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ON f-WISE ARC FORWARDING INDEX AND WAVELENGTH ALLOCATIONS IN FAULTY ALL-OPTICAL HYPERCUBES *

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Abstract. Motivated by the wavelength division multiplexing in all-optical networks, we consider the problem of finding an optimal (with respect to the least possible number of wavelengths) set of $f+1$ internally node disjoint dipaths connecting all pairs of distinct nodes in the binary r-dimensional hypercube, where $0 \leq f < r$. This system of dipaths constitutes a routing protocol that remains functional in the presence of up to f faults (of nodes and/or links). The problem of constructing such protocols for general networks was mentioned in [1]. We compute precise values of f-wise arc forwarding indexes and give (describe dipaths and color them) nearly optimal all-to-all f-fault tolerant protocols for the hypercube network. Our results generalize corresponding results from [1, 4, 14].

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1. Introduction

Motivation. Optics is a key technology in communication networks and it is expected to dominate important applications such as video conferencing, scientific visualization, real-time medical imaging, high speed super-computing, and distributed computing. A single optical wavelength supports rates of gigabytes-per-second, see *e.g.* [9,13]. Multiple laser beams that are propagated over the same fiber on distinct optical wavelengths can increase this capacity much further. This is achieved through WDM (Wavelength Division Multiplexing) [5], by partitioning the optical bandwidth into a large number of channels. It allows multiple data streams to be transferred concurrently (on different wavelengths) along the same optical fiber.

All-optical [15] communication networks are switched networks with reconfigurable wavelength selective optical switches, without wavelength converters, where different signals may travel on the same communication link (but on different wavelengths) into a node, and then exit from it on different links, keeping their original wavelengths. We refer the reader to [2] for an account of the theoretical problems and results obtained for this *all-optical model*. These systems provide all source-destination pairs with end-to-end transparent channels that are identified through a wavelength and a physical path. Maintaining the signal in optical form allows for high data transmission rates in these networks since there is no prohibitive overhead due to conversions to and from the electronic form.

The actual process of setting up switches and of assigning wavelengths is done using an electronic backbone control network. A solution consists of settings for the switches in the network (establishing a routing), and an assignment of wavelengths to the requests so that no directed link will carry two different signals on the same wavelength. One may wonder at the use of a relatively slow electronic network to set-up these high-speed connections. In fact, the major applications for such networks require connections that last for relatively long periods, once set-up.

Contribution of this work. The cost and feasibility of switching and amplification devices depend on the number of wavelengths they handle. It is evident that the number of wavelengths (optical bandwidth) is a costly source and thus a limiting factor. As pointed out in [1], in a network where links and/or nodes may fail it is important to establish routings for a required communication pattern, that guarantee fault-free transmission. The construction has to take into account the available capacity of the optical bandwidth. Hence it is important to construct fault-tolerant routings requiring the least possible number of wavelengths.

In this paper we will initiate the study of fault-tolerant routings in all-optical networks. We concentrate on one of the dominating topological structures for distributed systems — the binary hypercube, and we consider the *total exchange* (also called *all-to-all* or *gossiping*) communication process. We assume that up to f links/nodes may fail in the system, and describe nearly optimal (with respect to the number of wavelengths) all-to-all f-fault tolerant protocols for the

r-dimensional binary hypercube for any f, $0 \leq f < r$. For that purpose we introduce the notion of f-wise arc forwarding index, as a generalization of widely studied arc forwarding index introduced in [11], and compute its values for the hypercube. Our results generalize corresponding results of [1, 4, 14], where the problem was solved for $f = 0$. For other work concerning 0-fault tolerant protocols in all-optical networks see, *e.g.* [1–4,7,10,16,17]. A preliminary version of this paper has appeared in [12].

2. Model, notation, and results

The Optical network model. Our model for optical networks and the wavelength routing problem is adopted from [2]. An all-optical network is modeled as a symmetric directed graph $G = (V_G, A_G)$, where V_G is a set of vertices, and A_G is a set of arcs such that if $\langle u, v\rangle \in A_G$, then $\langle v, u\rangle \in A_G$. Let $P(x, y)$ denote a *dipath* (directed path) in G from the vertex x to the vertex y, *i.e.* a sequence of vertices $v_1, \ldots, v_{k+1}$ such that $\langle v_i, v_{i+1}\rangle \in A_G$ is an arc for $1 \leq i \leq k$, $v_1 = x$ and $v_{k+1} = y$. The number k is the length of the dipath $P(x, y)$. Let $\operatorname{dist}(x, y)$ denote the *distance* of x and y in G, that is the length of a shortest dipath $P(x, y)$. If $P(x, y)$ and $P(y, z)$ are two dipaths such that $P(x, y) \cap P(y, z) = \{y\}$, then the *composition* $P(x, y) \cdot P(y, z)$ is the dipath $P(x, y')$ followed by $P(y, z)$, where y' is the last but one vertex of $P(x, y)$.

The Wavelength routing problem. A *request* in G is an ordered pair of vertices $[x, y]$ in G corresponding to a message to be send from x to y. An *instance* I in G is a set of requests in G. We denote the fact that I is a subject to G by (G, I). An *f-wise routing* for an (G, I) is a set of internally vertex-disjoint dipaths

$$R = \{P_0(x, y), P_1(x, y), \ldots, P_f(x, y) : \ [x, y] \in I\}$$

where $P_i(x, y) \cap P_j(x, y) = \{x, y\}$ for all $i \neq j$. Let us note that if $f = 0$, then the 0-wise routing for an (G, I) is simply referred to as the routing for (G, I).

Let G be a symmetric digraph and I an instance in G. The *f-wise wavelength routing problem* for (G, I) consists of finding an f-wise routing R for (G, I) and of finding an assignment of a wavelength to each dipath from R, so that no two dipaths of R sharing an arc are assigned the same wavelength.

Communication instances. For a general network G and an arbitrary instance I, the problem of determining the minimum number of wavelengths in a solution to the wavelength routing problem for (G, I) is an NP-complete problem [6]. We consider widely used global communication instance, the *total exchange* (also called *all-to-all* or *gossiping*). Later, in the proofs we will decompose this instance into special instances, called *all-to-all-d* instances, in which only pairs of vertices of distance d are considered. Formally, these instances can be defined as follows:

- the *all-to-all* instance is $I_A = \{[x, y] : \ x, y \in V_G, x \neq y\}$,
- the *all-to-all-d* instance is $I_A^d = \{[x, y] : \ x, y \in V_G, \operatorname{dist}(x, y) = d\}$.

Routing parameters. A natural lower bound for the minimum number of wavelengths needed for a given instance on a network is its arc-congestion in the network. Formally, let G be a symmetric digraph, I_A the all-to-all instance in G, and R an f-wise routing for (G, I_A). The *load* of an arc $\alpha \in A_G$ in the routing R, denoted by $\vec{\pi}(G, R, \alpha)$, is the number of dipaths of R containing α. The load of a routing R, denoted by $\vec{\pi}(G, R)$, is the maximum of $\vec{\pi}(G, R, \alpha)$ over all arcs $\alpha \in A_G$. The *f-wise arc-forwarding index* $\vec{\pi}(G, f)$ is the minimum of $\vec{\pi}(G, R)$ over all possible f-wise routings R for (G, I_A). Note that $\vec{\pi}(G, 0)$ is the well-known arc-forwarding index of G, *cf.* [11].

In the wavelength routing problem, it is convenient to think of wavelengths as colors. Thus, let $\vec{\omega}(G, f)$ denote the smallest number of colors that are sufficient to solve the f-wise wavelength routing problem for (G, I_A) over all possible f-wise routings for (G, I_A). The parameter $\vec{\omega}(G, f)$ is called the *f-wise optical index* of G, and if $f = 0$, then it is the optical index of G.

To solve a given f-wise wavelength routing problem for (G, I) one has to use a number of wavelengths at least equal to the maximum number of dipaths having to share an arc, hence we have:

Proposition 2.1. *For any symmetric digraph G, and any $f \geq 0$, $\vec{\pi}(G, f) \leq \vec{\omega}(G, f)$.*

The directed hypercube. Let H_r denote the r-dimensional binary hypercube in which each edge is replaced by two opposite directed arcs. Let us note that the vertices of H_r are all binary strings of length r and an ordered pair of vertices, say $u = u_1 u_2 \ldots u_r$ and $v = v_1 v_2 \ldots v_r$, is joined by an arc $\langle u, v\rangle$ if, and only if the corresponding strings differ in precisely one position, say $1 \leq i \leq r$. Subsequently, we say that u and v differ in the *ith dimension*, that the corresponding arc is the i-th *dimension arc*, and sometimes we refer to v as $u(i)$. In other words:

$$v = u(i) \qquad \text{and} \qquad u = v(i).$$

Let the symbol $\prec$ denote the lexicographical order on the hypercube vertices. Now for each ordered pair $[u, v] \in I_A$, we have either $u \prec v$ or $v \prec u$. In order to distinguish what kind of pair we are dealing with, we define the *orientation* $\sigma(u, v) \in \{\prec, \succ\}$ of an ordered pair $[u, v] \in I_A$ as $\sigma(u, v) = \prec$, if $u \prec v$, and $\sigma(u, v) = \succ$, otherwise.

The fault tolerant model. We consider a fault tolerant model in which arbitrary set of vertices and/or arcs may fail. A routing for (G, I) is said to be *f-fault tolerant* if it can remain functional in the presence of up to f faulty vertices and/or arcs in G. A routing is *functional* as long as it contains a non-faulty communication dipath between each pair of non-faulty vertices from the instance I. The following simple observation shows that in order to construct f-fault tolerant routing protocols it is enough to describe f-wise routings.

Proposition 2.2. *For any symmetric digraph G, any instance I in G, and any $f \geq 0$, any f-wise routing for (G, I) is f-fault tolerant for (G, I).*

Our Results. First, we study the f-wise arc-forwarding index of H_r. For any integer f, $0 \le f < r$, we prove that

$$\vec{\pi}(H_r, f) = (f+1)2^{r-1} + \left\lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1) \binom{r}{i} \right\rceil.$$

The above result is obtained by proving a lower bound for $\vec{\pi}(H_r, f)$, and then describing an f-wise routing $\mathcal{R}_f$ for (H_r, I_A) whose load is equal to the lower bound for $\vec{\pi}(H_r, f)$.

Second, we give a coloring of dipaths of $\mathcal{R}_f$, which gives us an upper-bound for $\vec{\omega}(H_r, f)$. The number of colors in the coloring is nearly optimal, as we prove that

$$\vec{\pi}(H_r, f) \le \vec{\omega}(H_r, f) \le \begin{cases} (1+\varepsilon_r)\vec{\pi}(H_r, f), & \text{for } 0 \le f < \frac{r}{2}, \\ (\frac{3}{2}+\varepsilon_r)\vec{\pi}(H_r, f), & \text{for } \frac{r}{2} \le f < r, \end{cases}$$

where $\lim_{r\to\infty} \varepsilon_r = 0$. Because both the description of the layout $\mathcal{R}_f$ and also the description of the coloring of dipaths of $\mathcal{R}_f$ can be carried out in polynomial time, we conclude that we have given nearly optimal f-fault tolerant all-to-all protocols for all-optical hypercubes. Moreover, our results generalize corresponding results for the hypercube in [1, 4, 14]. Indeed, the results for hypercube in these papers can be obtained from ours by setting $f = 0$.

3. The f-wise arc-forwarding index of H_r

If a network possesses an f-wise routing for the instance I_A, then since any pair of vertices of the underling network must be connected with at least $f+1$ vertex/arc disjoint dipaths, the underling network must be $(f+1)$-vertex/arc connected. Since H_r is r-vertex/arc connected, we may hope to describe f-wise routings for H_r only for $f \in \{0, 1, \ldots, r-1\}$. Thus, through the rest of the paper we assume that $0 \le f < r$. We begin with giving a lower bound to the f-wise arc-forwarding index of H_r.

A bound for binomial coefficients. We use the following basic upper bound for binomial coefficients which can be found, for example, in [8].

$$\binom{r}{k} < 2^{r-\frac{1}{2}\log 3r+1}.$$

Lemma 3.1. *For any f, $0 \le f < r$,*

$$\vec{\pi}(H_r, f) \ge (f+1)2^{r-1} + \left\lceil \frac{2}{r} \sum_{i=1}^{f} (f-i+1) \binom{r}{i} \right\rceil.$$

Proof. We first lower-bound the total congestion count in any f-wise routing for (H_r, I_A). In any f-wise routing there are $f+1$ internally vertex disjoint dipaths joining any ordered pair of vertices. Consider any two vertices in H_r of distance d, $1 \le d \le r$. The total contribution of the corresponding $f+1$ dipaths to the congestion count is at least $d^2 + (f-d+1)(d+2)$ if $d \le f$, and if $d > f$, then the total contribution is at least $(f+1)d$. Since there are $2^r\binom{r}{d}$ distinct ordered pairs of vertices of distance d in H_r, the overall contribution is at least $2^r\binom{r}{d}((d+2)f-d+2)$ if $d \le f$, and is at least $2^r\binom{r}{d}(f+1)d$ if $d > f$.

Hence the total congestion count is at least

$$\sum_{i=1}^{f} 2^r\binom{r}{i}((i+2)f-i+2) + \sum_{i=f+1}^{r} 2^r\binom{r}{i}(f+1)i$$
$$= (f+1)r2^{2r-1} + 2^{r+1}\sum_{i=1}^{f}(f-i+1)\binom{r}{i} = L\,.$$

Since there are $r2^r$ arcs in H_r, the average congestion of any f-wise routing for (H_r, I_A) is at least $L/(r2^r)$. This means that

$$\vec{\pi}(H_r, f) \ge (f+1)2^{r-1} + \left\lceil \frac{2}{r}\sum_{i=1}^{f}(f-i+1)\binom{r}{i}\right\rceil. \qquad \square$$

In what follows, we describe the construction of an f-wise routing for (H_r, I_A) that attains the above lower bound. First, we give a basic idea of the construction. For every ordered pair of vertices $[u,v]$, we will describe $f+1$ internally vertex disjoint dipaths as follows. Assume the distance of u and v is d. Then, u and v differ in exactly d dimensions, and consequently, one can choose d shortest internally vertex-disjoint dipaths joining u to v. We will use these shortest dipaths in our routing in an order to be specified later. If $d < f+1$, we still need $f+1-d$ dipaths. The length of these dipaths will be $d+2$, and for every such dipath, we choose a dimension i, such that the dipath will have the form: $\langle u, u(i)\rangle \cdot P \cdot \langle v(i), v\rangle$, where P is the first (in the order given below) of the shortest dipaths from $u(i)$ to $v(i)$. We must choose i such that all the $f+1$ dipaths joining u and v (in our routing) will be internally vertex disjoint, and moreover, the congestion of the resulting routing will be as small as possible. To minimize the congestion, the routing must have the property that every arc of H_r plays the role of i almost the same number of times. To achieve this, we introduce and prove the existence of an (r,f)-system of f functions that will give us $f+1-d$ different dimensions for every pair of vertices $[u,v]$ of distance d, such that all the $f+1$ dipaths joining u to v will be internally vertex disjoint. In addition, the system will have the property that the load of every arc of H_r will be almost the same (such a system will be called regular). Now we are ready to present the detailed description of the routing.

The characteristic. We will treat some specific pairs of vertices as similar. For this reason, we introduce the notion of characteristic. For any pair of distinct vertices u and v of H_r, the *characteristic* $C = (c_1, c_2, \ldots, c_d)$, with $c_1 < c_2 < \cdots < c_d$, is the ordered list of all dimensions in which u and v differ. Let $\mathbf{C}$ denote the set of all characteristics in H_r, and $\mathbf{C}_d$ the set of all characteristics in H_r with d elements. Obviously, $|\mathbf{C}| = 2^r - 1$, and $|\mathbf{C}_d| = \binom{r}{d}$. For convenience, let $\mathbf{C}_{a,b} = \cup_{i=a}^{b} \mathbf{C}_i$.

Regular(r, f)-system of functions. Now we describe the above mentioned system of functions. The system will return the same value from $\{1, 2, \ldots, r\}$ for any ordered pair of vertices of H_r with the same characteristic and the same orientation. However, the returned values for $[u, v]$ and $[v, u]$ will be different (this technical requirement will be later used to minimize the number of wavelengths). The system consists of f functions $t_0, t_1, \ldots, t_{f-1}$, where t_i is defined on $\mathbf{C}_{1,f-i} \times \{\prec, \succ\}$. Thus, for any fixed characteristic $C \in \mathbf{C}_d$, we have functions $t_0, t_1, \ldots, t_{f-d}$ that are defined for $(C, \prec)$ and $(C, \succ)$.

Formally, an (r, f)-*system of functions* is a system of f functions $t_0, t_1, \ldots, t_{f-1}$, where $t_i : \mathbf{C}_{1,f-i} \times \{\prec, \succ\} \to \{1, \ldots, r\}$, and for each characteristic $C \in \mathbf{C}_d$, $1 \leq d \leq f$, and each orientation $\phi \in \{\prec, \succ\}$, the elements of C and the elements $t_0(C, \phi), \ldots, t_{f-d}(C, \phi)$ are all distinct. The idea behind using the orientation ϕ is that in this way we are able to describe two different dipaths connecting u and v: one for $[u, v]$ and another for $[v, u]$.

The *distribution* $e = (e_1, \ldots, e_r)$ of an (r, f)-system of functions $\{t_i\}_{i=0}^{f-1}$ is a vector of integers, where

$$e_j = |\{(x, i) : x \in \mathbf{C}_{1,f-i} \times \{\prec, \succ\}, \quad i = 0, \ldots, f-1, \quad \text{and} \quad t_i(x) = j\}| ,$$

for all $j = 1, \ldots, r$. Numbers e_j represent how many times functions $t_0, \ldots, t_{f-1}$ return the value j. We say that an (r, f)-system of functions is *regular*, if $\max_{j=1}^{r} e_j - \min_{j=1}^{r} e_j \leq 1$.

The following lemma guarantees the existence of a regular (r, f)-system of functions.

Lemma 3.2. *For any $0 \leq f < r$, there exists a regular (r, f)-system of functions satisfying*

$$t_i(C, \prec) \neq t_i(C, \succ), \quad \textit{for all } C \in \mathbf{C}_d,\ 1 \leq d \leq f,\ \textit{and } 0 \leq i \leq f - d. \tag{1}$$

Before we present the proof of the above lemma we need some more definitions.

The configuration. Let ρ be a *cyclic shift* on the set $\{1, \ldots, r\}$, *i.e.* $\rho(i) = i + 1$ for $i < r$ and $\rho(r) = 1$. Clearly, the map ρ can be extended to the set of all characteristics $\mathbf{C}$. Let us define the equivalence relation on $\mathbf{C}$ as follows: two characteristics $C_1, C_2 \in \mathbf{C}$ are equivalent, written $C_1 \sim C_2$, if there exists k such that $C_2 = \rho^k(C_1)$. This relation factorizes the set $\mathbf{C}$ (resp. $\mathbf{C}_d$) into the set of equivalence classes $\widehat{\mathbf{C}}$ (resp. $\widehat{\mathbf{C}}_d$). The equivalence classes are called *configurations*. One can imagine each configuration $\hat{C} \in \widehat{\mathbf{C}}$ as a ring with r beads colored by two colors, say black and white, see Fig. 1a). Note that if $\hat{C} \in \widehat{\mathbf{C}}_d$, then the ring representing $\hat{C}$ has exactly d black beads and $r - d$ white beads.

If we number beads clockwise starting from an arbitrary bead, then the black beads represent a characteristic. For example, in Fig. 1b) the characteristic $(3, 5, 6)$

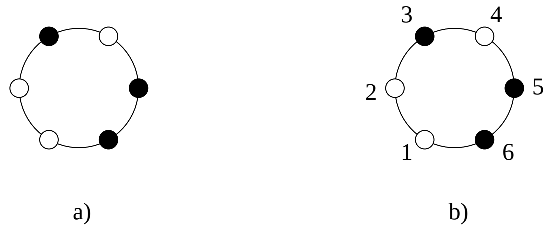

FIGURE 1. A configuration and a corresponding characteristic.

is depicted. It belongs to the equivalence class (configuration)

$$\hat{C} = \{(1,3,4),(2,4,5),(3,5,6),(1,4,6),(1,2,5),(2,3,6)\}\,.$$

If the rotation of the ring representing configuration $\hat{C}$ by m beads results in the original position of the ring, we say that $\hat{C}$ has *symmetry* m. Further, a configuration $\hat{C}$ is m-*symmetric*, if $m > 0$ is the smallest symmetry it has. It is easy to observe that any m-symmetric configuration contains exactly m characteristics and that $m \mid r$. Two numbers x and y are *congruent modulo* r, written $x \equiv y \pmod r$, if they give the same remainder after dividing by r.

Proof of Lemma 3.2. We first prove the existence of a regular (r, f)-system. Later, we show how it can be modified to satisfy (1). We prove a slightly stronger statement than in Lemma 3.2. We say that a system of functions has an (a, b)-*cyclic distribution* if its distribution $e = (e_1, \dots, e_r)$ for some $\alpha \geq 0$ satisfies:

$$e_a = e_{a+1} = \cdots = e_{b-1} = \alpha + 1 \text{ and } e_b = e_{b+1} = \cdots = e_r = e_1 = \cdots = e_{a-1} = \alpha$$

if $a \leq b$, and

$$e_a = e_{a+1} = \cdots = e_r = e_1 = \cdots = e_{b-1} = \alpha + 1 \text{ and } e_b = e_{b+1} = \cdots = e_{a-1} = \alpha$$

if $a > b$.

In what follows, we construct an (r, f)-system of functions that has $(1, b)$-cyclic distribution for some b. Such system is obviously regular.

Let $j_0 \in \{1, \dots, r\}$. Consider an m-symmetric configuration $\hat{C} \in \widehat{\mathbf{C}}_d$, $1 \leq d < r$, and an orientation $\phi \in \{\prec, \succ\}$. We first show how to define functions $\{t_i\}_{i=0}^{f-d}$ on $\hat{C} \times \{\phi\}$ such that they have (j_0, j_0')-cyclic distribution for some j_0'. Since $d < r$, there exists a characteristic $C_0 \in \hat{C}$ such that $j_0 \notin C_0$. Let $C_k = \rho^k(C_0)$ for $k = 1, \dots, m-1$; so $\hat{C} = \{C_0, \dots, C_{m-1}\}$.

Consider the numbered ring representing C_0. The ring has $r - d$ white beads. Since the ring is m-symmetric, each part of the ring with m beads has the same number of white beads. We divide the ring into $\frac{r}{m}$ disjoint parts, each consisting of m beads. Hence, each part contains exactly $w := \frac{m}{r}(r-d)$ white beads. Consider the part containing j_0. Let $j_1, \dots, j_{w-1} \in \{1, \dots, r\}$ be the corresponding numbers of remaining white beads in the part. Thus, $j_0, j_1, \ldots, j_{w-1}$ do not belong to C_0, and are all distinct modulo m. For notation convenience, we let $j_w = j_0$.

Let $f - d + 1 = p \cdot \frac{r}{m} + q$, where $0 \le q < \frac{r}{m}$, so

$$p = \left\lfloor \frac{m}{r}(f-d+1) \right\rfloor \le \frac{m}{r}(f-d+1) \le w,$$

where $p = w$ implies $q = 0$. Define the values of functions $t_0, \ldots, t_{f-d}$ on the set $\hat{C} \times \{\phi\} \subseteq \mathbf{C}_d \times \{\phi\}$ as follows:

$$t_{i+(l-1)\cdot\frac{r}{m}}(C_k, \phi) \equiv j_l + k + im \pmod{r}, \tag{2}$$

for $i = 0, \ldots, \frac{r}{m} - 1$, $l = 1, \ldots, p$, and if $q > 0$, let

$$t_{i+p\cdot\frac{r}{m}}(C_k, \phi) \equiv j_0 + k + im \pmod{r}, \tag{3}$$

for $i = 0, \ldots, q-1$. Note that since $p \le w$, all j_l's in (2) are defined, and if $q > 0$, then j_0 is different from all these values. Moreover, since the returned values of the (r, f)-system of functions must be from $\{1, \ldots, r\}$, the system $\{t_i\}_{i=0}^{f-d}$ is well defined using congruence relations in (2) and (3).

Now we show that the values of elements of C_k and the values

$$t_0(C_k, \phi), \ldots, t_{f-d}(C_k, \phi)$$

are all distinct for any $k = 0, \ldots, m-1$. Indeed, since for any $0 \le l < w$, $j_l \notin C_0$, we have also $(j_l + k) \bmod r \notin C_k$, and since $\hat{C}$ is m-symmetric, for all i, we have $(j_l + k + im) \bmod r \notin C_k$. Thus, $t_0(C_k, \phi), \ldots, t_{f-d}(C_k, \phi) \notin C_k$. Assume that $t_{i+(l-1)\cdot\frac{r}{m}}(C_k, \phi) = t_{i'+(l'-1)\cdot\frac{r}{m}}(C_k, \phi)$ for some $0 \le i, i' \le \frac{r}{m} - 1$, and $1 \le l, l' \le p$. (For the simplicity we consider only the case $q = 0$, the case $q > 0$ is left for the reader.) This implies that $j_l + im \equiv j_{l'} + i'm \pmod{r}$, and also $j_l \equiv j_{l'} \pmod{m}$. But j_l's are distinct modulo m, so $l = l'$ and the first equality simplifies to $(i - i')m \equiv 0 \pmod{r}$, which gives $i = i'$. Note that if $m = r$, then, by assumption, we must have $i = i' = 0$.

We next show that for each $l = 1, \ldots, p$, the functions $\{t_{i+(l-1)\cdot\frac{r}{m}}\}_{i=0}^{\frac{r}{m}-1}$ defined on $\hat{C} \times \{\phi\}$ contribute to the distribution by the r-vector $(1, \ldots, 1)$. This follows from the fact that the sequence

$$\begin{aligned}
&t_{0+(l-1)\cdot\frac{r}{m}}(C_0, \phi), t_{0+(l-1)\cdot\frac{r}{m}}(C_1, \phi), \ldots, t_{0+(l-1)\cdot\frac{r}{m}}(C_{m-1}, \phi),\\
&t_{1+(l-1)\cdot\frac{r}{m}}(C_0, \phi), t_{1+(l-1)\cdot\frac{r}{m}}(C_1, \phi), \ldots, t_{1+(l-1)\cdot\frac{r}{m}}(C_{m-1}, \phi),\\
&\qquad\vdots\\
&t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_0, \phi), t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_1, \phi), \ldots, t_{\frac{r}{m}-1+(l-1)\cdot\frac{r}{m}}(C_{m-1}, \phi)
\end{aligned}$$

is an arithmetic sequence modulo r with step 1. Indeed, it is easy to see that the difference modulo r between two consecutive elements in any line is 1, and the difference between the first element of a line and the last element of the previous line is

$$t_{(i+1)+(l-1)\cdot\frac{r}{m}}(C_0,\phi) - t_{i+(l-1)\cdot\frac{r}{m}}(C_{m-1},\phi) \equiv \\ (j_l+0+(i+1)m) - (j_l+(m-1)+im) = 1 \pmod{r}.$$

In the same sequel, one can show that the functions $\{t_{i+p\cdot\frac{r}{m}}\}_{i=0}^{q-1}$ on $\hat{C}\times\{\phi\}$ have (j_0, j_0')-cyclic distribution, where $j_0' \equiv j_0 + qm \pmod{r}$. Hence, the functions $\{t_i\}_{i=0}^{f-d}$ on $\hat{C}\times\{\phi\}$ have (j_0, j_0')-cyclic distribution, as claimed.

Now, let $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K$ be a sequence of all configurations, where $\hat{C}_i \subseteq \mathbf{C}_{1,f}$. Defining the (r,f)-system of functions is equivalent to defining it step by step on the following sequence

$$\hat{C}_1\times\{\prec\}, \hat{C}_2\times\{\prec\}, \ldots, \hat{C}_K\times\{\prec\}, \hat{C}_1\times\{\succ\}, \hat{C}_2\times\{\succ\}, \ldots, \hat{C}_K\times\{\succ\}\,.$$

By the above construction, we can define functions on $\hat{C}_1\times\{\prec\}$, such that they have $(1,a_1)$-cyclic distribution for some a_1. Then we define functions on $\hat{C}_2\times\{\prec\}$ such that they have (a_1,a_2)-cyclic distribution. Clearly, these functions on $\hat{C}_1\times\{\prec\}\cup\hat{C}_2\times\{\prec\}$ will have $(1,a_2)$-cyclic distribution. We continue inductively and construct an (r,f)-system of functions with $(1,b)$-cyclic distribution for some value b.

Finally, we show how a regular (r,f)-system of functions $\{t_i\}_{i=0}^{f-1}$ can be modified to satisfy (1). Let us observe that if we permute the values

$$t_0(C,\phi), \ldots, t_{f-d}(C,\phi)\,,$$

for fixed $C \in \mathbf{C}_d$, and $\phi \in \{\prec,\succ\}$, we get a regular system of functions, again. The result follows from the fact that for each $d = 1,\ldots,f-1$, and $C\in\mathbf{C}_d$, there exists a permutation of $t_0(C,\succ),\ldots,t_{f-d}(C,\succ)$ such that $t_i(C,\prec)\neq t_i(C,\succ)$ for $i=0,\ldots,f-d$. □

Description of the layout. Let $[u,v]\in I_A$ be an ordered pair of vertices of H_r with the characteristic $C=(c_1,c_2,\ldots,c_d)$. We define $f+1$ dipaths joining u to v as follows. For $i=0,1,\ldots,\min\{d-1,f\}$, we let $\mathcal{P}_i(u,v)=(u_0,u_1,\ldots,u_d)$, where $u_0=u$, $u_1=u_0(c_{i+1})$, $u_2=u_1(c_{i+2})$, $\ldots$, $u_{d-i}=u_{d-i-1}(c_d)$, $u_{d-i+1}=u_{d-i}(c_1)$, $u_{d-i+2}=u_{d-i+1}(c_2)$, $\ldots$, $u_d=u_{d-1}(c_i)$. Let us note that the i-th dipath is constructed so that from u we first use the c_{i+1}-th dimension, then the c_{i+2}-th dimension, and so on. These are the only shortest u to v dipaths in our layout. For $i=d,d+1,\ldots,f$, we let $\mathcal{P}_i(u,v)=\langle u,u'\rangle\cdot\mathcal{P}_0(u',v')\cdot\langle v',v\rangle$, where $u'=u(t_{i-d}(C,\sigma(u,v)))$, $v'=v(t_{i-d}(C,\sigma(u,v)))$, and $\{t_i\}_{i=0}^{f-1}$ is a regular (r,f)-system of functions.

Now, for every f, $0\le f<r$, we define the layout $\mathcal{R}_f$ for (H_r,I_A) as follows.

$$\mathcal{R}_f=\{\mathcal{P}_i(x,y),\ i=0,\ldots,f\ :\ [x,y]\in I_A\}.$$

Since the proof of Lemma 3.2 is constructive and the corresponding algorithm (which can be easily abstracted from the proof) is polynomial, there is a polynomial algorithm that for given r and f, returns the dipaths of the layout $\mathcal{R}_f$.

For convenience, we define the following sub-layouts of $\mathcal{R}_f$. First, for any $1 \le d \le r$, and $0 \le k \le f$, we define a sub-layout $R_{k,d}$ consisting of all k-th dipaths joining vertices at distance d in R_f. Thus,

$$\mathcal{R}_{k,d} = \{\mathcal{P}_k(x,y) \in \mathcal{R}_f : [x,y] \in I_A^d\}.$$

Obviously, $\mathcal{R}_f = \cup_{1\le d\le r} \cup_{0\le k\le f} \mathcal{R}_{k,d}$. Second, for any characteristic C and any $0 \le k \le f$, we define a sub-layout $R_k(C)$ consisting of all k-th dipaths joining pairs of vertices with the characteristic C. Thus,

$$\mathcal{R}_k(C) = \{\mathcal{P}_k(x,y) : \text{ the characteristic of } [x,y] \text{ is } C\}.$$

Routing properties of the layout $\mathcal{R}_f$.

Lemma 3.3. *The layout $\mathcal{R}_f$ is an f-wise routing in H_r for any $0 \le f < r$.*

Proof. It is enough to prove that the dipaths $\mathcal{P}_i(u,v)$, $i = 0,1,\dots,f$, are pairwise internally vertex disjoint for any pair of vertices $u = u_1u_2\dots u_r$ and $v = v_1v_2\dots v_r$ of characteristic $C = (c_1,c_2,\dots,c_d)$. Let $\sigma(u,v) = \prec$.

For any $0 \le i < j \le \min\{d-1,f\}$, the dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint. Indeed, every internal vertex, say $x = x_1x_2\dots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{c_{j+1}} = v_{c_{j+1}}$ and $x_{c_j} = u_{c_j}$, while every internal vertex, say $y = y_1y_2\dots y_r$, of $\mathcal{P}_i(u,v)$ has either $y_{c_{j+1}} = v_{c_{j+1}}$ and $y_{c_j} = v_{c_j}$, or $y_{c_{j+1}} = u_{c_{j+1}}$. The claim follows from the fact that $u_{c_j} \ne v_{c_j}$ and $u_{c_{j+1}} \ne v_{c_{j+1}}$(recall the definition of the characteristic).

For any $0 \le i \le \min\{d-1,f\}$ and $d \le j \le f$, the dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint. Note that if $f < d$, then this is trivially true. Now, every internal vertex, say $x = x_1x_2\dots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{t_{j-d}(C,\prec)} \ne u_{t_{j-d}(C,\prec)}$, while every internal vertex, say $y = y_1y_2\dots y_r$, of $\mathcal{P}_i(u,v)$ has $y_{t_{j-d}(C,\prec)} = u_{t_{j-d}(C,\prec)}$, since $t_{j-d}(C,\prec) \notin C$.

Finally, for any $d \le i < j \le f$, the dipaths $\mathcal{P}_i(u,v)$ and $\mathcal{P}_j(u,v)$ are internally vertex disjoint. (If $f < d$, then this is trivially true.) By the definition of the (r,f)-system of functions, the values $t_{i-d}(C,\prec)$, $t_{j-d}(C,\prec)$ are distinct and do not belong to C. Hence every internal vertex, say $x = x_1x_2\dots x_r$, of $\mathcal{P}_j(u,v)$ has $x_{t_{j-d}(C,\prec)} \ne u_{t_{j-d}(C,\prec)}$, while every internal vertex, say $y = y_1y_2\dots y_r$, of $\mathcal{P}_i(u,v)$ has $y_{t_{j-d}(C,\prec)} = u_{t_{j-d}(C,\prec)}$. □

Lemma 3.4. *For any f, $0 \le f < r$, it holds that*

$$\vec{\pi}(H_r,f) \le (f+1)2^{r-1} + \left\lceil \frac{2}{r}\sum_{i=1}^{f}(f-i+1)\binom{r}{i}\right\rceil.$$

Proof. We show that the layout $\mathcal{R}_f$ attains the upper bound. By Lemma 3.3, the proof will follow. We split the layout $\mathcal{R}_f$ into $(f+1)r$ sub-layouts $\mathcal{R}_{k,d}$, where

$0 \leq k \leq f$ and $1 \leq d \leq r$, and calculate the congestion for every sub-layout separately.

Let us fix d, $1 \leq d \leq r$. First, consider sub-layouts $\mathcal{R}_{k,d}$ for $k = 0, 1, \ldots, \min\{d-1, f\}$. For any such fixed k, all dipaths in $\mathcal{R}_{k,d}$ are the shortest dipaths. Fix a characteristic $C \in \mathbf{C}_d$, and consider $\mathcal{R}_k(C)$. Obviously, in the layout $\mathcal{R}_k(C)$, every arc of H_r is used at most once. More precisely, the congestion of an arc is one if, and only if the arc is an i-th dimension arc, where $i \in C$. Indeed, for any fixed C and any fixed k, fixing an i-th dimension arc ($i \in C$) determines whole dipath completely. This means that the congestion of an i-th dimension arc of H_r in the sub-layout $\mathcal{R}_{k,d}$ is equal to the total number of characteristic from $\mathbf{C}_d$ containing i. This number is equal to $\binom{r-1}{d-1}$. Since it does not depend on i, this means that the congestion of any arc of H_r in $\mathcal{R}_{k,d}$ is $\binom{r-1}{d-1}$.

Second, consider sub-layouts $\mathcal{R}_{k,d}$ for $k = d, d+1, \ldots, f$ (recall, d is fixed). If $f < d$, we are done. Otherwise, for any fixed k, all dipaths in $\mathcal{R}_{k,d}$ are of the form $\langle x, x' \rangle \cdot \mathcal{P}_0(x', y') \cdot \langle y', y \rangle$, where $x' = x(t_{k-d}(C, \sigma(x,y)))$, $y' = y(t_{k-d}(C, \sigma(x,y)))$, and $[x,y] \in I_A^d$ has characteristic C. We first calculate the congestion forced by dipaths $\mathcal{P}_0(x', y')$. One can see that $\cup_{[x,y] \in I_A^d} [x', y'] = I_A^d$. Indeed, x' and y' are of the same characteristic as x and y are, and moreover, $\sigma(x', y') = \sigma(x, y)$. Hence, $x = x'(t_{k-d}(C, \sigma(x', y')))$, $y = y'(t_{k-d}(C, \sigma(x', y')))$. As a consequence, we conclude that

$$\cup_{[x,y] \in I_A^d} \mathcal{P}_0(x', y') = \cup_{[x,y] \in I_A^d} \mathcal{P}_0(x, y) = \mathcal{R}_{0,d}\,.$$

Hence the load of any arc of H_r in $\mathcal{R}_{k,d}$ forced by dipaths $\mathcal{P}_0(x', y')$ is $\binom{r-1}{d-1}$. It follows that the load of any arc of H_r in $\mathcal{R}_f$ forced by the dipaths considered thus far is equal to $(f+1)2^{r-1}$, since $1 \leq d \leq r$. Finally, we consider the remaining arcs of $\cup_{1 \leq d \leq r} \cup_{d \leq k \leq f} \mathcal{R}_{k,d}$ (arcs of the form $\langle x, x' \rangle$ and $\langle y', y \rangle$). These arcs are exactly those, returned by functions $\{t_i\}_{i=0}^{f-1}$. Since $\{t_i\}_{i=0}^{f-1}$ is a regular (r, f)-system of functions, we know that any dimension of H_r is returned at most $\lceil \frac{2}{r} \sum_{i=1}^{f} (f - i + 1)\binom{r}{i} \rceil$ times. Indeed, for any fixed i, $1 \leq i \leq d$, there are $\binom{r}{i}$ distinct characteristics with i elements, and we apply $t_0, t_1, \ldots, t_{f-i}$ to every of these characteristics two times (once with $\prec$, and once with $\succ$) in the layout $\mathcal{R}_f$.

Hence, the load of any arc of H_r in the layout $\mathcal{R}_f$ is at most $(f+1)2^{r-1} + \lceil \frac{2}{r} \sum_{i=1}^{f} (f - i + 1)\binom{r}{i} \rceil$, as claimed. □

Corollary 3.5. *For any f, $0 \leq f < r$, it holds that*

$$\vec{\pi}(H_r, f) = (f+1)2^{r-1} + \left\lceil \frac{2}{r} \sum_{i=1}^{f} (f - i + 1)\binom{r}{i} \right\rceil .$$

The previous corollary determines the f-wise forwarding index of H_r. Moreover, it generalizes corresponding results (when $f = 0$) from [1, 4, 14], then $\vec{\pi}(H_r) = \vec{\pi}(H_r, 0) = 2^{r-1}$. Note that we have a nice closed formula also in the case $f = r-1$, then $\vec{\pi}(H_r, r-1) = (r+2) \cdot 2^{r-1} - 2$.

We have proved that $\mathcal{R}_f$, for $0 \le f < r$, is an f-wise routing, and thus, by Proposition 2.2, it is f-fault tolerant in H_r. In what follows, we show that $\mathcal{R}_f$ also requires nearly optimal number of wavelengths to solve the f-wise wavelength routing problem for H_r.

4. The f-wise optical index of H_r

Description of the coloring. Consider a characteristic $C \in \mathbf{C}_d$. It is easy to observe that if $k < d$, any two dipaths in $\mathcal{R}_k(C)$ are arc disjoint, so for $k < d$, we color all dipaths in $\mathcal{R}_k(C)$ by the same color. Denote by $\bar{C}$ the complement of characteristic C, *i.e.* $\bar{C} = \{1, \ldots, r\} \setminus C$. If $k < d$ and $k < r - d$, then any dipath in $\mathcal{R}_k(C)$ is arc disjoint with any dipath in $\mathcal{R}_k(\bar{C})$, since they use arcs of distinct dimensions. Hence, in this case we color all dipaths in $\mathcal{R}_k(C) \cup \mathcal{R}_k(\bar{C})$ by the same color.

Finally, we analyze the case $k \ge d$. Take any two distinct dipaths $\mathcal{P}_k(u_1, v_1)$, and $\mathcal{P}_k(u_2, v_2)$ from $\mathcal{R}_k(C)$, $C \in \mathbf{C}_d$, such that $\phi = \sigma(u_1, v_1) = \sigma(u_2, v_2)$. Now

$$\begin{aligned} \mathcal{P}_k(u_1, v_1) &= \langle u_1, u_1' \rangle \cdot \mathcal{P}_0(u_1', v_1') \cdot \langle v_1', v_1 \rangle \quad \text{and} \\ \mathcal{P}_k(u_2, v_2) &= \langle u_2, u_2' \rangle \cdot \mathcal{P}_0(u_2', v_2') \cdot \langle v_2', v_2 \rangle \,, \end{aligned}$$

where $u_1' = u_1(e)$, $v_1' = v_1(e)$, $u_2' = u_2(e)$, $v_2' = v_2(e)$, and $e = t_{k-d}(C, \phi)$. Clearly, $\mathcal{P}_0(u_1', v_1')$ and $\mathcal{P}_0(u_2', v_2')$ are arc-disjoint, since they are the shortest dipaths of the same characteristic C. Also arcs $\langle u_1, u_1' \rangle$, $\langle v_1', v_1 \rangle$ (resp. $\langle u_2, u_2' \rangle$, $\langle v_2', v_2 \rangle$) cannot be contained in the dipath $\mathcal{P}_0(u_2', v_2')$ (resp. $\mathcal{P}_0(u_1', v_1')$), since they are arcs of dimension $e \notin C$. Hence, the only possibility, where the above dipaths could intersect are arcs $\langle u_1, u_1' \rangle$, $\langle v_2', v_2 \rangle$ or arcs $\langle u_2, u_2' \rangle$, $\langle v_1', v_1 \rangle$. Assume, for instance, $\langle u_1, u_1' \rangle = \langle v_2', v_2 \rangle$. This implies $v_2' = u_1$, $u_1' = v_2$, and also $u_2' = v_1$, $v_1' = u_2$. We must have $\sigma(v_2, v_1) = \sigma(v_2', v_1') = \sigma(u_1, u_2)$, which is a contradiction. One can come to a similar contradiction with the second pair of arcs. So, the dipaths are arc-disjoint. Note, that the same holds without the assumption that the pairs $[u_1, v_1]$, $[u_2, v_2]$ have the same orientation, if we assume that $t_k(C, \prec) \neq t_k(C, \succ)$. According to Lemma 3.2, we can assume that the system of functions $\{t_i\}_{i=0}^{f-1}$ has property (1), and we can color all dipaths in $\mathcal{R}_k(C)$ by the same color, if $d < f$. If $d = f$, then $k = f$ and we need 2 colors for dipaths $\mathcal{P}_f(u, v) \in \mathcal{R}_f(C)$: one for corresponding dipaths with $u \prec v$ and another one for dipaths with $u \succ v$. This gives us the following upper bound on the f-wise optical index.

Theorem 4.1. *For any $0 \le f < \frac{r}{2}$, it holds that*

$$\begin{aligned} \vec{\omega}(H_r, f) &\le (f+1)2^{r-1} + \sum_{i=1}^{f} (f - i + 1) \binom{r}{i} + (1 - \delta_{f,0}) \cdot \binom{r}{f} \\ &\le \left(1 + \tfrac{4}{\sqrt{3r}}\right) \vec{\pi}(H_r, f) \,, \end{aligned}$$

and for any $\frac{r}{2} \leq f < r$*, it holds that*

$$\begin{aligned}\vec{\omega}(H_r, f) &\leq (2f + 2 - r/2)2^{r-1} - (f+1) + \frac{1}{2}\left\lceil\frac{r}{2}\right\rceil\binom{r}{\lceil r/2\rceil} + \binom{r}{f}\\ &\leq \left(\frac{3}{2} + \frac{2(r+5)}{3r(r+2)}\right)\vec{\pi}(H_r, f)\,.\end{aligned}$$

Here δ is the Kronecker δ, defined as follows $\delta_{i,j} = \begin{cases} 0 & \text{for } i \neq j,\\ 1 & \text{for } i = j.\end{cases}$

Proof. Let $S_k = \bigcup_{d=1}^{r} \mathcal{R}_{k,d}$. Consider, for instance, r odd. According to the description of coloring, to color paths in S_k, where $k \leq \frac{r-1}{2}$, we use $2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}$ colors: to color paths in $\cup_{d=\frac{r+1}{2}}^{r}\mathcal{R}_{k,d}$ we use 2^{r-1} colors; for paths in $\mathcal{R}_{k,d}$, where $k < d \leq \frac{r-1}{2}$, we use the same set of colors as for paths in $\mathcal{R}_{k,r-d}$ and for other paths we use new colors. If $k = d = f \neq 0$, then we have to use another set of $\binom{r}{f}$ colors. So for $f \leq \frac{r-1}{2}$, we color our layout with $\sum_{k=0}^{f}\left[2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}\right] + (f \neq 0)\cdot\binom{r}{f}$ colors, which gives us the first claim of the theorem.

Notice, that to color paths in S_k, where $k \geq \frac{r+1}{2}$, we can use the same number of colors as we used for coloring paths in $S_{\frac{r-1}{2}}$. This is $2^{r-1} + \sum_{i=1}^{\frac{r-1}{2}}\binom{r}{i} = 2\cdot 2^{r-1} - 1$ colors. All together we can color the layout $\mathcal{R}_f$, where $f \geq \frac{r+1}{2}$ with

$$\sum_{k=0}^{\frac{r-1}{2}}\left[2^{r-1} + \sum_{i=1}^{k}\binom{r}{i}\right] + \left(f + 1 - \frac{r+1}{2}\right)(2\cdot 2^{r-1} - 1) + \binom{r}{f} = \\ \left(2f + 2 - \frac{r}{2}\right)2^{r-1} - (f+1) + \frac{r+1}{4}\binom{r}{\frac{r+1}{2}} + \binom{r}{f}$$

colors, which gives the claim. The evaluation for even r is very similar. The last bounds in the statement of the theorem come from (3) and an observation that $\sum_{i=1}^{f}(f-i+1)\binom{r}{i} \leq f\cdot\binom{r}{f}$. $\square$

Note that for $(r-1)$-wise optical index of H_r we have a nice closed formula (similarly as we had for $(r-1)$-wise forwarding index) as an upper bound $\vec{\omega}(H_r, r-1) \leq \frac{3}{2}r\cdot 2^{r-1} + \frac{1}{2}\left\lceil\frac{r}{2}\right\rceil\binom{r}{\lceil r/2\rceil}$. Using Proposition 2.1, we obtain the following nearly optimal bounds on $\vec{\omega}(H_r, f)$.

Corollary 4.2. *For any* $0 \leq f < r$,

$$\vec{\pi}(H_r, f) \leq \vec{\omega}(H_r, f) \leq \begin{cases}(1+\varepsilon_r)\vec{\pi}(H_r, f), & \textit{for } 0 \leq f < \frac{r}{2},\\ (\frac{3}{2}+\varepsilon_r)\vec{\pi}(H_r, f), & \textit{for } \frac{r}{2} \leq f < r,\end{cases}$$

where $\lim_{r\to\infty}\varepsilon_r = 0$.

Concluding remarks. It is a well known open problem for 0-wise routings to show that $\vec{\pi}(G,0) = \vec{\omega}(G,0)$ for any symmetric digraph G. This is the case for many extensively studied interconnection networks and was recently proved also for the symmetric trees, *cf.* [7]. As we mentioned above, $\vec{\pi}(H_r,0) = \vec{\omega}(H_r,0)$. It is natural to ask the same question for f-wise routings:

Conjecture 4.3. Let G be a symmetric digraph with connectivity k. For any $0 \leq f < k$, $\vec{\pi}(G,f) = \vec{\omega}(G,f)$.

In particular, for hypercubes, our paper leaves a gap between $\vec{\pi}(G,f)$ and $\vec{\omega}(G,f)$ for $f > 0$. The computer tests show that for the 3-dimensional hypercube and $f = 0, 1, 2$, $\vec{\pi}(H_3,f) = \vec{\omega}(H_3,f)$ which supports our conjecture. This is also obvious for H_2.

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