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CORRIGENDUM: COMPLEXITY OF INFINITE WORDS ASSOCIATED WITH BETA-EXPANSIONS

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Abstract. We add a sufficient condition for validity of Proposition 4.10 in the paper Frougny *et al.* (2004). This condition is not a necessary one, it is nevertheless convenient, since anyway most of the statements in the paper Frougny *et al.* (2004) use it.

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1. Introduction

The aim of this note is to correct the mistake contained in our paper [2]. We shall use the notation of the paper and refer to the statements included in it.

We were pointed out [1] a counterexample to assertion (1) of Theorem 6.2 in the paper. The assertion says that the complexity of the fixed point u_β of the canonical substitution φ_β associated with a simple Parry number β with the Rényi expansion $d_\beta(1) = t_1 t_2 \cdots t_{m-1} 1$ is affine, namely $\mathcal{C}(n) = (m-1)n+1$. This statement is however true only under the condition used for assertion (2) of the theorem, namely that the Rényi expansion $d_\beta(1) = t_1 t_2 \cdots t_m$ satisfies

$$t_1 = t_2 = \cdots = t_{m-1} \qquad \text{or} \qquad t_1 > \max\{t_2, \ldots, t_{m-1}\}\,. \tag{$*$}$$

The mistake occurred due to a slip in the proof of Proposition 4.10. We show in this note that under the additional condition $(*)$ the proposition is valid.

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The corrected version of Proposition 4.10 of [2] is stated here as Proposition 2.2. At the end of this note we explain which statements of the paper [2] need to be equipped with condition $(*)$, as well.

Let us mention that the condition $(*)$ in Proposition 2.2 may be weakened. Nevertheless, we have chosen the condition in the form $(*)$, since anyway most of the statements in the paper [2] use it.

2. Proof of Proposition 4.10 of [2]

In order to prove Proposition 2.2 we need the following lemma.

Lemma 2.1. *Let $t_1 > \max\{t_2, \dots, t_{m-1}\}$. Let w be a right special factor of u_β with at least 3 distinct right extensions X, Y, Z, such that w contains a non-zero letter, wX is a left special factor and $X \neq 0$. Then there exists a word $\tilde{w}$ which is a right special factor of u_β with at least 3 distinct right extensions $\tilde{X}, \tilde{Y}, \tilde{Z}$ such that $\tilde{w}\tilde{X}$ is a left special factor, $\tilde{X} \neq 0$, and $wX = \varphi(\tilde{w}\tilde{X})$.*

Proof. The word w can be written as $w = w'U0^p$, where $U \neq 0$ and $p \geq 0$. Thus $U0^pX$, $U0^pY$, $U0^pZ$ are factors of u_β. Since at least one of X, Y, Z is ≥ 2, we can derive from Lemma 4.5 of [2] and condition $t_1 > \max\{t_2, \dots, t_{m-1}\}$ that $p < t_1$. Since $w'U$ is a left special factor, according to (ii) of Lemma 3.7 there exists a left special factor $\tilde{w}$ such that $w'U = \varphi(\tilde{w})$. Now

$$\begin{aligned} wX &= \varphi(\tilde{w})0^pX \\ wY &= \varphi(\tilde{w})0^pY \\ wZ &= \varphi(\tilde{w})0^pZ \end{aligned}$$

are distinct factors of u_β. Hence there must exist distinct letters $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ such that $\tilde{w}\tilde{X}$, $\tilde{w}\tilde{Y}$, $\tilde{w}\tilde{Z}$ are also factors of u_β. Moreover, since $X \neq 0$ and $p < t_1$, we have $\varphi(\tilde{X}) = 0^pX$, where $\tilde{X} \neq 0$. As $\varphi(\tilde{w}\tilde{X}) = wX$ is a left special factor, (ii) of Lemma 3.7 implies that $\tilde{w}\tilde{X}$ is a left special factor, which completes the proof. □

The following statement is the same as in Proposition 4.10 of [2], except the additional condition $(*)$.

Proposition 2.2. *Let $d_\beta(1)$ satisfies the condition $(*)$. Then for every maximal left special factor $v = v_0v_1 \cdots v_k$ containing a letter $v_j \neq 0$ there exists a maximal left special factor w and an $s \in \{t_1, t_2, \dots, t_{m-1}\}$ such that $v = \varphi(w)0^s$.*

Proof. Let $j = \max\{\, i \mid v_i \neq 0\}$. According to Lemma 3.7 there exists a left special factor $w = w_0w_1 \cdots w_\ell$ such that $v_0v_1 \cdots v_j = \varphi(w_0)\varphi(w_1) \cdots \varphi(w_\ell)$ and thus

$$v = v_0v_1 \cdots v_j0^s = \varphi(w_0)\varphi(w_1) \cdots \varphi(w_\ell)0^s\,, \qquad \text{where } s = k - j\,.$$

Since v is maximal, we can use Observation 4.2 and Corollary 4.6 to derive that $s \in \{t_1, t_2, \dots, t_{m-1}\}$.

It remains to show that w is a maximal left special factor of u_β. Assume that w is not maximal. We distinguish two cases according to which part of condition $(*)$ is satisfied.

- Let $t_1 = t_2 = \cdots = t_{m-1} =: t$. Since w is not maximal, then according to Lemma 4.9 there exists a left special factor wX, where $X \neq m-1$ or a left special factor $w(m-1)0$. However, then (ii) of Lemma 3.7 implies that $\varphi(wX) = \varphi(w)0^t(X+1)$, resp. $\varphi(w(m-1)0) = \varphi(w)0^{t_m+t}1$, is also a left special factor. Since $s = t$, the factor v is a proper prefix of both of them, which is a contradiction with the maximality of v.
- Let $t_1 > \max\{t_2, \ldots, t_{m-1}\}$. Since $v = \varphi(w)0^s$ is a maximal left special factor of u_β and w is not maximal, there exists a letter X such that wX is again a left special factor. Lemma 3.7 implies that $\varphi(wX)$ is also a left special factor. Since $v = \varphi(w)0^s$ may not be a proper prefix of $\varphi(wX)$, the condition $t_1 > \max\{t_2, \ldots, t_{m-1}\}$ implies $X \neq 0$.

  The maximality of the left special factor $v = \varphi(w)0^s$ implies also existence of distinct letters Y^*, Z^* such that $\varphi(w)0^sY^*$, $\varphi(w)0^sZ^*$ are factors of u_β and but they are not left special. There must exist distinct letters Y, Z such that wY, wZ are factors of u_β but not left special.

  We have thus shown that w is a right special factor with at least 3 distinct right extensions $X \neq 0$, Y, Z, where wX is a left special factor. Repeated use of Lemma 2.1 leads to a right special factor $w^{(0)} = 0^q$, for $q \geq 1$, which has at least 3 distinct right extensions $X^{(0)} \neq 0$, $Y^{(0)}$, $Z^{(0)}$, such that $w^{(0)}X^{(0)}$ is a left special factor of u_β. Lemma 4.5 implies that $X^{(0)} = 1$ and $q = t_1$. At least one letter among $Y^{(0)}$, $Z^{(0)}$ is non-zero, say $Y^{(0)}$. Then $Y^{(0)} \geq 2$, but then $w^{(0)}Y^{(0)} = 0^{t_1}Y^{(0)}$ is due to Lemma 4.5 not a factor of u_β, which is a contradiction. □

3. Conclusions

Proposition 4.10 was used in [2] for proving Corollary 4.11, second implication of Theorem 4.12, assertion (1) of Theorem 6.2 and Corollary 6.3. Therefore condition $(*)$ should be added in the mentioned statements as well.

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References

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