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ON THE STRUCTURE OF $(-\beta)$-INTEGERS

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Abstract. The $(-\beta)$-integers are natural generalisations of the β-integers, and thus of the integers, for negative real bases. When β is the analogue of a Parry number, we describe the structure of the set of $(-\beta)$-integers by a fixed point of an anti-morphism.

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1. Introduction

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form

$$\sum_{k=0}^{n-1} a_k \, (-\beta)^k,$$

where $(-\beta)$ is a negative real base with $\beta > 1$, the digits $a_k \in \mathbb{Z}$ satisfy certain conditions specified below, and $n \geq 0$. These numbers are called *$(-\beta)$-integers*, and have been recently studied by Ambrož *et al.* [1].

Before dealing with these numbers, we recall some facts about *β-integers*, which are the real numbers of the form

$$\pm \sum_{k=0}^{n-1} a_k \, \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \, \beta^k < \beta^m \quad \text{for all } 1 \leq m \leq n,$$

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i.e., $\sum_{k=0}^{n-1} a_k\,\beta^k$ is a greedy *β-expansion*. Equivalently, we can define the set of β-integers as

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n\ge 0} \beta^n\, T_\beta^{-n}(0),$$

where T_β is the *β-transformation*, defined by

$$T_\beta:\ [0,1) \to [0,1)\,, \quad x \mapsto \beta x - \lfloor \beta x \rfloor.$$

This map and the corresponding β-expansions were first studied by Rényi [20].

The notion of β-integers was introduced in the domain of quasicrystallography, see for instance [6], and the structure of the β-integers is very well understood now. We have $\beta\,\mathbb{Z}_\beta \subseteq \mathbb{Z}_\beta$, the set of distances between consecutive elements of $\mathbb{Z}_\beta$ is

$$\Delta_\beta = \{T_\beta^n(1^-) \mid n \ge 0\},$$

where $T_\beta^n(x^-) = \lim_{y\to x-} T_\beta^n(y)$, and the sequence of distances between consecutive elements of $\mathbb{Z}_\beta^+$ is coded by the fixed point of a substition, see [9] for the case when Δ_β is a finite set, that is when β is a *Parry number*. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found *e.g.* in [2–4, 11, 16].

Closely related to $\mathbb{Z}_\beta^+$ are the sets

$$S_\beta(x) = \bigcup_{n\ge 0} \beta^n\, T_\beta^{-n}(x) \qquad (x \in [0,1)),$$

which were used by Thurston [21] to define (fractal) tilings of $\mathbb{R}^{d-1}$ when β is a Pisot number of degree d, *i.e.*, a root > 1 of a polynomial $x^d + p_1 x^{d-1} + \cdots + p_d \in \mathbb{Z}[x]$ such that all other roots have modulus < 1, and an algebraic unit, *i.e.*, $p_d = \pm 1$. These tilings allow *e.g.* to determine the kth digit a_k of a number without knowing the other digits, see [15].

It is widely agreed that the greedy β-expansions are the natural representations of real numbers in a real base $\beta > 1$. For the case of negative bases, the situation is not so clear. Ito and Sadahiro [14] proposed recently to use the $(-\beta)$-*transformation* defined by

$$T_{-\beta}:\ \left[\frac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\right),\ x \mapsto -\beta x - \left\lfloor \frac{\beta}{\beta+1} - \beta x \right\rfloor,$$

see also [10]. This transformation has the important property that $T_{-\beta}(-x/\beta) = x$ for all $x \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$. Some instances are depicted in Figures 1, 3, 4 and 5.

The set of $(-\beta)$-integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n\ge 0} (-\beta)^n\, T_{-\beta}^{-n}(0).$$

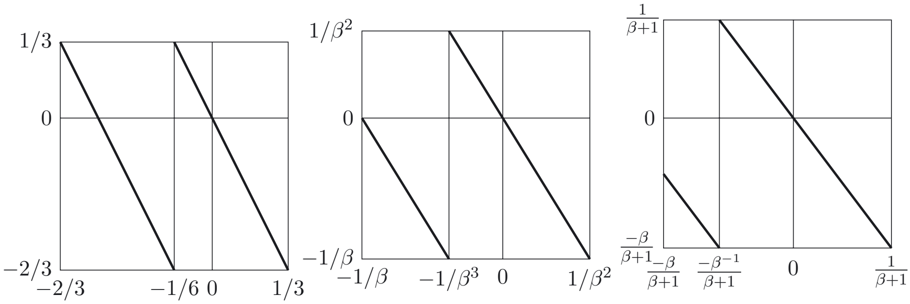

FIGURE 1. The $(-\beta)$-transformation for $\beta=2$ (left), $\beta=\frac{1+\sqrt{5}}{2}\approx 1.618$ (middle), and $\beta=\frac{1}{\beta}+\frac{1}{\beta^2}\approx 1.325$ (right).

These are the numbers

$$\sum_{k=0}^{n-1} a_k\,(-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta+1}\le \sum_{k=0}^{m-1} a_k\,(-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1\le m\le n.$$

Note that, in the case of β-integers, we have to add $-\mathbb{Z}_\beta^+$ to $\mathbb{Z}_\beta^+$ in order to obtain a set resembling $\mathbb{Z}$. In the case of $(-\beta)$-integers, this is not necessary because the $(-\beta)$-transformation allows to represent positive and negative numbers.

It is not difficult to see that $\mathbb{Z}_{-\beta}=\mathbb{Z}=\mathbb{Z}_\beta$ when $\beta\in\mathbb{Z}$, $\beta\ge 2$. Some other properties of $\mathbb{Z}_{-\beta}$ can be found in [1], mainly for the case when $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big)\le 0$ and $T_{-\beta}^{2n-1}\big(\frac{-\beta}{\beta+1}\big)\ge\frac{1-\lfloor\beta\rfloor}{\beta}$ for all $n\ge 1$. (Note that $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big)\in\big(\frac{1}{\beta+1}-\frac{\lfloor\beta\rfloor}{\beta},\frac{1-\lfloor\beta\rfloor}{\beta}\big)\cup\big(\frac{-\beta^{-1}}{\beta+1},0\big)$ implies $T_{-\beta}^{n+1}\big(\frac{-\beta}{\beta+1}\big)>0$).

The set

$$V_\beta=\left\{T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right)\mid n\ge 0\right\}$$

plays a similar role for $(-\beta)$-expansions as the set $\{T_\beta^n(1^-)\mid n\ge 0\}$ for β-expansions. If V_β is a finite set, then we call $\beta>1$ an *Yrrap number*. Note that these numbers are called *Ito–Sadahiro numbers* in [18], in reference to [14]. However, as the generalised β-transformations in [13] with $E=(1,\dots,1)$ are, up to conjugation by the map $x\mapsto\frac{1}{\beta+1}-x$, the same as our $(-\beta)$-transformations, these numbers were already considered by Góra and perhaps by other authors. Therefore, the neutral but intricate name $(-\beta)$-*numbers* was chosen in [17], referring to the original name β-numbers for Parry numbers [19]. The name Yrrap number, used in the present paper, refers to the connection with Parry numbers and to the fact that $T_{-\beta}$ is (locally) orientation-reversing.

For any Yrrap number $\beta\ge(1+\sqrt{5})/2$, we describe the sequence of $(-\beta)$-integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism (Thm. 3.2). Note that the orientation-reversing property of the map $x \mapsto -\beta x$ imposes the use of an anti-morphism instead of a morphism, and that anti-morphisms were considered in a similar context by Enomoto [8].

For $1 < \beta < \frac{1+\sqrt{5}}{2}$, we have $\mathbb{Z}_{-\beta} = \{0\}$, as already proved in [1]. However, our study still makes sense for these bases $(-\beta)$ because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n\to\infty} (-\beta)^n\, T_{-\beta}^{-n}(x) \qquad \left(x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)\right),$$

where the limit set consists of the numbers lying in all but finitely many sets $(-\beta)^n\, T_{-\beta}^{-n}(x)$, $n \geq 0$. Taking the limit instead of the union over all $n \geq 0$ implies that every $y \in \mathbb{R}$ lies in exactly one set $S_{-\beta}(x)$, $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, see Lemma 3.1. Note that $T_{-\beta}^2\left(\frac{-\beta^{-1}}{\beta+1}\right) \neq \frac{-\beta}{\beta+1}$ when $\beta \notin \mathbb{Z}$. Other properties of the $(-\beta)$-transformation for $1 < \beta < \frac{1+\sqrt{5}}{2}$ are exhibited in [17].

2. β-integers

In this section, we consider the structure of β-integers. The results are not new, but it is useful to state and prove them in order to understand the differences with $(-\beta)$-integers.

Recall that $\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\}$, and let Δ_β^* be the free monoid generated by Δ_β. Elements of Δ_β^* will be considered as words on the alphabet Δ_β, and the operation is the concatenation of words. The β-*substitution* is the morphism $\varphi_\beta : \Delta_\beta^* \to \Delta_\beta^*$, defined by

$$\varphi_\beta(x) = \underbrace{11\cdots 1}_{\lceil \beta x\rceil - 1 \text{ times}}\, T_\beta(x^-) \qquad (x \in \Delta_\beta).$$

Here, 1 is an element of Δ_β and not the identity element of Δ_β^* (which is the empty word). Recall that, as φ_β is a morphism, we have $\varphi_\beta(uv) = \varphi_\beta(u)\varphi_\beta(v)$ for all $u, v \in \Delta_\beta^*$. Since $\varphi_\beta^{n+1}(1) = \varphi_\beta^n(\varphi_\beta(1))$ and $\varphi_\beta(1)$ starts with 1, $\varphi_\beta^n(1)$ is a prefix of $\varphi_\beta^{n+1}(1)$ for every $n \geq 0$.

Theorem 2.1. *For any $\beta > 1$, the set of non-negative β-integers takes the form*

$$\mathbb{Z}_\beta^+ = \{z_k \mid k \geq 0\} \quad \textit{with} \quad z_k = \sum_{j=1}^k u_j,$$

where $u_1 u_2 \cdots$ is the infinite word with letters in Δ_β which has $\varphi_\beta^n(1)$ as prefix for all $n \geq 0$.

The set of differences between consecutive β-integers is Δ_β.

The main observation for the proof of the theorem is the following. We use the notation $|v| = k$ and $L(v) = \sum_{j=1}^k v_j$ for any $v = v_1 \cdots v_k \in \Delta_\beta^k$, $k \geq 0$.

Lemma 2.1. *For any $n \geq 0$, $1 \leq k \leq |\varphi_\beta^n(1)|$, we have*

$$T_\beta^n\left(\left[\frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n}\right)\right) = [0, u_k),$$

and $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$.

Proof. For $n = 0$, we have $|\varphi_\beta^0(1)| = 1$, $z_0 = 0$, $z_1 = 1$, $u_1 = 1$, thus the statements are true. Suppose that they hold for n, and consider

$$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1)\,\varphi_\beta(u_2)\,\cdots\,\varphi_\beta(u_{|\varphi_\beta^n(1)|}).$$

Let $1 \leq k \leq |\varphi_\beta^{n+1}(1)|$, and write $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1})\, v_1 \cdots v_i$ with $1 \leq j \leq |\varphi_\beta^n(1)|$, $1 \leq i \leq |\varphi_\beta(u_j)|$, *i.e.*, $v_1 \cdots v_i$ is a non-empty prefix of $\varphi_\beta(u_j)$.

For any $x \in (0, 1]$, we have $T_\beta(x^-) = \beta x - \lceil \beta x \rceil + 1$, hence $L(\varphi_\beta(x)) = \beta x$ for $x \in \Delta_\beta$. This yields that

$$z_k = L(u_1 \cdots u_k) = \beta\, L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$$

and $z_{k-1} = \beta z_{j-1} + i - 1$, hence

$$\left[\frac{z_{k-1}}{\beta}, \frac{z_k}{\beta}\right) = \left[z_{j-1} + \frac{i-1}{\beta}, z_{j-1} + \frac{i-1+v_i}{\beta}\right) \subseteq [z_{j-1}, z_{j-1} + u_j) = [z_{j-1}, z_j),$$

$$T_\beta^{n+1}\left(\left[\frac{z_{k-1}}{\beta^{n+1}}, \frac{z_k}{\beta^{n+1}}\right)\right) = T_\beta\left(\left[\frac{i-1}{\beta}, \frac{i-1+v_i}{\beta}\right)\right) = [0, v_i) = [0, u_k).$$

Moreover, we have $L(\varphi_\beta^{n+1}(1)) = \beta\, L(\varphi_\beta^n(1)) = \beta^{n+1}$, thus the statements hold for $n + 1$. □

Proof of Theorem 2.1. By Lemma 2.1, we have $z_{|\varphi_\beta^n(1)|} = \beta^n$ for all $n \geq 0$, thus $[0, 1)$ is split into the intervals $[z_{k-1}/\beta^n, z_k/\beta^n)$, $1 \leq k \leq |\varphi_\beta^n(1)|$. Therefore, Lemma 2.1 yields that

$$T_\beta^{-n}(0) = \{z_{k-1}/\beta^n \mid 1 \leq k \leq |\varphi_\beta^n(1)|\},$$

hence

$$\bigcup_{n \geq 0} \beta^n\, T_\beta^{-n}(0) = \{z_k \mid k \geq 0\}.$$

Since $u_k \in \Delta_\beta$ for all $k \geq 1$ and $u_{|\varphi^n(1)|} = T_\beta^n(1^-)$ for all $n \geq 0$, we have

$$\{z_k - z_{k-1} \mid k \geq 1\} = \{u_k \mid k \geq 1\} = \Delta_\beta. \qquad \square$$

For the sets $S_\beta(x)$, Lemma 2.1 gives the following corollary.

Corollary 2.1. *For any $x \in [0,1)$, we have*

$$S_\beta(x) = \{z_k + x \mid k \geq 0,\, u_{k+1} > x\} \subseteq x + S_\beta(0).$$

In particular, we have $S_\beta(x) - x = S_\beta(y) - y$ for all $x, y \in [0,1)$ with $(x,y] \cap \Delta_\beta = \emptyset$. From the definition of $S_\beta(x)$ and since $x \in \beta\, T_\beta^{-1}(x)$, we also get that

$$S_\beta(x) = \bigcup_{y \in T_\beta^{-1}(x)} \beta\, S_\beta(y) \qquad \bigl(x \in [0,1)\bigr).$$

This shows that $S_\beta(x)$ is the solution of a graph-directed iterated function system (GIFS) when β is a Parry number, *cf.* [15], Section 3.2.

3. $(-\beta)$-integers

We now turn to the discussion of $(-\beta)$-integers and sets $S_{-\beta}(x)$, $x \in \bigl[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$.

Lemma 3.1. *For any $\beta > 1$, $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, we have*

$$S_{-\beta}(x) = \bigcup_{n \geq 0} (-\beta)^n \left(T_{-\beta}^{-n}(x) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right) = \bigcup_{y \in T_{-\beta}^{-1}(x)} (-\beta)\, S_{-\beta}(y).$$

For any $y \in \mathbb{R}$, there exists a unique $x \in \bigl[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$ such that $y \in S_{-\beta}(x)$.
If $T_{-\beta}(x) = x$, then $S_{-\beta}(x) = \bigcup_{n\geq 0} (-\beta)^n\, T_{-\beta}^{-n}(x)$, in particular $S_{-\beta}(0) = \mathbb{Z}_{-\beta}$.

Proof. If $y \in S_{-\beta}(x)$, then we have $\frac{y}{(-\beta)^n} \in T_{-\beta}^{-n}(x)$ for all sufficiently large n, thus $y \in (-\beta)^n \bigl(T_{-\beta}^{-n}(x) \setminus \bigl\{\frac{-\beta}{\beta+1}\bigr\}\bigr)$ for some $n \geq 0$. On the other hand, $y \in (-\beta)^n \bigl(T_{-\beta}^{-n}(x) \setminus \bigl\{\frac{-\beta}{\beta+1}\bigr\}\bigr)$ for some $n \geq 0$ implies that $T_{-\beta}^m(\frac{y}{(-\beta)^m}) = T_{-\beta}^n(\frac{y}{(-\beta)^n}) = x$ for all $m \geq n$, thus $y \in S_{-\beta}(x)$. This shows the first equation. Since $x \in \bigl(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$ implies that $x \in (-\beta) \bigl(T_{-\beta}^{-1}(x) \setminus \bigl\{\frac{-\beta}{\beta+1}\bigr\}\bigr)$, we obtain that $S_{-\beta}(x) = \bigcup_{y \in T_{-\beta}^{-1}(x)} (-\beta)\, S_{-\beta}(y)$ for all $x \in \bigl[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$.

For any $y \in \mathbb{R}$, we have $y \in S_{-\beta}\bigl(T_{-\beta}^n\bigl(\frac{y}{(-\beta)^n}\bigr)\bigr)$ for all $n \geq 0$ such that $\frac{y}{(-\beta)^n} \in \bigl(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$, thus $y \in S_{-\beta}(x)$ for some $x \in \bigl[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$. To show that this x is unique, let $y \in S_{-\beta}(x)$ and $y \in S_{-\beta}(x')$ for some $x, x' \in \bigl[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\bigr)$. Then we have $y \in (-\beta)^n \bigl(T_{-\beta}^{-n}(x) \setminus \bigl\{\frac{-\beta}{\beta+1}\bigr\}\bigr)$ and $y \in (-\beta)^m \bigl(T_{-\beta}^{-m}(x') \setminus \bigl\{\frac{-\beta}{\beta+1}\bigr\}\bigr)$ for some $m, n \geq 0$, thus $x = T_{-\beta}^n\bigl(\frac{y}{(-\beta)^n}\bigr) = T_{-\beta}^m\bigl(\frac{y}{(-\beta)^m}\bigr) = x'$.

If $T_{-\beta}^n\bigl(\frac{-\beta}{\beta+1}\bigr) = x = T_{-\beta}(x)$, then $T_{-\beta}^{n+2}\bigl(\frac{-\beta^{-1}}{\beta+1}\bigr) = T_{-\beta}^{n+1}\bigl(\frac{-\beta}{\beta+1}\bigr) = T_{-\beta}(x) = x$ yields that $(-\beta)^n \frac{-\beta}{\beta+1} \in S_{-\beta}(x)$, which shows that $S_{-\beta}(x) = \bigcup_{n\geq 0} (-\beta)^n\, T_{-\beta}^{-n}(x)$ when $T_{-\beta}(x) = x$. $\square$

The first two statements of the following proposition can also be found in [1].

Proposition 3.1. *For any $\beta > 1$, we have $(-\beta)\,\mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$.*
If $\beta < (1+\sqrt{5})/2$, then $\mathbb{Z}_{-\beta} = \{0\}$.
If $\beta \geq (1+\sqrt{5})/2$, then

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n\,[-\beta, 1] = \big\{(-\beta)^n, (-\beta)^{n+1}\big\} \cup (-\beta)^{n+2}\left(T_{-\beta}^{-n-2}(0) \cap \left(\frac{-1}{\beta}, \frac{1}{\beta^2}\right)\right)$$

for all $n \geq 0$, in particular

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \begin{cases} \{-\beta, -\beta+1, \ldots, -\beta+\lfloor\beta\rfloor, 0, 1\} & \text{if } \beta^2 \geq \lfloor\beta\rfloor(\beta+1), \\ \{-\beta, -\beta+1, \ldots, -\beta+\lfloor\beta\rfloor-1, 0, 1\} & \text{if } \beta^2 < \lfloor\beta\rfloor(\beta+1). \end{cases}$$

Proof. The inclusion $(-\beta)\,\mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$ is a consequence of Lemma 3.1 and $0 \in T_{-\beta}^{-1}(0)$.

If $\beta < \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} < \frac{-\beta}{\beta+1}$, hence $T_{-\beta}^{-1}(0) = \{0\}$ and $\mathbb{Z}_{-\beta} = \{0\}$, see Figure 1 (right).

If $\beta \geq \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} \in T_{-\beta}^{-1}(0)$ implies $1 \in \mathbb{Z}_{-\beta}$, thus $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$. Clearly,

$$(-\beta)^{n+2}\left(T_{-\beta}^{-n-2}(0) \cap \left(\frac{-1}{\beta}, \frac{1}{\beta^2}\right)\right) \subseteq \mathbb{Z}_{-\beta} \cap (-\beta)^n\,(-\beta, 1).$$

To show the other inclusion, let $z \in (-\beta)^m\,T_{-\beta}^{-m}(0) \cap (-\beta)^n\,(-\beta, 1)$ for some $m \geq 0$. If $z \neq (-\beta)^m \frac{-\beta}{\beta+1}$, then $\frac{z}{(-\beta)^m} \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ and $\frac{z}{(-\beta)^{n+2}} \in \big(\frac{-1}{\beta}, \frac{1}{\beta^2}\big) \subseteq \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ imply that $T_{-\beta}^{n+2}\big(\frac{z}{(-\beta)^{n+2}}\big) = T_{-\beta}^{m}\big(\frac{z}{(-\beta)^{m}}\big) = 0$. If $z = (-\beta)^m \frac{-\beta}{\beta+1}$, then

$$\begin{aligned} T_{-\beta}^{n+2}\left(\frac{z}{(-\beta)^{n+2}}\right) &= T_{-\beta}^{n+2}\left(\frac{(-\beta)^{m-n-1}}{\beta+1}\right) = T_{-\beta}^{m+2}\left(\frac{-\beta^{-1}}{\beta+1}\right) \\ &= T_{-\beta}^{m+1}\left(\frac{-\beta}{\beta+1}\right) = T_{-\beta}(0) = 0, \end{aligned}$$

where we have used that $\frac{z}{(-\beta)^{n+2}} \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ implies $m \leq n$. Therefore, we have $z \in (-\beta)^{n+2}\,T_{-\beta}^{-n-2}(0)$ for all $z \in \mathbb{Z}_{-\beta} \cap (-\beta)^n\,(-\beta, 1)$.

Consider now $n = 0$, then

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \{-\beta, 1\} \cup \{z \in (-\beta, 1) \mid T_{-\beta}^2(z/\beta^2) = 0\}.$$

Since $\frac{-\lfloor\beta\rfloor}{\beta} \geq \frac{-\beta}{\beta+1}$ if and only if $\beta^2 \geq \lfloor\beta\rfloor(\beta+1)$, we obtain that

$$(-\beta)\,T_{-\beta}^{-1}(0) = \begin{cases} \{0, 1, \ldots, \lfloor\beta\rfloor\} & \text{if } \beta^2 \geq \lfloor\beta\rfloor(\beta+1), \\ \{0, 1, \ldots, \lfloor\beta\rfloor - 1\} & \text{if } \beta^2 < \lfloor\beta\rfloor(\beta+1). \end{cases}$$

If $T_{-\beta}^2(z/\beta^2) = 0$, then $z = -a_1\beta + a_0$ with $a_0 \in (-\beta)\,T_{-\beta}^{-1}(0)$, $a_1 \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$, and $\mathbb{Z}_{-\beta} \cap [-\beta, 1]$ consists of those numbers $-a_1\beta + a_0$ lying in $[-\beta, 1]$. $\square$

Proposition 3.1 shows that the maximal difference between consecutive $(-\beta)$-integers exceeds 1 whenever $\beta^2 < \lfloor\beta\rfloor(\beta+1)$, *i.e.*, $T_{-\beta}\big(\frac{-\beta}{\beta+1}\big) < 0$. For an example, this was also proved in [1]. In Examples 3.3 and 3.4, we see that the distance between two consecutive $(-\beta)$-integers can be even greater than 2, and the structure of $\mathbb{Z}_{-\beta}$ can be quite complicated. Therefore, we adapt a slightly different strategy as for $\mathbb{Z}_\beta$.

In the following, we always assume that the set

$$V'_\beta = V_\beta \cup \{0\} = \left\{ T_{-\beta}^n\left(\frac{-\beta}{\beta+1}\right) \;\middle|\; n \geq 0 \right\} \cup \{0\}$$

is finite, *i.e.*, β is an Yrrap number, and let β be fixed. For $x \in V'_\beta$, let

$$r_x = \min\left\{ y \in V'_\beta \cup \left\{\frac{1}{\beta+1}\right\} \;\middle|\; y > x \right\},\ \widehat{x} = \frac{x+r_x}{2},\ J_x = \{x\} \text{ and } \quad J_{\widehat{x}} = (x, r_x).$$

Then $\{J_a \mid a \in A_\beta\}$ forms a partition of $\big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, where

$$A_\beta = V'_\beta \cup \widehat{V'_\beta} \quad \text{with} \quad \widehat{V'_\beta} = \{\widehat{x} \mid x \in V'_\beta\}.$$

We have $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$ for every $x \in V'_\beta$, and the following lemma shows that the image of every $J_{\widehat{x}}$, $x \in V'_\beta$, is a union of intervals J_a, $a \in A_\beta$.

Lemma 3.2. *Let $x \in V'_\beta$ and write*

$$J_{\widehat{x}} \cap T_{-\beta}^{-1}(V'_\beta) = \{y_1, \ldots, y_m\}, \quad \textit{with} \quad x = y_0 < y_1 < \cdots < y_m < y_{m+1} = r_x.$$

For any $0 \leq i \leq m$, we have

$$T_{-\beta}\Big((y_i, y_{i+1})\Big) = J_{\widehat{x_i}} \quad \textit{with} \quad x_i = \lim_{y \to y_{i+1}-} T_{-\beta}(y),\ \textit{i.e.},\ \widehat{x_i} = T_{-\beta}\left(\frac{y_i + y_{i+1}}{2}\right),$$

and $\beta(y_{i+1} - y_i) = \lambda(J_{\widehat{x_i}})$, where λ denotes the Lebesgue measure.

Proof. Since $T_{-\beta}$ maps no point in (y_i, y_{i+1}) to $\frac{-\beta}{\beta+1} \in V'_\beta$, the map is continuous on this interval and $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$. We have $x_i \in V'_\beta$ since $x_i = T_{-\beta}(y_{i+1})$ in case $y_{i+1} < \frac{1}{\beta+1}$, and $x_i = \frac{-\beta}{\beta+1}$ in case $y_{i+1} = \frac{1}{\beta+1}$. Since $y_i = \max\{y \in T_{-\beta}^{-1}(V'_\beta) \mid y < y_{i+1}\}$, we obtain that $r_{x_i} = \lim_{y \to y_i+} T_{-\beta}(y)$, thus $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$. □

In view of Lemma 3.2, we define an anti-morphism $\psi_\beta : A_\beta^* \to A_\beta^*$ by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\widehat{x}) = \widehat{x_m}\, T_{-\beta}(y_m) \cdots \widehat{x_1}\, T_{-\beta}(y_1)\, \widehat{x_0} \qquad (x \in V'_\beta),$$

with m, x_i and y_i as in Lemma 3.2. Here, anti-morphism means that $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$ for all $u, v \in A_\beta^*$. Now, the last letter of $\psi_\beta(\widehat{0})$ is $\widehat{t}$, with $t = \max\{x \in V_\beta \mid x < 0\}$, and the first letter of $\psi_\beta(\widehat{t})$ is $\widehat{0}$. Therefore, $\psi_\beta^{2n}(\widehat{0})$ is a prefix of $\psi_\beta^{2n+2}(\widehat{0}) = \psi_\beta^{2n}(\psi_\beta^2(\widehat{0}))$ and $\psi_\beta^{2n+1}(\widehat{0})$ is a suffix of $\psi_\beta^{2n+3}(\widehat{0})$ for every $n \geq 0$.

Theorem 3.1. *For any Yrrap number $\beta \geq (1+\sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z_k \mid k \in \mathbb{Z},\, u_{2k} = 0\} \quad \text{with} \quad z_k = \begin{cases} \sum_{j=1}^{k} \lambda(J_{u_{2j-1}}) & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} \lambda(J_{u_{-2j+1}}) & \text{if } k \leq 0, \end{cases}$$

where $\cdots u_{-1}u_0u_1\cdots$ is the two-sided infinite word on the finite alphabet A_β such that $u_0 = 0$, $\psi_\beta^{2n}(\widehat{0})$ is a prefix of $u_1u_2\cdots$ and $\psi_\beta^{2n+1}(\widehat{0})$ is a suffix of $\cdots u_{-2}u_{-1}$ for all $n \geq 0$.

Note that $\cdots u_{-1}u_0u_1\cdots$ is a fixed point of ψ_β, with u_0 being mapped to u_0.

The following lemma is the analogue of Lemma 2.1. We use the notation

$$L(v) = \sum_{j=1}^{k} \lambda(J_{v_j}) \quad \text{if } v = v_1\cdots v_k \in A_\beta^k.$$

Note that $u_{2k} \in V'_\beta$ and $u_{2k+1} \in \widehat{V}'_\beta$ for all $k \in \mathbb{Z}$, thus $\lambda(J_{u_{2k}}) = 0$ for all $k \in \mathbb{Z}$.

Lemma 3.3. *For any $n \geq 0$, $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$, we have*

$$T_{-\beta}^n\left(\frac{z_{(-1)^n k}}{(-\beta)^n}\right) = u_{(-1)^n 2k}, \quad T_{-\beta}^n\left(\left(\frac{z_{(-1)^n k}}{(-\beta)^n}, \frac{z_{(-1)^n (k+1)}}{(-\beta)^n}\right)\right) = J_{u_{(-1)^n(2k+1)}},$$

and $z_{(-1)^n(|\psi_\beta^n(\widehat{0})|+1)/2} = (-1)^n\, L\big(\psi_\beta^n(\widehat{0})\big) = \lambda(J_{\widehat{0}})\,(-\beta)^n = r_0\,(-\beta)^n.$

Proof. The statements are true for $n = 0$ since $|\psi_\beta^0(\widehat{0})| = 1$, $z_0 = 0$, $z_1 = \lambda(J_{\widehat{0}}) = r_0$.

Suppose that they hold for even n, and consider

$$u_{-|\psi_\beta^{n+1}(\widehat{0})|}\cdots u_{-2}u_{-1} = \psi_\beta^{n+1}(\widehat{0}) = \psi_\beta\big(\psi_\beta^n(\widehat{0})\big) = \psi_\beta(u_{|\psi_\beta^n(\widehat{0})|})\cdots\psi_\beta(u_2)\psi_\beta(u_1).$$

Let $0 \leq k < |\psi_\beta^{n+1}(\widehat{0})|/2$, and write

$$u_{-2k-1}\cdots u_{-1} = v_{-2i-1}\cdots v_{-1}\,\psi_\beta(u_1\cdots u_{2j})$$

with $0 \leq j < |\psi_\beta^n(\widehat{0})|/2$, $0 \leq i < |\psi_\beta(u_{2j+1})|/2$, *i.e.*, $u_{-2i-1}\cdots u_{-1}$ is a suffix of $\psi_\beta(u_{2j+1})$.

By Lemma 3.2, we have $L(\psi_\beta(\widehat{x})) = \beta\,\lambda(J_{\widehat{x}})$ for any $x \in V'_\beta$. This yields that

$$-z_{-k-1} = \beta\, L(u_1\cdots u_{2j}) + L(v_{-2i-1}\cdots v_{-1}) = \beta\, z_j + L(v_{-2i-1}\cdots v_{-1})$$

and $-z_{-k} = \beta\, z_j + L(v_{-2i}\cdots v_{-1})$. By the induction hypothesis, we obtain that

$$T_{-\beta}^{n+1}\left(\frac{z_{-k}}{(-\beta)^{n+1}}\right) = T_{-\beta}^{n+1}\left(\frac{z_j}{(-\beta)^n} - \frac{L(v_{-2i}\cdots v_{-1})}{(-\beta)^{n+1}}\right)$$
$$= \begin{cases} T_{-\beta}(u_{2j}) = \psi_\beta(u_{2j}) = u_{-2k} & \text{if } i=0, \\ T_{-\beta}\big(x + L(v_{-2i}\cdots v_{-1})/\beta\big) = T_{-\beta}(y_i) = v_{-2i} = u_{-2k} & \text{if } i \geq 1, \end{cases}$$

where the y_i's are the numbers from Lemma 3.2 for $\widehat{x} = u_{2j+1}$, and

$$T_{-\beta}^{n+1}\left(\left(\frac{z_{-k}}{(-\beta)^{n+1}}, \frac{z_{-k-1}}{(-\beta)^{n+1}}\right)\right) = T_{-\beta}\big((y_i, y_{i+1})\big) = J_{v_{-2i-1}} = J_{u_{-2k-1}}.$$

Moreover, we have $L(\psi_\beta^{n+1}(\widehat{0})) = \beta\, L(\psi_\beta^n(\widehat{0})) = r_0 \beta^{n+1}$, thus the statements hold for $n+1$.

The proof for odd n runs along the same lines and is therefore omitted. □

Proof of Theorem 3.1. By Lemma 3.3, we have $z_{(-1)^n(|\psi_\beta^n(\widehat{0})|+1)/2} = r_0\,(-\beta)^n$ for all $n \ge 0$, thus $[0, r_0)$ splits into the intervals $\big(z_{(-1)^n k}(-\beta)^{-n}, z_{(-1)^n(k+1)}(-\beta)^{-n}\big)$ and points $z_{(-1)^n k}(-\beta)^{-n}$, $0 \le k < |\psi_\beta^n(\widehat{0})|/2$, hence

$$T_{-\beta}^{-n}(0) \cap [0, r_0) = \big\{z_{(-1)^n k}(-\beta)^{-n} \mid 0 \le k < |\psi_\beta^n(\widehat{0})|/2,\, u_{(-1)^n 2k} = 0\big\}.$$

Let $m \ge 1$ be such that $\beta^{2m} r_0 \ge \frac{1}{\beta+1}$. Then we have $\big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big) \subseteq (-\beta^{2m+1} r_0, \beta^{2m} r_0)$, and

$$T_{-\beta}^{-n}(0) \setminus \left\{\frac{-\beta}{\beta+1}\right\} \subseteq (-\beta)^{2m}\big(T_{-\beta}^{-n-2m}(0) \cap [0, r_0)\big) \cup (-\beta)^{2m+1}\big(T_{-\beta}^{-n-2m-1}(0) \cap [0, r_0)\big),$$

thus

$$\bigcup_{n\ge 0} (-\beta)^n \left(T_{-\beta}^{-n}(0) \setminus \left\{\frac{-\beta}{\beta+1}\right\}\right) = \bigcup_{n\ge 0} (-\beta)^n \left(T_{-\beta}^{-n}(0) \cap [0, r_0)\right) = \{z_k \mid k \in \mathbb{Z},\, u_{2k} = 0\}.$$

Together with Lemma 3.1, this proves the theorem. □

As in the case of positive bases, the word $\cdots u_{-1} u_0 u_1 \cdots$ also describes the sets $S_{-\beta}(x)$. Theorem 3.1 and Lemma 3.3 give the following corollary.

Corollary 3.1. *For any $x \in V_\beta'$, $y \in J_{\widehat{x}}$, we have*

$$S_{-\beta}(x) = \{z_k \mid k \in \mathbb{Z},\, u_{2k} = x\} \quad \text{and} \quad S_{-\beta}(y) = \{z_k + y - x \mid k \in \mathbb{Z},\, u_{2k+1} = \widehat{x}\}.$$

Lemma 3.1 and Corollary 3.1 imply that $S_{-\beta}(x)$ is the solution of a GIFS for any Yrrap number $\beta \ge (1+\sqrt{5})/2$, $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, *cf.* the end of Section 2.

Recall that our main goal is to understand the structure of $\mathbb{Z}_{-\beta}$ (in case $\beta \ge (1+\sqrt{5})/2$), *i.e.*, to describe the occurrences of 0 in the word $\cdots u_{-1} u_0 u_1 \cdots$ defined in Theorem 3.1 and the words between two successive occurrences. Let

$$R_\beta = \{u_{2k} u_{2k+1} \cdots u_{2s(k)-1} \mid k \in \mathbb{Z},\, u_{2k} = 0\} \qquad \text{with} \quad s(k) = \min\{j \in \mathbb{Z} \mid u_{2j} = 0,\, j > k\}$$

be the set of return words of 0 in $\cdots u_{-1} u_0 u_1 \cdots$.

Note that $s(k)$ is defined for all $k \in \mathbb{Z}$ since $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$ by Proposition 3.1.

For any $w \in R_\beta$, the word $\psi_\beta(w0)$ is a factor of $\cdots u_{-1}u_0u_1\cdots$ starting and ending with 0, thus we can write $\psi_\beta(w0) = w_1 \cdots w_m 0$ with $w_j \in R_\beta$, $1 \leq j \leq m$, and set

$$\varphi_{-\beta}(w) = w_1 \cdots w_m.$$

This defines an anti-morphism $\varphi_{-\beta} : R_\beta^* \to R_\beta^*$, which plays the role of the β-substitution.

Since $\cdots u_{-1}u_0u_1\cdots$ is generated by $u_1 = \widehat{0}$, as described in Theorem 3.1, we consider $w_\beta = u_0u_1 \cdots u_{2s(0)-1}$. We have

$$[0,1] = \left[0, \frac{1}{\beta+1}\right) \cup \left[\frac{1}{\beta+1}, 1\right], \quad T_{-\beta}\left((-\beta)^{-1}\left[\frac{1}{\beta+1}, 1\right]\right) = \left[\frac{-\beta}{\beta+1}, 0\right],$$

thus $L(w_\beta) = 1$ and

$$w_\beta = 0\,\widehat{0}\,x_1\,\widehat{x_1}\,\cdots\,x_m\,\widehat{x_m}\,x_{-\ell}\,\widehat{x_{-\ell}}\,\cdots x_{-1}\,\widehat{x_{-1}},$$

where the x_j are defined by $V_\beta' = \{x_{-\ell}, \ldots, x_{-1}, 0, x_1, \ldots, x_m\}$, $x_{-\ell} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_m$.

Theorem 3.2. *For any Yrrap number $\beta \geq (1+\sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z_k' \mid k \in \mathbb{Z}\} \quad \text{with} \quad z_k' = \begin{cases} \sum_{j=1}^{k} L(u_j') & \text{if } k \geq 0, \\ -\sum_{j=1}^{|k|} L(u_{-j}') & \text{if } k \leq 0, \end{cases}$$

where $\cdots u_{-2}'u_{-1}'u_1'u_2'\cdots$ is the two-sided infinite word on the finite alphabet R_β such that $\varphi_{-\beta}^{2n}(w_\beta)$ is a prefix of $u_1'u_2'\cdots$ and $\varphi_{-\beta}^{2n+1}(w_\beta)$ is a suffix of $\cdots u_{-2}'u_{-1}'$ for all $n \geq 0$.

The set of distances between consecutive $(-\beta)$-integers is

$$\Delta_{-\beta} = \{z_{k+1}' - z_k' \mid k \in \mathbb{Z}\} = \{L(w) \mid w \in R_\beta\}.$$

Note that the index 0 is omitted in $\cdots u_{-2}'u_{-1}'u_1'u_2'\cdots$ for reasons of symmetry.

Proof. The definitions of $\cdots u_{-1}u_0u_1\cdots$ in Theorem 3.1 and of $\varphi_{-\beta}$ imply that $\cdots u_{-2}'u_{-1}'\, u_1'u_2'\cdots$ is the derived word of $\cdots u_{-1}u_0u_1\cdots$ with respect to R_β, that is

$$u_k' = u_{|u_1'\cdots u_{k-1}'|}\cdots u_{|u_1'\cdots u_k'|-1}, \quad u_{-k}' = u_{-|u_{-k}'\cdots u_{-1}'|}\cdots u_{-|u_{1-k}'\cdots u_{-1}'|-1} \quad (k \geq 1)$$

with

$$\{|u_1'\cdots u_{k-1}'| \mid k \geq 1\} \cup \{-|u_{-k}'\cdots u_{-1}'| \mid k \geq 1\} = \{k \in \mathbb{Z} \mid u_k = 0\}.$$

Here, for any $v \in R_\beta^*$, $|v|$ denotes the length of v as a word in A_β^*, not in R_β^*. Since

$$z'_k = \sum_{j=1}^{k} L(u'_j) = \sum_{j=0}^{|u'_1\cdots u'_k|-1} \lambda(J_{u_j}) = \sum_{j=1}^{|u'_1\cdots u'_k|} \lambda(J_{u_j}), \quad z'_{-k} = -\sum_{j=1}^{k} L(u'_{-j}) = -\sum_{j=1}^{|u'_{-k}\cdots u'_{-1}|} \lambda(J_{u_{-j}})$$

for all $k \geq 0$, Theorem 3.1 yields that $\{z'_k \mid k \in \mathbb{Z}\} = \mathbb{Z}_{-\beta}$.

It follows from the definition of R_β that $\Delta_{-\beta} = \{L(w) \mid w \in R_\beta\}$.

It remains to show that R_β is a finite set. We first show that the restriction of ψ_β to $\widehat{V}'_\beta$ is primitive, which means that there exists some $m \geq 1$ such that, for every $x \in V'_\beta$, $\psi_\beta^m(\widehat{x})$ contains all elements of $\widehat{V}'_\beta$. The proof is taken from [13], Proposition 8. If $\beta > 2$, then the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}$ grow until they cover two consecutive discontinuity points $\frac{1}{\beta+1} - \frac{a+1}{\beta}, \frac{1}{\beta+1} - \frac{a}{\beta}$ of $T_{-\beta}$, and the next image covers all intervals $J_{\widehat{y}}$, $y \in V'_\beta$. If $\frac{1+\sqrt{5}}{2} < \beta \leq 2$, then $\beta^2 > 2$ implies that the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}^2$ grow until they cover two consecutive discontinuity points of $T_{-\beta}^2$. Since

$$\begin{aligned}
T_{-\beta}^2\left(\left(\frac{-\beta}{\beta+1}, \frac{\beta^{-2}}{\beta+1} - \frac{1}{\beta}\right)\right) &= \left(\frac{-\beta^3+\beta^2+\beta}{\beta+1}, \frac{1}{\beta+1}\right),\\
T_{-\beta}^2\left(\left(\frac{\beta^{-2}}{\beta+1} - \frac{1}{\beta}, \frac{-\beta^{-1}}{\beta+1}\right)\right) &= \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right),\\
T_{-\beta}^2\left(\left(\frac{-\beta^{-1}}{\beta+1}, \frac{\beta^{-2}}{\beta+1}\right)\right) &= \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right),\\
T_{-\beta}^2\left(\left(\frac{\beta^{-2}}{\beta+1}, \frac{1}{\beta+1}\right)\right) &= \left(\frac{-\beta}{\beta+1}, \frac{\beta^2-\beta-1}{\beta+1}\right),
\end{aligned}$$

the next image covers the fixed point 0, and further images grow until after a finite number of steps they cover all intervals $J_{\widehat{y}}$, $y \in V'_\beta$. The case $\beta = \frac{1+\sqrt{5}}{2}$ is treated in Example 3.1.

If $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \neq 0$ for all $n \geq 0$, then $T_{-\beta}^n$ is continuous at all points $x \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ with $T_{-\beta}^n(x) = 0$, thus $u_{2k} = 0$ is equivalent to $u_{2k+1} = \widehat{0}$ (see also Prop. 3.2 below). Hence we can consider the return words of $\widehat{0}$ in $\cdots u_{-1}u_0u_1\cdots$ instead of the return words of 0. Since $\psi_\beta^m(\widehat{x_0}\, x_1\, \widehat{x_2})$ has at least two occurrences of $\widehat{0}$ for all $x_0, x_1, x_2 \in V'_\beta$, there are only finitely many such return words. If $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) = 0$, then $\psi_\beta^n(x_0\, \widehat{x_1}\, x_2)$ starts and ends with 0 for all $x_0, x_1, x_2 \in V'_\beta$, hence R_β is finite as well. □

For details on derived words of primitive substitutive words, we refer to [7].

We remark that, for practical reasons, the set R_β can be obtained from the set $R = \{w_\beta\}$ by adding to R iteratively all return words of 0 which appear in $\psi_\beta(w0)$ for some $w \in R$ until R stabilises. The final set R is equal to R_β.

Now, we apply the theorems in the case of two quadratic examples.

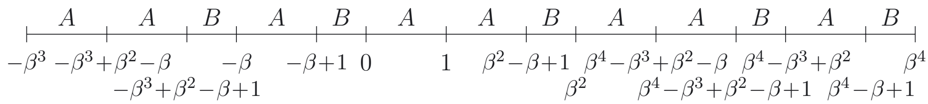

FIGURE 2. The $(-\beta)$-integers in $[-\beta^3, \beta^4]$, $\beta = (1+\sqrt{5})/2$.

Example 3.1. Let $\beta = \frac{1+\sqrt{5}}{2}$, *i.e.*, $\beta^2 = \beta + 1$, and $t = \frac{-\beta}{\beta+1} = \frac{-1}{\beta}$, then $V_\beta = V'_\beta = \{t, 0\}$. Since

$$J_{\widehat{t}} = (t, 0) = \left(t, \frac{-1}{\beta^3}\right) \cup \left\{\frac{-1}{\beta^3}\right\} \cup \left(\frac{-1}{\beta^3}, 0\right), \quad J_{\widehat{0}} = \left(0, \frac{1}{\beta^2}\right),$$

see Figure 1 (middle), the anti-morphism ψ_β on A_β^* is defined by

$$\psi_\beta : \quad t \mapsto 0, \quad \widehat{t} \mapsto \widehat{0}\, t\, \widehat{t}, \quad 0 \mapsto 0, \quad \widehat{0} \mapsto \widehat{t}.$$

Its two-sided fixed point $\cdots u_{-1} u_0 u_1 \cdots$ is

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}\, t\, \widehat{t}}_{\psi_\beta(\widehat{t}\,)} \underbrace{0}_{\psi_\beta(t)} \underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}\, t\, \widehat{t}}_{\psi_\beta(\widehat{t}\,)} \underbrace{0}_{\psi_\beta(t)} \underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})}$$
$$\underbrace{\dot{0}}_{\psi_\beta(\dot{0})} \underbrace{\widehat{0}\, t\, \widehat{t}}_{\psi_\beta(\widehat{t}\,)} \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}\, t\, \widehat{t}}_{\psi_\beta(\widehat{t}\,)} \underbrace{0}_{\psi_\beta(t)} \underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})} \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})} \underbrace{0}_{\psi_\beta(0)} \cdots,$$

where $\dot{0}$ marks the central letter u_0. The return word of 0 starting at u_0 is $w_\beta = 0\,\widehat{0}\,t\,\widehat{t}$. The image $\psi_\beta(w_\beta 0) = 0\ \widehat{0}\,t\,\widehat{t}\ 0\ \widehat{t}\ 0$ contains the return words w_β and $0\,\widehat{t}$. Since $\psi_\beta(0\,\widehat{t}\,0) = 0\ \widehat{0}\,t\,\widehat{t}\ 0$, there are no other return words of 0, *i.e.*, $R_\beta = \{A, B\}$ with $A = 0\,\widehat{0}\,t\,\widehat{t}$, $B = 0\,\widehat{t}$. Therefore, $\cdots u'_{-2} u'_{-1} u'_1 u'_2 \cdots$ is a two-sided fixed point of the anti-morphism

$$\varphi_{-\beta} : \quad A \mapsto AB, \quad B \mapsto A,$$

with

$$u'_{-13} \cdots u'_{-1}\ u'_1 \cdots u'_{21} = \\ AABAABABAABAB\ AABAABABAABAABABAABAB.$$

We have $\lambda(J_{\widehat{0}}) = \frac{1}{\beta^2}$, $\lambda(J_{\widehat{t}}) = \frac{1}{\beta}$, thus $L(A) = 1$, $L(B) = \frac{1}{\beta} = \beta - 1$, and some $(-\beta)$-integers are shown in Figure 2. Note that $(-\beta)^n$ can also be represented as $(-\beta)^{n+2} + (-\beta)^{n+1}$.

Example 3.2. Let $\beta = \frac{3+\sqrt{5}}{2}$, *i.e.*, $\beta^2 = 3\beta - 1$, then the $(-\beta)$-transformation is depicted in Figure 3, where $t_0 = \frac{-\beta}{\beta+1}$, $t_1 = T_{-\beta}(t_0) = \frac{\beta^2}{\beta+1} - 2 = \frac{-\beta^{-1}}{\beta+1}$,

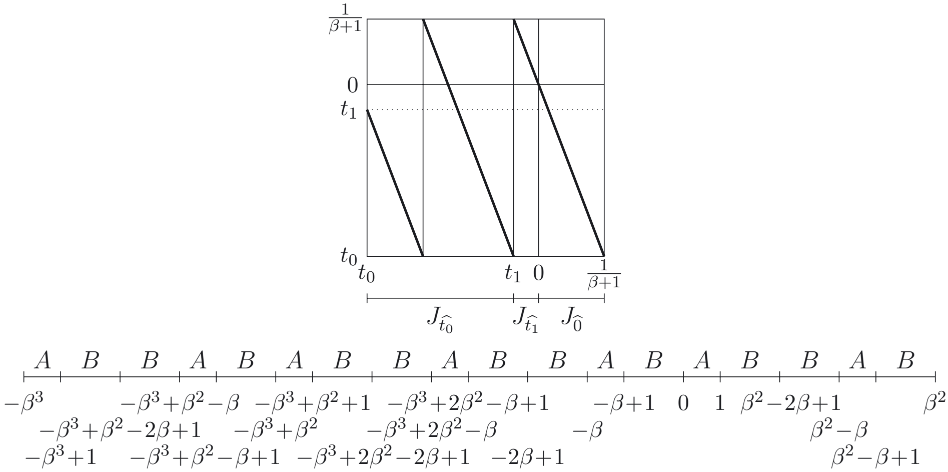

FIGURE 3. The $(-\beta)$-transformation and $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^2]$, $\beta = (3+\sqrt{5})/2$.

$T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0$. Therefore, $V'_\beta = \{t_0, t_1, 0\}$ and the anti-morphism ψ_β: $A_\beta^* \to A_\beta^*$ is defined by

$$\psi_\beta: \quad t_0 \mapsto t_1, \quad \widehat{t_0} \mapsto \widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}, \quad t_1 \mapsto t_0, \quad \widehat{t_1} \mapsto \widehat{0}, \quad 0 \mapsto 0, \quad \widehat{0} \mapsto \widehat{t_0}\, t_1\, \widehat{t_1},$$

which has the two-sided fixed point

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t_1})} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}}_{\psi_\beta(\widehat{t_0})} \underbrace{t_1}_{\psi_\beta(t_0)} \underbrace{\widehat{t_0}\, t_1\, \widehat{t_1}}_{\psi_\beta(\widehat{0})} \underbrace{\dot{0}}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t_1})} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}}_{\psi_\beta(\widehat{t_0})} \cdots,$$

where $\dot{0}$ marks the central letter u_0. We have $w_\beta = 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}$ and

$$\begin{aligned}
\psi_\beta: \quad & 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\,\widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0, \\
& 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\,\widehat{0}\, t_0\, \widehat{t_0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\,\widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0, \\
& 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\,\widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\,\widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0.
\end{aligned}$$

Note that $0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_0}\, t_1\, \widehat{t_1}$ and $0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}$ differ only by a letter in V'_β, and correspond therefore to intervals of the same length. Since the letters t_0 and t_1 are never mapped to 0, we identify these two return words. This means that $R_\beta = \{A, B\}$ with $A = 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}$, $B = 0\,\widehat{0}\, t_0\, \widehat{t_0}\, \{t_0, t_1\}\, \widehat{t_0}\, t_1\, \widehat{t_1}$, and

$$\cdots u'_{-2} u'_{-1}\ u'_1 u'_2 \cdots = \cdots ABBABABBABBAB\ ABBABABBABBAB \cdots$$

is a two-sided fixed point of the anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB, \quad B \mapsto ABB.$$

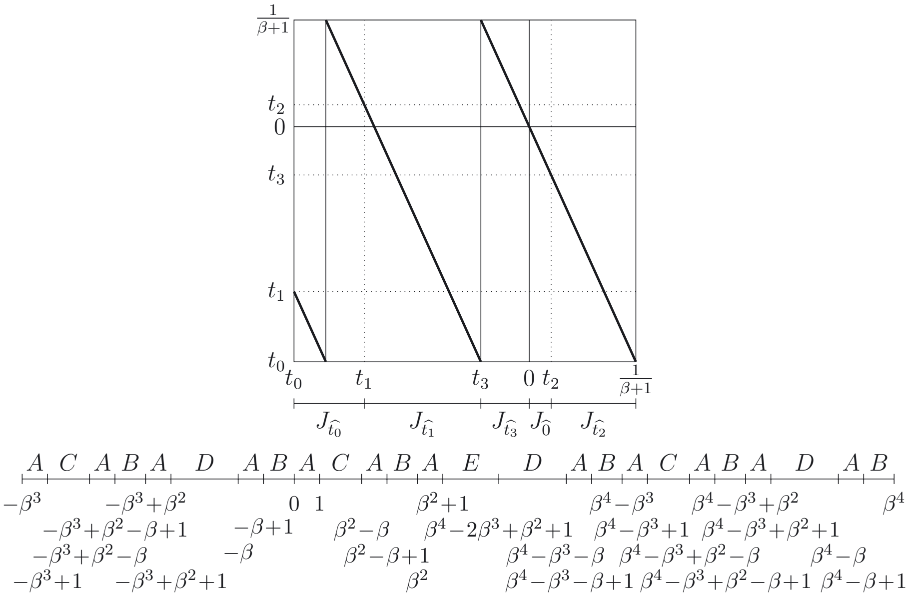

FIGURE 4. The $(-\beta)$-transformation and $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^4]$ from Example 3.3.

We have $L(A) = 1$, $L(B) = \beta - 1 > 1$, and some $(-\beta)$-integers are shown in Figure 3.

We remark that it is sufficient to consider the elements of $\widehat{V'_\beta}$ when one is only interested in $\mathbb{Z}_{-\beta}$. This is made precise in the following proposition.

Proposition 3.2. *Let β and $\cdots u_{-1}u_0u_1\cdots$ be as in Theorem 3.1, $t = \max\{x \in V_\beta \mid x < 0\}$. For any $k \in \mathbb{Z}$, $u_{2k} = 0$ is equivalent to $u_{2k-1} = \widehat{t}$ or $u_{2k+1} = \widehat{0}$.*

If $0 \notin V_\beta$ or the size of V_β is even, then $u_{2k} = 0$ is equivalent to $u_{2k-1} = \widehat{t}$.

If $0 \notin V_\beta$ or the size of V_β is odd, then $u_{2k} = 0$ is equivalent to $u_{2k+1} = \widehat{0}$.

Proof. Let $k \in \mathbb{Z}$ and $m \geq 0$ such that $z_k/\beta^{2m} \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. Then we have

- $u_{2k} = 0$ if and only if $T^{2m}_{-\beta}(z_k/\beta^{2m}) = 0$,
- $u_{2k-1} = \widehat{t}$ if and only if $\lim_{y\to z_k-} T^{2m}_{-\beta}(y/\beta^{2m}) = 0$,
- $u_{2k+1} = \widehat{0}$ if and only if $\lim_{y\to z_k+} T^{2m}_{-\beta}(y/\beta^{2m}) = 0$.

Thus $u_{2k} = 0$, $u_{2k-1} = \widehat{t}$ and $u_{2k+1} = \widehat{0}$ are equivalent when $T^{2m}_{-\beta}$ is continuous at z_k/β^{2m}. Assume from now on that z_k/β^{2m} is a discontinuity point of $T^{2m}_{-\beta}$. Then $T^{\ell}_{-\beta}(z_k/\beta^{2m}) = \frac{-\beta}{\beta+1}$ for some $1 \leq \ell \leq 2m$ and, if ℓ is minimal with this property,

$$\lim_{y\to z_k-} T^{2\lfloor \ell/2\rfloor+1}_{-\beta}(y/\beta^{2m}) = \frac{-\beta}{\beta+1} \quad \text{and} \quad \lim_{y\to z_k+} T^{2\lceil \ell/2\rceil}_{-\beta}(y/\beta^{2m}) = \frac{-\beta}{\beta+1}.$$

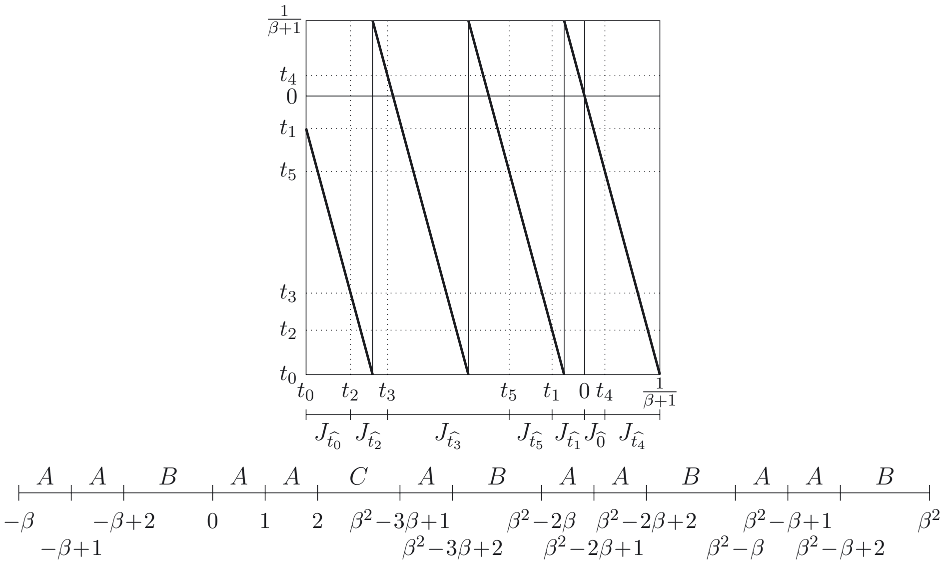

FIGURE 5. The $(-\beta)$-transformation and $\mathbb{Z}_{-\beta} \cap [-\beta, \beta^2]$ from Example 3.4.

Hence, if $0 \notin V_\beta$, we cannot have $u_{2k} = 0$, $u_{2k-1} = \widehat{t}$ or $u_{2k+1} = \widehat{0}$ at a discontinuity point, which proves the proposition in this case. If $0 \in V_\beta$, then $T_{-\beta}^{\#V_\beta - 1}\big(\frac{-\beta}{\beta+1}\big) = 0$, thus

- $T_{-\beta}^j(z_k/\beta^{2m}) = 0$ if and only if $j \geq \ell + \#V_\beta - 1$,
- $\lim_{y\to z_k-} T_{-\beta}^j(y/\beta^{2m}) = 0$ if and only if $j \geq 2\lfloor \ell/2 \rfloor + \#V_\beta$,
- $\lim_{y\to z_k+} T_{-\beta}^j(y/\beta^{2m}) = 0$ if and only if $j \geq 2\lceil \ell/2 \rceil + \#V_\beta - 1$.

Since $2\lfloor \ell/2 \rfloor \geq \ell - 1$ and $2\lceil \ell/2 \rceil \geq \ell$, we obtain $u_{2k} = 0$ whenever $u_{2k-1} = \widehat{t}$ or $u_{2k+1} = \widehat{0}$. If $\#V_\beta$ is even, then $u_{2k} = 0$ implies that $u_{2k-1} = \widehat{t}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2\lfloor \ell/2 \rfloor + \#V_\beta$. If $\#V_\beta$ is odd, then $u_{2k} = 0$ implies that $u_{2k+1} = \widehat{0}$ since $2m \geq \ell + \#V_\beta - 1$ implies that $2m \geq 2\lceil \ell/2 \rceil + \#V_\beta - 1$. This proves the proposition. □

By Proposition 3.2, it suffices to consider the anti-morphism $\widehat{\psi}_\beta : \widehat{V}_\beta'^* \to \widehat{V}_\beta'^*$ defined by

$$\widehat{\psi}_\beta(\widehat{x}) = \widehat{x_m} \cdots \widehat{x_1}\, \widehat{x_0} \quad \text{when} \quad \psi_\beta(\widehat{x}) = \widehat{x_m}\, T_{-\beta}(y_m) \cdots \widehat{x_1}\, T_{-\beta}(y_1)\, \widehat{x_0} \quad (x \in V_\beta').$$

Then $\Delta_{-\beta}$ is given by the set $\widehat{R}_\beta$ which consists of the return words of $\widehat{0}$ when $0 \notin V_\beta$ or the size of V_β is odd. When $0 \in V_\beta$ and the size of V_β is even, as in Example 3.1, then $\widehat{R}_\beta$ consists of the words $w\,\widehat{t}$ such that $\widehat{t}\,w$ is a return word of $\widehat{t}$.

Example 3.3. Let $\beta > 1$ with $\beta^3 = 2\beta^2 + 1$, *i.e.*, $\beta \approx 2.206$, and let $t_n = T^n_{-\beta}\big(\frac{-\beta}{\beta+1}\big)$ for $n \geq 0$. Then we have

$$t_0 = \frac{-\beta}{\beta+1}, \quad t_1 = \frac{\beta^2}{\beta+1} - 2 = \frac{\beta^{-1}-2}{\beta+1}, \quad t_2 = \frac{2\beta-1}{\beta+1} - 1 = \frac{\beta^{-2}}{\beta+1},$$
$$t_3 = \frac{-\beta^{-1}}{\beta+1}, \quad t_4 = \frac{1}{\beta+1} - 1 = t_0,$$

see Figure 4. The anti-morphism $\widehat{\psi}_\beta : \widehat{V}_\beta'^* \to \widehat{V}_\beta'^*$ is therefore defined by

$$\widehat{\psi}_\beta: \quad \widehat{t_0} \mapsto \widehat{t_2}\,\widehat{t_0}, \quad \widehat{t_1} \mapsto \widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\,\widehat{0}, \quad \widehat{t_3} \mapsto \widehat{0}\,\widehat{t_2}, \quad \widehat{0} \mapsto \widehat{t_3}, \quad \widehat{t_2} \mapsto \widehat{t_0}\,\widehat{t_1}.$$

Since $0 \notin V_\beta$, we consider return words of $\widehat{0}$ in the $\widehat{\psi}_\beta$-images of $\widehat{w}_\beta = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$:

$$\begin{aligned}
\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3} &\mapsto \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3},\\
\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3} &\mapsto \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\,,\\
\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3} &\mapsto \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\,,\\
\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3} &\mapsto \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\\
&\quad \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\,,\\
\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3} &\mapsto \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}\;\widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}.
\end{aligned}$$

Hence $\widehat{R}_\beta = \{A, B, C, D, E\}$ with $A = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$, $B = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$, $C = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$, $D = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$, $E = \widehat{0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_0}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_0}\,\widehat{t_1}\,\widehat{t_3}$, and $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\widehat{\varphi}_{-\beta} : \widehat{R}_\beta^* \to \widehat{R}_\beta^*$ given by

$$\widehat{\varphi}_{-\beta}: \quad A \mapsto AB\,, \quad B \mapsto AC\,, \quad C \mapsto AD\,, \quad D \mapsto AED\,, \quad E \mapsto ABD.$$

The $(-\beta)$-integers in $[-\beta^3, \beta^4]$ are represented in Figure 4, and we have

$$L(A) = 1, \quad L(B) = \beta - 1, \quad L(C) = \beta^2 - \beta - 1,$$
$$L(D) = \beta^2 - \beta \approx 2.659, \quad L(E) = \beta.$$

Note that $L(D) > \beta > 2$. Moreover, the cardinality of $\Delta_{-\beta}$ is larger than that of V_β, which in turn is larger than the algebraic degree d of β ($\#\Delta_{-\beta} = 5$, $\#V_\beta = 4$, $d = 3$).

Example 3.4. Let $\beta > 1$ with $\beta^6 = 3\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 - 2\beta - 1$, *i.e.*, $\beta \approx 3.695$, then the $(-\beta)$-transformation is depicted in Figure 5, where $t_n = T^n_{-\beta}\big(\frac{-\beta}{\beta+1}\big)$. We have $t_5 = \frac{-1}{\beta+1} = t_6$. The anti-morphism $\widehat{\psi}_\beta : \widehat{V}_\beta'^* \to \widehat{V}_\beta'^*$ is therefore given by

$$\begin{aligned}
\widehat{\psi}_\beta: \quad &\widehat{t_0} \mapsto \widehat{t_3}\,\widehat{t_5}\,, \quad &&\widehat{t_2} \mapsto \widehat{t_4}\,\widehat{t_0}\,\widehat{t_2}\,, \quad &&\widehat{t_3} \mapsto \widehat{t_5}\,\widehat{t_1}\,\widehat{0}\,\widehat{t_4}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1}\,\widehat{0}\,,\\
&\widehat{t_5} \mapsto \widehat{t_2}\,\widehat{t_3}\,, &&\widehat{t_1} \mapsto \widehat{0}\,\widehat{t_4}\,\widehat{t_0}\,, &&\widehat{0} \mapsto \widehat{t_5}\,\widehat{t_1}\,, \qquad \widehat{t_4} \mapsto \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}.
\end{aligned}$$

In order to deal with shorter words, we group the letters forming the words

$$a = \widehat{0}\,\widehat{t_4},\ b = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1},\ c = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5},\ d = \widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1},$$
$$e = \widehat{t_0}\,\widehat{t_2},\ f = \widehat{t_4},\ g = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3},\ h = \widehat{t_5}\,\widehat{t_1},$$

which correspond to the intervals $J_a = \big(0, \frac{1}{\beta+1}\big)$, $J_b = (t_0, 0)$, $J_c = (t_0, t_1)$, $J_d = (t_2, 0)$, $J_e = (t_0, t_3)$, $J_f = \big(t_4, \frac{1}{\beta+1}\big)$, $J_g = (t_0, t_5)$, $J_h = (t_5, 0)$, occurring in iterated images of J_a. The anti-morphism $\widehat{\psi}_\beta$ acts on these words by

$$\begin{aligned} \widehat{\psi}_\beta :\quad & a \mapsto b\,, && b \mapsto ababac\,, && c \mapsto dabac\,, && d \mapsto ababae\,, \\ & e \mapsto fc\,, && f \mapsto g\,, && g \mapsto habac\,, && h \mapsto ag. \end{aligned}$$

Since $\widehat{0}$ only occurs at the beginning of a, the return words of $\widehat{0}$ with their $\widehat{\psi}_\beta$-images are

$$\begin{aligned} ab &\mapsto ab\ ab\ acb\,, & aed &\mapsto ab\ ab\ aefcb\,, \\ acb &\mapsto ab\ ab\ acd\ ab\ acb\,, & aefcb &\mapsto ab\ ab\ acd\ ab\ acgfcb\,, \\ acd &\mapsto ab\ ab\ aed\ ab\ acb\,, & acgfcb &\mapsto ab\ ab\ acd\ ab\ \underbrace{acgh}_{=acb}\ ab\ acd\ ab\ acb. \end{aligned}$$

Therefore, $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\widehat{\varphi}_{-\beta} :\ \widehat{R}_\beta^* \to \widehat{R}_\beta^*$ which is defined by

$$\begin{aligned} \widehat{\varphi}_{-\beta} :\quad & A \mapsto AAB\,, && L(A) = 1\,, \\ & B \mapsto AACAB\,, && L(B) = \beta - 2 \approx 1.695\,, \\ & C \mapsto AADAB\,, && L(C) = \beta^2 - 3\beta - 1 \approx 1.569\,, \\ & D \mapsto AAE\,, && L(D) = \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104\,, \\ & E \mapsto AACAF\,, && L(E) = \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081\,, \\ & F \mapsto AACABACAB\,, && L(F) = \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12. \end{aligned}$$

4. Conclusions

With every Yrrap number $\beta \geq (1+\sqrt{5})/2$, we have associated an anti-morphism $\varphi_{-\beta}$ on a finite alphabet. The distances between consecutive $(-\beta)$-integers are described by a fixed point of $\varphi_{-\beta}$. In [1], the anti-morphism is described explicitely for each $\beta > 1$ such that $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \leq 0$ and $T_{-\beta}^{2n-1}\big(\frac{-\beta}{\beta+1}\big) \geq \frac{1-\lfloor\beta\rfloor}{\beta}$ for all $n \geq 1$. Examples 3.3 and 3.4 show that the situation can be quite complicated when this condition is not fulfilled. Although $\varphi_{-\beta}$ can be obtained by a simple algorithm, it seems to be difficult to find a priori bounds for the number of different distances between consecutive $(-\beta)$-integers or for their maximal value. Only the case of quadratic Pisot numbers β is completely solved; here, we know from [1, 14] that $\#V_\beta = \#\Delta_{-\beta} = 2$.

Recall that the maximal distance between consecutive β-integers is 1, and the number of different distances is equal to the cardinality of the set $\{T_\beta^n(1^-) \mid n \geq 0\}$. Example 3.3 shows that the $(-\beta)$-integers do not satisfy similar properties. By generalising Example 3.4 to $\beta > 1$ with $\beta^6 = (m+1)\beta^5 + m\beta^4 + m\beta^3 + \beta^2 - m\beta - 1$, $m \geq 2$, one sees that the maximal distance can be arbitrarily close to 4 for algebraic integers of degree 6 and $\#V_\beta = 6$.

In a forthcoming paper, we associate anti-morphisms $\varphi_{-\beta}$ on infinite alphabets with non-Yrrap numbers β, by considering the intervals occurring in the iterated $T_{-\beta}$-images of $\big(0, \frac{1}{\beta+1}\big)$, *cf.* Example 3.4, and we show that the distances between consecutive $(-\beta)$-integers can be unbounded, *e.g.* for $\beta > 1$ satisfying $\frac{-\beta}{\beta+1} = \sum_{k=1}^{\infty} a_k(-\beta)^{-k}$ where $a_1a_2\cdots = 31232\,1\,2\,31232\,2\cdots$ is a fixed point of the morphism $3 \mapsto 31232,\ 2 \mapsto 2,\ 1 \mapsto 1$. For Yrrap numbers β, this implies that there is no bound for the distance between consecutive $(-\beta)$-integers which is independent of β. However, large distances occur probably only far away from 0 and when $\#V_\beta$ is large, and it would be interesting to quantify these relations.

Another topic that is worth investigating is the structure of the sets $S_{-\beta}(x)$ for $x \neq 0$, and of the corresponding tilings when β is a Pisot unit. A related question is whether $\mathbb{Z}_{-\beta}$ can be given by a cut and project scheme, *cf.* [5,12].

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