < \delta_e \ \&\ \varphi_{e,s}(t) \leq 1 - h(s)(1-x_s) + \delta_e\right), \qquad (6)$$

then define $x_{s+1} := 1-h(s)(1-x_s)+2\delta_e$. Otherwise, let $x_{s+1} := 1-h(s)(1-x_s)$. It is not difficult to see that (x_s) is increasing and converges h-monotonically to some x. Furthermore, if φ_e is a total increasing function such that $|\varphi_e(n)-\varphi_e(n+1)| \leq 2^{-(n+1)}$ for all n, then $\lim_{n\to\infty} \varphi_e(n) \neq x$. That is, x is a non-computable h-mc real. □

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