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THE FIBONACCI AUTOMORPHISM OF FREE BURNSIDE GROUPS

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Abstract. We prove that the Fibonacci morphism is an automorphism of infinite order of free Burnside groups for all odd $n \geq 665$ and even $n = 16k \geq 8000$.

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1. Introduction

The question of study of automorphisms of free Burnside groups was stated by Ol'shanskii in the Kourovka Notebook [7]. The first results were obtained by Cherepanov in [4,5] and by Atabekyan in [2,3]. In paper [4] it was proved that the Fibonacci morphism is an automorphism of infinite order of free Burnside groups for all odd $n > 10^{10}$ and even $n = 16k \geq 8000$.

This paper shows that the bound of odd n can be decreased from $n > 10^{10}$ to $n \geq 665$.

Consider an automorphism $\varphi : F_2 \to F_2$ of the absolutely free group F_2 of rank two with free generators $\{a, b\}$, given on generators by formulae

$$\varphi : a \mapsto b, \;\; \varphi : b \mapsto ab.$$

This automorphism is called after Fibonacci since the lengths of words $\varphi^k(a)$ are equal to corresponding members of the numerical Fibonacci sequence. If we consider the sequence of mirror copies of words $\varphi^k(a)$, we obtain the iterations of the following morphism

$$h : a \mapsto b, \;\; \varphi : b \mapsto ba.$$

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This morphism is also called after Fibonacci. All the statements of this paper that we prove for the first morphism hold for the second morphism either.

Automorphism φ naturally induces an automorphism of free the Burnside group $B(2,n)$, which we denote by the same letter φ. Let us remember that a free Burnside group $B(2,n)$ is the quotient $F_2/{F_2}^n$, where ${F_2}^n$ is the subgroup generated by all possible n-powers of elements of F_2. Obviously the group $B(2,n)$ has a presentation $B(2,n) = \langle a_1, a_2 | A^n = 1,\ \textit{for all words}\, A = A(a_1, a_2)\rangle$.

Theorem 1.1. *For arbitrary odd $n \geq 665$ and arbitrary even $n = 16k \geq 8000$ the Fibonacci automorphism φ has infinite order in the group $Aut(B(2,n))$.*

Theorem 1.1 strengthens the similar result of paper [4], decreasing the bound of odd n to $n \geq 665$.

To prove Theorem 1.1 we prove the following result that is individually interesting.

Proposition 1.2. *For any natural k no forth power of a non-empty word occurs in a cyclic word $\varphi^k(a)$.*

As usual, by a cyclic word we mean a word written on a circle without fixing its start. Proposition 1.2 strengthens one of the results of paper [6] by Karhumäki, where a similar statement is proved without the assumption that the word $\varphi^k(a)$ is cyclic. Our proof of Proposition 1.2 does not depend on paper [6] by Karhumäki.

Proposition 1.2 also strengthens the Lemma 1.3 of paper [4] by Cherepanov, according to which no 24th power of a non-empty word occurs in a cyclic Fibonacci sequence. Bearing on paper [6] by Karhumäki in [9] Mignosi and Pirillo proved the following interesting result:

Proposition 1.3. *The Fibonacci infinite word contains no fractional power with an exponent grater than $2 + ((\sqrt{5}+1)/2)$ and, for any real number $\varepsilon > 0$, it contains a fractional power with an exponent grater than $2 + ((\sqrt{5}+1)/2) - \varepsilon$.*

Theorem 1.1 implies

Corollary 1.4. *For arbitrary odd $n \geq 665$ the quotient group*

$$Aut(B(2,n))/Inn(B(2,n))$$

is infinite.

Proof. According to the famous theorem of S. I. Adian (see Thm. VI.3.4 of [1]) the center of $B(m,n)$ is trivial for $n \geq 665$ and $m > 1$. Therefore $Inn(B(2,n))$ is isomorphic to $B(2,n)$. Since any inner automorphism of the group $B(2,n)$ has a finite order, from Theorem 1.1 it follows that for any natural number l each automorphism φ^l is not inner. Hence the quotient $Aut(B(2,n))/Inn(B(2,n))$ is infinite. □

2. The Proof of Proposition 1.2

We adhere to the following notions and notations of monograph [1].

Definition 2.1. The word A is called *primitive* if it cannot be presented in a form D^r for $r > 1$.

We say that the word E occurs in a word X, if there exist words R and Q such that $X \doteq REQ$ holds. If the word R (word Q) is empty, then E is a prefix (suffix) of X. If X is a word over the alphabet that does not contain the letter $*$ and $X \doteq REQ$, the word $R * E * Q$ is called an occurrence of word E in a word X. E is called a base of the occurrence $R * E * Q$. For a given word X we denote by $\overline{X}$ the cyclic word generated by X, that is the word X written on a circle without fixing its start. For a given word X by $\partial(X)$ we denote the length of X, that is the number of its letters over the alphabet $\{a, b\}$. For the equality by definition of two words or two occurrences in a same word we use the symbol $\rightleftharpoons$.

Consider an automorphism $\varphi : a \mapsto b, \varphi : b \mapsto ab$ of the group $B(2, n)$. Let us first write out a few images

$$\varphi^k(a) : a \mapsto b \mapsto ab \mapsto bab \mapsto \underbrace{ab}\,\underbrace{bab} \mapsto \underbrace{bab}\,\underbrace{abbab} \mapsto \underbrace{abbab}\,\underbrace{bababbab}.$$

Denote

$$X_0 \rightleftharpoons \varphi^0(a) = a,\ X_k \rightleftharpoons \varphi^k(a),\ k = 1, 2, \dots$$

Since $X_{k+1} = X_{k-1} \cdot X_k$, the lengths of words of the sequence X_k, $k = 1, 2, \dots$ form a Fibonacci sequence. Let us denote

$$A \rightleftharpoons X_k, B \rightleftharpoons X_{k-1}, C \rightleftharpoons X_{k-2}, D \rightleftharpoons X_{k-3},$$

$$E \rightleftharpoons X_{k-4}, F \rightleftharpoons X_{k-5}, G \rightleftharpoons X_{k-6}, H \rightleftharpoons X_{k-7}.$$

Then $A = CDC, B = DC, C = ED$ and $X_{k+1} = BA = DCCDC$.

Let us recall the following statements from [1], that we often refer to.

Lemma 2.2 (see Lem. I.2.2 in [1])**.** *If $AB \doteq BA$, then there exists a word D, such that $A \doteq D^t, B \doteq D^r$, for some $t, r \geq 0$.*

Lemma 2.3 (see Lem. I.2.9 in [1])**.** *Suppose $A^tA_1 \doteq B^rB_1$, where $\partial(A^tA_1) \geq \partial(AB)$, A_1 is a suffix of A, B_1 - a prefix of B. If A is a primitive word, then for some k, $B \doteq A^k$ holds.*

Lemma 2.4 (see Lem. IV.2.16 in [1])**.** *If no elementary α-power of rank 1 occurs in a word X, then $X \overset{\alpha}{\sim} Y \Rightarrow X \doteq Y$.*

Definition 2.5 (see Def. I.4.34 in [1])**.** Suppose $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$, where

$$X \in \mathcal{A} \Leftrightarrow X \in \mathcal{R}_{\alpha-1} \& Norm(\alpha, X, 9) = \varnothing.$$

Elements of the set $\mathcal{A}$ are called *absolutely reduced*.

From Propositions 1.2 and 1.3 follows

Corollary 2.6. *For any natural k the cyclic word $\overline{\varphi^k(a)}$ contains no fractional power $2 + ((\sqrt{5}+1)/2)$ of a non-empty word.*

Proof. Let us prove it by induction on k. The base of induction is obvious. Suppose the statement is true for all natural $l \le k$ and prove it for $k+1$. Since all cyclic shifts of the word $X_{k+1} = DCCDC$ occur in a Fibonacci word $X_{k+3} = DCCDCCDCDCCDC$ that contains no $2+((\sqrt{5}+1)/2)$ fractional power according to proposition 1.3, the word $\overline{X_{k+1}}$ contains no fractional $2+((\sqrt{5}+1)/2)$ power of a non-empty word either. □

Lemma 2.7. *None of the words $\overline{X_k}$ is a proper power, that is $\overline{X_k} \neq Z^t, t \ge 2$, for $k = 1, 2, \dots$*

Proof. Since the cyclic shift of a proper power is itself a proper power, it is enough to prove that X_k is not a proper power. We prove this by induction on k. For $k \le 5$ the proof is obvious. Suppose that the lemma is proved for all numbers $l \le k$, and prove it for $k+1$. We can assume that the word Z is primitive. Let $X_{k+1} = DCCDC = Z^t, t \ge 2$. Then $CDCDC = (CD)^2C = {Z_1}^t$ for some cyclic shift Z_1 of Z. Since $t \ge 2$, $\partial(Z_1) + \partial(CD) < \partial(CDCDC)$ and by Lemma 2.3 we obtain $\overline{B} = CD = {Z_1}^p$. Therefore $B = Z^p$ and $A = Z^q$, $p, q \ge 1$, $p \neq q$. This contradicts the inductive assumption. □

Lemma 2.8. *If $\overline{X_k}$ contains a power Z^t then $t < 4$.*

Proof. The proof is by induction on k. For $k \le 5$ the proof is obvious since the word $\overline{X_5} = \overline{bababbab}$ does not contain Z^4. For $k = 6$ we have the word $\overline{X_6} = \overline{abbabbababbab}$ that does not contain a subword Z^4 with $\partial(Z) \le 3$. For $k = 7$ no word Z^4 with $\partial(Z) \le 3$ occurs in $\overline{X_7} = \overline{bababbababbabbababbab}$. The word $\overline{X_7}$ does not contain Z^4 with $\partial(Z) = 4, 5$ either. Let $k \ge 7$. Suppose the statement is proved for all $l \le k$ and prove it for $k+1$. Let Z^4 occur in $\overline{X_{k+1}} = \overline{DCCDC}$. Z^4 does not occur in a word $\overline{A} = \overline{CDC}$ by inductive assumption. Since D is a suffix of C, any subword of word $\overline{DCCDC}$ of length three over the alphabet $\{C, D\}$ occurs in $\overline{CDC}$. Therefore Z^4 does not occur in a subword of word $\overline{DCCDC}$ of length three over the alphabet $\{C, D\}$.

Let us prove that Z^4 does not occur in a subword of $\overline{DCCDC}$ of length four over the alphabet $\{C, D\}$ either. Let us write out subwords of length four of word $\overline{DCCDC}$ over the alphabet $\{C, D\}$:

$$DCCD, CCDC, CDCD, DCDC, CDCC.$$

Since the word $DCDC$ is a suffix of $CCDC$, it is enough to consider the case of the occurrence of Z^4 in words $DCCD, CCDC, CDCC, CDCD$.

I. Suppose Z^4 occurs in $DCCD$. Then it contains the base CC of the occurrence $D * CC * D$. We have obvious inequalities $2\partial(C) < 4\partial(Z) < 2\partial(D) + 2\partial(C)$. Therefore $\partial(Z) < \partial(C)$ and $\partial(Z) + \partial(C) < 2\partial(C)$. By Lemma 2.3 we obtain

$C = Z^p, p \geq 1$. According to Lemma 2.7 we have $p \ngeq 2$, and using $\partial(D) < \partial(C) = \partial(Z)$ we obtain $p \neq 1$.

II. Suppose Z^4 occurs in $CDCC = EDDEDED$. Then it contains the base $DED = DC$ of the occurrence $ED * DED * ED$. We have $\partial(Z) < \partial(C)$. Let us consider the following cases:

(1) If Z^4 contains the suffix E of the base of the occurrence $ED * DEDE * D$, then from $\partial(Z) < \partial(D) + \partial(E) = \partial(C)$ follows $(DE)^2 = {Z_1}^{t_1} {Z_1}', t_1 \geq 2$ for some cyclic shift Z_1 of Z. According to Lemma 2.3 we obtain $DE = {Z_1}^p, p \geq 1$. Hence $ED = C = {Z_2}^p$. By Lemma 2.7 we have $p \ngeq 2$ and by $\partial(Z) < \partial(C)$ the inequality $p \neq 1$ holds.

Thus, we can assume that Z^4 is contained in the base $EDDEDE$ of the occurrence $*EDDEDE * D$ in a word $CDCC$ and, at the same time, contains the base of the occurrence $ED * DED * ED$.

(2) If Z^4 contains the base $DDED$ of the occurrence $E * DDED * ED$, then from equality $DDED = FEDED = FCC$, where F is a suffix of C, and inequality $\partial(Z) < \partial(C)$ follows that ${Z_1}'{Z_1}^{t_1} = FC^2, t_1 \geq 2$ for some cyclic shift Z_1 of Z. Therefore $C = ED = {Z_2}^p$, for some cyclic shift Z_2 of Z. Since $\partial(CDCC) < 4\partial(C)$, the case $p = 1$ is impossible, and $p > 1$. This contradicts the Lemma 2.7.

(3) Thus, one can assume that Z^4 occurs in the base $DDEDE$ of occurrence $E * DDEDE * D$ and at the same time does not contain the prefix D and suffix E of that base. Then Z^4 occurs in $\overline{A} = \overline{EDDED}$. This contradicts the inductive assumption.

III. Now let Z^4 occur in $CDCD = EDDEDD$. Then it contains the base $DED = DC$ of the occurrence $ED * DED * D$. First note that $Z^t \neq CDCD$ holds, since in the contrary case $\partial(Z) < \partial(CD)$, hence $CD = Z^p, p \geq 2$. Thus, $B = DC = {Z_1}^p$ for some cyclic shift Z_1 of Z. This is a contradiction to inductive assumption. According to the case above the word Z^4 does not occur in the base $DDED$ of the occurrence $E * DDED * D$ since that base is equal to the base of the occurrence $E * DDED * C$ in an already considered word $CDCC$. Let us consider the following cases:

(1) If Z^4 contains the base of the occurrence $EF * EFEEFE * FE$ in a word $CDCD$ then $EFEEFE = (C)^2 = {Z_1}^{t_1}{Z_1}'$ and in view of $\partial(Z_1) < \partial(C)$ we have $C = {Z_1}^p$. This contradicts the Lemma 2.7 for $p \geq 2$ and the inequality $\partial(D) < \partial(C)$ for $p = 1$.

(2) Thus Z^4 occurs in the base $FEEFEFE$ of the occurrence $EFE* FEEFEFE*$ in a word $CDCD$ and contains the base of the occurrence $EFE * FEEFE * FE$. Then $\partial(Z) < \partial(E) + \partial(F)$ holds. If Z^4 contains the base of the occurrence $EFE * FEEFEF * E$ then the word $E(EF)^2 = GFEFEF$ is periodic with period EF and $E(EF)^2 = {Z_1}'{Z_1}^{t_1}$. According to Lemma 2.3 we have $EF = {Z_1}^p$ and $FE = {Z_2}^p = D$. In view of Lemma 2.7 $p \ngeq 2$ holds and by $\partial(Z) < \partial(E) + \partial(F)$ we have $p \neq 1$.

(3) It remains to consider the case when Z^4 strictly occurs in the base of the occurrence $EF * EFEEFEF * E$ and at the same time does not contain the prefix E and suffix F of that base. Then $\partial(Z) < \partial(FE)$ and $EEFE = G(FE)^2 = {Z_1}'{Z_1}^{t_1}$ where $t_1 \geq 2$. According to Lemma 2.3 we have $D = FE = {Z_2}^p$ for some cyclic shift Z_2 of Z, in spite of Lemma 2.7.

IV. Now suppose Z^t occurs in a word $CCDC$. Then it contains the base $EDD = CD$ of the occurrence $ED * EDD * ED$ in a word $CCDC$. Since $\partial(D) < \partial(C)$ we have $\partial(Z) < \partial(C)$. Therefore ${Z_1}'{Z_1}^{t_1} = EDD = CD$ for some cyclic shift Z_1 of Z, where $t_1 \geq 2$, E is a suffix of D and $\partial(Z_1) < \partial(D) + \partial(E)$. According to Lemma 2.3 we obtain that $D = {Z_1}^p, p \geq 1$. By Lemma 2.7 we have $p = 1$ and $Z_1 = D$. The base of the occurrence $ED * EDD * ED$ in a word $CCDC$ is non-continuable to the right relative to period $Z_1 = D$ since the first letters of words D and E are different by definition of words X_k. Then Z^t occurs in a word $\overline{A} = \overline{CCD} = \overline{EDEDD}$ that contradicts the inductive assumption.

Thus we have proved that no word of form Z^4 occurs in a subword of length four of cyclic word $\overline{DCCDC}$ over the alphabet $\{C, D\}$. Let us now prove that it does not occur in a cyclic word $\overline{DCCDC}$ either. Assuming the contrary we obtain that Z^t contains one of the words CCD, CDC, DCC and DCD. In view of obvious inequalities $2\partial(C) < 2\partial(D) + \partial(C)$ and $\partial(C) + \partial(D) < 2\partial(C) + \partial(D)$ we obtain that more than half of the word Z^4 occurs in one of the following words CCD, CDC, DCC, DCD.

(1) Let Z^4 contain the base of the occurrence

$$D * CCD * C = D * EHGGFEFEFE * C$$

in a word $DCCDC$. Since $\partial(D) + \partial(EH) + \partial(C) \leq \partial(GGFEFEFE)$, we have $GG(FE)^3 = {Z_1}'{Z_1}^{t_1}$ for some cyclic shift Z_1 of Z, where $t_1 \geq 2$. Using the Lemma 2.3 we obtain that $D = FE = {Z_2}^p$ for some cyclic shift Z_1 of Z. By Lemma 2.7 we have $p = 1$ and $D = Z_2$. Since the first letters of C and D are different, the word $D * EHGGFEFEFE * C$ is non-continuable to the right relative to period $D = Z_2$. Then D^4 is a suffix of the occurrence $*DCCD*C = *DEFGFEFEFE*C$. Therefore FG is a suffix of the word $D = FE = FGF$, hence $FG = GF$. By Lemma 2.2 we get $E = T^p$ for some word T and $p \geq 2$. This contradicts the Lemma 2.7.

(2) Suppose Z^4 contains the base of the occurrence $C * DCC * D$ in a cyclic word $CDCCD$. Since D is a suffix of C, we have $DCC = {Z_1}'{Z_1}^{t_1}$ for some cyclic shift Z_1 of Z, and, obviously, $t_1 \geq 2$ holds. Then, according to Lemma 2.3, the word $C = {Z_2}^p$ is a proper power. By Lemma 2.7 we have $p = 1$ and $C = Z_2$. But since C is non-continuable to the right and to the left side one cannot count the word C^4 because of the inequality $\partial(D) < \partial(C)$, we obtain a contradiction.

(3) Let Z^4 contain the base of the occurrence

$$C * CDC * D = EFE * GHGF(EFE)^2 * FE$$

or

$$D * CDC * C = FE * GHGF(EFE)^2 * EFE$$

in a cyclic word $CCDCD$ or $DCDCC$ respectively. One has $E = GF = GHG$. In view of the obvious inequality

$$\partial(F(EFE)^2) > \partial(EFE) + \partial(FE) + \partial(E)$$

by Lemma 2.3 and Lemma 2.7 we obtain $C = EFE = Z_2$. The suffix C of the base of the occurrence $C * CDC * D$ is not continuable to the right relative to period C because the first letters of words C and D are different. In view of the inequality $\partial(D) < \partial(C)$ the base of the occurrence $*CCDC * D$ does not end with word C^4. Now let Z^4 contain the base of the occurrence $D * CDC * C$. Consider the maximal power of the word C that occurs in a word $DCDCC$, where one C of that power coincides with the suffix C of the base of the occurrence $D * CDC * C$. We can continue the occurrence $D * CDC * C$ to the right relative to period $C = Z_2$. Now let us count from right to left the maximal power of C that occurs in a word $DCDCC$. We have the equalities

$$DCDCC = FEEFEFEEFEEFE = FEEFCCC.$$

It is obvious that the equality $EEF = EFE = C$ has to hold, and therefore $EF = FE$. Then $D = FE = T^p, p \geq 2$ that contradicts the Lemma 2.7.

(4) Finally suppose Z^4 contains the base of the occurrence

$$C * DCD * C = EF * EDCD * C$$

in a cyclic word $CDCDC$. Let us repeat the reasoning of case one. Having changed only the occurrence $D*CCD*C$ by the occurrence $EF*EDCD*C = EF*CCD*C$ in a word $CDCDC$, we can assume, that Z^4 does not contain the base of the occurrence $EF*CCD*C$. Therefore Z^4 occurs in a base of the occurrence $EF * EDCDC*$ and at the same time contains the base of the occurrence

$$EFE * DCD * C = EFE * GFEDD * C = EFE * GDDD * ED.$$

We have the inequality $2\partial(D) + \partial(E) \leq \partial(GD^3)$. Then $GD^3 = {Z_1}'{Z_1}^{t_1}$ for some cyclic shift Z_1 of Z, and $t_1 \geq 2$. Hence $D = {Z_2}^p$. By Lemma 2.7 we have $p = 1$ and $D = Z_2$. Since the first letters of words C and D are different, the occurrence $EFE * GFEDD * C$ is not continuable to the right relative to period D. If D^4 is a suffix of the base of occurrence $*CDCD * C = *CFGFEDD * C$ then from right to left we read D^2, $D = FE$, and the equality $FG = GF = E$ must hold. Therefore $E = T^p, p \geq 2$ that contradicts the Lemma 2.7. Thus we proved that no word of form $Z^t, t \geq 4$ can occur in a word $\overline{X_{k+1}} = \overline{DCCDC}$. The Lemma is proved completely. □

It remains to note that the Lemma 2.8 is a reformulation of Proposition 1.2.

3. Proof of the main result

Now turn to the proof of Theorem 1.1.

Proof. Suppose that φ has a finite order in $Aut(B(2,n))$, that is $\varphi^k = id$. Then, particularly $\varphi^k(a) = a$, that is the word $X_k = a$. Therefore $a^{-1}X_k$ is equal to the empty word in $B(2,n)$. Consider two possible cases:

(1) If X_k starts with the letter b, then $a^{-1}X_k$ is not reducible and obviously contains no forth power of a non-empty word by Proposition 1.2 since all letters in X_k are positive.
(2) If X_k starts with the letter a, then we reduce $a^{-1}a$ and the result $X_k{}'$ contains no fourth power by Lemma 1.2. Hence, by definition 2.5 the irreducible word $a^{-1}X_k$ is absolutely reduced for odd $n \geq 665$ and according to Lemma 2.4 it cannot be equal to the empty word in $B(2,n)$. This proves the Theorem 1.1 for odd n.

To prove the Theorem 1.1 for even $n = 16k \geq 8000$ (as in [4]) we use Theorem 2(i) of [8] according to which if a non-empty freely non-reducible word X is equal to one in $B(m,n)$, then X contains a non-empty subword of the form $A^{(n/2)-1240}$. Again by Proposition 1.2 the word $a^{-1}X_k$ contains no subword of the form $A^{n/2-1240}$, hence it cannot be equal to the empty word in $B(2,n)$. □

Appendix

The author thanks the Referee for suggesting the following much shorter proof of Proposition 1.2 using some well-known properties of Fibonacci words.

Proof. Let h denote the Fibonacci morphism given by

$$h : a \mapsto b, \ \ h : b \mapsto ba.$$

It is well-known that the reversal of $h(b)$ is a conjugate of $h(b)$ for all $k \geq 0$. Thus the square of the reversal of $h^{n-1}(b)$ is a factor of the cube $h^{n-1}(b)^3$, which is a factor of the Fibonacci infinite word $\lim_{k\to\infty} h^k(b)$ for all $n \geq 3$. Therefore the square of $h^{n-1}(b)$ does not contain 4th powers (see [6], Thm. 2). The claim now follows from the fact that the word $\varphi^n(a)$ equals the reversal of $h^{n-1}(b)$, which can be proved by induction. □

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