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INFINITELY MANY SOLUTIONS FOR ASYMPTOTICALLY LINEAR PERIODIC HAMILTONIAN ELLIPTIC SYSTEMS *

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Abstract. This paper is concerned with the following periodic Hamiltonian elliptic system

$$\begin{cases} -\Delta\varphi + V(x)\varphi = G_\psi(x,\varphi,\psi) \text{ in } \mathbb{R}^N, \\ -\Delta\psi + V(x)\psi = G_\varphi(x,\varphi,\psi) \text{ in } \mathbb{R}^N, \\ \varphi(x) \to 0 \text{ and } \psi(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

Assuming the potential V is periodic and 0 lies in a gap of $\sigma(-\Delta + V)$, $G(x,\eta)$ is periodic in x and asymptotically quadratic in $\eta = (\varphi,\psi)$, existence and multiplicity of solutions are obtained *via* variational approach.

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1. Introduction and main results

Consider the following Hamiltonian elliptic system

$$\begin{cases} -\Delta\varphi + V(x)\varphi = G_\psi(x,\varphi,\psi) \text{ in } \mathbb{R}^N, \\ -\Delta\psi + V(x)\psi = G_\varphi(x,\varphi,\psi) \text{ in } \mathbb{R}^N, \\ \varphi(x) \to 0 \text{ and } \psi(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{ES}$$

where $N \ge 1$, $\varphi,\psi : \mathbb{R}^N \to \mathbb{R}$, $V \in C(\mathbb{R}^N,\mathbb{R})$ and $G \in C^1(\mathbb{R}^N \times \mathbb{R}^2,\mathbb{R})$.

For the case of a bounded domain, assuming $V \equiv 0$, there are a number of papers concerned with the systems like or similar to (ES). For example, see Benci and Rabinowitz [8], De Figueiredo and Ding [11], De Figueiredo and Felmer [12], Hulshof and Van de Vorst [17] and their references for superlinear systems and earlier works, see Kryszewski and Szulkin [18] and the references therein for asymptotically linear systems, see Pistoia and Ramos [22] and their references for a singularly perturbed problem. Recently, De Figueiredo *et al.* [14] treated the system with $G(x,\varphi,\psi) = F(\varphi)+H(\psi)$, and a nontrivial solution was obtained *via* an Orlitz space approach.

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There are several authors who considered the systems on the whole space $\mathbb{R}^N$. But most of them focused on the case $V \equiv 1$, which is not only radial but also periodic. The main difficulty of such type of problems is the lack of the compactness of the Sobolev embedding. An usual way to overcome the difficulty is imposing a radial symmetry assumption on the nonlinearities and working on the radially symmetric function space, which possesses a compact embedding. By this means, De Figueiredo and Yang [13] obtained a positive radially symmetric solution which decays exponentially to 0 at infinity. Their results were generalized by Sirakov [25] in a different way. Later, Bartsch and De Figueiredo [6] proved that the system admits infinitely many radial as well as non-radial solutions if G is even in z. Li and Yang [21] proved, *via* a generalized linking theorem, that (ES) has a positive ground state solution for $V \equiv 1$ and an asymptotically quadratic nonlinearity $G(x,\varphi,\psi) = F(\varphi)+H(\psi)$, and based on this result they obtained a positive solution for $G(x,\varphi,\psi) = \int_0^{\varphi} f(x,t)\mathrm{d}t + \int_0^{\psi}(x,s)\mathrm{d}s$ if $f(x,\varphi)$ and $g(x,\psi)$ have autonomous limits $\bar{f}(\varphi)$ and $\bar{h}(\psi)$ at infinity. Very recently, Ding and Lin [16] considered semiclassical problems for systems of Schrödinger equations with subcritical and critical nonlinearities.

Another usual way is avoiding the indefinite character of the original functional by using the dual variational method, see for instance Ávila and Yang [4,5], Alves *et al.* [3], Yang [28] and the references therein.

In this paper, we consider a periodic asymptotically linear elliptic system with 0 lying in a gap of $\sigma(S)$, where $S := -\Delta + V$ is the Schrödinger operator, and $\sigma(S)$ denotes the spectrum of the operator S. We must face two kinds of indefiniteness: one comes from the system itself and the other comes from each equation in the system. As to our knowledge, there is no multiplicity result for the asymptotically linear systems. The purpose of this paper is to obtain the existence and multiplicity of solutions. Since there is no radial assumption, we have to work on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Thanks to the periodic assumption, we can prove a weak version of the Cerami condition similar to Coti-Zelati and Rabinowitz [9,10]. By establishing a proper variational framework, we can obtain the multiplicity results *via* the critical point theory of strongly indefinite functional, which were developed recently by Bartsch and Ding [7].

More precisely, for the potential V, we assume

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each x_i for $i = 1, \ldots, N$.

It is well known that, under (V_0), S is semibounded from below and the spectrum $\sigma(S) = \sigma_{\mathrm{cont}}(S)$ is a union of closed intervals (see Reed and Simon [23]), where $\sigma_{\mathrm{cont}}(S)$ denotes the continuous spectrum of the operator S. The relationship between 0 and $\sigma(S)$ is important to our approach. So we assume in addition that

(V_1) 0 lies in a gap of $\sigma(S)$.

By (V_1), there holds

$$\bar{\Lambda} := \sup[\sigma(S) \cap (-\infty, 0)] < 0 < \underline{\Lambda} := \inf[\sigma(S) \cap (0, \infty)].$$

In what follows, we use the notation $\eta =: (\varphi, \psi)$. For the nonlinearity, we assume

- (G_0) $G \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0,\infty))$ is 1-periodic in each x_i for $i = 1, \ldots, N$;
- (G_1) $G(x,\eta) = o(|\eta|^2)$ as $|\eta| \to 0$.
- (G_2) $|G_\eta(x,\eta) - G_\infty(x)\eta|/|\eta| \to 0$ as $|\eta| \to \infty$, where $G_\infty \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each x_i for $i = 1, \ldots, N$;
- (G_3) $G_0 := \inf_{x\in\mathbb{R}^N} G_\infty(x) > \Lambda$, where $\Lambda := \max\{\underline{\Lambda}, -\bar{\Lambda}\}$;
- (G_4) $\hat{G}(x,\eta) > 0$ if $\eta \neq 0$, and $\hat{G}(x,\eta) \to \infty$ as $|\eta| \to \infty$, where $\hat{G}(x,\eta) = \frac{1}{2}G_\eta(x,\eta)\eta - G(x,\eta)$;
- (G_4') there exists $\delta_0 \in (0, \Lambda_0)$ such that $\hat{G}(x,\eta) \geq \delta_0$ whenever $|G_\eta(x,\eta)| \geq (\Lambda_0 - \delta_0)|\eta|$, where $\Lambda_0 := \min\{-\bar{\Lambda}, \underline{\Lambda}\}$;
- (G_5) $\hat{G}(x,\eta) \geq 0$, and there exists $\delta_1 > 0$ such that $\hat{G}(x,\eta) > 0$ if $0 < |\eta| \leq \delta_1$.

Remark 1.1. There are some functions which satisfy (G_0)–(G_5). For example, $G(x,\eta) = a(x)|\eta|^2(1 - \frac{1}{\ln(e+|\eta|)})$, where $\inf a(x) > \Lambda$ and is 1-periodic in each x_i for $i = 1, \ldots, N$.

Observe that, due to the periodicity of V, G, if (φ,ψ) is a solution of (ES), then so is $(a * \varphi, b * \psi)$ for each $a, b \in \mathbb{Z}^N$, where $(a * \varphi)(x) = \varphi(x + a)$ and $(b * \psi)(x) = \psi(x + b)$. Two solutions (φ_1, ψ_1) and (φ_2, ψ_2) are said to be geometrically distinct if $a * \varphi_1 \neq \varphi_2$ and $b * \psi_1 \neq \psi_2$ for all $a, b \in \mathbb{Z}^N$. Our main result is the following:

Theorem 1.1. *Let (V_0)–(V_1), (G_0)–(G_3), (G_5) and (G_4) or (G_4') be satisfied. Then (ES) has a least energy solution. If additionally $G(x, \eta)$ is even in η then (ES) has infinitely many geometrically distinct solutions.*

The paper is organized as follows. In Section 2, we set up the framework in which we study the variational problem associated to (ES). The linking structure of the functional will be discussed in Section 3. Some properties of $(C)_c$ sequences will be showed in Section 4. The proof of Theorem 1.1 announced above will be given in the last section.

2. Variational setting

Below by $|\cdot|_q$ we denote the usual L^q-norm, c or c_i stand for different positive constants. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, we always choose the equivalent norm $\|(x,y)\|_{X\times Y} = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$ on the product space $X \times Y$. In particular, if X and Y are two Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, we choose the inner product $((x,y),(w,z))_{X\times Y} = (x,w)_X + (y,z)_Y$ on the product space $X \times Y$.

Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of S. Assumption (V_1) implies an orthogonal decomposition:

$$L^2 := L^2(\mathbb{R}^N, \mathbb{R}) = L^+ \oplus L^-, \quad z = z^- + z^+,$$

where $L^- = E_0 L^2$ and $L^+ = (Id - E_0)L^2$. Denoting by $|S|$ the absolute value of S and its square root operator is

$$|S|^{1/2} = \int_{-\infty}^{\infty} |\lambda| \mathrm{d}E(\lambda) : \mathcal{D}(|S|^{1/2}) \to L^2,$$

where

$$\mathcal{D}(|S|^{1/2}) = \left\{ u \in L^2 | \int_{-\infty}^{\infty} |\lambda| \mathrm{d}(E(\lambda)u, u)_{L^2} < \infty \right\}.$$

Let $H = \mathcal{D}(|S|^{1/2})$ be the Hilbert space with the inner product

$$(u,v)_H = (|S|^{1/2}u, |S|^{1/2}v)_{L^2}$$

and the corresponding norm $\|u\|_H = (u,u)_H^{1/2}$. There is an induced decomposition

$$H = H^- \oplus H^+, \quad H^\pm = H \cap L^\pm,$$

which is orthogonal with respect to the inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_H$. Then for any $u \in H$, $u = u^+ + u^-$, $u^\pm \in H^\pm$, there holds

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 = \|u^+\|_H^2 - \|u^-\|_H^2. \tag{2.1}$$

Let $E = H \times H$ with the inner product

$$((u,v),(\varphi,\phi)) = (u,\varphi)_H + (v,\psi)_H$$

and the corresponding norm

$$\|(u,v)\| = [\|u\|_H^2 + \|v\|_H^2]^{1/2}.$$

Recall that $E \hookrightarrow L^p(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for $p \in [2, 2^*]$ and $E \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for $p \in [2, 2^*)$, where 2^* is the Sobolev critical exponent. On E we define the following functional

$$I(\eta) = I(\varphi, \psi) = \int_{\mathbb{R}^N} \nabla\varphi\nabla\psi + V(x)\varphi\psi - \int_{\mathbb{R}^N} G(x, \eta),$$

for $\eta = (\varphi, \psi) \in E$. Our hypotheses imply that $I \in C^1(E)$ and a standard argument shows that its critical points are weak solutions of (ES).

Setting

$$E^+ = H^+ \times H^-, \quad E^- = H^- \times H^+,$$

then for any $z = (u, v) \in E$, we have

$$z = z^+ + z^-, \text{ where } z^+ = (u^+, v^-),\ z^- = (u^-, v^+).$$

Clearly, E^+ and E^- are orthogonal with respect to the inner products $(\cdot,\cdot)_{L^2\times L^2}$ and $(\cdot,\cdot)$. Hence $E = E^+ \oplus E^-$. Now we introduce a change of variable

$$\begin{cases} \varphi = \dfrac{u+v}{\sqrt{2}}, \\ \psi = \dfrac{u-v}{\sqrt{2}}, \end{cases} \tag{2.2}$$

and set $H(x,z) = H(x,u,v) := G(x, \frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$, where here and in what follows we write $z = (u,v)$ and $|z| = (|u|^2 + |v|^2)^{1/2}$.

The assumptions on G imply that H satisfies

- (H_0) $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0,\infty))$ is 1-periodic in each x_i for $i = 1, \ldots, N$;
- (H_1) $H(x,z) = o(|z|^2)$ as $|z| \to 0$.
- (H_2) $|H_z(x,z) - G_\infty(x)z|/|z| \to 0$ as $|z| \to \infty$, where G_∞ is given in (G_2);
- (H_3) $G_0 := \inf\limits_{x\in\mathbb{R}^N} G_\infty(x) > \Lambda$;
- (H_4) $\hat{H}(x,z) > 0$ if $z \neq 0$, and $\hat{H}(x,z) \to \infty$ as $|z| \to \infty$;
- (H_4') there exists $\delta_0 \in (0, \Lambda_0)$ such that $\hat{H}(x,z) \geq \delta_0$ whenever $|H_z(x,z)| \geq (\Lambda_0 - \delta_0)|z|$;
- (H_5) $\hat{H}(x,z) \geq 0$, and there exists $\delta_1 > 0$ such that $\hat{H}(x,z) > 0$ if $0 < |z| \leq \delta_1$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\varphi\nabla\psi + V(x)\varphi\psi &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2 - |\nabla v|^2 - V(x)v^2) \\ &= \frac{1}{2}(\|u^+\|_H^2 - \|u^-\|_H^2 - \|v^+\|_H^2 + \|v^-\|_H^2) \\ &= \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2).\end{aligned}$$

Thus, we have an equivalent functional

$$\Phi(z) = \Phi(u,v) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z), \tag{2.3}$$

where $\Psi(z) = \int_{\mathbb{R}^N} H(x,z)$. It is obvious that $z = (u,v)$ is a critical point of Φ if and only if $((u+v)/\sqrt{2}, (u-v)/\sqrt{2})$ is a critical point of I. In what follows, we shall seek for the critical points of Φ under the assumptions on H. The functional Φ is strongly indefinite; such type of functionals have appeared extensively in the study of differential equations *via* critical point theory, see for example [19,26,27] and the references therein.

3. Linking structure

In this section, we discuss the linking structure of Φ.

Lemma 3.1. *Suppose* (H_0)–(H_2) *are satisfied. Then there is a* $\rho > 0$ *such that* $\kappa := \inf \Phi(\partial B_\rho \cap E^+) > 0$.

Proof. Observe that, given $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|H_z(x,z)| \le \varepsilon|z| + C_\varepsilon|z|^{p-1} \tag{3.1}$$

and

$$|H(x,z)| \le \varepsilon|z|^2 + C_\varepsilon|z|^p \tag{3.2}$$

for all (x,z), where $p > 2$. Now, the conclusion follows in a standard way. □

By (H_3), we can take a number γ such that

$$\Lambda < \gamma < G_0. \tag{3.3}$$

Since $\sigma(S)$ is absolutely continuous, the subspace $Y_1 := (E_\gamma - E_0)L^2$ and $Y_2 := (E_0 - E_{-\gamma})L^2$ are infinite dimensional subspaces of H^+ and H^-, respectively. Recall that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of S. Then

$$\Lambda|u|_2^2 \le \|u\|_{E_0}^2 \le \gamma|u|_2^2 \text{ for all } u \in Y_1,$$

$$\Lambda|v|_2^2 \le \|v\|_{E_0}^2 \le \gamma|v|_2^2 \text{ for all } v \in Y_2.$$

Set $W_0 := Y_1 \times Y_2$, then W_0 is an infinite dimensional subspace of E^+ and

$$\Lambda|w|_2^2 \le \|w\|_{E_0}^2 \le \gamma|w|_2^2 \text{ for all } w \in W_0. \tag{3.4}$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences so that

$$\underline{\Lambda} = \alpha_0 < \alpha_1 < \alpha_2 < \ldots \le \gamma,$$

$$\bar{\Lambda} = \beta_0 > \beta_1 > \beta_2 > \ldots \ge \max\{-\gamma, \inf\sigma(S)\}.$$

For each $n \in \mathbb{N}$, take an element $e_n \in (E_{\alpha_n} - E_{\alpha_{n-1}})L^2$ with $\|e_n\| = 1$ and define $Y_1^n := \operatorname{span}\{e_1, \ldots, e_n\}$. Similarly, take $f_n \in (E_{\beta_{n-1}} - E_{\beta_n})L^2$ with $\|f_n\| = 1$ and define $Y_2^n := \operatorname{span}\{f_1, \ldots, f_n\}$. Then $W_n := Y_1^n \times Y_2^n$ is a increasing sequence of finite dimensional subspaces of E^+. For any subspace W_n of W_0 set $E_n = E^- \oplus W_n$.

Lemma 3.2. *Let (H_0) and (H_2)–(H_3) be satisfied and $\rho > 0$ be given by Lemma 3.1. Then $\sup\Phi(E_n) < \infty$, and there is a sequence $R_n > 0$ such that $\sup\Phi(E_n\backslash B_n) < \inf\Phi(B_\rho)$, where $B_n := \{z \in E_n : \|z\| \le R_n\}$.*

Proof. It is sufficient to prove that $\Phi(z) \to -\infty$ in E_n as $\|z\| \to \infty$. If not, then there are $M > 0$ and $\{z_j\} \subset E_n$ with $\|z_j\| \to \infty$ such that $\Phi(z_j) \ge -M$ for all j. Denote $y_j := z_j/\|z_j\|$, passing to a subsequence if necessary, $y_j \rightharpoonup y$, $y_j^- \rightharpoonup y^-$ and $y_j^+ \to y^+$. Since $\Psi(z) \ge 0$,

$$\begin{aligned}\frac{1}{2}(\|y_j^+\|^2 - \|y_j^-\|^2) &\ge \frac{1}{2}(\|y_j^+\|^2 - \|y_j^-\|^2) - \frac{\Psi(z_j)}{\|z_j\|^2} \\ &= \frac{\Phi(z_j)}{\|z_j\|^2} \ge \frac{-M}{\|z_j\|^2},\end{aligned} \tag{3.5}$$

from where it follows that

$$\frac{1}{2}\|y_j^-\|^2 \le \frac{1}{2}\|y_j^+\|^2 + \frac{M}{\|z_j\|^2}. \tag{3.6}$$

We claim that $y^+ \ne 0$. Indeed, if not, (3.6) yields that $\|y_j^-\| \to 0$. Thus $\|y_j\| \to 0$, which contradicts with $\|y_j\| = 1$. Since

$$\begin{aligned}\|y^+\|^2 - \|y^-\|^2 - \int_{\mathbb{R}^N} G_\infty(x)|y|^2 &\le \|y^+\|^2 - \|y^-\|^2 - G_0|y|_2^2 \\ &\le -(G_0 - \gamma)|y^+|_2^2 < 0,\end{aligned}$$

then there exists $\Omega \subset \mathbb{R}^N$ such that

$$\|y^+\|^2 - \|y^-\|^2 - \int_\Omega G_\infty(x)|y|^2 < 0. \tag{3.7}$$

Setting $R(x,z) := H(x,z) - \frac{1}{2}G_\infty(x)|z|^2$, then $|R(x,z)| \le c|z|^2$ for some $c > 0$, $R(x,z)/|z|^2 \to 0$ as $|z| \to \infty$ uniformly in x. By Lebesgue's dominated convergence theorem we have

$$\lim_{j\to\infty} \int_\Omega \frac{|R(x,z_j)|}{\|z_j\|^2} = \lim_{j\to\infty} \int_\Omega \frac{|R(x,z_j)|}{|z_j|^2}|y_j|^2 = 0. \tag{3.8}$$

Thus (3.5), (3.7)–(3.8) imply that

$$\begin{aligned}
0 &\le \lim_{j\to\infty} \left[\frac{1}{2}(\|y_j^+\|^2 - \|y_j^-\|^2) - \int_\Omega \frac{H(x,z_j)}{\|z_j\|^2}\right] \\
&= \lim_{j\to\infty} \left[\frac{1}{2}(\|y_j^+\|^2 - \|y_j^-\|^2) - \frac{1}{2}\int_\Omega G_\infty(x)|y_j|^2 - \int_\Omega \frac{R(x,z_j)}{\|z_j\|^2}\right] \\
&\le \frac{1}{2}(\|y^+\|^2 - \|y^-\|^2 - \int_\Omega G_\infty(x)|y|^2) < 0.
\end{aligned}$$

Now the desired conclusion follows from this contradiction. □

As a consequence, we have:

Lemma 3.3. *Let (H_0) and (H_2)–(H_3) be satisfied and $\kappa > 0$ be given by Lemma 3.1. Then letting $e \in W_0$ with $\|e\| = 1$, there is $R_1 > \rho$ such that $\Phi|_{\partial Q} \le \kappa$, where $Q := \{z = z^- + se : z^- \in E^-, s \ge 0, \|z\| \le R_1\}$.*

4. The $(C)_c$-sequence

Lemma 4.1. *Suppose that (H_0)–(H_2) and (H_4) or (H_4') are satisfied. Then any $(C)_c$-sequence of Φ is bounded.*

Proof. Let $\{z_j\}$ be such that $\Phi(z_j) \to c$ and $(1 + \|z_j\|)\Phi'(z_j) \to 0$. Suppose to the contrary that $\{z_j\}$ is unbounded. Setting $y_j := z_j/\|z_j\|$, then $\|y_j\| = 1$. Without loss of generality, we can assume that $y_j \rightharpoonup y$ in E. Observe that for j large

$$C \ge \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}^N} \hat{H}(x,z_j), \tag{4.1}$$

and

$$\begin{aligned}
\Phi'(z_j)(z_j^+ - z_j^-) &= \|z_j\|^2 - \int_{\mathbb{R}^N} H_z(x,z_j)(z_j^+ - z_j^-) \\
&= \|z_j\|^2 \left[1 - \int_{\mathbb{R}^N} \frac{H_z(x,z_j)(y_j^+ - y_j^-)}{\|z_j\|}\right].
\end{aligned} \tag{4.2}$$

Suppose that (H_4) holds. Set for $r \ge 0$,

$$g(r) := \inf\left\{\hat{H}(x,z) | x \in \mathbb{R}^N \text{ and } z \in \mathbb{R}^2 \text{ with } |z| \ge r\right\}.$$

By (H_4), $g(r) \to \infty$ as $r \to \infty$.

For $0 \le a < b$, let

$$\Omega_j(a,b) = \{x \in \mathbb{R}^N | a \le |z_j(x)| < b\}$$

and

$$C_a^b = \inf\left\{\frac{\hat{H}(x,z)}{|z|^2} | x \in \mathbb{R}^N \text{ with } a \le |z(x)| \le b\right\}.$$

One has

$$\hat{H}(x, z_j(x)) \geq C_a^b |z_j(x)|^2 \text{ for all } x \in \Omega_j(a,b).$$

It follows from (4.1) that

$$\begin{aligned} C &\geq \int_{\Omega_j(0,a)} \hat{H}(x,z_j) + \int_{\Omega_j(a,b)} \hat{H}(x,z_j) + \int_{\Omega_j(b,\infty)} \hat{H}(x,z_j) \\ &\geq \int_{\Omega_j(0,a)} \hat{H}(x,z_j) + C_a^b \int_{\Omega_j(a,b)} |z_j|^2 + g(b)|\Omega_j(b,\infty)|. \end{aligned} \tag{4.3}$$

Using (4.3) one has

$$|\Omega_j(b,\infty)| \leq \frac{C}{g(b)} \to 0 \tag{4.4}$$

as $b \to \infty$ uniformly in x, and for any fixed $0 < a < b$,

$$\int_{\Omega_j(a,b)} |y_j|^2 = \frac{1}{\|z_j\|^2} \int_{\Omega_j(a,b)} |z_j|^2 \leq \frac{C}{C_a^b \|z_j\|^2} \to 0 \tag{4.5}$$

as $j \to \infty$. It follows from (4.4) that, for any $s \in [2, 2^*)$

$$\int_{\Omega_j(b,\infty)} |y_j|^s \leq \left(\int_{\Omega_j(b,\infty)} |y_j|^{2^*} \right)^{\frac{s}{2^*}} |\Omega_j(b,\infty)|^{\frac{2^*-s}{2^*}} \leq c|\Omega_j(b,\infty)|^{\frac{2^*-s}{2^*}} \to 0 \tag{4.6}$$

as $b \to \infty$ uniformly in j. In virtue of (4.5), for any $s \in [2, 2^*)$, there holds

$$\int_{\Omega_j(a,b)} |y_j|^s \leq c \left(\int_{\Omega_j(a,b)} |y_j|^2 \right)^{(2^*-s)/(2^*-2)} \to 0 \ as\ j \to \infty. \tag{4.7}$$

From (4.2),

$$\int_{\mathbb{R}^N} \frac{H_z(x,z_j)(y_j^+ - y_j^-)|y_j|}{|z_j|} \to 1. \tag{4.8}$$

Let $0 < \varepsilon < 1/3$. By (H_1) there is $a_\varepsilon > 0$ such that

$$|H_z(x,z)| < \frac{\varepsilon}{c}|z|$$

for all $|z| \leq a_\varepsilon$. Consequently,

$$\begin{aligned} \int_{\Omega_j(0,a_\varepsilon)} \frac{H_z(x,z_j)(y_j^+ - y_j^-)|y_j|}{|z_j|} &\leq \int_{\Omega_j(0,a_\varepsilon)} \frac{\varepsilon}{c}|y_j^+ - y_j^-||y_j| \\ &\leq \frac{\varepsilon}{c}|y_j|_2^2 < \varepsilon \end{aligned} \tag{4.9}$$

for all j.

By (H_1) and (H_2), there is some $C > 0$ such that

$$|H_z(x,z)| \leq C|z| \tag{4.10}$$

for all (x,z). By (4.6) and Hölder inequality, we can take large b_ε such that

$$\begin{aligned}\int_{\Omega_j(b_\varepsilon,\infty)}\frac{H_z(x,z_j)(y_j^+-y_j^-)|y_j|}{|z_j|}&\le C\int_{\Omega_j(b_\varepsilon,\infty)}|y_j^+-y_j^-||y_j|\\&\le C|\Omega_j(b_\varepsilon,\infty)|^{\frac{1}{N}}\left(\int_{\Omega_j(b_\varepsilon,\infty)}|y_j^+-y_j^-|^2\right)^{\frac12}\left(\int_{\Omega_j(b_\varepsilon,\infty)}|y_j|^{2^*}\right)^{\frac{1}{2^*}}\le\varepsilon\end{aligned}\tag{4.11}$$

for all j. By (4.5) there is j_0 such that

$$\begin{aligned}\int_{\Omega_j(a_\varepsilon,b_\varepsilon)}\frac{H_z(x,z_j)(y_j^+-y_j^-)|y_j|}{|z_j|}&\le C\int_{\Omega_j(a_\varepsilon,b_\varepsilon)}|y_j^+-y_j^-||y_j|\\&\le C|y_j|_2\Big(\int_{\Omega_j(a_\varepsilon,b_\varepsilon)}|y_j|^2\Big)^{\frac12}\le\varepsilon\end{aligned}\tag{4.12}$$

for all $j\ge j_0$. By (4.9) and (4.10)–(4.12), one has

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\frac{H_z(x,z_j)(z_j^+-z_j^-)}{\|z_j\|^2}\le 3\varepsilon<1,$$

which contradicts (4.8).

In the case that (H_4') holds. By the Lions' concentration compactness principle, only two cases needed to be considered: $\{y_j\}$ is vanishing or $\{y_j\}$ is nonvanishing.

If $\{y_j\}$ is vanishing, then by the vanishing lemma we have $y_j\to 0$ in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$ for $p\in(2,2^*)$. In virtue of (H_4'), set

$$\Omega_j:=\left\{x\in\mathbb{R}^N|\frac{|H_z(x,z_j(x))|}{|z_j(x)|}\le\Lambda_0-\delta_0\right\}\cdot$$

Then $\Lambda_0|y_j|_2^2\le\|y_j\|^2=1$ and we have

$$\begin{aligned}\left|\int_{\Omega_j}\frac{H_z(x,z_j)(y_j^+-y_j^-)}{\|z_j\|}\right|&=\left|\int_{\Omega_j}\frac{H_z(x,z_j)(y_j^+-y_j^-)|y_j|}{|z_j|}\right|\\&\le(\Lambda_0-\delta_0)|y_j|_2^2\\&\le\frac{\Lambda_0-\delta_0}{\Lambda_0}\end{aligned}$$

for all j. This, jointly with (4.8), implies that for $\Omega_j^c=\mathbb{R}^N\backslash\Omega_j$

$$\lim_{j\to\infty}\int_{\Omega_j^c}\frac{H_z(x,z_j)(y_j^+-y_j^-)|y_j|}{|z_j|}>1-\frac{\Lambda_0-\delta_0}{\Lambda_0}=\frac{\delta_0}{\Lambda_0}\cdot$$

On the other hand, by (4.10) there holds for an arbitrarily fixed $s\in(2,2^*)$,

$$\begin{aligned}\int_{\Omega_j^c}\frac{H_z(x,z_j)(y_j^+-y_j^-)|y_j|}{|z_j|}&\le C\int_{\Omega_j^c}|y_j^+-y_j^-||y_j|\\&\le C|y_j|_2|\Omega_j^c|^{(s-2)/2s}|y_j|_s.\end{aligned}$$

Since $|y_j|_s \to 0$ and $|y_j|_2$ is bounded, one has $|\Omega_j^c| \to \infty$. From (H_4'), $\hat{H}(x,z) \geq \delta_0$ on Ω_j^c and hence

$$\int_{\mathbb{R}^N} \hat{H}(x,z_j) \geq \int_{\Omega_j^c} \hat{H}(x,z_j) \geq \delta_0 |\Omega_j^c| \to \infty,$$

contrary to (4.1).

If $\{y_j\}$ is nonvanishing, *i.e.*, there exist $\alpha > 0$, $R < \infty$ and $\{a_j\} \subset \mathbb{R}^N$ such that

$$\liminf_{j\to\infty} \int_{B(a_j,R)} |y_j|^2 \geq \alpha.$$

Set $\tilde{z}_j(x) = z_j(x+a_j)$, $\tilde{y}_j(x) = y_j(x+a_j)$ and $\eta_j(x) = \eta(x+a_j)$ for each $\eta = (\varphi,\psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. Now from (H_2) there holds

$$\begin{aligned}
\Phi'(z_j)\eta_j &= (z_j^+ - z_j^-, \eta_j) - (G_\infty(x)z_j, \eta_j)_{L^2\times L^2} - \int_{\mathbb{R}^N} R_z(x,z_j)\eta_j \\
&= \|z_j\| \left[(y_j^+ - y_j^-, \eta_j) - (G_\infty(x)y_j, \eta_j)_{L^2\times L^2} - \int_{\mathbb{R}^N} R_z(x,z_j)\eta_j \frac{|y_j|}{|z_j|} \right] \\
&= \|z_j\| \left[(\tilde{y}_j^+ - \tilde{y}_j^-, \eta) - (G_\infty(x)\tilde{y}_j, \eta)_{L^2\times L^2} - \int_{\mathbb{R}^N} R_z(x,\tilde{z}_j)\eta \frac{|\tilde{y}_j|}{|\tilde{z}_j|} \right]
\end{aligned}$$

and hence

$$(\tilde{y}_j^+ - \tilde{y}_j^-, \eta) - (G_\infty(x)\tilde{y}_j, \eta)_{L^2\times L^2} - \int_{\mathbb{R}^N} R_z(x,\tilde{z}_j)\eta \frac{|\tilde{y}_j|}{|\tilde{z}_j|} = o(1). \tag{4.13}$$

Since $\|\tilde{y}_j\| = \|y_j\| = 1$, up to a subsequence we may assume that $\tilde{y}_j \rightharpoonup \tilde{y}$ in E, $\tilde{y}_j \to \tilde{y}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N) \times L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\tilde{y}_j(x) \to \tilde{y}(x)$ a.e. in $\mathbb{R}^N$. Clearly, $\tilde{y} \neq 0$. Observe that

$$\left| \int_{\mathbb{R}^N} R_z(x,\tilde{z}_j)\eta \frac{|\tilde{y}_j|}{|\tilde{z}_j|} \right| \leq \left| \int_{\mathbb{R}^N} \frac{R_z(x,\tilde{z}_j)\eta|\tilde{y}|}{|\tilde{z}_j|} \right| + \left| \int_{\mathbb{R}^N} \frac{R_z(x,\tilde{z}_j)\eta|\tilde{y}_j - \tilde{y}|}{|\tilde{z}_j|} \right| = I + II.$$

Since $\frac{R_z(x,\tilde{z}_j)\eta|\tilde{y}|}{|\tilde{z}_j|} \to 0$ a.e. in $\mathbb{R}^N$, it follows from the dominated convergence theorem that $I \to 0$. By local compactness of embedding, one has

$$II = \left| \int_{\mathrm{supp}\eta} \frac{R_z(x,\tilde{z}_j)\eta|\tilde{y}_j - \tilde{y}|}{|\tilde{z}_j|} \right| \leq C|\eta|_2 |\tilde{y}_j - \tilde{y}|_{L^2(\mathrm{supp}\eta)} \to 0.$$

Combining the above two estimates for I and II, we have

$$\int_{\mathbb{R}^N} R_z(x,\tilde{z}_j)\eta \frac{|\tilde{y}_j|}{|\tilde{z}_j|} \to 0,$$

and letting $j \to \infty$ in (4.13) we get

$$(\tilde{y}^+ - \tilde{y}^-, \eta) - (G_\infty(x)\tilde{y}, \eta)_{L^2\times L^2} = 0, \text{ for each } \eta = (\varphi,\psi) \in C_0^\infty(\mathbb{R}^N). \tag{4.14}$$

Let $\tilde{y} = (\zeta,\xi)$, then $\tilde{y}^+ - \tilde{y}^- = (\zeta^+ - \zeta^-, \xi^- - \xi^+)$. Thus from (4.14) we have

$$(\zeta^+ - \zeta^-, \varphi)_H + (\xi^- - \xi^+, \psi)_H - (G_\infty(x)\zeta, \varphi)_{L^2} - (G_\infty(x)\xi, \psi)_{L^2} = 0,$$

which implies that

$$\begin{cases} -\Delta\zeta + V(x)\zeta = G_\infty\zeta, \\ \Delta\xi + V(x)\xi = G_\infty\xi. \end{cases}$$

Hence ζ is an eigenfunction of $S_1 := -\Delta + (V - G_\infty)$ and ξ is an eigenfunction of $S_2 := -\Delta + (G_\infty - V)$, which contradicts with the fact that S_i has only continuous spectrum for $i = 1, 2$ since $V - G_\infty$ is 1-periodic. Therefore $\{z_j\}$ is bounded in E. $\square$

Let $\{z_j\} \subset E$ be a $(C)_c$-sequence of Φ, by Lemma 3.1, it is bounded, up to a subsequence, we may assume $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^p_{loc} for $p \in [2, 2^*)$ and $z_j(x) \to z(x)$ a.e. on $\mathbb{R}^N$. Plainly, z is a critical point of Φ.

Recall that a mapping f from a Banach space X to another Banach space Y is called a BL-split, if for every sequence $\{x_n\}$ in X with $x_n \rightharpoonup x$ it holds that $f(x_n) - f(x_n - x) \to f(x)$ in Y (see Ackermann [2]). We adopt here a cut-off technique developed in Ackermann [1,2]. Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \le 1$, $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{z}_j(x) = \eta(2|x|/j)z(x)$, then $\tilde{z}_j \to z$ in E. In a similar way to Lemma 3.2 in Ackermann [1] (see also Lem. 4.4 in Ding and Jeanjean [15]), we can obtain the following:

Lemma 4.2. *Under the assumptions of Theorem 1.1, $\Psi(\cdot)$ and $\Psi'(\cdot)$ are both BL-splits.*

Let $\mathcal{K} := \{z \in E | \Phi'(z) = 0, z \neq 0\}$ be the set of nontrivial critical points of Φ.

Lemma 4.3. *Under the assumptions of Lemma 4.1, the following two conclusions hold*

(1) $\nu := \inf\{\|z\| : z \in \mathcal{K}\} > 0$;

(2) $\theta := \inf\{\Phi(z) | z \in \mathcal{K}\} > 0$.

Proof. (1) For any $z \in \mathcal{K}$, there holds

$$0 = \Phi'(z)(z^+ - z^-) = \|z\|^2 - \int_{\mathbb{R}^N} H_z(x, z)(z^+ - z^-),$$

jointly with (3.1), which implies that

$$\|z\|^2 \le \varepsilon|z|_2^2 + C_\varepsilon|z|_p^p \le c\varepsilon\|z\|_2^2 + cC_\varepsilon\|z\|^p,$$

where $p \in (2, 2^*)$. Choose ε small enough, hence

$$0 < \left(\frac{1 - c\varepsilon}{cC_\varepsilon}\right)^{\frac{1}{p-2}} \le \|z\|$$

for each $z \in \mathcal{K}$.

(2) Suppose to the contrary that there exist a sequence $\{z_j\} \in \mathcal{K}$ such that $\Phi(z_j) \to 0$. By (1), $\|z_j\| \ge \nu$. Clearly, $\{z_j\}$ is a $(C)_0$-sequence of Φ, and hence is bounded by Lemma 4.1. Moreover, $\{z_j\}$ is nonvanishing. By the invariance under translation of Φ, we can assume, up to a translation, that $z_j \rightharpoonup z \in \mathcal{K}$. Since $z = (u, v)$ is a solution of (ES), $|z(x)| \to 0$ as $|x| \to \infty$. Thus there is a bounded domain $\Omega \subset \mathbb{R}^N$ with positive measure such that $0 < |z(x)| < \delta_1$ for $x \in \Omega$ by (H_5). Then

$$o(1) = \Phi(z_j) = \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}^N} \hat{H}(x, z_j) \ge \int_\Omega \hat{H}(x, z_j),$$

letting $j \to \infty$ yields

$$0 \ge \lim_{j\to\infty} \int_\Omega \hat{H}(x, z_j) \ge \int_\Omega \hat{H}(x, z) > 0.$$

This ends the proof. $\square$

In the following lemma we discuss further the $(C)_c$-sequence. Let $[l]$ denote the integer part of $l \in \mathbb{R}$. The following lemma is standard (see Coti-Zelati and Rabinowitz [9,10] and Séré [24]).

Lemma 4.4. *Under the assumptions of Theorem 1.1, let $\{z_j\} \subset E$ be a $(C)_c$-sequence of Φ. Then either*
(i) $z_j \to 0$ *(and hence $c = 0$), or*
(ii) $c \geq \theta$ *and there exist a positive integer $l \leq [c/\theta]$, $y_1, \dots, y_l \in \mathcal{K}$ and sequences $\{a_j^i\} \subset \mathbb{Z}^N$, $i = 1, 2, \dots, l$, such that, after extraction of a subsequence of $\{z_j\}$,*

$$\Big\| z_j - \sum_{i=1}^{l} a_j^i * y_i \Big\| \to 0,$$

$$\sum_{i=1}^{l} \Phi(y_i) = c$$

and for $i \neq k$,

$$|a_j^i - a_j^k| \to \infty$$

as $j \to \infty$.

5. Proof of Theorem 1.1

In this section we prove our Theorem 1.1. First, we recall some terminology from Bartsch and Ding [7]. Let E be a Banach space with direct sum $E = X \oplus Y$ and corresponding projections P_X, P_Y onto X, Y. Let $\mathcal{S} \subset X^*$ be a dense subset, for each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R},\ p_s(u) = |s(x)| + \|y\| \text{ for } u = x + y \in E.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the topology induced by the semi-norm family $\{p_s\}$, and by w^* denote the weak*-topology on E^*. Now, some notations are needed. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E | \Phi(u) \geq a\}$, $\Phi^b = \{u \in E | \Phi(u) \leq b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Recall that a set $\mathcal{A} \subset E$ is said to be a $(C)_c$-attractor if for any $\varepsilon, \delta > 0$ and any $(C)_c$-sequence $\{z_j\}$ there is j_0 such that $z_j \in U_\varepsilon(\mathcal{A} \cap \Phi_{c-\delta}^{c+\delta})$ for $j \geq j_0$. Given an interval $I \subset \mathbb{R}$, $\mathcal{A}$ is said to be a $(C)_I$-attractor if it is a $(C)_c$-attractor for all $c \in I$. Φ is said to be weakly sequentially lower semicontinuous if for any $u_j \rightharpoonup u$ in E one has $\Phi(u) \leq \liminf_{j\to\infty} \Phi(u_j)$, and Φ' is said to be weakly sequentially continuous if $\lim_{j\to\infty} \Phi'(u_j)w = \Phi'(u)w$ for each $w \in E$. Suppose

- (Φ_0) for any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous;
- (Φ_1) for any $c > 0$, there exists $\xi > 0$ such that $\|u\| < \xi \|P_Y u\|$ for all $u \in \Phi_c$;
- (Φ_2) there exists $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho \cap Y) > 0$, where $S_\rho := \{u \in E : \|u\| = \rho\}$;
- (Φ_3) there exists a finite-dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that we have for $E_0 := X \oplus Y_0$ and $B_0 := \{u \in E_0 : \|u\| \leq R\}$ that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_\rho \cap Y)$;
- (Φ_4) there is an increasing sequence $Y_n \subset Y$ of finite-dimensional subspaces and a sequence $\{R_n\}$ of positive numbers such that, letting $E_n = X \oplus Y_n$ and $B_n = B_{R_n} \cap E_n$, $\sup \Phi(E_n) < \infty$ and $\sup \Phi(E_n \setminus B_n) < \inf \Phi(B_\rho \cap Y)$;
- (Φ_5) for any interval $I \subset (0, \infty)$ there is a $(C)_I$-attractor A with $P_X A$ bounded and $\inf\{\|P_Y(z-w)\| : z, w \in A, P_Y(z-w) \neq 0\} > 0$.

Now we state two critical point theorems which will be used later (see Bartsch-Ding [7]).

Theorem 5.1. *Let (Φ_0)–(Φ_2) be satisfied and suppose there are $R > \rho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi(\partial Q) \leq \kappa$ where $Q := \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. Then Φ has a $(C)_c$-sequence with $\kappa \leq c \leq \sup \Phi(Q)$.*

Remark 5.1. If we set

$$\Gamma_{Q,S} := \{h \in C([0,1] \times Q, E) : h \text{ satisfies } (h_1) - (h_5)\},$$

where

(h_1) $h : [0,1] \times (Q, \mathcal{T_S}) \to (E, \mathcal{T_S})$ is continuous;
(h_2) $h(0,u) = u$ for all $u \in Q$;
(h_3) $\Phi(h(t,u)) \le \Phi(u)$ for all $t \in [0,1]$, $u \in Q$;
(h_4) $h([0,1] \times \partial Q) \cap (S_\rho \cap Y) = \emptyset$;
(h_5) each $(t,u) \in [0,1] \times Q$ has a $\mathcal{T_S}$-open neighbourhood W such that the set $\{v - h(s,u) : (s,v) \in W \times([0,1] \times Q)\}$ is contained in a finite-dimensional subspace of E.

Then c has minimax characterization $c = \inf\limits_{h\in\Gamma_{Q,S}} \sup\limits_{u\in Q} \Phi(h(1,u))$.

Theorem 5.2. *Assume Φ is even with $\Phi(0) = 0$, let (Φ_0)–(Φ_2) and (Φ_4)–(Φ_5) be satisfied. Then Φ has possesses an unbounded sequence of positive critical values.*

Lemma 5.3. Φ *defined in* (2.3) *satisfies* (Φ_0).

Proof. We first show that Φ_a is $\mathcal{T_S}$-closed for every $a \in \mathbb{R}$. Consider a sequence $\{z_j\} \subset \Phi_a$ which $\mathcal{T_S}$-converges to $z \in E$, and write $z_j = z_j^- + z_j^+$, $z = z^- + z^+$. Then $z_j^+ \to z^+$ in norm topology and hence $\{z_j^+\}$ is bounded in norm topology. Observe that there exists $C > 0$ such that

$$\|z_j^-\|^2 = \|z_j^+\|^2 - 2\Phi(z_j) - 2\int_{\mathbb{R}^N} H(x, z_j) \le C$$

since $H(t,x,z) \ge 0$. This implies the boundedness of $\{z_j^-\}$ and hence $z_j^- \rightharpoonup z^-$. Therefore we have $z_j \rightharpoonup z$. It is easy to check that Ψ is weakly sequentially lower semi-continuous by the fact that $E \hookrightarrow L^p_{\text{loc}}$ for $p \in [2, 2^*)$ and the Fatou's Lemma, from where it follows that

$$a \le \lim_{n\to\infty} \Phi(z_j) \le \Phi(z),$$

so $z \in \Phi_a$ and hence Φ_a is $\mathcal{T_S}$-closed.

Next we show that $\Phi' : (\Phi_c, \mathcal{T_S}) \to (E^*, w^*)$ is continuous. It suffices to show that Ψ' has the same property, recall $\Psi(z) := \int_{\mathbb{R}^N} H(x,z)$. Suppose $z_j \to z$ in $\mathcal{T_S}$ topology. Then $z_j \to z$ in $L^p_{\text{loc}}(\mathbb{R}^N) \times L^p_{\text{loc}}(\mathbb{R}^N)$ for $p \in [2, 2^*)$. It is obvious that

$$\Psi'(z_j)\eta = \int_{\mathbb{R}^N} H_z(x, z_j)\eta \to \int_{\mathbb{R}^N} H_z(x,z)\eta = \Psi'(z)\eta$$

as $n \to \infty$ for all $\eta = (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. Now using the density of $C_0^\infty(\mathbb{R}^N)$ in E_0 we can obtain the desired conclusion. □

Lemma 5.4. Φ *satisfies* (Φ_1).

Proof. For any $c > 0$ and $z \in \Phi_c$, using the fact that $H \ge 0$ one has

$$0 < c \le \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2).$$

This yields $\|z^-\| < \|z^+\|$, and hence $\|z\| \le \sqrt{2}\|z^+\|$. □

Proof of Theorem 1.1. **1. Existence of a nontrivial solution.** With $X = E^-$ and $Y = E^+$ the condition (Φ_0) holds by Lemma 5.3 and (Φ_1) holds by Lemma 5.4. Lemma 3.1 implies (Φ_2). Lemma 3.3 shows that Φ possesses the linking structure of Theorem 5.1. Therefore, using Theorem 5.1, there exists a sequence $\{z_j\} \subset E$ such that $\Phi(z_j) \to c \geq \kappa$ and $(1 + \|z_j\|)\Phi'(z_j) \to 0$. By Lemma 4.1, $\{z_j\}$ is bounded. Now by the Lions' concentration compactness principle, either vanishing or nonvanishing occurs for the functions $|z_j|^2$. If vanishing occurs, by the vanishing lemma,

$$\int_{\mathbb{R}^N} H_z(x, z_j)z_j = o(1), \ \int_{\mathbb{R}^N} H(x, z_j) = o(1).$$

Then

$$o(1) = \Phi'(z_j)z_j = \frac{1}{2}(\|z_j^+\|^2 - \|z_j^-\|^2) - \int_{\mathbb{R}^N} H_z(x, z_j)z_j = \frac{1}{2}(\|z_j^+\|^2 - \|z_j^-\|^2) + o(1),$$

which implies that $\frac{1}{2}(\|z_j^+\|^2 - \|z_j^-\|^2) = o(1)$. Hence $\Phi(z_j) \to 0$ as $j \to \infty$. This contradicts to $\Phi(z_j) \to c \geq \kappa > 0$ as $j \to \infty$. So nonvanishing occurs, *i.e.*, there exist $\alpha > 0$, $R < \infty$ and $\{a_j\} \subset \mathbb{R}^N$ such that

$$\liminf_{j\to\infty} \int_{B(a_j,R)} |z_j|^2 \geq \alpha.$$

Setting $\tilde{z}_j(x) = z_j(x + a_j)$, by the invariance under translation of Φ, $\{\tilde{z}_j\}$ is a $(C)_c$-sequence of Φ and $\tilde{z}_j \rightharpoonup \tilde{z}$. From

$$\liminf_{j\to\infty} \int_{B(0,R)} |\tilde{z}_j|^2 \geq \alpha > 0,$$

we see that $\tilde{z} \neq 0$, and hence $\tilde{z}$ is a nontrivial critical point of Φ.

2. Existence of a least energy solution. Claim that $\theta := \inf\{\Phi(z)|z \in \mathcal{K}\}$ is achieved.

In fact, the process of part 1 shows that $\mathcal{K}$ is nonempty and hence θ is finite. Let $\{z_j\} \subset \mathcal{K}$ be a minimizing sequence for θ. Clearly, $\{z_j\}$ is a $(C)_\theta$-sequence of Φ, hence is bounded by Lemma 4.1. From (1) of Lemma 4.3, $\|z_j\| \geq \nu > 0$, one can rule out the case of vanishing. Hence $\{z_j\}$ is nonvanishing. Similarly, passing to a translation, $\tilde{z}_j$ has a nonzero weak limit $\tilde{z}_1 \in \mathcal{K}$ and $\tilde{z}_j(x) \to \tilde{z}_1(x)$ a.e. in $\mathbb{R}^N$, where $\tilde{z}_j$ is given as in the part 1 of the proof. By Fatou's lemma,

$$\begin{aligned}\theta &= \lim_{j\to\infty} \Phi(z_j) = \lim_{j\to\infty} \Phi(\tilde{z}_j)\\ &\geq \liminf_{j\to\infty} \int_{\mathbb{R}^N} \hat{H}(x, \tilde{z}_j)\\ &\geq \int_{\mathbb{R}^N} \hat{H}(x, \tilde{z}_1) = \Phi(\tilde{z}_1),\end{aligned}$$

from where it follows that θ is achieved by $\tilde{z}_1 \in \mathcal{K}$. This is equivalent to say that $I_{inf} := \{I(\eta) : I'(\eta) = 0, \ \eta \neq 0\}$ is attained by $\tilde{\eta} := ((\tilde{u} + \tilde{v})/\sqrt{2}, (\tilde{u} - \tilde{v})/\sqrt{2})$, where we write $\tilde{z}_1 = (\tilde{u}, \tilde{v})$.

Remark 5.2. If one strengthen (H_5) to

(H_6) $\hat{H}(x, z) \geq 0$ and $\hat{H}(x, z) > 0$ if $z \neq 0$,

then using the minimax characterization of c in Remark 5.1, one can show that $c = \theta$ by a Brouwer degree argument, see Lemma 3.4 in Li and Yang [21] or Proposition 3.8 in Li and Szulkin [20] for more details. Hence $\tilde{z}$ is also a least energy solution. But in general, we do not know whether the two critical points $\tilde{z}$ and $\tilde{z}_1$ are the same or not. (H_6) guarantees that δ_1 in (H_5) is large enough, and the latter ensures that the two critical values c and θ coincide with each other.

3. Multiplicity. Φ is even provided $G(x,\eta)$ is even in η. Lemma 3.2 says that Φ satisfies (Φ_4). Next, we assume

$$\mathcal{K}/\mathbb{Z}^N \text{ is a finite set.} \tag{5.1}$$

In fact, if (5.1) is false, then the last conclusion of Theorem 1.1 holds automatically. In the sequel, we assume (5.1) holds. Let $\mathcal{F}$ be a set consisting of arbitrarily chosen representatives of the $\mathbb{Z}^N$-orbits of $\mathcal{K}$. Then $\mathcal{F}$ is a finite set by (5.1), and since Φ' is odd we may assume that $\mathcal{F} = -\mathcal{F}$. If $z \in \mathcal{K}$, then $\Phi(z) \geq \theta$ by (2) of Lemma 4.3. Hence there exists $\theta \leq \vartheta$ such that

$$\theta \leq \min_{\mathcal{F}} \Phi = \min_{\mathcal{K}} \Phi \leq \max_{\mathcal{K}} \Phi \leq \max_{\mathcal{F}} \Phi \leq \vartheta.$$

For $l \in \mathbb{N}$ and a finite set $\mathcal{B} \subset E$ we define

$$[\mathcal{B}, l] := \left\{ \sum_{i=1}^{j} a_i * z_j | 1 \leq j \leq l,\ a_i \in \mathbb{Z}^N,\ z_j \in \mathcal{B} \right\}.$$

As in Coti-Zelati and Rabinowitz [9,10],

$$\inf \{\|z - z'\| : z, z' \in [\mathcal{B}, l]\} > 0. \tag{5.2}$$

Now we check (Φ_5). Given a compact interval $I \subset (0, \infty)$ with $d := \max I$ we set $l = [d/\theta]$ and $\mathcal{A} = [\mathcal{F}, l]$. We have $P^+[\mathcal{F}, l] = [P^+\mathcal{F}, l]$. Thus from (5.2)

$$\inf \left\{ \|z_1^+ - z_2^+\| : z_1, z_2 \in \mathcal{A}, z_1^+ \neq z_2^+ \right\} > 0.$$

In addition, $\mathcal{A}$ is a $(C)_I$-attractor by Lemma 4.4 and $\mathcal{A}$ is bounded because $\|z\| \leq l \max\{\|\bar{z}\| : \bar{z} \in \mathcal{F}\}$ for all $z \in \mathcal{A}$. Therefore, by Theorem 5.2, Φ has a unbounded sequence of critical values which contradicts with the assumption (5.1), and hence Φ has infinitely many geometrically distinct nontrivial critical points. Therefore, our multiplicity result follows. This completes the proof. □

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