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A BOUND FOR THE ω-EQUIVALENCE PROBLEM OF POLYNOMIAL D0L SYSTEMS

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Abstract. We give a bound for the ω-equivalence problem of polynomially bounded D0L systems which depends only on the size of the underlying alphabet.

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1. Introduction

Infinite words generated by iterated morphisms are widely studied in combinatorics of words and language theory, see [8]. Culik II and Harju [1] have shown that equivalence is decidable for infinite words generated by D0L systems. This is one of the deepest results concerning iterated morphisms.

While the ω-equivalence problem for D0L systems is known to be decidable, very little is known about bounds for the problem. Here, by a bound we understand an integer computable from two given D0L systems which indicates how many initial terms in the sequences have to be compared with respect to the prefix order to decide whether the systems are ω-equivalent. No such bounds have been explicitly given in the general case. It is an open problem whether there exists a bound depending only on the cardinality of the alphabet. Indeed, no such bound is known even for the D0L sequence equivalence problem.

In this paper we give a bound for the ω-equivalence problem of polynomially bounded D0L systems which depends only on the cardinality of the alphabet. To obtain this result we first use elementary morphisms (see [3]), and then apply the recently established bound for the sequence equivalence problem of polynomially bounded D0L systems (see [6]).

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It is assumed that the reader is familiar with the basics concerning D0L systems, see [7, 8]. For infinite D0L words see also [2, 4, 5].

2. Definitions and results

We use standard language-theoretic notation and terminology. In particular, the *cardinality* of a finite set X is denoted by $\mathrm{card}(X)$ and the *length* of a word $w \in X^*$ is denoted by $|w|$. By definition, the length of the *empty word* ε equals zero. If w is a nonempty word, $\mathrm{first}(w)$ is the first letter of w. If $w \in X^*$ and $x \in X$, then $|w|_x$ is the number of occurrences of the letter x in the word w. If $w \in X^*$ and $Z \subseteq X$ we denote

$$|w|_Z = \sum_{z \in Z} |w|_z.$$

If $w \in X^*$, the set $\mathrm{alph}(w)$ is defined by

$$\mathrm{alph}(w) = \{x \in X \mid |w|_x \geq 1\}.$$

Two words $u, v \in X^*$ are called *comparable* if one of them is a prefix of the other.

A D0L *system* is a triple $G = (X, h, w)$ where X is a finite alphabet, $h : X^* \longrightarrow X^*$ is a morphism and $w \in X^*$ is a word. The *sequence* $S(G)$ generated by G consists of the words

$$w, h(w), h^2(w), h^3(w), \ldots$$

The *language* $L(G)$ of G is defined by

$$L(G) = \{h^n(w) \mid n \geq 0\}.$$

A D0L system $G = (X, h, w)$ is called *polynomially bounded* (or *polynomial*) if there exists a polynomial $P(n)$ such that

$$|h^n(w)| \leq P(n) \qquad \text{for all } n \geq 0.$$

A D0L system $G = (X, h, w)$ is polynomially bounded if and only if there does not exist a letter $x \in \mathrm{alph}(L(G))$ such that for some $n \geq 1$ we have $|h^n(x)|_x \geq 2$ (see [9]).

Suppose $G = (X, h, w)$ is a D0L system such that w is a prefix of $h(w)$ and $L(G)$ is infinite. Then we denote by $\omega(G)$ the unique infinite word having prefix $h^n(w)$ for all $n \geq 0$. If, on the other hand, $G = (X, h, w)$ is a D0L system such that w is not a prefix of $h(w)$ or $L(G)$ is finite, we say that $\omega(G)$ does not exist.

Suppose $G = (X, h, w)$ is a D0L system such that $\omega(G)$ exists. Then, if v is a prefix of $\omega(G)$ and $|w| \leq |v|$, the word v is a proper prefix of $h(v)$.

Let $G_i = (X, h_i, w_i)$, $i = 1, 2$, be D0L systems. G_1 and G_2 are called *sequence equivalent* if $S(G_1) = S(G_2)$. G_1 and G_2 are called *ω-equivalent* if both $\omega(G_1)$ and $\omega(G_2)$ exist and are equal.

Next, if m is a positive integer, denote

$$A(m) = 4^{m+2}((m+2)!+1)^{(m+2)^2}.$$

The following result is proved in [6].

Theorem 1. *Let m be a positive integer. If $G_i = (X, h_i, w)$, $i = 1, 2$, are polynomially bounded D0L systems and card$(X) \leq m$, then*

$$S(G_1) = S(G_2)$$

if and only if

$$h_1^n(w) = h_2^n(w) \text{ for all } 0 \leq n \leq A(m).$$

The purpose of this paper is to prove a similar result for the ω-equivalence problem of polynomially bounded D0L systems.

Let $G = (X, h, a)$ be a D0L system such that $a \in X$. Denote $X_1 = X - \{a\}$. G is called a *1-system* if

1. $h(a) \in aX_1^*$;
2. $h(x) \in X_1^*$ if $x \in X_1$;
3. $L(G)$ is infinite;
4. if $x \in X_1$, then x occurs infinitely many times in $\omega(G)$.

In the preliminary section of [1] Culik II and Harju show that in studying the ω-equivalence problem for D0L systems it suffices to consider 1-systems.

Let $G_i = (X, h_i, a)$, $i = 1, 2$, be 1-systems. We say that G_1 and G_2 satisfy the *growth condition* if there do not exist integers $1 \leq s \leq \text{card}(X)^2$, $1 \leq j_1, \dots, j_s \leq 2$ and a letter $x \in X$ such that

$$|h_{j_s} \dots h_{j_1}(x)|_x \geq 2. \tag{1}$$

If m is a positive integer, denote

$$C(m) = 2(m!+1)^{m+1}$$

and

$$\omega(m) = 2(A(m+1)+2)(C(m)+1).$$

Now we can state the main result.

Theorem 2. *Let m be a positive integer. If $G_i = (X, h_i, a)$, $i = 1, 2$, are polynomially bounded 1-systems and card$(X) \leq m$, then*

$$\omega(G_1) = \omega(G_2)$$

if and only if

(i) G_1 *and* G_2 *satisfy the growth condition;*

(ii) *all words in the set*

$$\{h_{i_n} \dots h_{i_1}(a) \mid 0 \le n \le \omega(m), i_1, \dots, i_n \in \{1,2\}\}$$

are comparable.

It is not difficult to see that to check condition (ii) in Theorem 2 only $2\omega(m)$ comparisons are needed.

If $G_i = (X, h_i, a)$, $i = 1, 2$, are polynomially bounded ω-equivalent 1-systems then the growth and comparability conditions of Theorem 2 hold. Indeed, the ω-equivalence implies the comparability condition while the growth condition follows by Lemma 7 in [4]. In the following sections we prove that the conditions of Theorem 2 are also sufficient for the ω-equivalence.

3. Simplification of 1-systems

In this section we simplify 1-systems by using elementary morphisms as in [3]. First we recall some results from [6].

Let $h : X^* \longrightarrow X^*$ be a morphism. The set of *cyclic letters* is defined by

$$\mathbf{CYCLIC}(h) = \{x \in X \mid |h(x)|_x \ge 1\}\cdot$$

The relation $\le_h$ on X is defined by setting

$$x \le_h y$$

for $x, y \in X$ if and only if there is $n \ge 0$ such that

$$|h^n(x)|_y \ge 1.$$

If $Z \subseteq X$, a letter $z \in Z$ is called $\le_h$-minimal in Z if $x \le_h z$ holds for no $x \in Z - \{z\}$.

Lemma 3. *Let* $G = (X, h, w)$ *be a polynomially bounded D0L system such that* $\mathbf{CYCLIC}(h) = X$. *Then the relation* $\le_h$ *is a partial order on alph*$(L(G))$.

Proof. See [6]. □

Let now $h_i : X^* \longrightarrow X^*$, $i = 1, 2$, be morphisms. Then the triple (f, p_1, p_2) *simplifies the pair* (h_1, h_2) if the following conditions hold:

1. there is an alphabet Y such that $f : X^* \longrightarrow Y^*$ and $p_i : Y^* \longrightarrow X^*$, $i = 1, 2$, are morphisms;
2. there exist sequences $i_{11}, \dots, i_{1k}$ and $i_{21}, \dots, i_{2k}$ of elements from $\{1, 2\}$ such that

$$h_1 h_{i_{11}} \dots h_{i_{1k}} = p_1 f, \qquad h_2 h_{i_{21}} \dots h_{i_{2k}} = p_2 f; \tag{2}$$

3. the morphisms p_i and fp_i, $i = 1, 2$, are elementary;
4. **CYCLIC**$(fp_i) = Y$, $i = 1, 2$.

Note that $k, i_{11}, \ldots, i_{1k}, i_{21}, \ldots, i_{2k}$ are not uniquely determined by the triple (f, p_1, p_2). Any value of k such that there exist $i_{11}, \ldots, i_{1k}, i_{21}, \ldots, i_{2k}$ satisfying (2) is called an *index* of the triple (f, p_1, p_2). Whenever we consider a triple (f, p_1, p_2) simplifying a pair (h_1, h_2) it is tacitly assumed that Y, k and $i_{11}, \ldots, i_{1k}, i_{21}, \ldots, i_{2k}$ are as above.

A morphism $h : X^* \longrightarrow X^*$ is called *nontrivial* if $h(X) \neq \{\varepsilon\}$. If $h_i : X^* \longrightarrow X^*$, $i = 1, 2$, are morphisms, the pair (h_1, h_2) is called *nontrivial* if all products of h_1 and h_2 are nontrivial.

Lemma 4. *Let m be a positive integer. If X is an alphabet with at most m letters, $h_i : X^* \longrightarrow X^*$, $i = 1, 2$, are morphisms and the pair (h_1, h_2) is nontrivial, then there exists a triple (f, p_1, p_2) simplifying the pair (h_1, h_2) and having index k such that $2m \leq k \leq C(m)$.*

Proof. See [6]. □

The following lemmas study in detail the simplification of 1-systems.

Lemma 5. *Let $G_i = (X, h_i, a)$, $i = 1, 2$, be 1-systems and let (f, p_1, p_2) simplify the pair (h_1, h_2). Denote first$(f(a)) = c$ and $Y_1 = Y - \{c\}$. Then*

(i) $fp_i(c) \in cY_1^*$;
(ii) $fp_i(y) \in Y_1^*$ *if* $y \in Y_1$, $i = 1, 2$.

Proof. Because G_1 and G_2 are 1-systems, $p_i f(a) \in aX_1^*$ and $p_i f(x) \in X_1^*$ if $x \in X_1$. Hence $p_i(c) \in aX_1^*$ implying $f(a) \in cY_1^*$ and $f(x) \in Y_1^*$ if $x \in X_1$. Therefore $fp_i(c) \in cY_1^*$.

Let $y \in Y_1$. Because $|fp_i(y)|_y \geq 1$ there is a letter $x \in X$ such that $|f(x)|_y \geq 1$. If $x \in X_1$ then $p_i f(x) \in X_1^*$ and $p_i(y) \in X_1^*$. If $x = a$ then $y \in \text{alph}(c^{-1} f(a))$ and $p_i(c^{-1} f(a)) \in X_1^*$ implying again that $p_i(y) \in X_1^*$. It follows that $fp_i(y) \in Y_1^*$. □

Lemma 6. *Let $G_i = (X, h_i, a)$, $i = 1, 2$, be 1-systems and let (f, p_1, p_2) simplify the pair (h_1, h_2). Assume that the words $p_1 fp_2 f(a)$ and $p_2 fp_1 f(a)$ are comparable and that the D0L systems (Y, fp_i, y), for $y \in Y$ and $i = 1, 2$, are polynomially bounded. Denote first$(f(a)) = c$ and $Y_1 = Y - \{c\}$. Suppose e is $\leq_{fp_1}$-minimal in Y_1 and $\leq_{fp_2}$-minimal in Y_1. Denote $g_1 = fp_1 fp_2$, $g_2 = fp_2 fp_1$ and $Z = \{x \in X \mid |f(x)|_e \geq 1\}$. Then*

(i) $|fp_i(e)|_e = 1$, $i = 1, 2$;
(ii) $|fp_i(y)|_e = 0$ *if* $y \in Y_1 - \{e\}$, $i = 1, 2$;
(iii) $|g_1(c)|_e = |g_2(c)|_e$;
(iv) $|p_1 fp_2(c)|_Z = |p_2 fp_1(c)|_Z$;
(v) $|p_1 fp_2(e)|_Z = |p_2 fp_1(e)|_Z = 1$;
(vi) $|p_1 fp_2(y)|_Z = |p_2 fp_1(y)|_Z = 0$ *if* $y \in Y_1 - \{e\}$.

Proof. Because $e \in \mathbf{CYCLIC}(fp_i)$ and (Y, fp_i, e) is polynomially bounded we have $|fp_i(e)|_e = 1$. Because e is $\le_{fp_i}$-minimal in Y_1 we have $|fp_i(y)|_e = 0$ if $y \in Y_1 - \{e\}$. This implies also

$$\begin{aligned}|g_1(c)|_e &= |fp_1fp_2(c)|_e = |fp_1(c)|_e + |fp_2(c)|_e\\ &= |fp_2(c)|_e + |fp_1(c)|_e = |fp_2fp_1(c)|_e = |g_2(c)|_e.\end{aligned}$$

To prove (iv) observe that one of the words $p_1fp_2(c)$ and $p_2fp_1(c)$ is a prefix of the other, say $p_1fp_2(c) = p_2fp_1(c)v$ where $v \in X_1^*$. Then

$$|fp_1fp_2(c)|_e = |fp_2fp_1(c)|_e + |f(v)|_e.$$

By (iii) it follows that $|v|_Z = 0$. Hence $|p_1fp_2(c)|_Z = |p_2fp_1(c)|_Z$.

Finally, because

$$|fp_1fp_2(e)|_e = |fp_2fp_1(e)|_e = 1$$

and

$$|fp_1fp_2(y)|_e = |fp_2fp_1(y)|_e = 0$$

if $y \in Y_1 - \{e\}$, we get (v) and (vi). □

Lemma 7. *Let $G_i = (X, h_i, a)$, $i = 1, 2$, be 1-systems and let (f, p_1, p_2) having index $k \ge 2card(X)$ simplify the pair (h_1, h_2). Denote first$(f(a)) = c$ and $Y_1 = Y - \{c\}$. Assume that all words in the set*

$$\{h_{i_n} \dots h_{i_1}(a) \mid 0 \le n \le k + 2\}$$

are comparable. Then alph$(p_if(a)) = X$ and alph$(fp_ifp_j(c)) = Y$ for all $i, j \in \{1, 2\}$.

Proof. First, we claim that $p_if(a)$ is a prefix of $\omega(G_1)$ for $i = 1, 2$. If not, let w_i be the longest common prefix of $p_if(a)$ and $\omega(G_1)$. Then $h_1(w_i)$ is a common prefix of $h_1p_if(a)$ and $\omega(G_1)$ which is longer than w_i. This contradicts the assumption that $h_1p_if(a)$ and $p_if(a)$ are comparable.

A similar argument shows that $p_if(a)$ is also a prefix of $\omega(G_2)$ for $i = 1, 2$. Because G_i, $i = 1, 2$, are 1-systems we have

$$\text{alph}(h_1^n(a)) = \text{alph}(h_2^n(a)) = X$$

for $n \ge \text{card}(X)$. Because p_if contains at least card(X) terms equal to h_1 or at least card(X) terms equal to h_2 when it is regarded as a product of h_1 and h_2, this implies that

$$\text{alph}(p_if(a)) = X, \quad i = 1, 2.$$

Hence alph$(p_i(Y)) = X$, $i = 1, 2$. Because fp_1 is elementary alph$(fp_1(Y)) = Y$ implying that alph$(f(X)) = Y$. Therefore

$$Y \subseteq \text{alph}(f(X)) \subseteq \text{alph}(fp_if(a)) \subseteq \text{alph}(fp_ifp_j(c))$$

for $i, j \in \{0, 1\}$. □

4. Reduction to the sequence equivalence

The following two lemmas are the most essential step in the deduction of Theorem 2 from Theorem 1.

Let a, X_1 and Z be as in Lemma 6. If $u \in aX_1^*ZX_1^*$, let $\alpha(u)$ be the longest prefix of u belonging to aX_1^*Z.

Lemma 8. *We continue with the notations and assumptions of Lemma 6. Suppose* $|p_1fp_2(c)|_Z = |p_2fp_1(c)|_Z \geq 1$. *If* n *is a positive integer and the words* $p_1fp_2g_1^n(c)$ *and* $p_2fp_1g_2^n(c)$ *are comparable then*

$$\alpha(p_1fp_2g_1^n(c)) = \alpha(p_2fp_1g_2^n(c)). \tag{3}$$

Conversely, if (3) holds for all $n \geq 1$*, then* $\omega(G_1) = \omega(G_2)$.

Proof. Let n be a positive integer. Because $|p_1fp_2(c)|_Z = |p_2fp_1(c)|_Z \geq 1$, the words $p_1fp_2g_1^n(c)$ and $p_2fp_1g_2^n(c)$ belong to $aX_1^*ZX_1^*$. By Lemma 6 we have

$$\begin{aligned} |p_1fp_2g_1^n(c)|_Z &= |p_1fp_2(c)|_Z + |g_1^n(c)|_e \\ &= |p_1fp_2(c)|_Z + n|g_1(c)|_e \\ &= |p_2fp_1(c)|_Z + n|g_2(c)|_e \\ &= |p_2fp_1(c)|_Z + |g_2^n(c)|_e \\ &= |p_2fp_1g_2^n(c)|_Z. \end{aligned} \tag{4}$$

Assume then that the words $p_1fp_2g_1^n(c)$ and $p_2fp_1g_2^n(c)$ are comparable. By (4) we have (3).

Conversely, assume that (3) holds for all $n \geq 1$. Denote $H_1 = (X, p_1fp_2f, a)$ and $H_2 = (X, p_2fp_1f, a)$. By (4) the languages $L(H_1)$ and $L(H_2)$ are infinite because $|g_1(c)|_e \geq 1$. Hence $\omega(H_1)$ and $\omega(H_2)$ exist. Equations (3) and (4) together imply that $\omega(H_1)$ and $\omega(H_2)$ have arbitrarily long common prefixes. Therefore $\omega(H_1) = \omega(H_2)$. To prove that $\omega(G_1) = \omega(G_2)$ assume $\omega(G_1)$ and $\omega(G_2)$ are not equal and let w be the longest common prefix of $\omega(G_1)$ and $\omega(G_2)$. Then $h_1(w)$ and $h_2(w)$ are not comparable. On the other hand, because w is also a prefix of $\omega(H_1) = \omega(H_2)$, the words $p_1fp_2f(w)$ and $p_2fp_1f(w)$ are comparable and have prefixes $h_1(w)$ and $h_2(w)$, respectively. This contradiction proves that $\omega(G_1) = \omega(G_2)$. □

Lemma 9. *We again continue with the notations and assumptions of Lemma 6 and suppose that* $|p_1fp_2(c)|_Z = |p_2fp_1(c)|_Z \geq 1$. *Assume that the D0L systems* $(X, p_1fp_2f, p_1fp_2(c))$ *and* $(X, p_2fp_1f, p_2fp_1(c))$ *are polynomially bounded. If (3) holds for* $1 \leq n \leq A(1 + card(X)) + 1$ *then (3) holds for all* $n \geq 1$.

Proof. The claim follows by Theorem 1 because the sequences obtained from

$$(\alpha(p_1fp_2g_1^n(c)))_{n\geq 1}$$

and

$$(\alpha(p_2 f p_1 g_2^n(c)))_{n\geq 1}$$

by barring the last letter of every term are polynomially bounded D0L sequences over an alphabet with at most $1 + \text{card}(X)$ letters. □

Next, we show that if we simplify polynomially bounded 1-systems satisfying the growth condition then the resulting D0L systems are polynomially bounded, too.

Lemma 10. *Suppose $G_i = (X, h_i, a)$, $i = 1, 2$, are 1-systems satisfying the growth condition. Then there do not exist integers $s \geq 1$, $1 \leq j_1, \ldots, j_s \leq 2$ and a letter $x \in X$ such that (1) holds.*

Proof. Suppose on the contrary that there exist integers $s \geq 1$, $1 \leq j_1, \ldots, j_s \leq 2$ and a letter $x \in X$ such that (1) holds. Choose s, $j_1, \ldots, j_s$ and x so that s is as small as possible. By considering the derivation tree of $h_{j_s} \ldots h_{j_1}(x)$ it is seen that there exist a nonnegative integer t, letters $x_0, \ldots, x_t \in X$ and pairs $(x_{t+1}, \overline{x}_{t+1}), \ldots, (x_s, \overline{x}_s) \in X \times X$ satisfying the following conditions:

(i) $x_0 = x$ and $x_s = \overline{x}_s = x$;
(ii) $h_{j_{\gamma+1}}(x_\gamma) \in X^* x_{\gamma+1} X^*$ for $0 \leq \gamma \leq s-1$;
(iii) $h_{j_{\gamma+1}}(\overline{x}_\gamma) \in X^* \overline{x}_{\gamma+1} X^*$ for $t+1 \leq \gamma \leq s-1$;
(iv) $h_{j_{t+1}}(x_t) \in X^* x_{t+1} X^* \overline{x}_{t+1} X^*$.

By the minimality of s, no letter appears twice in the sequence $x_0, \ldots, x_t$ and no pair appears twice in the sequence $(x_{t+1}, \overline{x}_{t+1}), \ldots, (x_s, \overline{x}_s)$. Furthermore, none of $(x_1, x_1), \ldots, (x_t, x_t)$ appears among the pairs. But then

$$s \leq \text{card}(X)^2,$$

which is a contradiction because G_1 and G_2 satisfy the growth condition. □

Lemma 11. *Suppose $G_i = (X, h_i, a)$, $i = 1, 2$, are 1-systems satisfying the growth condition. Let (f, p_1, p_2) simplify the pair (h_1, h_2) and denote first$(f(a)) = c$. Then the D0L systems $(X, p_1 f p_2 f, x)$, $(X, p_2 f p_1 f, x)$, $(Y, f p_i, c)$ and $(Y, f p_1 f p_2, c)$ for $x \in X$, $i = 1, 2$, are polynomially bounded.*

Proof. The claims follow by Lemma 10. □

Now we are ready to prove Theorem 2. For that purpose let m be a positive integer and let $G_i = (X, h_i, a)$, $i = 1, 2$, be 1-systems such that $\text{card}(X) \leq m$. Assume that G_1 and G_2 satisfy the growth condition and that all words in the set

$$\{h_{i_n} \ldots h_{i_1}(a) \mid 0 \leq n \leq \omega(m), i_1, \ldots, i_n \in \{1, 2\}\}$$

are comparable. To conclude the proof of Theorem 2 it suffices to show that $\omega(G_1) = \omega(G_2)$.

First, by Lemma 4 there is a triple (f, p_1, p_2) which simplifies the pair (h_1, h_2) and has index k such that $2m \leq k \leq C(m)$. As before, denote $\text{first}(f(a)) = c$, $Y_1 = Y - \{c\}$, $g_1 = f p_1 f p_2$ and $g_2 = f p_2 f p_1$. By Lemma 11 the D0L systems

(X, p_1fp_2f, x), (X, p_2fp_1f, x), (Y, fp_i, c) and (Y, fp_1fp_2, c) for $x \in X$, $i = 1, 2$, are polynomially bounded. Also, by Lemma 7, the D0L systems (Y, fp_i, y) are polynomially bounded for $y \in Y$, $i = 1, 2$. Further, by Lemmas 3 and 7 the relations $\leq_{fp_1fp_2}$ and $\leq_{fp_i}$, $i = 1, 2$, are partial orders on Y.

Now, let $e \in Y_1$ be $\leq_{fp_1fp_2}$-minimal in Y_1. Such a letter exists because $\leq_{fp_1fp_2}$ is a partial order on a finite set. Then e is also $\leq_{fp_i}$-minimal in Y_1 for $i = 1, 2$. Indeed, if $y \in Y_1$ and $|fp_1(y)|_e \geq 1$ then $|fp_1fp_2(y)|_e \geq |fp_1(y)|_e \geq 1$ because $|fp_2(y)|_y \geq 1$. Similarly, if $y \in Y_1$ and $|fp_2(y)|_e \geq 1$ we have $|fp_1fp_2(y)|_e \geq |fp_1(e)|_e \geq 1$. In both cases, because e is $\leq_{fp_1fp_2}$-minimal in Y_1, necessarily $y = e$ which proves that e is $\leq_{fp_i}$-minimal in Y_1 for $i = 1, 2$.

Next, denote $Z = \{x \in X \mid |f(x)|_e \geq 1\}$. By Lemma 7 we have $|p_1fp_2(c)|_Z = |p_2fp_1(c)|_Z \geq 1$. Now we are in a position to apply Lemmas 8 and 9. By assumption, the words $p_1fp_2g_1^n(c)$ and $p_2fp_1g_2^n(c)$ are comparable for $0 \leq n \leq A(m+1)+1$. By Lemma 8 we have (3) for $1 \leq n \leq A(m+1)+1$. By Lemma 9 we get (3) for all $n \geq 1$. Then $\omega(G_1) = \omega(G_2)$ by Lemma 8.

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