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POLYNOMIALS OVER THE REALS IN PROOFS OF TERMINATION: FROM THEORY TO PRACTICE *

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Abstract. This paper provides a framework to address termination problems in term rewriting by using orderings induced by algebras over the reals. The generation of such orderings is parameterized by concrete *monotonicity requirements* which are connected with different classes of termination problems: termination of rewriting, termination of rewriting by using dependency pairs, termination of innermost rewriting, top-termination of infinitary rewriting, termination of context-sensitive rewriting, etc. We show how to define term orderings based on algebraic interpretations over the real numbers which can be used for these purposes. From a practical point of view, we show how to automatically generate *polynomial* algebras over the reals by using constraint-solving systems to obtain the coefficients of a polynomial in the domain of the real or rational numbers. Moreover, as a consequence of our work, we argue that software systems which are able to generate constraints for obtaining polynomial interpretations over the naturals which prove termination of rewriting (*e.g.*, AProVE, C*i*ME, and TTT), are potentially able to obtain suitable interpretations over the reals by just solving the constraints in the domain of the real or rational numbers.

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1. Introduction

Monotonicity in term rewriting has to do with the ability of the rewrite relation to "reproduce" any rewriting step within arbitrary syntactic contexts, *i.e.*, if a term t rewrites to a term s, then for all k-ary symbols f, *all* arguments

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$i \in \{1, \ldots, k\}$, and terms $t_1, \ldots, t_k$, the term $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k)$ rewrites into $f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$. Accordingly, orderings $>$ for proving termination of rewriting (*i.e.*, the absence of infinite rewrite sequences) are also required to be *monotonic*. Recently, however, *non-monotonic* term orderings have received an increasing attention as suitable formal tools for proving termination of both rewriting and some restricted forms of rewriting. For instance, the dependency pairs method for proving termination of rewriting, and proofs of termination of innermost rewriting can benefit from them (see [1, 15]).

In this setting, a possible way to specify *monotonicity requirements* is the use of a *replacement map* μ which associates, for each k-ary symbol f of the signature, the argument positions $\mu(f) \subseteq \{1, \ldots, k\}$ of f which we will eventually call *monotonic* arguments. In fact, there is a form of rewriting, called context-sensitive rewriting (*CSR* [32, 33]), where, given a replacement map μ, rewritings in a term $f(t_1, \ldots, t_k)$ are allowed only on the arguments t_i with $i \in \mu(f)$. The following example motivates the use of *CSR* in term rewriting and programming:

Example 1. The following TRS $\mathcal{R}$:

```
take(s(n),cons(x,xs))       -> cons(x,take(n,xs))
take(0,xs)                  -> nil
incr(cons(x,xs))            -> cons(s(x),incr(xs))
pairNs                      -> cons(0,incr(oddNs))
oddNs                       -> incr(pairNs)
zip(nil,xs)                 -> nil
zip(x,nil)                  -> nil
zip(cons(x,xs),cons(y,ys))  -> cons(pair(x,y),zip(xs,ys))
tail(cons(x,xs))            -> xs
repItems(cons(x,xs))        -> cons(x,cons(x,repItems(xs)))
repItems(nil)               -> nil
```

can be used to approximate the value of $\pi/2$ by means of the so-called *Wallis' product*: $\frac{\pi}{2} = lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}$. The expression

```
zip(repItems(tail(pairNs)),tail(repItems(oddNs)))
```

produces the previous fractions and the function `take` can be used to obtain appropriate approximations.

Let $\mu(\texttt{cons}) = \{1\}$ and $\mu(f) = \{1, \ldots, k\}$ for all other k-ary symbols f. The μ-termination of $\mathcal{R}$ (*i.e.*, termination of *CSR* under the replacement map μ) can also be proved by using a polynomial interpretation (see Ex. 15 below). This knowledge can be used to obtain the desired normal forms whenever they exist, or to approximate infinite normal forms (see [32, 33]).

Termination of *CSR* is fully captured by the so-called *μ-reduction orderings* [43], *i.e.*, well-founded and stable orderings $>$ which fulfill the monotonicity requirements expressed by the replacement map μ. We refer the reader to [20, 32, 33, 36] for further details, applications, and references about *CSR* and termination of *CSR*.

Recently, a number of connections between replacement maps μ (specifying monotonicity requirements) and termination problems / proofs which can be addressed by using μ-reduction orderings have been discovered and investigated:

Replacement map	Termination problem
$\mu_\top$	Termination of rewriting
$\mu \sqsubset \mu_\top$	Termination of *CSR* (proper cases)
$\mu^{can}_{\mathcal{R}} \sqsubseteq \mu$	Top-termination
$\mu^{inn}_{\mathcal{R}}$	Termination of innermost rewriting
$\mu_\bot$	Termination of rewriting using dependency pairs.

Here, $\mu_\top$ is the replacement map specifying full monotonicity in all arguments of all function symbols. In this case, *CSR* and ordinary rewriting coincide. Then, the $\mu_\top$-reduction orderings are exactly the standard *reduction* orderings which can be used to prove termination of rewriting [13, 44]. If μ does not contain all argument indices for some symbol f (written $\mu \sqsubset \mu_\top$), then we are in the proper case for termination of *CSR*. The *canonical* replacement map $\mu^{can}_{\mathcal{R}}$ for a TRS $\mathcal{R}$ is the most restrictive replacement map which ensures that the (positions of) non-variable subterms of the left-hand sides of the rules of $\mathcal{R}$ are replacing [32, 33]. If, for each symbol f, $\mu(f)$ contains at least all indices in $\mu^{can}_{\mathcal{R}}(f)$ (written $\mu^{can}_{\mathcal{R}} \sqsubseteq \mu$), then a proof of μ-termination of *CSR* is a proof of *top-termination*[1] [34]. The replacement map $\mu^{inn}_{\mathcal{R}}$ is obtained for a given TRS $\mathcal{R}$ as explained in [15], Definition 3; then, the $\mu^{inn}_{\mathcal{R}}$-termination of $\mathcal{R}$ implies that $\mathcal{R}$ is innermost terminating [15], Corollary 11. Finally, $\mu_\bot$ is the replacement map introducing no monotonicity constraint. Then, the $\mu_\bot$-reduction orderings can be used (together with suitable quasi-orderings[2]) as part of a *reduction pair* in proofs of termination of rewriting by using Arts and Giesl's dependency pairs technique (see below).

In this paper, we investigate the use of algebras over the reals as a suitable way to generate μ-reduction orderings satisfying the monotonicity constraints specified by a replacement map μ when addressing termination problems.

Contributions of the paper

Orderings induced by algebras over the reals

Term orderings can be obtained by giving appropriate algebraic interpretations to the function symbols of a signature. In this approach, given an interpretation domain A ordered by $>_A$, each k-ary symbol f of the signature is given a mapping $[f] : A^k \to A$. Then, the interpretation of symbols is homomorphically extended to terms t where variable symbols are interpreted as variables ranging in A, and a strict ordering $>$ on terms (*i.e.*, a transitive and irreflexive relation) is defined by $t > s$ if the interpretation of t is bigger (according to $>_A$) than the interpretation

[1] A TRS is *top-terminating* if no infinitary reduction sequence performs infinitely many rewrites at topmost position Λ [14].

[2] A quasi-ordering is a reflexive and transitive relation.

of s for all possible valuations in A of the variables in t and s. In Section 3, we show how to define term (quasi-) orderings based on algebraic interpretations over the real numbers. Given interpretations of the symbols f as real functions $[f] : A^k \to A$ for some subset A of the real numbers, and a positive real number δ, a well-founded and stable (strict) ordering $>_\delta$ on terms is defined as follows: for all terms t, s, $t >_\delta s$ if, for all possible valuations in A, the difference between the interpretation of t and that of s is *at least* δ. Monotonicity requirements expressed by a replacement map μ can be ensured by requiring that, for all symbols f and $i \in \mu(f)$, the partial derivative of $[f]$ w.r.t. the i-th argument x_i is not below 1.

Termination of rewriting using dependency pairs

The introduction of the dependency pairs approach [1] into the field of termination of rewriting has brought new applications of non-monotonic orderings. In this approach, the left-hand sides l and the right-hand sides r of the rules $l \to r$ of the TRS are compared by using a *weakly* monotonic and stable *quasi-ordering* $\succeq$. The price to pay is that we have to further consider the so-called *dependency pairs* associated to the TRS. The components of each dependency pair have to be compared by means of a stable and well-founded (but no necessarily monotonic or weakly monotonic!) ordering $\sqsupset$. If $\succeq$ and $\sqsupset$ satisfy a given compatibility property, then $(\succeq, \sqsupset)$ is called a reduction pair. In Section 4, we show how to use our techniques to generate reduction pairs which are suitable to be used together with the dependency pairs approach for proving termination of rewriting.

Polynomial interpretations over the reals

Termination of rewriting is undecidable (even for TRSs containing only one rule [10]) and lot of research has been devoted to develop methods and heuristics to achieve proofs of termination in restricted (and mechanizable) cases [1, 12, 28, 30, 40]. Polynomial interpretations and the corresponding reduction orderings [30] are well-suited to achieve automatic or semiautomatic proofs of termination of rewriting [6, 9, 18, 30, 39]. Although polynomial interpretations have several limitations regarding their ability for proving termination of rewrite systems (see, *e.g.*, [7, 23]) the introduction of the dependency pairs approach has brought new applications of basic techniques (like polynomial orderings).

In the usual approach, each k-ary symbol f is given a polynomial $[f]$ on k variables with *non-negative integer coefficients* [5, 6, 9, 30, 37, 44]. Real coefficients are allowed in other approaches [11, 12, 18, 39] but additionally requiring a subterm property (*i.e.*, $[f](x_1, \ldots, x_i, \ldots, x_k) > x_i$ for all k-ary symbols f and $i \in \{1, \ldots, k\}$) to guarantee *well-foundedness* of the induced ordering, which cannot be guaranteed by the ordering $>_\mathbb{R}$ over the reals, which is not well-founded. The use of such polynomials, however, is quite restrictive regarding their ability to introduce non-monotonicity in the corresponding orderings: in the first case, the possibilities for introducing non-monotonicity in a given argument of a function symbol are basically restricted to drop (*i.e.*, fix a zero coefficient for) some monomials that refer to this argument. In the second case (closer to ours), requiring the subterm property implies monotonicity of the induced ordering in the current practical frameworks (see below).

In Section 5, we apply the framework in Section 3 to the systematic and practic use of *real* coefficients in polynomial intepretations: by using the generic approach above, we can associate a well-founded and stable ordering $>_\delta$ to an arbitrary polynomial interpretation over the reals, *i.e.*, a collection of polynomials $[f] \in \mathbb{R}[x_1, \ldots, x_k]$, for each k-ary symbol f, whose coefficients are real numbers. The monotonicity constraints can be selectively ensured as explained above. No subterm property is required. We show how to *obtain* a suitable δ when attempting a proof of compatibility of $>_\delta$ with a set of pairs of terms (*e.g.*, rewrite rules or dependency pairs); this will be essential for the automatic generation of such polynomial interpretations. We also discuss how to deal with polynomials with negative coefficients in our framework. Finally, we prove that polynomial interpretations over the reals provide a strictly more powerful framework for introducing non-monotonicity in the computed orderings.

Automatic generation of polynomials over the reals

Polynomial interpretations play a prominent role in the *implementation* of several existing tools for automatically proving termination of rewriting like AProVE [21], C*i*ME [8], and TTT [26]. In Section 6, we discuss how to automatically obtain polynomial interpretations over the reals satisfying a concrete monotonicity specification μ whose induced ordering $>_\delta$ is compatible with, *e.g.*, a set of rewrite rules or a set of dependency pairs. The procedure is quite simple, in fact, and there is no need to consider any explicit value for δ. As usual, we consider *parametric* polynomial interpretations whose indeterminate coefficients are intended to be *real* instead of natural numbers. Then, we impose a number of *constraints* which, according to our results, ensure the appropriate properties of the computed polynomial interpretation. We obtain a set of arithmetic constraints which can be sent to a constraint-solving system to obtain the coefficients by solving the constraints in the domain of the real or rational numbers. For instance, we illustrate our development by using the constraint-solving system CON'*FLEX* [38] to obtain polynomial interpretations over the reals.

Implementation

In Section 7, we explain the implementation of our techniques as part of the tool MU-TERM [35]. The tool automatically generates the constraints on the indeterminate coefficients of the polynomial interpretations as described in Section 6 and solves them to obtain the corresponding polynomial interpretation.

Related work

Section 8 discusses some related work. Polynomials over the real numbers were proposed by Dershowitz [11, 12] as an alternative to Lankford's polynomials over the naturals [30]. Giesl and Steinbach have investigated how to implement the use of such polynomials in proofs of termination of rewriting [18,39]. In Section 8 we show that if their methods solve a termination problem then it can also be solved by using our method. The usual frameworks for proving termination by using polynomials with natural coefficients are also subsumed by our technique.

Moreover, we show that, although relying in different formal grounds, the constraints generated by the methods which focus on polynomials with non-negative *integer* coefficients (following the original Lankford's approach, see, *e.g.*, [9] for a recent discussion), are actually so close to ours that it is possible to use the set of constraints generated in this way to obtain a polynomial interpretation over the reals by *just interpreting the indeterminate coefficients as real coefficients instead of natural ones*. Thus, software systems (like, *e.g.*, AProVE, C*i*ME, or TTT) which are able to generate constraints for obtaining polynomial interpretations over the naturals which prove termination of rewriting, are potentially able to obtain suitable interpretations over the reals by just solving the constraints in the domain of the real or rational numbers!

Section 9 concludes and points to some future work.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ be the sets of natural, integer, rational and real numbers, respectively; given one of such sets N and $z \in N$, we let $N_z = \{x \in N \mid x \geq z\}$ and $N_{>z} = \{x \in N \mid x > z\}$.

Orderings

A binary relation R on A is *terminating* (or well-founded) if there is no infinite sequence $a_1 \; R \; a_2 \; R \; a_3 \cdots$. A transitive and reflexive relation $\succeq$ on A is a quasi-ordering. A transitive and irreflexive relation $>$ on A is an ordering. Given $f : A^k \to A$ and $i \in \{1, \ldots, k\}$, we say that $>$ (respectively $\succeq$) is (weakly) monotonic on the i-th argument of f if, whenever $x > y$ (respectively $x \succeq y$), we have $f(x_1, \ldots, x_{i-1}, x, \ldots, x_k) > f(x_1, \ldots, x_{i-1}, y, \ldots, x_k)$ (respectively $f(x_1, \ldots, x_{i-1}, x, \ldots, x_k) \succeq f(x_1, \ldots, x_{i-1}, y, \ldots, x_k)$) for all $x, y, x_1, \ldots, x_k \in A$.

Signatures and terms

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, *i.e.*, a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. We denote the empty chain by Λ. Given positions p, q, we denote its concatenation as $p.q$. If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. The set of positions of a term t is $\mathcal{P}os(t)$. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s. The symbol labelling the root of t is denoted as $root(t)$.

Algebraic interpretations

Term orderings can be obtained by giving appropriate interpretations to the function symbols of a signature. Given a signature $\mathcal{F}$, an $\mathcal{F}$-algebra is a pair $\mathcal{A} = (A, \mathcal{F}_A)$, where A is a set and $\mathcal{F}_A$ is a set of mappings $f_A : A^k \to A$ for each $f \in \mathcal{F}$ where $k = ar(f)$. We say that $\mathcal{A}$ is an $\mathcal{F}$-algebra over the reals (resp. rationals, integers, naturals) if $A \subseteq \mathbb{R}$ (resp. $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$). For a given valuation mapping $\alpha : \mathcal{X} \to A$, the evaluation mapping $[\alpha] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to A$ is inductively defined by $[\alpha](x) = \alpha(x)$ if $x \in \mathcal{X}$ and $[\alpha](f(t_1, \ldots, t_k)) = f_A([\alpha](t_1), \ldots, [\alpha](t_k))$ for $x \in \mathcal{X}$, $f \in \mathcal{F}$, $t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Given a term t with $\mathcal{V}ar(t) = \{x_1, \ldots, x_n\}$, we write $[t]$ to denote the *function* $F_t : A^n \to A$ given by $F_t(a_1, \ldots, a_n) = [\alpha_{(a_1,\ldots,a_n)}](t)$ for each tuple $(a_1, \ldots, a_n) \in A^n$, where $\alpha_{(a_1,\ldots,a_n)}(x_i) = a_i$ for $1 \leq i \leq n$.

An *ordered* $\mathcal{F}$-algebra, is a triple $(A, \mathcal{F}_A, >_A)$, where $(A, \mathcal{F}_A)$ is a $\mathcal{F}$-algebra and $>_A$ is a (strict) ordering on A. Then, we can define an ordering $>$ on terms given by $t > s$ if and only $[\alpha](t) >_A [\alpha](s)$, for all $\alpha : \mathcal{X} \to A$. If $>_A$ is well-founded, then $>$ also is and the algebra is said to be well-founded [44], Section 6.2.1.

Rewrite systems

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position p), written $t \xrightarrow{p}_{\mathcal{R}} s$ (or just $t \to s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $\rho : l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ.

Given $\mathcal{R} = (\mathcal{F}, R)$, we take $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Termination of rewriting

A TRS is terminating if $\to$ is terminating. The problem of proving termination of a TRS is equivalent to finding a well-founded, stable, and monotonic (strict) ordering $>$ on terms (*i.e.*, a *reduction ordering*) which is *compatible* with the rules of the TRS, *i.e.*, such that $l > r$ for all rules $l \to r$ of the TRS. Here, *monotonic* means that, for all k-ary symbol f and $i \in \{1, \ldots, k\}$, $>$ is monotonic on the i-th argument of f, when f is viewed as a mapping $f : \mathcal{T}(\mathcal{F}, \mathcal{X})^k \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. *Stable* means that, whenever $t > s$, we have $\sigma(t) > \sigma(s)$ for all terms t, s and substitutions σ.

Given a TRS $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ the set $\mathsf{DP}(\mathcal{R})$ of dependency pairs of $\mathcal{R}$ consists of the pairs $\langle t, s\rangle$ as follows: if $f(t_1, \ldots, t_m) \to r \in R$ and $r = C[g(s_1, \ldots, s_n)]$ for some defined symbol $g \in \mathcal{D}$ and $s_1, \ldots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, then $\langle F(t_1, \ldots, t_m), G(s_1, \ldots, s_n)\rangle \in \mathsf{DP}(\mathcal{R})$, where F and G are new fresh symbols (called *tuple* symbols) associated to defined symbols f and g respectively [1]. Let $\widehat{\mathcal{F}}$ be the set of tuple symbols associated to symbols in $\mathcal{F}$ (actually in $\mathcal{D}$).

A reduction pair $(\succeq, \sqsupset)$ consists of a stable and weakly monotonic quasi-ordering $\succeq$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\succeq \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succeq \subseteq \sqsupset$ [16, 29]. Note that *monotonicity is not required* for $\sqsupset$. A TRS $\mathcal{R}$ is terminating if and only if there is a reduction pair $(\succeq, \sqsupset)$ such that $l \succeq r$ for all $l \to r \in R$ and $t \sqsupset s$ for all $\langle t, s\rangle \in \mathsf{DP}(\mathcal{R})$.

Replacement maps and context-sensitive rewriting

A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [32]. We let $\mu_\perp$ (resp. $\mu_\top$) be $\mu_\perp(f) = \varnothing$ (resp. $\mu_\top(f) = \{1, \ldots, ar(f)\}$) for all $f \in \mathcal{F}$.

The set of *μ-replacing positions* $\mathcal{P}os^\mu(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^\mu(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^\mu(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^\mu(t|_i)$, if $t \notin \mathcal{X}$. In *context-sensitive rewriting* (*CSR* [32]), we (only) contract *replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_\mu s$, if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^\mu(t)$.

Example 2. Consider $\mathcal{R}$ and μ as in Example 1. Then, we have:

$$\underline{\texttt{pairsNs}} \hookrightarrow_\mu \texttt{cons(0,incr(}\underline{\texttt{oddNs}}\texttt{))} \not\hookrightarrow_\mu \texttt{cons(0,incr(incr(pairsNs)))}$$

Since $2.1 \notin \mathcal{P}os^\mu(\texttt{cons(0,incr(oddNs))})$, redex `oddNs` cannot be μ-rewritten.

A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_\mu$ is terminating. Termination of *CSR* is fully captured by the so-called *μ-reduction orderings* [43], *i.e.*, well-founded, stable orderings $>$ which are *μ-monotonic*, *i.e.*, for all $f \in \mathcal{F}$ and $i \in \mu(f)$, $>$ is monotonic in the i-th argument of f. Then, a TRS $\mathcal{R} = (\mathcal{F}, R)$ is μ-terminating if and only if there is a μ-reduction ordering $>$ which is *compatible* with the rules of $\mathcal{R}$, *i.e.*, for all $l \to r \in R$, $l > r$ [43], Proposition 1.

3. Algebras over the reals and reduction orderings

In this paper we are interested in using real functions over real numbers to define term (quasi-) orderings which are useful in proofs of termination. Given a signature $\mathcal{F}$, $A \subseteq \mathbb{R}_0$, and an $\mathcal{F}$-algebra over the reals $\mathcal{A} = (A, \mathcal{F}_A)$, consider $\gtrsim$ given by

$$t \gtrsim s \Leftrightarrow \forall \alpha : \mathcal{X} \to A, [\alpha](t) - [\alpha](s) \geq_{\mathbb{R}} 0$$

for all $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. We have that $\gtrsim$ is an stable quasi-ordering on terms:

Proposition 1. *Let $\mathcal{F}$ be a signature, $A \subseteq \mathbb{R}$ and $\mathcal{A} = (A, \mathcal{F}_A)$ be an $\mathcal{F}$-algebra. Then, $\gtrsim$ is a stable quasi-ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

Proof. Reflexivity and transitivity follow by that of partial order $\geq_{\mathbb{R}}$. Stability of $\gtrsim$ is also easy: let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\mathcal{X}_t = \mathcal{V}ar(t)$, $\mathcal{X}_s = \mathcal{V}ar(s)$. Consider a substitution $\sigma : \mathcal{X}_t \cup \mathcal{X}_s \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ and an arbitrary mapping $\alpha : \mathcal{X} \to A$. Let $\alpha_0 : \mathcal{X} \to A$ be as follows: $\alpha_0(x) = [\alpha](\sigma(x))$ if $x \in \mathcal{X}_t \cup \mathcal{X}_s$ and $\alpha_0(x) = \alpha(x)$ otherwise. Thus, $[\alpha](\sigma(t)) = [\alpha_0](t)$ and $[\alpha](\sigma(s)) = [\alpha_0](s)$. Since $t \gtrsim s$, we have $[\alpha_0](t) - [\alpha_0](s) \geq 0$. Therefore, $[\alpha](\sigma(t)) - [\alpha](\sigma(s)) \geq 0$, *i.e.*, $\sigma(t) \gtrsim \sigma(s)$. $\square$

The ordering $(\mathbb{R}, >_{\mathbb{R}})$ is *not* well-founded, even if we consider bounded subsets $A \subseteq \mathbb{R}_m$ for some $m \in \mathbb{R}$. However, as in [17, 27], given $\delta \in \mathbb{R}_{>0}$, we use the following (strict) ordering on the set of real numbers: $\forall x, y \in \mathbb{R}$,

$$x >_{\mathbb{R},\delta} y \text{ if } x - y \geq_{\mathbb{R}} \delta.$$

Let $\mathcal{F}$ be a signature, $A \subseteq \mathbb{R}$, and $\mathcal{A} = (A, \mathcal{F}_A)$ be an $\mathcal{F}$-algebra. Now, given $\delta \in \mathbb{R}_{>0}$ we define the relation $>_\delta$ on terms by

$$t >_\delta s \Leftrightarrow \forall \alpha : \mathcal{X} \to A, [\alpha](t) - [\alpha](s) \geq_{\mathbb{R}} \delta.$$

Given $m \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we say that $f_A : A^k \to A$ is *m-bounded* if $f_A(x_1, \ldots, x_k) \geq m$ for all $x_1, \ldots, x_k \in A$. If there exists $m \in \mathbb{R}$ such that f_A is m-bounded for all $f \in \mathcal{F}$, then we say that $\mathcal{A} = (A, \mathcal{F}_A)$ is m-bounded. Thus, we have the following.

Theorem 1. *Let $\mathcal{F}$ be a signature, $A \subseteq \mathbb{R}$, $m \in \mathbb{R}$, $\mathcal{A} = (A, \mathcal{F}_A)$ be an m-bounded $\mathcal{F}$-algebra, and $\delta \in \mathbb{R}_{>0}$. Then, $>_\delta$ is a well-founded and stable (strict) ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$.*

Proof. Checking transitivity is easy; irreflexivity is a consequence of well-foundedness, which we prove as follows: assume that there is an infinite sequence

$$t_1 >_\delta t_2 >_\delta \cdots >_\delta t_n >_\delta \cdots$$

Since $[\alpha](t_i) - [\alpha](t_{i+1}) \geq \delta > 0$ for all $\alpha : \mathcal{X} \to A$, we have that, for an arbitrary valuation α_0, we can write: $[\alpha_0](t_{i+1}) \leq [\alpha_0](t_i) - \delta$ for $i \geq 1$. Let $M = [\alpha_0](t_1)$ and $n = 2 + \lceil \frac{M-m}{\delta} \rceil$. Then, $[\alpha_0](t_n) \leq [\alpha_0](t_1) - (n-1) \times \delta = M - (1 + \lceil \frac{M-m}{\delta} \rceil) \times \delta \leq M - \frac{M-m}{\delta} \times \delta - \delta = m - \delta < m$ thus contradicting the m-boundedness of $\mathcal{A}$. Stability of $>_\delta$ is proved as in Proposition 1. $\square$

Requiring m-boundedness in Theorem 1 is necessary: the algebra $(\mathbb{R}, \{a_{\mathbb{R}}, c_{\mathbb{R}}\})$ given by $a_{\mathbb{R}} = 2$ and $c_{\mathbb{R}}(x) = -x^2$ is not m-bounded for any $m \in \mathbb{R}$ and the following infinite decreasing sequence of terms is possible:

$$a >_1 c(a) >_1 c(c(a)) >_1 \cdots >_\delta c^n(a) >_1 \cdots$$

With $\delta = 0$, well-foundedness of $>_\delta$ is not guaranteed: Consider now $A = \mathbb{R}_0$, $a_A = 1$ and $c_A(x) = \frac{1}{2}x$; the algebra $(A, \{a_A, c_A\})$ is 0-bounded but the previous infinite decreasing sequence is also possible with $>_0$. Also, requiring a positive but "local" δ to compare two given terms does not work: consider $>'$ given by

$$t >' s \Leftrightarrow \exists \delta \in \mathbb{R}_{>0}, \forall \alpha : \mathcal{X} \to A, [\alpha](t) - [\alpha](s) \geq \delta.$$

Since $[c^n(a)] = \frac{1}{2^n}$ for all $n \geq 0$, the previous infinite sequence is also possible with $>'$. Finally, for similar reasons, using $\delta = 0$ and $>$ instead of $\geq$ to compare $[\alpha](t)$ and $[\alpha](s)$ in Theorem 1 does not work either.

Remark 1. For an m-bounded $\mathcal{F}$-algebra $(A, \mathcal{F}_A)$ over the reals, we can consider $(A', \mathcal{F}_{A'})$ where $A' = [m, +\infty[\cap A$ and the mappings in $\mathcal{F}_{A'}$ are the restrictions to A' of the mappings in $\mathcal{F}_A$. This generates the same ordering $>_\delta$ on terms which, in fact, is the ordering on terms induced by the (well-founded) ordered algebra $(A', \mathcal{F}_{A'}, >_{\mathbb{R},\delta})$. On the other hand, if $\mathcal{A} = (A, \mathcal{F}_A)$ is an $\mathcal{F}$-algebra with $A \subseteq \mathbb{R}_m$ for some $m \in \mathbb{R}$, then $(A, \mathcal{F}_A)$ is m-bounded.

In order to use an ordering $>_\delta$ (induced by an m-bounded algebra) for proving termination of rewriting, we have to further ensure that $>_\delta$ is *monotonic*. The following example shows the use of Theorem 1 to prove termination of TRSs.

Example 3. Consider the TRS $\mathcal{R} = (\mathcal{F}, R)$ [44], Example 6.2.22:

$\mathtt{f(f(x))} \to \mathtt{f(g(f(x)))}$

and the 0-bounded algebra $(A, \mathcal{F}_A)$, where $A = \mathbb{R}_0$,

$$\mathtt{f}_A(x) = \lceil x \rceil + \frac{1}{2} \quad \text{and} \quad \mathtt{g}_A(x) = \lfloor x \rfloor$$

(here, $\lceil x \rceil$ is the least integer above –or equal to– x and $\lfloor x \rfloor$ is the integer part of x). Note that $>_1$ is monotonic: if $t >_1 s$, then $[\alpha](t) \geq [\alpha](s) + 1$ for all valuations $\alpha : \mathcal{X} \to A$; hence, since $\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$ and $\lceil x + 1 \rceil = \lceil x \rceil + 1$, we have $[\alpha](\mathtt{f}(t)) - [\alpha](\mathtt{f}(s)) = \lceil [\alpha](t) \rceil - \lceil [\alpha](s) \rceil \geq \lceil [\alpha](s) + 1 \rceil - \lceil [\alpha](s) \rceil = 1$. Similarly, $[\alpha](\mathtt{g}(t)) - [\alpha](\mathtt{g}(s)) = \lfloor [\alpha](t) \rfloor - \lfloor [\alpha](s) \rfloor \geq \lfloor [\alpha](s) + 1 \rfloor - \lfloor [\alpha](s) \rfloor = 1$. Thus, by Theorem 1, $>_1$ is a reduction ordering. On the other hand, we have:

$$\begin{array}{rcl} [\alpha](\mathtt{f(f(x))}) & = & \lceil \lceil \alpha(x) \rceil + \frac{1}{2} \rceil + \frac{1}{2} = \lceil \alpha(x) \rceil + \frac{3}{2}, \text{ and} \\ [\alpha](\mathtt{f(g(f(x)))}) & = & \lceil \lfloor (\lceil \alpha(x) \rceil + \frac{1}{2}) \rfloor \rceil + \frac{1}{2} = \lceil \alpha(x) \rceil + \frac{1}{2}. \end{array}$$

Therefore:

$$[\alpha](\mathtt{f(f(x))}) - [\alpha](\mathtt{f(g(f(x)))}) = \lceil \alpha(x) \rceil + \frac{3}{2} - \left(\lceil \alpha(x) \rceil + \frac{1}{2} \right) = 1.$$

Then, $\mathtt{f(f(x))} >_1 \mathtt{f(g(f(x)))}$ and $\mathcal{R}$ is terminating.

3.1. Monotonicity

The interpretation in Example 3 can also be used to show that monotonicity does not need to be preserved when the value of δ is changed to either some $\delta' < \delta$ or some $\delta' > \delta$.

Example 4. Consider again the interpretation in Example 3. We have that

(1) $>_\delta$ is not monotonic if $\delta = \frac{1}{2} + n$ for some $n \in \mathbb{N}$: given a valuation α, $[\alpha](f^{n+1}(x)) = \lceil \alpha(x) \rceil + n + \frac{1}{2} = \lceil \alpha(x) \rceil + \delta \geq \alpha(x) + \delta \geq [\alpha](x)$, *i.e.*, $t = f^{n+1}(x) >_\delta x = s$, but $[\alpha](g(t)) = \lfloor \lceil [\alpha](x) \rceil + \delta \rfloor = \lceil [\alpha](x) \rceil + n$ and $[\alpha](g(s)) = \lfloor [\alpha](x) \rfloor$. Since the valuation α which assigns the value 0 to x verifies $[\alpha](g(t)) = n$ and $[\alpha](g(s)) = 0$, it follows that $g(t) \not>_\delta g(s)$, *i.e.*, $>_\delta$ is not monotonic in the argument of g.

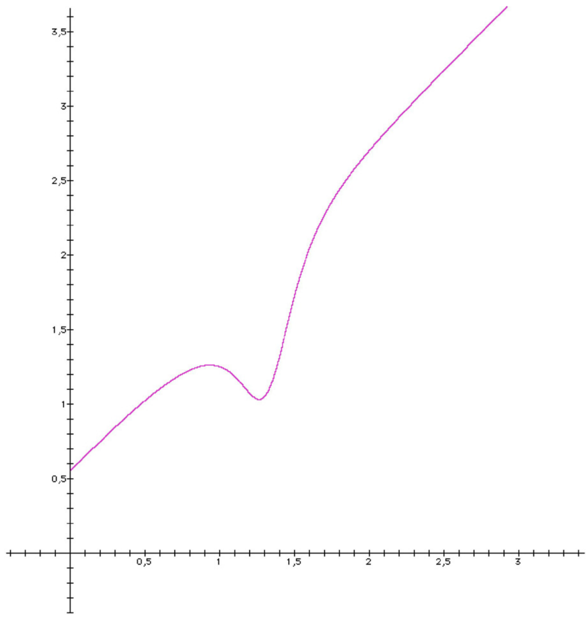

FIGURE 1. Detail of the function $\frac{3}{4} + x - \frac{1}{1+(x^{\frac{8}{3}}-2)^2}$ in Example 5.

(2) $>_\delta$ is monotonic if $\delta = n$ for any $n \in \mathbb{N}_1$: the proof is similar to that given in Example 3 for $>_1$.

The interpretation in Example 3 is based on non-continuous functions. The following example shows that even with continuous and differentiable functions, monotonicity of $>_\delta$ can highly depend on the selected value for δ.

Example 5. Let $\mathcal{F} = \{a, f, g\}$, where a is a constant and f and g are unary symbols. Let $A = \mathbb{R}_1$ and consider the 1-bounded algebra $\mathcal{A} = (A, \mathcal{F}_A)$, where

$$\begin{array}{rcl} a_A & = & \sqrt[16]{2^3} = \sqrt[16]{8} \simeq 1.1388 \\ f_A(x) & = & x^2 \\ g_A(x) & = & \dfrac{3}{4} + x - \dfrac{1}{1+(x^{\frac{8}{3}}-2)^2}. \end{array}$$

It is possible to see that $>_{\frac{5}{9}}$ is monotonic. Let $\delta = \sqrt[8]{8} - \sqrt[16]{8} \approx 0.1581$. Now, $>_\delta$ is *not* monotonic. Figure 1 is helpful to understand why monotonicity of $>_\delta$ in the first argument of $g_\mathcal{A}$ depends on the value of δ: the graph of $g_\mathcal{A}$ is almost a straight line except a "hole" roughly between 1 and 1.6. If the value of δ is big enough, then this hole is 'skipped' when comparing terms with $>_\delta$ and no lack of monotonicity is observed.

The following theorem provides a sufficient condition to ensure monotonicity of $>_\delta$ for an arbitrary δ (in a wide class of interpretations).

Theorem 2. *Let $\mathcal{F}$ be a signature and $A \subseteq \mathbb{R}$. Let $\mathcal{A} = (A, \mathcal{F}_A)$ be an $\mathcal{F}$-algebra, $f \in \mathcal{F}$, and $1 \leq i \leq k = ar(f)$ be such that f_A is continuous and differentiable in its i-th argument. If $\frac{\partial f_A(x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 1$, then for all $\delta \in \mathbb{R}_{>0}$, $>_\delta$ is monotonic in the i-th argument of f.*

Proof. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ be such that $t >_\delta s$ for an arbitrary $\delta \in \mathbb{R}_{>0}$. We have to prove that

$$f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) >_\delta f(t_1, \ldots, t_{i-1}, s, \ldots, t_k),$$

i.e., that, for all $\alpha : \mathcal{X} \to A$,

$$[\alpha]f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) - [\alpha]f(t_1, \ldots, t_{i-1}, s, \ldots, t_k) \geq \delta$$

equivalently, that, for all $\alpha : \mathcal{X} \to A$,

$$f_A(a_1, \ldots, a_{i-1}, [\alpha](t), \ldots, a_k) - f_A(a_1, \ldots, a_{i-1}, [\alpha](s), \ldots, a_k) \geq \delta$$

where $a_i = [\alpha](t_i)$ for $1 \leq i \leq k$. Given an arbitrary valuation α, we have $[\alpha](t) - [\alpha](s) \geq \delta$, *i.e.*, $[\alpha](t) > [\alpha](s)$. By the Mean Value Theorem, there is $z \in \mathbb{R}$ satisfying $[\alpha](s) < z < [\alpha](t)$ such that

$$\frac{\partial f_A}{\partial x_i}(a_1, \ldots, a_{i-1}, z, \ldots, a_k) = \frac{f_A(a_1, \ldots, a_{i-1}, [\alpha](t), \ldots, a_k) - f_A(a_1, \ldots, a_{i-1}, [\alpha](s), \ldots, a_k)}{[\alpha](t) - [\alpha](s)}.$$

By hypothesis,

$$\frac{\partial f_A}{\partial x_i}(a_1, \ldots, a_{i-1}, z, \ldots, a_k) \geq 1.$$

Thus, $f_A(a_1, \ldots, a_{i-1}, [\alpha](t), \ldots, a_k) - f_A(a_1, \ldots, a_{i-1}, [\alpha](s), \ldots, a_k) \geq [\alpha](t) - [\alpha](s) \geq \delta$. Thus, $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) >_\delta f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$. $\square$

The following example shows that, in general, the "only if" direction of Theorem 2 does not hold.

Example 6. Consider the signature $\mathcal{F} = \{a, f\}$ consisting of a constant symbol a and a unary symbol f. Let $\mathcal{A} = (\mathbb{R}_0, \mathcal{F}_{\mathbb{R}_0})$ be an $\mathcal{F}$-algebra where $f_\mathcal{A}(x) = 0$ and $a_\mathcal{A} = 0$. Then, for all valuation mapping α and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we have $[\alpha](t) = 0$. Therefore, for all $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $\delta > 0$, we have $t \not>_\delta s$ and $>_\delta$ is vacuously monotonic in the argument of f. However, $\frac{\partial f_A}{\partial x} = 0$.

Example 5 shows that monotonicity of $>_\delta$ for the corresponding interpretation depends on the concrete value of δ. Note that $\frac{\partial g_A}{\partial x} = 1 + \frac{\frac{16}{3}x^{\frac{5}{3}}(x^{\frac{8}{3}}-2)}{(1+(x^{\frac{8}{3}}-2)^2)^2} \not\geq 1$ for some values $x < \sqrt[8]{8} \simeq 1.2968$. Thus, requiring partial derivatives greather than or equal to 1 in Theorem 2 is necessary to ensure that monotonicity of $>_\delta$ for the interpretation does not depend on the selected δ. However, even with Theorem 2, the appropriate choice of δ for proving termination of TRSs is tricky.

Example 7. Consider the following TRS:

$\mathtt{f(f(a))} \to \mathtt{f(g(f(a)))}$

and the 1-bounded $\mathcal{F}$-algebra of Example 5. Note that $>_{\frac{5}{9}}$ is a reduction ordering on terms. Now,

$$\mathtt{f}_A(\mathtt{f}_A(\mathtt{a}_A)) = \sqrt[4]{8}$$

and

$$\mathtt{f}_A(\mathtt{g}_A(\mathtt{f}_A(\mathtt{a}_A))) = \left(\frac{3}{4} + \sqrt[8]{8} - 1\right)^2 = \left(\sqrt[8]{8} - \frac{1}{4}\right)^2 = \sqrt[4]{8} - \frac{1}{2}\sqrt[8]{8} + \frac{1}{16}.$$

Since

$$\begin{array}{rcl}\mathtt{f}_A(\mathtt{f}_A(\mathtt{a}_A)) - \mathtt{f}_A(\mathtt{g}_A(\mathtt{f}_A(\mathtt{a}_A))) & = & \sqrt[4]{8} - \left(\sqrt[4]{8} - \frac{1}{2}\sqrt[8]{8} + \frac{1}{16}\right) \\ & = & 0.58592 \geq \frac{5}{9}\end{array}$$

we conclude that $\mathtt{f(f(a))} >_{\frac{5}{9}} \mathtt{f(g(f(a)))}$ thus proving termination of $\mathcal{R}$.

However, if we take $A' = \mathbb{R}_{\frac{3}{2}}$ instead of $A = \mathbb{R}_1$ in the interpretation of Example 5, then $\frac{\partial g_{A'}}{\partial x} \geq 1$ (because this holds for all $x \geq \sqrt[8]{8}$), *i.e.*, monotonicity of $>_\delta$ is ensured for all $\delta > 0$. Now, however, we give $[\mathtt{a}]$ some real number above $\frac{3}{2}$; then there is no $\delta > 0$ such that $[\mathtt{f(f(a))}] >_\delta [\mathtt{f(g(f(a)))}]$.

The following result provides a sufficient condition ensuring weak monotonicity of $\gtrsim$ in the argument of a function symbol.

Proposition 2. *Let $\mathcal{F}$ be a signature and $A \subseteq \mathbb{R}$. Let $\mathcal{A} = (A, \mathcal{F}_A)$ be an $\mathcal{F}$-algebra, $f \in \mathcal{F}$, and $1 \leq i \leq k = ar(f)$ be such that f_A is continuous and differentiable in its i-th argument. If $\frac{\partial f_A(x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 0$, then $\gtrsim$ is weakly monotonic in the i-th argument of f.*

Proof. Similar to the proof of Theorem 2. □

The previous examples show that the use of arbitrary real functions for inducing reduction orderings can be quite involved regarding the selection of an appropriate value for δ. Now we investigate how to work without making δ *explicit*.

3.2. Avoiding the choice of δ

When using an algebra over the reals $\mathcal{A}$ to induce a well-founded and stable ordering $>_\delta$ which can be used in proofs of termination, we would like to disregard from choosing any value for δ. Theorem 2 provides quite a simple way to do it regarding the necessary checking of total or partial monotonicity of $>_\delta$: if the partial derivatives of each function w.r.t. the monotonic arguments are greather than or equal to 1, then we can use any (positive) value for δ. The next proposition provides a basis to avoid the explicit specification of δ when checking *compatibility* of the ordering with a set of pairs of terms (*e.g.*, the rules of a TRS or its dependency pairs).

In the following, when considering an algebra over the reals $\mathcal{A} = (A, \mathcal{F}_A)$ and terms $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, with $\mathcal{V}ar(t) \cup \mathcal{V}ar(s) = \{x_1, \ldots, x_n\}$, we write $[t] - [s]$ to denote the *function* $F_{t,s} : A^n \to \mathbb{R}$ given by $F_{t,s}(a_1, \ldots, a_n) = [\alpha_{(a_1,\ldots,a_n)}](t) - [\alpha_{(a_1,\ldots,a_n)}](s)$ for each tuple $(a_1, \ldots, a_n) \in A^n$, where $\alpha_{(a_1,\ldots,a_n)}(x_i) = a_i$ for $1 \leq i \leq n$.

Proposition 3. *Let $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$ be a finite set of pairs of terms and $\mathcal{A} = (A, \mathcal{F}_A)$ be an algebra over the reals. If for each $(t, s) \in T$, there is $\delta_{t,s} \in \mathbb{R}_{>0}$ such that $F_{t,s} = [t] - [s]$ is $\delta_{t,s}$-bounded in A, then $\delta = min(\{\delta_{t,s} \mid (t, s) \in T\})$ is positive and $t >_\delta s$ for all $(t, s) \in T$.*

Proof. Since $\delta_{t,s} > 0$ for all $(t, s) \in T$ and T is finite, δ is well-defined and, moreover, $\delta > 0$. Whenever $\delta_1 \geq \delta_2$, we have that $[t] - [s] \geq \delta_1$ (*i.e.*, $t >_{\delta_1} s$) implies that $[t] - [s] \geq \delta_2$, *i.e.*, $t >_{\delta_2} s$. Hence, the conclusion follows. $\square$

Let's illustrate the use of Proposition 3 in proofs of termination of *CSR*.

Example 8. Consider the following TRS $\mathcal{R}$ borrowing the well-known Toyama's example:

```
c -> a      f(a,b,x) -> f(x,x,x)
c -> b
```

together with $\mu(\mathtt{f}) = \{1, 3\}$. Let $\mathcal{A} = (\mathbb{R}_1, \mathcal{F}_{\mathbb{R}_1})$, where $\mathcal{F}_{\mathbb{R}_1}$ is:

$$\begin{array}{rclcrcl} [\mathtt{f}](x,y,z) & = & x + xy^{-1} + zy^{-1} + z = \frac{(xy+z)+(x+yz)}{y} & \quad & [\mathtt{a}] & = & 2 \\ [\mathtt{c}] & = & 3 & & [\mathtt{b}] & = & 1. \end{array}$$

We have:

$$\frac{\partial[\mathtt{f}]}{x} = 1 + y^{-1} \geq 1 \qquad \frac{\partial[\mathtt{f}]}{z} = 1 + y^{-1} \geq 1.$$

Thus, by Theorem 2, $>_\delta$ is μ-monotonic for each $\delta \in \mathbb{R}_{>0}$. Regarding the rules of $\mathcal{R}$, we have:

$$\begin{array}{rcl} [\mathtt{f(a,b,x)}] - [\mathtt{f(x,x,x)}] & = & 2x + 4 - (2x + 2) = 2 > 0 \\ [\mathtt{c}] - [\mathtt{a}] & = & 3 - 2 = 1 > 0 \\ [\mathtt{c}] - [\mathtt{b}] & = & 3 - 1 = 2 > 0. \end{array}$$

As in the proof of Proposition 3, we let $\delta = \min(\{2, 1\}) = 1$. Then, $>_1$ is compatible with the rules of $\mathcal{R}$. Since $\mathcal{A}$ is 1-bounded, by Theorem 1 this proves the μ-termination of $\mathcal{R}$. Note that *no* given δ is needed in the proof!

The results in this section provide a general framework to prove termination by using (quasi-) orderings induced by algebras over the reals. Of course, in order to achieve automatic proofs we need to *generate* such algebraic interpretations rather than relying on a given one. As we will see in the following sections, when considering polynomial interpretations, we can automatically generate them to fulfill the conditions in Proposition 1, Theorem 2, Proposition 2, and Proposition 3, thus avoiding any explicit choice of δ in the implementation of automatic proofs of termination.

4. Termination of TRSs using dependency pairs

As explained in Section 2, when using the dependency pairs method [1], we can prove termination of a TRS $\mathcal{R}$ by showing that there is a reduction pair $(\succeq, \sqsupset)$ such that the *lhs* and *rhs* of each rule of $\mathcal{R}$ are comparable by using $\succeq$ whereas the components of each dependency pair are comparable by using $\sqsupset$.

Example 9. Consider the TRS $\mathcal{R}$ which is part of the TPDB[3]:

```
f(f(x))    -> f(g(f(g(f(x)))))
f(g(f(x))) -> f(g(x)).
```

Termination of $\mathcal{R}$ can be proved by finding a reduction pair $(\succeq, \sqsupset)$ such that:

`f(f(x))`	$\succeq$	`f(g(f(g(f(x)))))`	`F(f(x))`	$\sqsupset$	`F(g(f(g(f(x)))))`
`f(g(f(x)))`	$\succeq$	`f(g(x))`	`F(f(x))`	$\sqsupset$	`F(g(f(x)))`
			`F(f(x))`	$\sqsupset$	`F(x)`
			`F(g(f(x)))`	$\sqsupset$	`F(g(x))`

where `F` is the tuple symbol that corresponds to `f`.

Now we show that algebras over the reals are useful to generate reduction pairs.

Proposition 4. *Let $\mathcal{F}$ be a signature, $m \in \mathbb{R}$, $\delta \in \mathbb{R}_{>0}$, $A \subseteq \mathbb{R}$, and $\mathcal{A} = (A, \mathcal{F}_A)$ be an m-bounded $\mathcal{F}$-algebra such that $\gtrsim$ is weakly monotonic. Then, $(\gtrsim, >_\delta)$ is a reduction pair.*

Proof. By Proposition 1 and by hypothesis, $\gtrsim$ is a stable, weakly monotonic quasi-ordering. By Theorem 1, $>_\delta$ is a well-founded and stable ordering. We also have $\gtrsim \circ >_\delta \subseteq >_\delta$: if there is u such that $[\alpha](t) - [\alpha](u) \geq 0$ and $[\alpha](u) - [\alpha](s) \geq \delta$ for all $\alpha : \mathcal{X} \to A$, then $[\alpha](t) - [\alpha](u) + [\alpha](u) - [\alpha](s) = [\alpha](t) - [\alpha](s) \geq 0 + \delta = \delta$, *i.e.*, $t >_\delta s$. Thus, $(\gtrsim, >_\delta)$ is a *reduction pair.* □

The absence of monotonicity requirements in reduction pairs (apart from weak monotonicity) corresponds to the use of the *least* replacement map $\mu_\perp(f) = \varnothing$ for all $f \in \mathcal{F}$ which expresses *no* monotonicity requirements for $>_\delta$. Now, we can define an algebra over the reals which makes $\gtrsim$ and $>_\delta$ compatible with the rules and the dependency pairs of Example 9.

Example 10. The polynomial interpretation $(\mathbb{R}_0, \mathcal{F}_{\mathbb{R}_0})$ with:

$$\begin{array}{rcl@{\qquad}rcl} [\mathtt{f}](x) & = & x + 4 & [\mathtt{F}](x) & = & x \\ [\mathtt{g}](x) & = & \frac{1}{2}x & & & \end{array}$$

defines a reduction pair $(\gtrsim, >_1)$ which proves termination of $\mathcal{R}$ in Example 9.

[3] Termination Problems Data Base, see `http://www.lsi.upc.es/~albert/tpdb.html` and also `http://www.lri.fr/~marche/wst2004-competition/tpdb/Rubio/aoto.trs`.

In fact, Arts and Giesl already noticed that the polynomials used with dependency pairs *do not necessarily depend on all their arguments* [1], page 142, *i.e.*, they can induce non-monotonic orderings. We will see that the use of positive real coefficients *below the unit* (*i.e.*, between 0 and 1) to introduce non-monotonicity in the corresponding term orderings makes a difference which cannot be simulated by using polynomials over the naturals (see Prop. 8 below). In the proof of termination of Example 10, the difference is noticeable in that the proof which uses polynomials over the rationals is pretty simple and automatic[4], but it becomes more involved (in the sense that more elaborated techniques are required) or impossible when more traditional base orderings are used in combination with the dependency pairs approach. Other interesting examples can also be given.

Example 11. Consider the following TRS [2] Example 3.42, (originally due to Alfons Geser):

```
half(0) -> 0
half(s(0)) -> 0
half(s(s(x))) -> s(half(x))
lastbit(0) -> 0
lastbit(s(0)) -> s(0)
lastbit(s(s(x))) -> lastbit(x)
conv(0) -> cons(nil,0)
conv(s(x)) -> cons(conv(half(s(x))),lastbit(s(x)))
```

The set of dependency pairs for $\mathcal{R}$ is:

$$\begin{array}{lrll}
\langle & \mathtt{HALF(s(s(x)))} & , \mathtt{HALF(x)} & \rangle \\
\langle & \mathtt{LASTBIT(s(s(x)))} & , \mathtt{LASTBIT(x)} & \rangle \\
\langle & \mathtt{CONV(s(x))} & , \mathtt{CONV(half(s(x)))} & \rangle \\
\langle & \mathtt{CONV(s(x))} & , \mathtt{HALF(s(x))} & \rangle \\
\langle & \mathtt{CONV(s(x))} & , \mathtt{LASTBIT(s(x))} & \rangle
\end{array}$$

The following polynomial interpretation:

$$\begin{array}{rclrcl}
[\mathtt{half}](x) & = & \frac{1}{2}x & [\mathtt{cons}](x,y) & = & 0 \\
[\mathtt{0}] & = & 0 & [\mathtt{nil}](x) & = & 0 \\
[\mathtt{s}](x) & = & x+1 & [\mathtt{HALF}](x) & = & x \\
[\mathtt{lastbit}](x) & = & 1 & [\mathtt{LASTBIT}](x) & = & x \\
[\mathtt{conv}](x) & = & 0 & [\mathtt{CONV}](x) & = & 2x
\end{array}$$

defines a reduction pair $(\gtrsim, >_1)$ which proves termination of $\mathcal{R}$. Termination of $\mathcal{R}$ is proved in [2] by transforming the dependency pairs using *narrowing* [1,2].

Many refinements in the use of dependency pairs are possible (see, *e.g.*, [1,16,24,42]). They are intended to remove or simplify as many rules and dependency pairs as possible before trying to use a reduction pair to compare the

[4] For instance, MU-TERM can be used to obtain this proof.

corresponding components. The use of non-monotonic interpretations over the reals for generating such reduction pairs complements these improvements, although more research should be done to fix the best conditions for using all together.

5. Polynomial interpretations

Let $\mathbb{A}$ be a subring of a commutative ring $\mathbb{B}$. Let $x_1, \ldots, x_n \in \mathbb{B}$. For each n-tuple $(r_1, \ldots, r_n) = (r) \in \mathbb{N}^n$ (called a multi-index), we use vector notation, letting $(x) = (x_1, \ldots, x_n)$, and $\pi_r(x) = x_1^{r_1} \cdots x_n^{r_n}$ [31]. Such products π_r (for each multi-index $r \in \mathbb{N}^n$) are called (primitive) monomials in n variables over $\mathbb{B}$. $\sum_{i=1}^n r_i$ is the *degree* of the monomial; if $r_1 = r_2 = \cdots = r_n = 0$, then π_r is the constant monomial 1.

A polynomial P in n variables over $\mathbb{B}$ with coefficients in $\mathbb{A}$ is the sum $P = \sum_{r\in\mathbb{N}^n} a_r \cdot \pi_r(x)$, where $a_r \in \mathbb{A}$, of finitely many monomials in n variables over $\mathbb{B}$. The set of such polynomials is denoted by $\mathbb{A}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are distinct variables. When variables $x_1, \ldots, x_n$ range on $\mathbb{B}$, P induces a (polynomial) function $P(x_1, \ldots, x_n) : \mathbb{B}^n \to \mathbb{B}$. In the following, we will write $\pi_r \in P$ (or just $\pi \in P$) to express that π_r is a monomial of a polynomial P such that $a_r \neq 0$. Moreover, for a given monomial $\pi_r \in P$ we let $\mathrm{coef}(\pi_r) = a_r$, $deg_i(\pi_r) = r_i$ for $1 \leq i \leq n$, and $\mathcal{I}_\pi^+ = \{i \in \{1, \ldots, n\} \mid r_i \neq 0\}$.

The interpretation of a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ by a polynomial interpretation $(A, \mathcal{F}_A)$ yields a polynomial $[t]$ in n variables $x_1, \ldots, x_n$ where $\mathcal{V}ar(t) = \{x_1, \ldots, x_n\}$.

Remark 2. Note that, when using polynomial functions to build a polynomial algebraic interpretation $\mathcal{A} = (A, \mathcal{F}_A)$ for a signature $\mathcal{F}$, we have to guarantee that $\mathcal{A}$ is actually an algebra by ensuring that, for all k-ary symbols $f \in \mathcal{F}$, and $x_1, \ldots, x_k \in A$, $[f](x_1, \ldots, x_k) \in A$.

5.1. Implicit δ in polynomial interpretations

Continuing the discussion in Section 3.2, when polynomial interpretations are considered, we have a simple way to ensure the conditions in Proposition 3.

Proposition 5. *Let $\alpha \in \mathbb{R}_0$ be such that $\alpha \in A \subseteq \mathbb{R}_\alpha$, and $P \in \mathbb{R}_0[x_1, \ldots, x_n]$ be a polynomial without negative coefficients. Then $\beta = P(\alpha, \ldots, \alpha)$ is the minimum of P in A^n, i.e., for all $x_1, \ldots, x_n \in A$, $P(x_1, \ldots, x_n) \geq \beta$.*

Proof. The product of non-negative real numbers is monotone, *i.e.*, $\forall x, x', y, y' \in \mathbb{R}_0$, whenever $x \leq x'$ and $y \leq y'$, we have $x \cdot y \leq x' \cdot y'$. Let $\pi \in P$. Since $\alpha \geq 0$ and $\mathrm{coef}(\pi) > 0$, we have that $\mathrm{coef}(\pi) \cdot \pi(\alpha, \ldots, \alpha) \leq \mathrm{coef}(\pi) \cdot \pi(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in A$. Since the addition of real numbers is monotone, we further have $\beta = P(\alpha, \ldots, \alpha) = \sum_{\pi\in P} \mathrm{coef}(\pi) \cdot \pi(\alpha, \ldots, \alpha) \leq \sum_{\pi\in P} \mathrm{coef}(\pi) \cdot \pi(x_1, \ldots, x_n) = P(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in A$. □

Proposition 5 does not hold for arbitrary (even positive) polynomials.

Example 12. The following polynomial [3], Section 6.5.4:

$$P(x,y) = (xy-1)^2 + x^2 = x^2y^2 - 2xy + x^2 + 1$$

is positive for all $x, y \in \mathbb{R}$. Let's show that there is no $\beta > 0$ such that $P(x,y) \geq \beta$ for all $x, y \geq 0$: Since $P(\frac{1}{2}, 1) = \frac{1}{2}$, we can assume that $0 < \beta \leq \frac{1}{2}$. Then, $P(\beta, \frac{1}{\beta}) = \beta^2 < \beta$.

Proposition 5 is interesting when we *require* that the polynomial $P_{t,s}(x_1, \ldots, x_n) = [t] - [s]$ (with indeterminate coefficients!) which is obtained from the rules or dependency pairs of a TRS has positive coefficients as a suitable way to ensure positiveness of $P_{t,s}$ (see Sect. 6 below). The following corollary of Propositions 3 and 5 formalizes this point.

Corollary 1. *Let $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$ be a finite set of pairs of terms. Let $\alpha \in \mathbb{R}_0$ be such that $\alpha \in A \subseteq \mathbb{R}_\alpha$, and $\mathcal{A} = (A, \mathcal{F}_A)$ be a polynomial algebra. Given $(t,s) \in T$, let $P_{t,s}(x_1, \ldots, x_n) = [t] - [s]$. If for all $(t,s) \in T$, $P_{t,s}$ contains no negative coefficient and $\delta_{t,s} = P_{t,s}(\alpha, \ldots, \alpha) > 0$, then $\delta = min(\{\delta_{t,s} \mid (t,s) \in T\}$ satisfies $t >_\delta s$ for all $(t,s) \in T$.*

5.2. Polynomial interpretations with negative coefficients

Due to their potential for introducing non-monotonicity in the generated orderings, we are going to consider polynomials P possibly containing real negative coefficients, *i.e.*, $P \in \mathbb{R}[x_1, \ldots, x_n]$.

Example 13. Consider the TRS $\mathcal{R}$:

```
g(x) -> h(x)        h(d) -> g(c)
c    -> d
```

together with $\mu(\mathtt{g}) = \mu(\mathtt{h}) = \varnothing$ [43], Example 1. The μ-termination of $\mathcal{R}$ is proved with the following polynomial interpretation:

$$\begin{array}{rclcrcl} [\mathtt{g}](x) & = & x^2 - 3x + 4 & \quad & [\mathtt{c}] & = & 1 \\ [\mathtt{h}](x) & = & x^2 - 3x + 3 & & [\mathtt{d}] & = & 0. \end{array}$$

The use of negative coefficients in the interpretation is crucial in this example.

The following result imposes some general restrictions on the structure of (m-bounded) polynomials containing such *negative* coefficients: if an m-bounded polynomial P contains a negative monomial π containing a variable x raised to r, then either P contains a positive monomial with x raised to $r' > r$, or π is "contained" in a 0-bounded part of P.

Proposition 6. *Let $m \in \mathbb{R}$, $A \subseteq \mathbb{R}_0$ be unbounded, $P \in \mathbb{R}[x_1, \ldots, x_n]$ be m-bounded in A, and $\pi \in P$ be such that $\mathcal{I}^+_\pi \neq \varnothing$. If* $\mathrm{coef}(\pi) < 0$, *then, for all $i \in \mathcal{I}^+_\pi$, either*

(1) *there is a monomial $\pi' \in P$ satisfying* $\mathrm{coef}(\pi') > 0$ *and $deg_i(\pi') > r_i$; or*

(2) *there is* $P' \in \mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ *such that* P' *is* 0*-bounded in* A, $\frac{\pi}{x_i^{r_i}} \in P'$, *and* $P' \cdot x_i^{r_i} \subseteq P$,

where $r_i = deg_i(\pi)$.

Proof. We proceed by contradiction. Let $i \in \mathcal{I}_\pi^+$ and assume that, for all monomial $\pi' \in P$, we have $\text{coef}(\pi') \leq 0$ whenever $deg_i(\pi') > r_i$. Moreover, let $P' \cdot x_i^{r_i}$ be the polynomial obtained by taking all monomials $\pi' \in P$ such that $deg_i(\pi') = r_i$ (in particular, π belongs to such a polynomial). Then, $P' \in \mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, $\frac{\pi}{x_i^{r_i}} \in P'$, and we can assume that P' is not 0-bounded in A. Thus, there are $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n \in A$ such that

$$P'(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) < 0.$$

We obtain that $P(z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n)$ can be written as follows:

$$a_n x_i^n + \cdots + a_{r_i+1} x_i^{r_i+1} + a_{r_i} x_i^{r_i} + a_{r_i-1} x_i^{r_i-1} + \cdots + a_1 x_i + a_0$$

where $a_n, \ldots, a_{r_i+1} \leq 0$ (because all monomials $\pi' \in P$ with $deg_i(\pi') > r_i$ satisfy $\text{coef}(\pi') \leq 0$ and we take $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n \geq 0$) and

$$a_{r_i} = P'(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) < 0.$$

Let $Q(x_i) = a_{r_i} x_i^{r_i} + a_{r_i-1} x_i^{r_i-1} + \cdots + a_1 x_i + a_0 \in \mathbb{R}[x_i]$; then, for all $x_i \geq 0$,

$$P(z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n) \leq Q(x_i).$$

Since $a_{r_i} < 0$, we have that $lim_{x \to +\infty} Q(x) = -\infty$, *i.e.*, for all $M \in \mathbb{R}$, there is $x \in \mathbb{R}_{\geq 0}$ such that, for all $y \geq x$, $Q(y) < M$. In particular, since A is not bounded, there is $x \in A$ such that $Q(x) < m$. Then,

$$P(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \leq Q(x) < m.$$

This contradicts m-boundedness of P in A. □

The following example shows the need of the two cases in Proposition 6:

Example 14. The polynomial $P(x, y) = x^2 - 2xy + y^2 = (x - y)^2$ is 0-bounded in $\mathbb{R}_0$ (actually in $\mathbb{R}$). Note that the negative monomial $-2xy$ does not satisfy 2 in Proposition 6. On the other hand, $P(x, y) = xy - y$ is 0-bounded in $[1, +\infty[$ and it does not satisfy 1 in Proposition 6.

5.3. Polynomial interpretations and non-monotonic orderings

The μ-reduction orderings can also be defined by means of m-bounded $\mathcal{F}$-algebras over the reals. Well-foundedness and stability of $>_\delta$ is already ensured by Theorem 1. The μ-monotonicity requirements can be guaranteed by using Theorem 2; Proposition 3 can be eventually used to avoid an explicit δ (see Ex. 8).

We show that the computation of non-monotonic orderings can greatly benefit both from the use of polynomial interpretations over $\mathbb{N}$ (instead of $\mathbb{N}_1$) and $\mathbb{Q}_0$ (instead of $\mathbb{N}$).

Proposition 7. *There is a TRS which can be proved μ-terminating by using a polynomial interpretation over $\mathbb{N}$ whereas it cannot be proved by using a polynomial interpretation over $\mathbb{N}_1$.*

Proof. Consider the TRS $\mathcal{R}$:

```
if(true,x,y)  -> x       f(x) -> if(x,c,f(true))
if(false,x,y) -> y
```

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{if}) = \{1, 2\}$ [43], Example 5. The μ-termination of $\mathcal{R}$ can be proved by using the ordering $>_1$ induced by the following polynomial interpretation over $\mathbb{N}$:

$$\begin{array}{rcl@{\qquad}rcl} [\mathtt{f}](x) & = & 3x+2 & [\mathtt{true}] & = & 0 \\ [\mathtt{if}](x,y,z) & = & xz+x+y+1 & [\mathtt{false}] & = & 1. \\ [\mathtt{c}] & = & 0. & & & \end{array}$$

Assume that there is a polynomial interpretation $(\mathbb{N}_1, \mathcal{F}_{\mathbb{N}_1})$ and some $\delta > 0$ such that $>_\delta$ is a μ-reduction ordering which is compatible with the rules of $\mathcal{R}$. Such an interpretation includes a polynomial $[\mathtt{if}](x, y, z)$ which must contain a monomial π containing z; otherwise, compatibility of $>_\delta$ with the second rule for `if` would be impossible. Since coef(π) is a positive natural number, we have coef(π) ≥ 1 and hence, since all coefficients in polynomials are natural numbers and $x, y, z \geq 1$, $\frac{\partial[f]}{\partial z} \geq \frac{\partial \pi}{\partial z} \geq 1$. Thus, by Theorem 2, $>_\delta$ is monotonic in the third argument of $[\mathtt{if}]$. Hence, $>_\delta$ is a reduction ordering compatible with $\mathcal{R}$. But this is not possible, since $\mathcal{R}$ is not terminating. □

Proposition 8. *There is a TRS $\mathcal{R}$ which can be proved μ-terminating by using a polynomial interpretation over $\mathbb{Q}_0$ whereas it cannot be proved terminating by using a polynomial interpretation over $\mathbb{N}$.*

Proof. Consider the TRS $\mathcal{R}$:

```
zeros   -> 0:zeros
tl(x:y) -> y
```

together with $\mu(\mathtt{:}) = \{1\}$. The μ-termination of $\mathcal{R}$ can be proved by using the ordering $>_1$ induced by the following polynomial interpretation:

$$\begin{array}{rcl@{\qquad}rcl} [\mathtt{zeros}] & = & 2 & x\,[\mathtt{:}]\,y & = & x + \frac{1}{2}y \\ [\mathtt{0}] & = & 0 & [\mathtt{tl}](x) & = & 2x+1. \end{array}$$

Note that $>_1$ is μ-monotonic: since $\frac{\partial[\mathtt{:}]}{\partial x} = 1 \geq 1$ and $\frac{\partial[\mathtt{tl}]}{\partial x} = 2 \geq 1$, by Theorem 2, μ-monotonicity follows.

In order to see that there is no polynomial interpretation over the naturals which prove μ-termination of $\mathcal{R}$, note that the polynomial [:] for ":" should, then, contain a monomial $a\,x^m y^n$ with $m \in \mathbb{N}$, $a, n \in \mathbb{N}_1$; otherwise, the rule `tl(x:y) -> y` cannot be oriented (because [`tl`] would not receive any information about the contents of variable `y`). Consider two cases for m:

(1) $m = 0$. In this case, for the rule `zeros -> 0:zeros` we would have

$$[\texttt{zeros}] = c > [\texttt{0:zeros}] \geq a\,c^n$$

which is not possible if $a, c, n \geq 1$, because $a\,c^n \geq c$.

(2) $m > 0$. We consider two cases:

(a) Constant `0` is interpreted as 0. This means that variables can also take value 0. In this case, when considering valuations α such that $\alpha(\texttt{x}) = 0$, for the rule `tl(x:y) -> y` we would have $[\alpha](\texttt{tl(x:y)}) = [\texttt{tl}](0[\texttt{:}]\alpha(y)) = [\texttt{tl}](P_{\texttt{:}}(\alpha(y)))$, where $P_{\texttt{:}}$ is the polynomial in a single variable y obtained from [`:`] by removing all monomials of positive degree in x. Since

$$[\alpha](\texttt{tl(x:y)}) = [\texttt{tl}](P_{\texttt{:}}(\alpha(y))) > \alpha(y) = [\alpha](\texttt{y})$$

$P_{\texttt{:}}$ must contain a monomial of positive degree (in y). This means that $x[\texttt{:}]y$ contains at least a monomial of positive degree in y and degree 0 in x. By reasoning as in 1 above, we conclude that such a polynomial interpretation cannot deal with the rule `zeros -> 0:zeros`.

(b) Constant `0` is interpreted as $d > 0$. Then, for `zeros -> 0:zeros` we would need to have: $[\texttt{zeros}] = c > [\texttt{0:zeros}] \geq a\,d^m c^n$, which is not possible if $a, c, d, m, n \geq 1$, because $a\,d^m c^n \geq c$.

□

The proof of Proposition 8 shows why coefficients ranging between 0 and 1 are useful: by giving "`:`" a polynomial interpretation [`:`] whose second (non-replacing) argument contributes as half of its value we can deal with recursive calls in right-hand sides (as in the rule `zeros -> 0:zeros`) whilst sufficient information is still kept to be used in left-hand sides (as in the rule `tl(x:y) -> y`). A similar reasoning shows that the TRS in Example 1, whose polynomial termination is proved below by means of a polynomial interpretation over the rationals, cannot be proved μ-terminating by using polynomials over the natural numbers.

Example 15. The μ-termination of $\mathcal{R}$ in Example 1 can be proved by using the following polynomial interpretation:

$$\begin{array}{rclcrcl}
[\texttt{nil}] & = & 1 & & [\texttt{zip}](x,y) & = & 2x+2y \\
[\texttt{pairNs}](x) & = & 3 & & [\texttt{take}](x,y) & = & x+y+2 \\
[\texttt{cons}](x,y) & = & x+\frac{1}{4}y+1 & & [\texttt{tail}](x) & = & 4x \\
[\texttt{pair}](x,y) & = & x+y & & [\texttt{repItems}](x) & = & 2x \\
[\texttt{0}] & = & 0 & & [\texttt{oddNs}](x) & = & 5 \\
[\texttt{s}](x) & = & x & & [\texttt{incr}](x) & = & x+1
\end{array}$$

which can be computed automatically by using MU-TERM.

6. Automatic generation of polynomial interpretations with real coefficients

Polynomial interpretations are well-suited to *mechanize* the proofs of termination [30]. In [9], Contejean, Marché, Tomás, and Urbain explain how to proceed for *searching* a polynomial interpretation over the naturals (if any) which proves termination of a TRS $\mathcal{R}$.

Regarding the domain A of the computed polynomial algebra, Contejean *et al.* fix $A = \mathbb{N}$ and prove that every proof of termination obtained by using a polynomial algebra $(\mathbb{N}_n, \mathcal{F}_{\mathbb{N}_n})$ for a given $n > 0$ can be also obtained by a polynomial algebra $(\mathbb{N}, \mathcal{F}_{\mathbb{N}})$ by a simple translation [9], Proposition 3.12.

In their approach, polynomials $[f]$ only contain non-negative integer coefficients, *i.e.*, $[f] \in \mathbb{N}[x_1, \ldots, x_k]$ for each $f \in \mathcal{F}$. In order to generate them, they associate a *parametric* polynomial to each $f \in \mathcal{F}$. A parametric polynomial is a polynomial whose coefficients are *variables* whose values have to be found in some way. In order to fix values for the indeterminate coefficients (thus obtaining a polynomial reduction ordering $\succ$ which proves termination of $\mathcal{R}$), the following constraints are imposed [9], Section 4.1:

(1) For each $f \in \mathcal{F}$, all coefficients in $[f]$ are non-negative integers. Since the domain of the interpretation is $\mathbb{N}$, this guarantees that the polynomial functions actually define a proper (well-founded) algebra (see Rem. 2).
(2) For each k-ary symbol $f \in \mathcal{F}$ and $i \in \{1, \ldots, k\}$, $[f]$ contains a monomial x_i with $\mathrm{coef}(x_i) \geq 1$. This guarantees *monotonicity* of $\succ$.
(3) For each $l \to r \in R$, $[l] - [r] \geq 1$, which actually means that *all coefficients in each polynomial* $P_{l,r} = [l] - [r] - 1$ *are non-negative*, *i.e.*, the constraint $[l] - [r] \geq 1$ actually generates a constraint $\mathrm{coef}(\pi) \geq 0$ for each monomial $\pi \in P_{l,r}$. This guarantees *compatibility* of $\succ$ with the rules of $\mathcal{R}$.

The obtained *Diophantine constraints* can be solved by using existing algorithms.

As discussed above, non-monotonic polynomial interpretations can be useful in many termination problems. In our setting, we discuss two main termination problems: the μ-termination of TRSs and termination of TRSs by using dependency pairs. In the following, we discuss how to automatically compute non-monotonic polynomial interpretations over the reals which are useful for this purposes.

6.1. Proof of polynomial μ-termination of *CSR*

In order to obtain a polynomial proof of μ-termination of a TRS $\mathcal{R} = (\mathcal{F}, R)$, we will use parametric polynomial interpretations whose indeterminate coefficients are intended to be *real* instead of natural numbers, *i.e.*, $[f] \in \mathbb{R}[x_1, \ldots, x_k]$ for each k-ary symbol $f \in \mathcal{F}$.

Our interpretation domain is $\mathbb{R}_0$. As in Contejean *et al.*'s work, this choice is actually justified by [9], Proposition 3.12, because it is easy to check that the proof remains valid when we use polynomial interpretations with real coefficients and the ordering $>_\delta$ between terms (for any $\delta > 0$). On the other hand, Corollary 1 also suggests to use $\mathbb{R}_0$, where we are able to easily avoid fixing an explicit value for δ. Proposition 7 also motivates the inclusion of 0 in the interpretation. As mentioned in Remark 1, this choice makes our polynomial interpretation 0-bounded. For this, however, we have to ensure that all polynomials receiving values in $\mathbb{R}_0$ also return values in $\mathbb{R}_0$ (Rem. 2).

Our parametric polynomials are intended to have real (possibly negative) coefficients. Then, we impose restrictions on the indeterminate coefficients to (try to) compute a polynomial interpretation inducing a μ-reduction ordering $>_\delta$ which proves the μ-termination of $\mathcal{R}$. As discussed in Section 5.1, however, when using polynomial interpretations we can easily avoid fixing a concrete δ. According to this, the set of constraints is obtained by requiring the polynomial interpretations to satisfy the following conditions which ensure that we are using a μ-reduction ordering (items 1 and 2) which is compatible with $\mathcal{R}$ (item 3):

(1) Properly defined polynomial interpretation over $\mathbb{R}_0$: $[f](x_1,\ldots,x_k) \geq 0$ for all k-ary symbols $f \in \mathcal{F}$ and $x_1,\ldots,x_k \geq 0$. This also guarantees 0-boundedness (Rem. 2). By Theorem 1, $>_\delta$ is well-founded and stable for any $\delta > 0$, in particular, for the *implicit* δ which can be obtained from the computed interpretation (see item 3 below).

(2) μ-monotonicity: $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 1$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and $x_1,\ldots,x_k \geq 0$. By Theorem 2, this guarantees the μ-monotonicity of $>_\delta$ for *all* $\delta > 0$.

(3) Compatibility with the rules of $\mathcal{R}$: For each $l \to r \in R$, we require $[l]-[r] > 0$ for all $x_1,\ldots,x_n \geq 0$, where $x_1,\ldots,x_n$ are the variables in $\mathcal{V}\mathrm{ar}(l) \cup \mathcal{V}\mathrm{ar}(r)$. More precisely: we impose that the constant coefficient $\pi_{0\cdots 0}$ of $P_{l,r} = [l]-[r]$ is positive, and *all other coefficients in* $P_{l,r}$ *are non-negative*. Thus, the constraint $[l]-[r] > 0$ is actually implied by the conjunction of a constraint $\mathrm{coef}(\pi_{0\cdots 0}) > 0$ and constraints $\mathrm{coef}(\pi) \geq 0$ for each $\pi \in P_{l,r}$ such that $\pi \neq \pi_{0\cdots 0}$. By Corollary 1, this guarantees the *compatibility* of $>_\delta$ for the *implicit* $\delta = \min(\{\mathrm{coef}(\pi_{0\cdots 0}) \mid \pi_{0\cdots 0} \in P_{l,r}, l \to r \in R\})$.

This set of constraints is intended to be solved *in the domain of the real numbers*. In contrast to [9], we do not deal with Diophantine constraints. Although such polynomial constraints over the reals are decidable [41], the difficulty of the procedure depends on the degree and composition of the parametric polynomials that we use for this. As in [9], we consider three classes of polynomials which are well-suited for automatization of termination proofs: linear [30], simple, and simple-mixed [39] polynomial interpretations. Under the conditions described above, these polynomial interpretations do *not* admit negative coefficients (this is justified below). We describe a new class of interpretations, called *2-simple-mixed*, which are well-suited for this purpose.

In the following, we investigate how to proceed with these polynomial interpretations. Note, however, that only the treatment of items 1 and 2 above will actually vary depending on the concrete shape of the polynomials used in the interpretation. The treatment of item 3 is the same in all cases.

6.1.1. *Linear polynomial interpretations*

A polynomial $P \in \mathbb{R}[x_1,\ldots,x_n]$ is *linear*, if $P = a_nx_n + a_{n-1}x_{n-1} + \cdots + a_1x_1 + a_0$. Note that, since we have to guarantee that $[f](x_1,\ldots,x_k) \geq 0$ for all $f \in \mathcal{F}$ and $x_1,\ldots,x_k \geq 0$, the independent coefficient a_0 of $[f]$ cannot be negative; otherwise, $[f](0,\ldots,0) = a_0 < 0$. Moreover, according to Proposition 6, negative coefficients cannot be used in any other monomial in a linear polynomial interpretation. Therefore, we must impose $[f] \in \mathbb{R}_0[x_1,\ldots,x_n]$. Regarding μ-monotonicity, following Theorem 2 we fix $\frac{\partial[f]}{\partial x_i} = a_i \geq 1$ for each $i \in \mu(f)$.

Following the generic method in Section 6.1, each k-ary symbol $f \in \mathcal{F}$ is interpreted as a linear polynomial $[f] = a_kx_k + \cdots + a_1x_1 + a_0$ where

(1) $a_0 \in \mathbb{R}_0$ and $a_i \in \mathbb{R}_0$ if $i \notin \mu(f)$;
(2) $a_i \in \mathbb{R}_1$ if $i \in \mu(f)$.

The following example shows how these ideas work in practice.

Example 16. Consider the TRS $\mathcal{R}$:

```
nats     -> adx(zeros)     adx(x:y)  -> incr(x:adx(y))
zeros    -> 0:zeros        incr(x:y) -> s(x):incr(y)
hd(x:y) -> x               tl(x:y)   -> y
```

together with $\mu(\texttt{:}) = \mu(\texttt{s}) = \varnothing$ and $\mu(\texttt{incr}) = \mu(\texttt{adx}) = \mu(\texttt{hd}) = \mu(\texttt{tl}) = \{1\}$ [20], Section 1. We are going to prove the μ-termination of $\mathcal{R}$. The symbols of the signature are given parametric *linear* polynomials:

$$\begin{array}{rclcrcl}
[\texttt{nats}] &=& a_0 & \quad & [\texttt{incr}](x) &=& f_1x + f_0\\
[\texttt{adx}](x) &=& b_1x + b_0 && [\texttt{s}](x) &=& g_1x + g_0\\
[\texttt{zeros}] &=& c_0 && [\texttt{hd}](x) &=& h_1x + h_0\\
x\,[\texttt{:}]\,y &=& d_{10}x + d_{01}y + d_{00} && [\texttt{tl}](x) &=& i_1x + i_0\\
[\texttt{0}] &=& e_0 &&&&
\end{array}$$

where, according to the procedure described above, we have the following:

(1) Constraints due to the proper definition of the interpretation.

$$\begin{array}{cccc}
\boxed{a_0 \geq 0} & \boxed{d_{00} \geq 0} & \boxed{e_0 \geq 0} & \boxed{g_1 \geq 0}\\
\boxed{b_0 \geq 0} & \boxed{d_{01} \geq 0} & \boxed{f_0 \geq 0} & \boxed{h_0 \geq 0}\\
\boxed{c_0 \geq 0} & \boxed{d_{10} \geq 0} & \boxed{g_0 \geq 0} & \boxed{i_0 \geq 0}
\end{array}$$

(2) Constraints due to μ-monotonicity.

$$\boxed{b_1 \geq 1} \quad \boxed{f_1 \geq 1} \quad \boxed{h_1 \geq 1} \quad \boxed{i_1 \geq 1}$$

(3) Constraints due to compatibility with rules:
 (a) Compatibility with the rule `nats -> adx(zeros)`:

$$\boxed{a_0 - b_1c_0 - b_0 > 0}$$

(b) Compatibility with the rule `zeros -> 0:zeros`:

$$\boxed{c_0 - d_{10}e_0 - d_{01}c_0 - d_{00} > 0}$$

(c) Compatibility with the rule `incr(x:y) -> s(x):incr(y)`:

$$f_1(d_{10}x + d_{01}y + d_{00}) + f_0 - (d_{10}(g_1x + g_0) + d_{01}(f_1y + f_0) + d_{00}) > 0.$$

Now we collect the coefficients accompanying each variable $x_1, \ldots, x_k$ to obtain a constraint $A_kx_k + \cdots + A_1x_1 + A_0 > 0$. In this case:

$$(f_1d_{10} - d_{10}g_1)x + (f_1d_{01} - d_{01}f_1)y + (f_1d_{00} + f_0 - d_{10}g_0 - d_{01}f_0 - d_{00}) > 0.$$

Then, according to item 3 in Section 6.1, we have now $A_k \geq 0 \wedge \cdots \wedge A_1 \geq 0 \wedge A_0 > 0$. For the constraint above, we obtain

$$\boxed{f_1d_{10} - d_{10}g_1 \geq 0} \wedge \boxed{f_1d_{01} - d_{01}f_1 \geq 0} \wedge \boxed{f_1d_{00} + f_0 - d_{10}g_0 - d_{01}f_0 - d_{00} > 0}$$

(d) Compatibility with `adx(x:y) -> incr(x:adx(y))`, `hd(x:y) -> x`, and `tl(x:y) -> y` similarly yield the remainder of constraints.

Now, the set of computed constraints can be solved as a set of constraints over the reals by an appropriate system. With CON'*FLEX*, we obtain a solution leading to the following polynomial interpretation over the reals:

$$\begin{array}{rcl@{\qquad}rcl}
[\mathtt{nats}] & = & 1.740 & [\mathtt{incr}](x) & = & 1.000x + 0.251 \\
[\mathtt{adx}](x) & = & 1.100x + 0.658 & [\mathtt{s}](x) & = & 0.300x + 0.224 \\
[\mathtt{zeros}] & = & 0.932 & [\mathtt{hd}](x) & = & 2.322x + 0.300 \\
x\,[\mathtt{:}]\,y & = & 0.431x + 0.431y + 0.288 & [\mathtt{tl}](x) & = & 2.322x + 0.300 \\
[\mathtt{0}] & = & 0.300 & & &
\end{array}$$

which proves the μ-termination of $\mathcal{R}$.

Now, we can obtain the "hidden" δ which is implicitly used for ensuring that the polynomial interpretation computed in Example 16 actually proves termination of $\mathcal{R}$ in the example. According to Corollary 1 (with $\alpha = 0$):

$$\begin{array}{rcl}
\delta_{\mathtt{nats},\mathtt{adx(zeros)}} & = & 0,0568 \\
\delta_{\mathtt{zeros},\mathtt{0:zeros}} & = & 0,113008 \\
\delta_{\mathtt{incr(x:y)},\mathtt{s(x):incr(y)}} & = & 0,046275 \\
\delta_{\mathtt{adx(x:y)},\mathtt{incr(x:adx(y))}} & = & 0,152202 \\
\delta_{\mathtt{hd(x:y)},\mathtt{x}} & = & 0,968736 \\
\delta_{\mathtt{tl(x:y)},\mathtt{y}} & = & 0,968736.
\end{array}$$

Therefore,

$$\delta = \min(\{0,0568, 0,113008, 0,046275, 0,152202, 0,968736, 0,968736\}) = 0,046275.$$

6.1.2. *Simple and simple-mixed polynomial interpretations*

A polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ is *simple* if no exponent is greater than 1, *i.e.*, for each monomial $\pi \in P$ and $i \in \mathcal{I}^+_\pi$, we have $deg_i(\pi) = 1$. The polynomial P is *simple-mixed* if no exponent is greater than 1 or $n = 1$ and $P = a_2 x_1^2 + a_0$ [39].

Note that, since we have to guarantee that $[f](x_1, \ldots, x_k) \geq 0$ for all $f \in \mathcal{F}$ and $x_1, \ldots, x_k \geq 0$ again the independent coefficient cannot be negative. Moreover, by using Proposition 6, it is not difficult to see that we cannot use negative coefficients in any other monomial in simple polynomial interpretations: since item 1 in Proposition 6 does not apply, the only possibility would be to have (by item 2) each negative monomial π contained in a 0-bounded polynomial P' which contains a negative coefficient $\mathrm{coef}(\pi) = \mathrm{coef}(\pi')$ for a monomial $\pi' = \frac{\pi}{x_i} \in P'$ for some $i \in \mathcal{I}^+_\pi$. Note that π' contains exactly the same variables as π but x_i. Now, we can repeatedly apply Proposition 6 to P' and π' to finally conclude that there is a polynomial $P^{''}$ which is 0-bounded in $\mathbb{R}_0$ and contains a negative constant $\mathrm{coef}(\pi)$. Since $P^{''}(0, \ldots, 0) = \mathrm{coef}(\pi)$, this contradicts 0-boundedness of $P^{''}$ in $\mathbb{R}_0$. A similar reasoning applies to simple-mixed interpretations, because polynomials with quadratic monomials have no linear monomial.

Regarding μ-monotonicity: since $[f]$ has no negative coefficients, we just need to fix $\mathrm{coef}(\pi_i) \geq 1$ if $\pi_i \in [f]$ is the monomial x_i for each $i \in \mu(f)$. This is justified as follows: by Theorem 2 we can impose $\frac{\partial [f](x_1, \ldots, x_i, \ldots, x_k)}{\partial x_i} \geq 1$ for each $i \in \mu(f)$. Since $\frac{\partial [f]}{\partial x_i}(0, \ldots, 0) = \mathrm{coef}(\pi_i)$, it must be $\mathrm{coef}(\pi_i) \geq 1$. On the other hand, since $\frac{\partial [f](x_1, \ldots, x_i, \ldots, x_k)}{\partial x_i}$ has no negative coefficient, by Proposition 5, for all $x_1, \ldots, x_k \in \mathbb{R}_0$, $\frac{\partial [f]}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_k) \geq \mathrm{coef}(\pi_i) \geq 1$.

Hence, following the generic method in Section 6.1, we assume that each k-ary symbol $f \in \mathcal{F}$ is interpreted as a simple or simple-mixed polynomial $[f]$ where

(1) $\mathrm{coef}(\pi) \geq 0$ for all $\pi \in [f]$, and
(2) $\mathrm{coef}(\pi_i) \geq 1$ if $\pi_i \in [f]$ is the monomial x_i and $i \in \mu(f)$.

6.1.3. *Use of polynomials with negative coefficients*

The polynomials in the previous sections do not admit negative coefficients in any monomial. According to Proposition 6, if we want to use negative coefficients in some monomials (as, *e.g.*, in Ex. 13), we have to consider, at least, the following class of polynomials.

Definition 1 (2-simple-mixed polynomial)**.** *A polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ is 2-simple-mixed if each monomial $\pi \in P$ satisfies either:*

(1) *$deg_i(\pi) \leq 1$ for all $i \in \{1, \ldots, k\}$; or*
(2) *$deg_i(\pi) = 2$ for some $i \in \{1, \ldots, k\}$ and $deg_j(\pi) = 0$ for all $j \in \{1, \ldots, k\} - \{i\}$.*

Note that simple-mixed polynomials are also 2-simple mixed. The polynomials in Example 13 are 2-simple mixed.

Regarding the automatic generation of such polynomials, we note that, since we still have to guarantee that $[f](x_1, \ldots, x_k) \geq 0$ for all $f \in \mathcal{F}$ and $x_1, \ldots, x_k \geq 0$, again the constant coefficient cannot be negative. On the other hand, by Proposition 6, $\mathrm{coef}(\pi) \geq 0$ if $\pi \in [f]$ contains a variable x_i raised to 2 (*i.e.*, $deg_i(\pi) = 2$); moreover, $\mathrm{coef}(\pi) > 0$ if some other $\pi' \in [f]$ contains x_i (*i.e.*, $i \in \mathcal{I}^+_\pi \cap \mathcal{I}^+_{\pi'}$ and $\deg_i(\pi') = 1$) and satisfies $\mathrm{coef}(\pi') < 0$ (otherwise, we could reason as done in Section 6.1.2 for simple polynomial interpretations).

The following well-known fact can be used to guarantee 0-boundedness of a quadratic polynomial:

Observation 1. *Let $P(x) = Ax^2 + Bx + C$. Then, $P(x) \geq 0$ for all $x \geq 0$ if and only if either*

(1) $A \geq 0 \wedge B \geq 0 \wedge C \geq 0$*; or*
(2) $A > 0 \wedge B < 0 \wedge 4AC - B^2 \geq 0$.

The idea, now, is to use Observation 1 to determine the value (and sign) of coefficients. We do *not* need to fix any coefficient to be negative. This arises as a consequence of the second item in Observation 1.

Regarding μ-monotonicity, note that, for every k-ary symbol $f \in \mathcal{F}$ and $i \in \{1, \ldots, k\}$, $\frac{\partial[f]}{\partial x_i}$ is a simple polynomial. Thus, as discussed in Section 6.1.2, in order to guarantee that $\frac{\partial[f]}{\partial x_i} \geq 1$ (equivalently $\frac{\partial[f]}{\partial x_i} - 1 \geq 0$) for all $i \in \mu(f)$, we require that polynomial $\frac{\partial[f]}{\partial x_i} - 1$ contains no negative coefficient, *i.e.*, for all $\pi \in [f]$, $i \in \mu(f) \cap \mathcal{I}^+_\pi$, and $\pi' \in [f]$ such that $i \in \mathcal{I}^+_{\pi'}$, we let $\mathrm{coef}(\pi') \geq 1$ if $\pi' = x_i$, or $\mathrm{coef}(\pi') \geq 0$ otherwise. Note that μ-monotonicity restrictions can force some coefficients to be non-negative in the interpretation. Thus, in order to reduce the size of the (disjunctive) constraint, it makes sense to first consider μ-monotonicity restrictions and then eventually discard other possibilities which are actually forbidden.

Example 17. Consider the TRS $\mathcal{R}$ and μ as in Example 13. We look for a 2-simple mixed polynomial interpretation:

$$\begin{array}{rclcrcl} [\mathtt{g}](x) & = & a_2x^2 + a_1x + a_0 & \quad & [\mathtt{c}] & = & c_0 \\ [\mathtt{h}](x) & = & b_2x^2 + b_1x + b_0 & & [\mathtt{d}] & = & d_0 \end{array}$$

which proves the μ-termination of $\mathcal{R}$. Again, we use Theorems 1 and 2 to generate a set of constraints on the unknown coefficients. Observation 1 is also used to guarantee 0-boundedness of the interpretation.

(1) Constraints due to the monotonicity: no constraint ($\mu(f) = \varnothing$ for all symbols f).
(2) Constraints due to the proper definition of the interpretation (use Observation 1):
   (a) $[\mathtt{g}](x) \geq 0$: $a_2x^2 + a_1x + a_0 \geq 0$ becomes

$$\boxed{a_2 \geq 0 \wedge a_1 \geq 0 \wedge a_0 \geq 0} \quad \vee \quad \boxed{a_2 > 0 \wedge a_1 < 0 \wedge 4a_2a_0 - a_1^2 \geq 0}$$

   Note that we obtain a *disjunction* of a conjunction of constraints. Each such disjunctive component generates an independent set of restrictions. Note that we do *not* need to conjecture any coefficient to be negative. This arises as a consequence of the use of Observation 1 (*e.g.*, for a_1 in the second member of the previous disjunction).

(b) $[\mathtt{h}](x) \geq 0$ for all $x \geq 0$: $b_2x^2 + b_1x + b_0 \geq 0$ becomes:

$$\boxed{b_2 \geq 0 \wedge b_1 \geq 0 \wedge b_0 \geq 0} \quad \vee \quad \boxed{b_2 > 0 \wedge b_1 < 0 \wedge 4b_2b_0 - b_1^2 \geq 0}$$

(c) $[\mathtt{c}] \geq 0$: $\boxed{c_0 \geq 0}$

(d) $[\mathtt{d}] \geq 0$: $\boxed{d_0 \geq 0}$

(3) Constraints due to compatibility with rules: proceed as in Example 16.

Among the four sets of restrictions which we obtain from the previous disjuntions, the following one:

$$\begin{array}{ll}
a_2 > 0 & d_0 \geq 0 \\
a_1 < 0 & a_2 - b_2 \geq 0 \\
4a_2a_0 - a_1^2 \geq 0 & a_1 - b_1 \geq 0 \\
b_2 > 0 & a_0 - b_0 > 0 \\
b_1 < 0 & c_0 - d_0 > 0 \\
4b_2b_0 - b_1^2 \geq 0 & b_2d_0^2 + b_1d_0 + b_0 - a_2c_0^2 - a_1c_0 - a_0 > 0 \\
c_0 \geq 0 &
\end{array}$$

leads to a solution which can be directly obtained by using CON'*FLEX*. The obtained solution yields the following 2-simple mixed polynomial interpretation:

$$\begin{array}{rclcrcl}
[\mathtt{g}](x) & = & 0.090x^2 - 0.800x + 2.000 & & [\mathtt{c}] & = & 1.850 \\
[\mathtt{h}](x) & = & 0.090x^2 - 0.800x + 1.990 & & [\mathtt{d}] & = & 0.550
\end{array}$$

which proves the μ-termination of $\mathcal{R}$ for the (implicit) $\delta = 0.010$.

6.2. Proof of termination using dependency pairs

When proving termination of a TRS $\mathcal{R} = (\mathcal{F}, R)$ by using the dependency pairs method, we have to obtain a polynomial $\mathcal{F} \cup \widehat{\mathcal{F}}$-algebra inducing a reduction pair $(\gtrsim, >_\delta)$ where $\gtrsim$ is compatible with the rules of $\mathcal{R}$ and $>_\delta$ is compatible with the dependency pairs in $\mathsf{DP}(\mathcal{R})$.

Following the discussion in Section 6.1 and taking into account Section 4, the set of constraints is obtained by requiring the polynomial interpretations to satisfy the following constraints:

(1) Properly defined polynomial interpretation over $\mathbb{R}_0$: $[f](x_1, \ldots, x_k) \geq 0$ for all $f \in \mathcal{F} \cup \widehat{\mathcal{F}}$ and $x_1, \ldots, x_k \geq 0$. This guarantees 0-boundedness, and well-foundedness of $>_\delta$ for any $\delta > 0$, in particular, for the *implicit* δ which can be computed as shown in item 4 below.

(2) Weak monotonicity of $\gtrsim$: $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 0$ for all k-ary symbol $f \in \mathcal{F} \cup \widehat{\mathcal{F}}$, $i \in \{1,\ldots,k\}$ and $x_1,\ldots,x_k \geq 0$. By Proposition 2, this guarantees weak monotonicity of $\gtrsim$.

(3) Compatibility of $\gtrsim$ with the rules of $\mathcal{R}$: For each $l \to r \in R$, we impose $P_{l,r} = [l] - [r] \geq 0$ for all $x_1,\ldots,x_n \geq 0$, where $x_1,\ldots,x_n$ are the variables in $\mathcal{V}\mathrm{ar}(l) \cup \mathcal{V}\mathrm{ar}(r)$. In particular, we can impose that *all coefficients in* $P_{l,r}$ *are non-negative*: $\mathrm{coef}(\pi) \geq 0$ for each $\pi \in P_{l,r}$. By Proposition 5, this guarantees the *compatibility* of $\gtrsim$ with the rules of the TRS.

(4) Compatibility with the dependency pairs in $\mathcal{R}$: For each $\langle t, s \rangle \in \mathsf{DP}(\mathcal{R})$, we impose $P_{t,s} = [t] - [s] > 0$ for all $x_1,\ldots,x_n \geq 0$, where $x_1,\ldots,x_n$ are the variables in $\mathcal{V}ar(t) \cup \mathcal{V}ar(s)$. More precisely: $\mathrm{coef}(\pi_{0\cdots 0}) > 0$ and $\mathrm{coef}(\pi) \geq 0$ for each $\pi \in P_{t,s}$ such that $\pi \neq \pi_{0\cdots 0}$. By Corollary 1, this guarantees the *compatibility* of $>_\delta$ for the *implicit* δ given by

$$\delta = \min(\{\mathrm{coef}(\pi_{0\cdots 0}) \mid \pi_{0\cdots 0} \in P_{t,s}, \langle t, s \rangle \in \mathsf{DP}(\mathcal{R})\}).$$

Example 18. Consider the TRS $\mathcal{R}$ and dependency pairs $\mathsf{DP}(\mathcal{R})$ of Example 9. We are going to prove termination of $\mathcal{R}$ by computing an appropriate reduction pair $(\gtrsim, >_\delta)$ for the rules in $\mathcal{R}$ and the dependency pairs in $\mathsf{DP}(\mathcal{R})$. Both signature and tuple symbols are interpreted as *linear* polynomials:

$$\begin{array}{rclcrcl} [\mathtt{f}](x) & = & a_1 x + a_0 & \qquad & [\mathtt{F}](x) & = & c_1 x + c_0. \\ [\mathtt{g}](x) & = & b_1 x + b_0. & & & & \end{array}$$

(1) Constraints due to the proper definition of the interpretation. All being linear polynomials, we require non-negative coefficients everywhere:

$$\boxed{a_0 \geq 0} \qquad \boxed{b_0 \geq 0} \qquad \boxed{c_0 \geq 0}$$
$$\boxed{a_1 \geq 0} \qquad \boxed{b_1 \geq 0} \qquad \boxed{c_1 \geq 0}$$

(2) Constraints due to weak monotonicity of $\gtrsim$. In this case, they are subsumed by the previous ones.

(3) Constraints due to compatibility of $\gtrsim$ with the rules of $\mathcal{R}$. Compatibility with the rule `f(f(x)) -> f(g(f(g(f(x)))))` yields

$$a_1(a_1 x + a_0) + a_0 - (a_1(b_1(a_1(b_1(a_1 x + a_0) + b_0) + a_0) + b_0) + a_0) \geq 0$$

which becomes:

$$\boxed{a_1^2 - b_1^2 a_1^3 \geq 0} \quad \wedge \quad \boxed{a_1 a_0 - b_1^2 a_1^2 a_0 - b_1 a_1^2 b_0 - b_1 a_1 a_0 - a_1 b_0 \geq 0}$$

and similarly for the rule `f(g(f(x))) -> f(g(x))`.

(4) Constraints due to compatibility of $>_\delta$ with the dependency pairs of $\mathcal{R}$: Compatibility with the dependency pair $\langle \mathtt{F(f(x))}, \mathtt{F(g(f(x)))} \rangle$ yields:

$$c_1(a_1 x + a_0) + c_0 - (c_1(b_1(a_1 x + a_0) + b_0) + c_0) > 0$$

which becomes: $\boxed{c_1a_1 - c_1b_1a_1 \geq 0} \quad \wedge \quad \boxed{c_1a_0 - c_1b_1a_0 - c_1b_0 > 0}$

and similarly with the dependency pairs $\langle \mathtt{F(f(x))}, \mathtt{F(g(f(g(f(x)))))}\rangle$, $\langle \mathtt{F(f(x))}, \mathtt{F(x)}\rangle$, and $\langle \mathtt{F(g(f(x)))}, \mathtt{F(g(x))}\rangle$.

Now, we solve the constraints by using CON'*FLEX*. The computed solution corresponds to the following polynomial interpretation:

$$\begin{array}{rcl} [\mathtt{f}](x) & = & x + 0.001 \\ [\mathtt{g}](x) & = & 0.001x \end{array} \qquad [\mathtt{F}](x) \;=\; 0.001x$$

which proves termination of $\mathcal{R}$. We can compute the implicit δ which is used here: we only need to consider the dependency pairs (no δ is necessary for $\gtrsim$):

$$\begin{array}{rclrcl} \delta_{\mathtt{F(f(x)),F(g(f(g(f(x)))))}} & = & 9,98999 \times 10^{-7} & \delta_{\mathtt{F(f(x)),F(x)}} & = & 10^{-6} \\ \delta_{\mathtt{F(f(x)),F(g(f(x)))}} & = & 9,99 \times 10^{-7} & \delta_{\mathtt{F(g(f(x))),F(g(x))}} & = & 10^{-9} \end{array}$$

Therefore, $\delta = \min(\{9,98999 \times 10^{-7}, 9,99 \times 10^{-7}, 10^{-6}, 10^{-9}\}) = 10^{-9}$.

7. Implementation

We have implemented most of the techniques described in the previous section as part of the tool MU-TERM. The tool automatically generates the constraints on the indeterminate coefficients of the polynomial interpretations as described in Sections 6.1.1, 6.1.2, and 6.2 (negative coefficients are not supported yet).

In order to solve such constraints, the tool can provide a textual version of them. The tool can also use C*i*ME as an *auxiliary* tool to solve the constraints. Our choice of C*i*ME is motivated by the availability of a language for expressing constraints, and commands for solving them. This is present in C*i*ME but currently missing (or unavailable) in other termination tools which (internally) may use similar constraint solvers for dealing with polynomial orderings (*e.g.*, AProVE). In contrast with, *e.g.*, CON'*FLEX*, C*i*ME is available on several platforms, including Linux, Windows, and Mac OS X. Another weak point of CON'*FLEX* is that, due to the techniques used in its implementation, it often provides *approximated* solutions which have to been checked before taking them as definitive. This is avoided in our implementation by solving the constraints in the domain of the *rational* numbers, which permit exact arithmetic manipulation.

Although C*i*ME solves Diophantine inequations yielding non-negative integers as solutions, the use of rational numbers is easily made compatible with this limitation: rational coefficients are processed by splitting them into a pair of natural numbers (a non-negative numerator and a positive denominator) and then transforming the constraints to "remove" quotients as explained below. Negative coefficients have not been implemented yet (although we sketch below how to deal with them in C*i*ME). For these reasons, MU-TERM only computes polynomials with *rational* (non-negative) coefficients. Thus, we take $[f] \in \mathbb{Q}_0[x_1, \ldots, x_k]$ for each

k-ary symbol $f \in \mathcal{F}$. We also take into account some efficiency issues: in C*i*ME, natural coefficients are managed as a whole, whereas rational coefficients are not allowed. Thus, the more rational coefficients, the more processing will be needed and more unknown variables will be obtained (two per rational coefficient). Thus, we furnish MU-TERM with three main *generation modes*:

(1) *No rationals*: corresponding to the constraints generated as explained in Section 6 and can be (externally) solved as restrictions over the reals (in $\mathbb{R}_0$) or directly solved by C*i*ME as restrictions over the naturals.
(2) *Rationals and integers*: here, since rational coefficients are intended to introduce non-monotonicity, we only use them with arguments $i \notin \mu(f)$.
(3) *All rationals*: where all coefficients of polynomials are intended to be rational numbers.

Since we use C*i*ME to finally obtain our polynomial interpretations, and the second mode represents the most efficient option still enabling the generation of rational coefficients for non-monotonic arguments of symbols, the default generation mode of MU-TERM is this one; the following example shows how does it work.

Example 19. Consider $\mathcal{R}$ and μ as in Example 16. In order to solve the set of constraints obtained in the example, since $\mu(\texttt{:}) = \varnothing$, we further assume $d_{10} = \frac{p_{10}}{q_{10}}$ and $d_{01} = \frac{p_{01}}{q_{01}}$ for $p_{10}, p_{01} \in \mathbb{N}$ and $q_{10}, q_{01} \in \mathbb{N}_1$. Analogously, since $\mu(\texttt{s}) = \varnothing$, for $[\texttt{s}]$ we let $g_1 = \frac{p_1}{q_1}$ for $p_1 \in \mathbb{N}$ and $q_1 \in \mathbb{N}_1$. We take all other coefficients to be natural numbers. Then, our parametric polynomial is:

$$\begin{array}{rclrcl}
[\texttt{nats}] & = & a_0 & [\texttt{incr}](x) & = & f_1x + f_0 \\
[\texttt{adx}](x) & = & b_1x + b_0 & [\texttt{s}](x) & = & \frac{p_1}{q_1}x + g_0 \\
[\texttt{zeros}] & = & c_0 & [\texttt{hd}](x) & = & h_1x + h_0 \\
x\,[\texttt{:}]\,y & = & \frac{p_{10}}{q_{10}}x + \frac{p_{01}}{q_{01}}y + d_{00} & [\texttt{tl}](x) & = & i_1x + i_0 \\
[\texttt{0}] & = & e_0. & & &
\end{array}$$

According to this, we just need to transform the constraints in Example 16 as follows:

(1) Constraints due to the proper definition of the interpretation. Since C*i*ME solves the constraints in $\mathbb{N}$, no negative coefficient is possible. Thus, we can remove all constraints here, since they are implicitly satisfied.
(2) Constraints due to μ-monotonicity. Since we choose to use rational coefficients these constraints remain unchanged.
(3) Constraints due to compatibility with rules:
    (a) Compatibility with the rule `nats -> adx(zeros)`: It only involves monotonic coefficients, so it does not change: $\boxed{a_0 - b_1c_0 - b_0 > 0}$
    (b) Compatibility with the rule `zeros -> 0:zeros`: The constraint $c_0 - d_{10}e_0 - d_{01}c_0 - d_{00} > 0$ becomes

$$\boxed{q_{10}q_{01}c_0 - q_{01}p_{10}e_0 - q_{10}p_{01}c_0 - q_{10}q_{01}d_{00} > 0}$$

after replacing d_{10} by $\frac{p_{10}}{q_{10}}$ and d_{01} by $\frac{p_{01}}{q_{01}}$, and multiplying both members of the constraint by the denominators q_{10} and q_{01}.

(c) Constraints for compatibility with rules `incr(x:y) -> s(x):incr(y)`, `adx(x:y) -> incr(x:adx(y))`, `hd(x:y) -> x`, and `tl(x:y) -> y` are obtained in a similar way.

(4) Constraints ensuring positive denominators in rational coeficients

$$\boxed{q_{10} > 0} \qquad \boxed{q_{01} > 0} \qquad \boxed{q_1 > 0}$$

Now, this set of constraints can be solved as a set of Diophantine inequations using the C*i*ME system. The obtained solution yields the polynomial interpretation:

$$\begin{array}{rclrclrcl} [\mathtt{nats}] & = & 5 & [\mathtt{incr}](x) & = & x+1 & x\,[\mathtt{:}]\,y & = & x+\frac{1}{2}y \\ [\mathtt{adx}](x) & = & x+3 & [\mathtt{s}](x) & = & 0 & [\mathtt{tl}](x) & = & 2x+1 \\ [\mathtt{zeros}] & = & 1 & [\mathtt{hd}](x) & = & x+1 & [\mathtt{0}] & = & 0 \end{array}$$

which proves the μ-termination of $\mathcal{R}$.

Regarding the use of C*i*ME for solving constraints involving *negative* coefficients, we can process the constraints over the reals as follows: consider a constraint $c < 0$ specifying a negative coefficient c. We let $c = -z_c$ to obtain a restriction $z_c > 0$; we also replace occurrences of c by $-z_c$ in all arithmetic restrictions (and eventually change the sign of the arithmetic component, if necessary). Then, we obtain a new set of inequations having no constraint $c < 0$ which C*i*ME could now solve.

Example 20. Consider the set of constraints obtained in Example 17 where we let $a_1 = -y_1$ and $b_1 = -z_1$ to transform the constraints $a_1 < 0$ and $b_1 < 0$ into $y_1 > 0$ and $z_1 > 0$. We also transform all other constraints containing occurrences of a_1 or b_1 accordingly. We obtain the following transformed constraint:

$$\begin{array}{ll} a_2 > 0 & d_0 \geq 0 \\ y_1 > 0 & a_2 - b_2 \geq 0 \\ 4a_2a_0 - y_1^2 \geq 0 & z_1 - y_1 \geq 0 \\ b_2 > 0 & a_0 - b_0 > 0 \\ z_1 > 0 & c_0 - d_0 > 0 \\ 4b_2b_0 - z_1^2 \geq 0 & b_2d_0^2 - z_1d_0 + b_0 - a_2c_0^2 + y_1c_0 - a_0 > 0 \\ c_0 \geq 0 & \end{array}$$

which can now be solved by C*i*ME. The obtained solution yields the following 2-simple mixed polynomial interpretation:

$$\begin{array}{rclrcl} [\mathtt{g}](x) & = & x^2 - 3x + 4 & [\mathtt{c}] & = & 1 \\ [\mathtt{h}](x) & = & x^2 - 3x + 3 & [\mathtt{d}] & = & 0 \end{array}$$

which proves the μ-termination of $\mathcal{R}$.

8. Related work

The use of polynomials in proofs of termination of rewriting has been investigated in two main settings:

(1) polynomials with non-negative integer coefficients (see, [5,6,9,30,44]) and
(2) polynomials with real coefficients where a subterm property is required to guarantee well-foundedness of the induced term ordering (see, *e.g.*, [11,12,18,39]).

Recent variants to this general setting include the use of polynomials with negative integer coefficients [19,25]. Let's consider these kinds of polynomial interpretations in connection with our framework.

8.1. Polynomials with non-negative integer coefficients

When considering polynomials P having no negative integer coefficients, well-foundedness of the induced polynomial ordering $>$ comes for free due to the use of a well-founded domain $(\mathbb{N}, >_{\mathbb{N}})$. Since polynomial interpretations over the natural numbers can also be seen as 0-bounded algebras over the real numbers whose domain is just $A = \mathbb{N} \subseteq \mathbb{R}$, the ordering $>_1$ induced by such a 0-bounded algebra obviously coincides with the usual polynomial ordering.

In most cases, the carrier set of the algebra is restricted to be $\mathbb{N}_1$ (or even $\mathbb{N}_2$ as suggested in [44]) in such a way that the corresponding polynomial functions are actually mappings $[f] : \mathbb{N}_1^k \to \mathbb{N}_1$ for each k-ary function f [5], Definition 5.3.4. Note that, for the purpose of defining non-monotonic polynomial interpretations, the use of $\mathbb{N}$ as the carrier is an important way to achieve non-monotonic polynomial interpretations (see Prop. 7).

Remark 3. Surprisingly, the constraints generated by the methods which focus on polynomials with non-negative integer coefficients (see, *e.g.*, [9]), are actually so close to ours that it is possible to use the set of constraints generated in this way to obtain a polynomial interpretation over the reals by *just interpreting the indeterminate coefficients as real numbers instead of naturals*.

For instance, item 1 in the beginning of Section 6 specifies that all coefficients in such polynomials must be non-negative integers; any solution *over the reals* to this kind of constraints would immediately satisfy the constraint specified in item 1 in Section 6.1 regarding non-negativeness of our polynomials with *real* coefficients.

Item 2 in the beginning of Section 6 specifies that each polynomial $[f]$ for a k-ary symbol f must contain a monomial x_i satisfying $\mathrm{coef}(x_i) \geq 1$. If the indeterminate coefficients of the parametric polynomial $[f]$ are intended now to be *real* numbers, then the satisfaction of this constraint (again, over the reals) implies that $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 1$: indeed, since $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i}$ has no negative coefficient, for all $x_1, \ldots, x_k$, $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq a_i \geq 1$ as required in item 2 of Section 6.1.

Finally, item 3 in the beginning of Section 6 becomes item 3 in Section 6.1 without further considerations.

Interestingly, this means that software systems (like, *e.g.*, AProVE, C*i*ME, or TTT) which are able to generate constraints for obtaining polynomial interpretations over the naturals, are potentially able to obtain interpretations over the reals by just solving the *same* constraints in the domain of the real or rational numbers!

8.2. Polynomials with real coefficients

Polynomials over the real were proposed by Dershowitz [11,12] as an alternative to Lankford's polynomials over the naturals [30]. In contrast to Lankford's, however, the subterm property (*i.e.*, $[f](x_1,\ldots,x_i,\ldots,x_k) >_{\mathbb{R}} x_i$ for all k-ary symbols f, $1 \leq i \leq k$, and $x_1,\ldots,x_k \in \mathbb{R}$) is explicitly required to ensure well-foundedness. This property, in fact, implies monotonicity when only *simple* polynomials with non-negative coefficients are allowed. In the following proposition, $>$ is the term ordering induced by a polynomial algebra whose domain is ordered by $>_{\mathbb{R}}$.

Proposition 9. *Let $\mathcal{F}$ be a signature, $0 \in A \subseteq \mathbb{R}_0$, and $(A,\mathcal{F}_A)$ be a polynomial $\mathcal{F}$-algebra where $[f] \in \mathbb{R}_0[x_1,\ldots,x_k]$ is a simple polynomial for each k-ary symbol $f \in \mathcal{F}$. If, for all k-ary symbols $f \in \mathcal{F}$, $i \in \{1,\ldots,k\}$, and $x_1,\ldots,x_k \geq 0$, $[f](x_1,\ldots,x_i,\ldots,x_k) >_{\mathbb{R}} x_i$ then, $>$ is monotonic and, for all $\delta \in \mathbb{R}_{>0}$, $>_\delta$ is monotonic.*

Proof. Consider an arbitrary k-ary symbol $f \in \mathcal{F}$ with k positive and an arbitrary $i \in \{1,\ldots,k\}$. By the subterm property, we have that $[f](0,\ldots,0) = \mathrm{coef}(\pi_{0\cdots 0}) > 0$, where $\pi_{0\cdots 0}$ is the constant coefficient of $[f]$ which we label c_0 in the remainder of the proof. By the subterm property and since $[f]$ is a simple polynomial, we also have, for all $c \in \mathbb{R}_{>0}$, $[f](0,\ldots,c,\ldots,0) = c_i \cdot c + c_0 >_{\mathbb{R}} c$, where c_i is the coefficient of the monomial $x_i \in [f]$. Therefore, $c_i \cdot c + c_0 >_{\mathbb{R}} c \Leftrightarrow c_i >_{\mathbb{R}} 1 - \frac{c_0}{c}$ which is only possible (for all positive c) if $c_i \geq_{\mathbb{R}} 1$. Then, since all coefficients in $[f]$ are non-negative, for each monomial $\pi \in [f]$, we have that, whenever $x >_{\mathbb{R}} y$, for all $x_1,\ldots,x_k \in \mathbb{R}_0$, $\pi(x_1,\ldots,x,\ldots,x_k) \geq_{\mathbb{R}} \pi(x_1,\ldots,y,\ldots,x_k)$ and, since $c_i \geq_{\mathbb{R}} 1$, $[f](x_1,\ldots,x,\ldots,x_k) >_{\mathbb{R}} [f](x_1,\ldots,y,\ldots,x_k)$. Now consider terms t,s such that $t > s$; then $[\alpha](t) >_{\mathbb{R}} [\alpha](s)$ for all valuations $\alpha : \mathcal{X} \to A$. Therefore, $f(t_1,\ldots,t,\ldots,t_k) > f(t_1,\ldots,s,\ldots,t_k)$ for all terms $t_1,\ldots,t_k$, *i.e.*, $>$ is monotonic.

Since $[f]$ only has non-negative coefficients, this is also the case for all partial derivatives of $[f]$. Therefore $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq c_i \geq 1$ and, by Theorem 2, for all $\delta \in \mathbb{R}_{>0}$, $>_\delta$ is monotonic. $\square$

Proposition 9 does not hold for simple-mixed polynomial interpretations: consider $\mathcal{F} = \{a, f\}$ where $[a] = 0$ and $[f](x) = \frac{1}{4}x^2 + 2$. Then, $>$ has the subterm property but, *e.g.*, $>_2$ is not monotonic: $[f(a)] = 2$, hence $[f(a)] - [a] = 2$, *i.e.*, $f(a) >_2 a$; however, $[f(f(a))] = 3$, and $[f(f(a))] - [f(a)] = 1$, *i.e.*, $f(f(a)) \not>_2 f(a)$.

Remark 4. As a consequence of Proposition 9, imposing the subterm property to achieve well-foundedness is *not* compatible with defining non-monotonic orderings by means of simple polynomial interpretations where negative coefficients are not allowed and $\mathbb{R}_0$ is taken as the interpretation domain. On the other hand, as shown in [19], Example 3, requiring a μ-subterm property (*i.e.*, restricted to the arguments $i \in \mu(f)$ for each $[f]$) does *not* guarantee well-foundedness anymore.

Giesl has shown how to implement the use of polynomials over the reals *à la Dershowitz* in proofs of termination of rewriting [18]. In [18], Theorem 1, Giesl gives the following criterion for proving termination with polynomials with real coefficients: for all rules $l \to r$ in the TRS, constant symbols c, k-ary symbols f in the signature, and $i \in \{1, \ldots, k\}$, he obtains a set of constraints as follows[5]:

$$\begin{array}{llll} (1) & [l] - [r] > 0 & (3) & \frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 0 \\ (2) & [f](x_1, \ldots, x_i, \ldots, x_k) - x_i \geq 0 & (4) & [c] \geq 0. \end{array}$$

Then, he applies two *differentiation* rules to transform this set of constraints until all rule variables are removed. If there is a solution for the resulting set of constraints, then the TRS is terminating.

Hong and Jakuš have investigated the power of this technique regarding tests of positiveness of polynomials [22]. By using their results, in particular Theorems 1 and 2, we conclude that *all polynomials involved in the constraints* (1)–(4) *above must have non-negative coefficients.* We can then see that if Giesl's method solves a termination problem, then it can also be solved by using our method: regarding the constraint $[l] - [r] > 0$, according to item 3 in Section 6.1, we would have exactly the same constraint and, moreover, our assumption of non-negativeness of the coefficients in $[l] - [r]$ and the presence of a constant positive coefficient, which is not explicit in Giesl's method, is actually *implicit* according to Hong and Jakuš' results. Consider now the constraint $[f](x_1, \ldots, x_i, \ldots, x_k) - x_i \geq 0$, where $P_f(x_1, \ldots, x_k) = [f](x_1, \ldots, x_i, \ldots, x_k) - x_i$ is a polynomial with non-negative coefficients. Note that $\frac{\partial P_f(x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i}$ has only non-negative coefficients. Therefore, $\frac{\partial P_f}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_k) \geq 0$ for all $x_1, \ldots, x_k \geq 0$. Since $\frac{\partial P_f(x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} = \frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} - 1$, we have that $\frac{\partial [f]}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_k) \geq 1$ holds for all $x_1, \ldots, x_k \geq 0$, as required by item 2 in Section 6.1. Note that, again according to Hong and Jakuš', our first explicit assumption (item 1 in Section 6.1) is implicit in Giesl's method; in fact, it is a consequence of constraints (3) and (4).

Steinbach's approach to the generation of polynomial algebras over the reals [39] is also a particular case of ours: Steinbach uses polynomial interpretations with coefficients either zero or not below 1, and where for each variable x_i there is a nonzero coefficient in the polynomial; the interpretation domain is $\mathbb{R}_1$ [39], Definition 1. Therefore, all constraints generated according to item 1 in Section 6.1 (in this case $[f](x_1, \ldots, x_k) \geq 1$ for all $x_1, \ldots, x_k \geq 1$) immediately hold. Under the previous conditions it is also obvious that, for all k-ary function symbols f and $i \in \{1, \ldots, k\}$, $\frac{\partial [f](x_1,\ldots,x_i,\ldots,x_k)}{\partial x_i} \geq 1$ for all $x_1, \ldots, x_k \in \mathbb{R}_1$. Thus, monotonicity of

[5] Giesl's formulation considers a real number μ which is a lower bound of the interpretation of all ground terms. This corresponds to using $\mathbb{R}_\mu$ as the interpretation domain (see Sect. 6). Here, following the discussion in Section 6, we take $\mu = 0$ to simplify the comparison.

$>_\delta$ is also guaranteed. Finally, regarding 3 in Section 6.1, Steinbach also imposes that all coefficients in polynomial $P_{l,r} = [l] - [r]$ are non-negative and that, at least *one* of them (no necessarily the constant monomial) is positive. Since the interpretation domain is $\mathbb{R}_1$, this corresponds to take, as required by Corollary 1, $\alpha = 1$ and $\delta_{l,r} = P_{l,r}(1, \ldots, 1) = \sum_{\pi \in P_{l,r}} \text{coef}(\pi) \geq \text{coef}(\pi_+) > 0$, where π_+ is the monomial of $P_{l,r}$ whose coefficient must be positive.

8.3. Polynomials with negative integer coefficients

The use of polynomial interpretations to generate μ-reduction orderings has been first investigated in [19]. In [19], only polynomials with integer coefficients are considered. Although taking explicit benefit from the use of negative coefficients (see Ex. 13), the restrictions imposed in [19] for coefficients of the considered polynomial interpretations are the usual ones (*i.e.*, $[f] \in \mathbb{Z}[x_1, \ldots, x_k]$, but the induced polynomials are actually used as functions $[f](x_1, \ldots, x_k) \in \mathbb{N}^k \to \mathbb{N}$). Again, it is not difficult to see that the orderings induced by the polynomial interpretations in [19] correspond to the ordering $>_1$ induced by the polynomial interpretation viewed as a 0-bounded algebra over $\mathbb{N} \subseteq \mathbb{R}$. Thus, the framework in [19] is strictly included in the new framework presented in this paper. For instance, it does not apply to polynomial interpretations like that of Example 15 where *rational* coefficients occur in the polynomials.

Recently also, Hirokawa and Middeldorp have proposed the use of linear polynomials with negative integer coefficient to define algebraic interpretations [25]. As remarked in Section 6.1.1, this would lead to a ill-defined polynomial algebra. The authors, then, use such polynomials to define 'almost' polynomial functions $[f]$ which always take the value 0 in the problematic cases; therefore, again $[f](x_1, \ldots, x_k) \in \mathbb{N}^k \to \mathbb{N}$. The obtained interpretation is a 0-bounded algebra over the naturals which is a particular case of algebra over the reals. In [25] the authors explain how to use these kind of interpretations for implementing proofs of termination of rewriting. Their techniques have been implemented as part of the tool TTT. Hirokawa and Middeldorp only allow for integer coefficients in polynomials. Thus, their technique could also benefit from our results.

9. Conclusions

We have shown how to obtain a stable quasi-ordering $\gtrsim$ and a stable and well-founded ordering $>_\delta$ on terms by using a given interpretation of function symbols as real functions (Prop. 1 and Th. 1). Given a monotonicity specification expressed by a replacement map μ, we have shown how to guarantee that the ordering $>_\delta$ fulfills these monotonicity constraints (Th. 2). We have also given a sufficient condition for weak monotonicity of $\gtrsim$ (Prop. 2). As remarked in the introduction, this permits to cover different termination problems. We have shown that the general framework applies to different kinds of real functions in algebras, including non-continuous functions (Ex. 3), irrational functions (Ex. 5), polynomial fractions

(Ex. 8), etc. Regarding proofs of termination of (unrestricted) rewriting, the methods presented here are also helfpul when the lack of monotonicity plays a crucial role for the use of term orderings. This is the case of the dependency pairs method, where non-monotonic (but well-founded and stable) orderings can be used in proofs of termination as part of a *reduction pair*. We have proven that our techniques are well-suited for defining such reduction pairs (Prop. 4). We have shown that, in fact, termination proofs based on the dependency pairs approach can greatly benefit from the systematic use of reduction pairs defined in this way (Exs. 10 and 11).

We have described how to automatically obtain polynomial interpretations over the reals according to our general framework. We deal with very general *polynomial* interpretations which can be characterized by the use of $\mathbb{R}_0$ as carrier set, and the use of polynomials with *real*, possibly negative, coefficients. We have proved that these mechanisms are actually useful when trying to avoid *monotonicity* if, according to the underlying replacement map, it is not required (Props. 7 and 8). We have given conditions to avoid the specification of any concrete value for δ (Cor. 1). We have also introduced a new class of polynomial interpretations (which we call 2-simple mixed polynomial interpretations) which are well-suited for including negative coefficients which can help to set up the desired monotonicity requirements. We have shown how to automatically generate such polynomial interpretations using negative coefficients. Our extended class of polynomial interpretations provides quite a powerful tool for proving termination of *CSR*. For instance, all examples of termination of *CSR* in [20] (the most recent paper on the topic) have been proved terminating now by using polynomial interpretations, see:

`http://www.dsic.upv.es/~slucas/csr/termination/examples`

The techniques described in Section 6 have been implemented in MU-TERM.

We have also shown that the usual practical frameworks for proving termination by using polynomials with natural coefficients and polynomials with real coefficients are subsumed by our techniques (Sect. 8). Moreover, we have proven that our framework is strictly more powerful that of [19] (Prop. 8). Finally, according to the comparison drawn with related work, we can say that our results provide a formal basis allowing existing tools (like, *e.g.*, AProVE, C*i*ME, or TTT) which are able to generate constraints for obtaining polynomial interpretations over the naturals in proofs of termination of TRSs, to smoothly move towards the use of interpretations over the reals by just solving the obtained constraints in the domain of the real or rational numbers (Rem. 3).

9.1. Future work

From a theoretical point of view, an interesting question is: what are the limitations of the approach? Examples 3 and 7 show that algebras over the reals apply to prove termination of non-simply terminating TRSs (see [44], Prop. 6.3.26(iv)); thus, what would be the position of these orderings in a termination hierarchy possibly extending that of [44], Section 6.3?

Another exciting one is the following: is there a TRS which can be proved terminating by using a reduction ordering based on a polynomial interpretation over the reals (or rationals) but which *cannot* be proved terminating by using a polynomial interpretation over the naturals? Regarding proofs of μ-termination for "proper" replacement maps $\mu \sqsubset \mu_\top$, the answer is "yes" (Prop. 8); regarding proofs of termination of rewriting, this is an open problem.

Our work shows that μ-reduction orderings based on algebras over the reals can be uniformly used for different termination problems depending on the monotonicity restrictions that we are considering. Other μ-reduction orderings (*e.g.*, the context-sensitive recursive path ordering, CSRPO [4]) could also be suitable for implementing the necessary comparisons. Investigating the use of CSRPO in proofs of termination when using the dependency pairs approach is also interesting.

Regarding the practical use of the generic framework presented in this paper, we plan to investigate new families of real functions which could be well-suited for automatization purposes. We will focus on those functions which provide mechanisms for loosing monotonicity in some arguments. For instance, polynomial fractions could give a first starting point.

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