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LIPSCHITZ MODULUS IN CONVEX SEMI-INFINITE OPTIMIZATION *VIA* D.C. FUNCTIONS*

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Abstract. We are concerned with the Lipschitz modulus of the optimal set mapping associated with canonically perturbed convex semi-infinite optimization problems. Specifically, the paper provides a lower and an upper bound for this modulus, both of them given exclusively in terms of the problem's data. Moreover, the upper bound is shown to be the exact modulus when the number of constraints is finite. In the particular case of linear problems the upper bound (or exact modulus) adopts a notably simplified expression. Our approach is based on variational techniques applied to certain difference of convex functions related to the model. Some results of [M.J. Cánovas *et al.*, *J. Optim. Theory Appl.* (2008) Online First] (which go back to [M.J. Cánovas, *J. Global Optim.* **41** (2008) 1–13] and [Ioffe, *Math. Surveys* **55** (2000) 501–558; *Control Cybern.* **32** (2003) 543–554]) constitute the starting point of the present work.

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1. Introduction

This paper aims to quantify the Lipschitzian behavior of the optimal set mapping for a parametric family of convex semi-infinite optimization problems. Our focus is on the *sharp Lipschitz constant*, according to the terminology used for ordinary linear programming problems by Li [19] (see also Klatte and Thiere [17] and Robinson [22]). Specifically, the paper provides a lower and an upper bound on this constant, called *Lipschitz modulus* in our context. The upper bound turns out to be the exact modulus in the case of problems with finitely many constraints. The starting point is the expression for this constant established in [5] in terms of the *regular subdifferential* of certain distance functions (see Thm. 2.2), where some results of Ioffe [13,14] are applied (see [4], Thm. 3). The main original contribution of the present paper consists of deriving formulae

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(for the referred bounds) which are given exclusively in terms of the problem's data. To do that, we shall make use of some tools related to subdifferential calculus of *difference of convex functions* (*d.c. functions*, for short).

Now we introduce the parametrized model we are dealing with. This is the canonically perturbed convex programming problem, in $\mathbb{R}^n$,

$$P(c,b): \quad \begin{array}{l} \text{Inf } f(x) + \langle c, x\rangle \\ \text{s.t. } g_t(x) \leq b_t, \ \ t \in T, \end{array} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $c \in \mathbb{R}^n$, $\langle .,. \rangle$ represents the usual inner product in $\mathbb{R}^n$, the index set, T, is a compact metric space, $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are given convex functions, $(t,x) \mapsto g_t(x)$ is assumed to be continuous on $T \times \mathbb{R}^n$, and $b \in C(T,\mathbb{R})$ (*i.e.*, $t \mapsto b_t$ is also continuous). Note that ordinary convex programming problems are included in our model just by considering T finite. The pair $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ is regarded as the parameter to be perturbed. The topology in the parameter space, $\mathbb{R}^n \times C(T,\mathbb{R})$, is derived from the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\}, \tag{1.2}$$

where $\mathbb{R}^n$ is equipped with any given norm $\|\cdot\|$ and $\|b\|_\infty := \max_{t\in T}|b_t|$. The corresponding dual norm in $\mathbb{R}^n$ is given by $\|u\|_* := \max\{\langle u, x\rangle \mid \|x\| \leq 1\}$, and d_* denotes the related distance.

Associated with the parametrized problem $P(c,b)$, we consider the *optimal set mapping*, $\mathcal{F}^* : \mathbb{R}^n \times C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$, which assigns to each parameter $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ the *optimal set* – set of (global) optimal solutions – of $P(c,b)$; *i.e.*,

$$\mathcal{F}^*(c,b) := \arg\min\{f(x) + \langle c,x\rangle \mid g_t(x) \leq b_t, \ \ t \in T\}.$$

Now we recall the well-known Karush-Kuhn-Tucker (KKT) optimality conditions in our framework and, to this aim, we introduce the necessary notation. Associated with each $b \in C(T,\mathbb{R})$, we denote by $\sigma(b)$ the corresponding constraint system; *i.e.*, $\sigma(b) := \{g_t(x) \leq b_t, t \in T\}$, and $\mathcal{F}(b)$ represents the *feasible set* of $\sigma(b)$. We say that $\sigma(b)$ satisfies the *Slater constraint qualification* (SCQ, for short) if there exists $x^0 \in \mathbb{R}^n$ such that $g_t(x^0) < b_t$, for all $t \in T$. When $\sigma(b)$ satisfies SCQ, the condition '$x \in \mathcal{F}^*(c,b)$' is equivalent to the KKT conditions (see [11], Chap. 7):

$$x \in \mathcal{F}(b) \ \text{ and } \ (\partial f(x) + c) \cap \text{cone}\left(\bigcup\nolimits_{t\in T_b(x)} (-\partial g_t(x))\right) \neq \emptyset, \tag{1.3}$$

where

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\};$$

i.e., $T_b(x)$ is the set of active indices at x for $\sigma(b)$, whereas ∂ represents the ordinary subdifferential in convex analysis, and cone(X) is the convex cone generated by the set X. It is assumed that cone(X) always contains the zero-vector, 0_n, and so cone$(\emptyset) = \{0_n\}$. Actually, SCQ is only needed in the implication '$x \in \mathcal{F}^*(c,b) \Rightarrow$ (1.3)'.

Appealing to Carathéodory's theorem one can replace $T_b(x)$ in (1.3) by some subset $D \subset T_b(x)$ with $|D| \leq n$. When the only possibility is $|D| = n$, we say that the *extended Nürnberger condition* (ENC, for short) holds. Formally, ENC is satisfied at a given $((\overline{c},\overline{b}),\overline{x}) \in \text{gph}(\mathcal{F}^*)$, the graph of $\mathcal{F}^*$, if

$$\begin{array}{c} \sigma(\overline{b}) \ \text{ satisfies SCQ and there is no } D \subset T_{\overline{b}}(\overline{x}) \\ \text{with } \ |D| < n \text{ such that } (\partial f(\overline{x}) + \overline{c}) \cap \text{cone}\left(\bigcup_{t\in D}(-\partial g_t(\overline{x}))\right) \neq \emptyset. \end{array}$$

See [3,5] for motivation, details, and consequences of this condition (some of them are gathered in Sect. 5). When confined to the linear case (f and g_t, $t \in T$, being linear functions), ENC turns out to be equivalent to the one introduced by Nürnberger in [21] for characterizing the strong uniqueness of optimal solutions in a neighborhood of the nominal parameter.

ENC constitutes a specification of KKT conditions with strong consequences in relation to stability. Along the paper, it is a crucial assumption under which $\mathcal{F}^*$ exhibits a nice Lipschitzian behavior. In particular, ENC at $((\overline{c},\overline{b}),\overline{x}) \in \text{gph}(\mathcal{F}^*)$ implies *strong Lipschitz stability* of $\mathcal{F}^*$ at this point (see Lem. 5 and Thm. 10 in [3]), which can be read as single-valuedness and Lipschitz continuity of $\mathcal{F}^*$ near $\left(\overline{c},\overline{b}\right)$ (the convex-valuedness of $\mathcal{F}^*$ allows us to say 'single-valuedness' instead of 'local single-valuedness'). In our context of problems (1.1) (even in more general contexts, see for instance [15]), *strong Lipschitz stability* of $\mathcal{F}^*$ at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ is equivalent to *pseudo-Lipschitz* property of $\mathcal{F}^*$ at this point, also equivalent to the *metric regularity* of its inverse $\left(\mathcal{F}^*\right)^{-1}$ at $\left(\overline{x},\left(\overline{c},\overline{b}\right)\right)$. The last property can be stated as follows: there exist a constant $\kappa \geq 0$ and some associated neighborhoods U of $\overline{x}$ and V of $\left(\overline{c},\overline{b}\right)$ such that

$$d\left(x,\mathcal{F}^*\left(c,b\right)\right) \leq \kappa d\left(\left(c,b\right),\left(\mathcal{F}^*\right)^{-1}\left(x\right)\right), \tag{1.4}$$

for all $x \in U$ and all $(c,b) \in V$. Here, as usual, we adopt the convention $d(x,\varnothing) = +\infty$. The infimum of the (Lipschitz) constants $\kappa \geq 0$ verifying (1.4) (for some associated neighborhoods) is called *modulus of metric regularity* which, due to single-valuedness of $\mathcal{F}^*$ near $\left(\overline{c},\overline{b}\right)$, coincides with the *Lipschitz modulus* of $\mathcal{F}^*$ at $\left(\overline{c},\overline{b}\right)$ and can be written as follows:

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \limsup_{\substack{(c,b),\left(\widetilde{c},\widetilde{b}\right)\to\left(\overline{c},\overline{b}\right)\\ (c,b)\neq\left(\widetilde{c},\widetilde{b}\right)}} \frac{\left\|\mathcal{F}^*\left(c,b\right)-\mathcal{F}^*\left(\widetilde{c},\widetilde{b}\right)\right\|}{\left\|\left(c,b\right)-\left(\widetilde{c},\widetilde{b}\right)\right\|}. \tag{1.5}$$

In (1.5) we are using, for the sake of simplicity, the same notation for the set $\mathcal{F}^*\left(c,b\right)$ and for its unique element, provided that (c,b) is close enough to $\left(\overline{c},\overline{b}\right)$.

Under ENC, [5], Theorem 5.1 (recalled in Thm. 2.2) provides an expression for $\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b})$ in terms of the *regular subdifferential* (see Sect. 2) of the functions f_b, $b \in C\left(T,\mathbb{R}\right)$, given by

$$f_b\left(x\right) := d\left(b,\widetilde{\mathcal{G}}\left(x\right)\right), \tag{1.6}$$

where $\widetilde{\mathcal{G}} : \mathbb{R}^n \rightrightarrows C\left(T,\mathbb{R}\right)$ is defined as follows:

$$\widetilde{\mathcal{G}}\left(x\right) := \left\{b' \in C\left(T,\mathbb{R}\right) : x \in \mathcal{F}^*\left(\overline{c},b'\right)\right\};$$

i.e., f_b assigns to each $x \in \mathbb{R}^n$ the distance from b to the set of parameters b' such that x is optimal for $P\left(\overline{c},b'\right)$. Among other consequences of ENC, a remarkable fact is that perturbations on c are negligible (note that c remains fixed at $\overline{c}$ in the definitions of f_b and $\widetilde{\mathcal{G}}$). Moreover, under ENC, $f_b\left(x\right)$ is finite for (x,b) close enough to $\left(\overline{x},\overline{b}\right)$ (see Thm. 2.1(iv)).

As commented above, our goal in this paper is to approach $\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b})$ *via* a procedure that only involves the problem's data (*i.e.*, functions f and g_t's). The main intermediate step consists of obtaining a representation of f_b in terms of these data (see Thm. 3.1). As a consequence of this representation, we provide a lower bound of the Lipschitz modulus (Prop. 4.1). After that, appealing additionally to Theorem 4.1, we deduce the upper bound, and show that it equals $\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b})$ when T is finite.

In summary, the structure of the paper is as follows: Section 2 collects the preliminary concepts and results needed later. In Section 3, under ENC, we provide a representation of f_b in terms of certain d.c. functions, which rely directly on the data of $P\left(\overline{c},b\right)$ (see Thm. 3.1). Section 4 is divided into two subsections, Sections 4.1 and 4.2, which are respectively concerned with the lower and the upper bound (or exact value) on the Lipschitz modulus. The latter appeals to some subdifferential calculus for the referred d.c. functions. This bound admits a notable simplification when confined to the particular case of linear problems. Finally, Section 5 is intended to emphasize the scope of the results of the present paper, as well as, to provide extra motivation about the crucial assumption of ENC, examples and an application to functional approximation.

2. Notation and basic concepts

In this section we provide further notation and some preliminary results. Given $\varnothing \neq X \subset \mathbb{R}^k$, we denote by $\operatorname{co}(X)$ the *convex hull* of X. If y is a point in any metric space, we denote by $B_\delta(y)$ the open ball centered at y with radius δ, whereas the corresponding closed ball is represented by $\overline{B}_\delta(y)$. As usual, $|X|$ denotes the cardinality of a set X, and $X \backslash Y := \{x \in X : x \notin Y\}$.

As commented above, ENC plays a decisive role in our analysis of the Lipschitzian behavior of $\mathcal{F}^*$, and provides nice stability results for both $\mathcal{F}^*$ and $\widetilde{\mathcal{G}}$, the latter by means of a certain stability of the indices involved in KKT conditions (see [5]). Theorem 2.1 below gathers some of the main consequences of this property, which are used in the present paper. It appeals to the following notation: associated with a point $x \in \mathbb{R}^n$ we consider the set

$$\mathcal{D}(x) := \{D \subset T : |D| = n \text{ and } (\partial f(x) + \overline{c}) \cap \operatorname{cone}(\cup_{t\in D}(-\partial g_t(x))) \neq \varnothing\},$$

and, given $x \in \mathcal{F}(b)$ and $\delta \geq 0$, we consider

$$T_b^\delta(x) := \{t \in T \mid g_t(x) \geq b_t - \delta\} \text{ and } \mathcal{T}_b^\delta(x) := \{D \in \mathcal{D}(x) : D \subset T_b^\delta(x)\}.$$

For simplicity we write $\mathcal{T}_b^0(x) = \mathcal{T}_b(x)$ if $\delta = 0$.

Observe that the sets $\mathcal{D}(x)$ and $\mathcal{T}_b^\delta(x)$ involve $\overline{c}$. Nevertheless, for the sake of simplicity, since $\overline{c}$ remains fixed throughout the paper, it has been omitted in the notations.

Theorem 2.1. *For the convex program* (1.1)*, let* $\left((\overline{c}, \overline{b}), \overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$*. If* ENC *is satisfied at* $\left((\overline{c}, \overline{b}), \overline{x}\right)$*, then the following conditions hold:*

(i) [3], Proposition 9(i). *There exists a neighborhood* W *of* $\left((\overline{c}, \overline{b}), \overline{x}\right)$ *such that* ENC *is satisfied at any* $((c,b),x) \in W \cap \operatorname{gph}(\mathcal{F}^*)$.

(ii) [3], Proposition 9(ii). *There exist* $u \in \partial f(\overline{x})$ *as well as some* $u_i \in -\partial g_{t_i}(\overline{x})$, $t_i \in T_{\overline{b}}(\overline{x})$, *and some* $\lambda_i > 0$ *for* $i \in \{1, \ldots, n\}$, *such that* $\{u_1, \ldots, u_n\}$ *is a basis of* $\mathbb{R}^n$ *and*

$$u + \overline{c} = \Sigma_{i=1}^n \lambda_i u_i.$$

(iii) [3], Theorem 10. $\mathcal{F}^*$ *is* strongly Lipschitz stable *at* $\left((\overline{c}, \overline{b}), \overline{x}\right)$.

(iv) [5], Theorem 4.3. $\widetilde{\mathcal{G}}$ *is lower semicontinuous at* $(\overline{x}, \overline{b})$ *(in the sense of* [20]*, Def.* 1.63(i)*);* i.e., *for all neighborhood* V *of* $\overline{b}$ *there exists a neighborhood* U *of* $\overline{x}$ *such that*

$$\widetilde{\mathcal{G}}(x) \cap V \neq \varnothing \text{ for all } x \in U.$$

(v) [5], Theorem 4.2. *There exists* $\delta_0 > 0$ *such that, for every* $\delta \in]0, \delta_0]$, *there are neighborhoods* U *and* V *of* $\overline{x}$ *and* $\overline{b}$, *respectively, verifying*

$$\varnothing \neq \mathcal{T}_b(x) \subset \mathcal{T}_{\overline{b}}^\delta(\overline{x}) \subset \mathcal{D}(x) \quad \text{for all } x \in U \text{ and all } b \in V.$$

Theorem 2.2 provides the expression for $\operatorname{lip}\mathcal{F}^*(\overline{c}, \overline{b})$ which constitutes, as we announced above, the immediate antecedent of the present work. It refers to some variational notions.

Consider a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $z \in \mathbb{R}^n$ where $\varphi(z)$ is finite. A vector $v \in \mathbb{R}^n$ is called a *regular subgradient* of φ at z, written $v \in \widehat{\partial}\varphi(z)$, if

$$d\varphi(z)(w) := \liminf_{\tau \searrow 0,\ w' \to w} \frac{\varphi(z + \tau w') - \varphi(z)}{\tau} \geq \langle v, w \rangle, \text{ for all } w \in \mathbb{R}^n. \tag{2.1}$$

Here $d\varphi(z)(w)$ is called *(lower) subderivative* of φ at z for w (see [24], Defs. 8.1 and 8.3, Exe. 8.4).

We denote by $|\nabla\varphi|(z)$ the *strong slope* (see [7]) of φ at z which is given by

$$|\nabla\varphi|(z) := \limsup_{y\to z,\ y\neq z} \frac{(\varphi(z)-\varphi(y))^+}{\|z-y\|},$$

where $\alpha^+ := \max\{\alpha, 0\}$ is the positive part of α.

The *regular subdifferential* of φ at z, $\widehat{\partial}\varphi(z)$, is a closed convex subset of $\mathbb{R}^n$, which satisfies

$$d_*(0_n, \widehat{\partial}\varphi(z)) \geq |\nabla\varphi|(z). \tag{2.2}$$

Moreover, if φ is a proper convex function, then, according to [24], Proposition 8.12, $\widehat{\partial}\varphi(z)$ coincides with the ordinary subdifferential set in convex analysis, $\partial f(z)$.

Theorem 2.2 ([5], Thm. 5.1)**.** *Assume that* ENC *is satisfied at* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$, *and let* f_b, $b \in C(T,\mathbb{R})$, *be defined as in* (1.6)*. Then we have*

$$\begin{aligned}\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) &= \limsup_{\substack{(z,b)\to\left(\overline{x},\overline{b}\right)\\ f_b(z)>0}} \left(d_*(0_n, \widehat{\partial} f_b(z))\right)^{-1}\\ &= \limsup_{\substack{(z,b)\to\left(\overline{x},\overline{b}\right)\\ f_b(z)>0}} \left(|\nabla f_b|(z)\right)^{-1}.\end{aligned}$$

At this point we recall some tools used in Section 4, when approaching $\widehat{\partial} f_b(z)$ by sets having a more operative representation.

Definition 2.1. If the function $\varphi:\mathbb{R}^n\to\mathbb{R}$ can be decomposed as the difference of two convex functions $\varphi_1,\varphi_2:\mathbb{R}^n\to\mathbb{R}$, *i.e.*,

$$\varphi(x) = \varphi_1(x) - \varphi_2(x), \text{ for all } x\in\mathbb{R}^n, \tag{2.3}$$

then it is called a *d.c. function (difference of convex functions)*, and (2.3) is one of its possible *d.c. decompositions.*

We shall prove that each function f_b, with b near $\overline{b}$, can be represented as the infimum of certain d.c. functions in a certain neighborhood of $\overline{x}$ (see Sect. 3), which motivates the following comments. First, given a non-empty set S and a family of functions $f^s:\mathbb{R}^n\to\mathbb{R}$, $s\in S$, we consider the infimum function in a certain open set $U\subset\mathbb{R}^n$

$$f(z) := \inf\{f^s(z),\ s\in S\}, \text{ for all } z\in U.$$

Then, as an immediate consequence of the definitions, one has

$$\widehat{\partial} f(z) \subset \bigcap_{s\in I(z)} \widehat{\partial} f^s(z), \tag{2.4}$$

provided that the set $I(z) := \{s\in S \mid f^s(z) = f(z)\}$ is non-empty. In our case, $\widehat{\partial} f_b(z)$ will be contained in an intersection of sets as in (2.4) where, additionally, the functions playing the role of f^s are d.c. functions. So, the next paragraphs are devoted to gather some statements about the regular subdifferential of d.c. functions that are used in the paper.

If $\varphi:\mathbb{R}^n\to\mathbb{R}$ is a d.c. function, with d.c. decomposition $\varphi=\varphi_1-\varphi_2$, it is obvious that the *one-sided directional derivative*

$$\varphi'(x;w) := \lim_{\tau\searrow 0} \frac{\varphi(x+\tau w)-\varphi(x)}{\tau}$$

exists and is finite everywhere, due to the fact that

$$\varphi'(x;w) = \varphi_1'(x;w) - \varphi_2'(x;w).$$

Moreover, since φ_1 and φ_2 are finite-everywhere convex functions, thus locally Lipschitz continuous ([23], Thm. 10.4), it follows that φ is also locally Lipschitz continuous. Then, one can easily check that 'liminf' in (2.1) can be replaced by 'lim' (*i.e.*, the ordinary limit exists) and, in addition,

$$\mathrm{d}\varphi(x)(w) = \varphi'(x;w).$$

In particular, φ is *semidifferentiable,* following the terminology of [24], Definition 7.20. Moreover, the equality

$$\varphi'(x;w) = \max_{u\in\partial\varphi_1(x)} \langle u,w\rangle - \max_{v\in\partial\varphi_2(x)} \langle v,w\rangle,$$

saying that $\varphi'(x;\cdot)$ is the difference of two (finite) sublinear functions, implies that φ is *quasidifferentiable* in the sense of Demyanov and Rubinov [8]. Now, applying Proposition 4.1 in [9] and Exercise 8.4 in [24], we obtain, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}\widehat{\partial}\varphi(x) &= \partial\varphi_1(x) \dot{-} \partial\varphi_2(x) := \{z\in\mathbb{R}^n \mid z+\partial\varphi_2(x)\subset\partial\varphi_1(x)\} \\ &= \bigcap_{v\in\partial\varphi_2(x)} \{\partial\varphi_1(x) - v\},\end{aligned} \tag{2.5}$$

the sets $\partial\varphi_1(x)$ and $\partial\varphi_2(x)$ being the ordinary subdifferential sets in convex analysis.

3. Representation of f_b *via* d.c. functions

The main goal of this section consists of describing f_b in terms of the problem's data (objective function and constraint system of (1.1)). Specifically, we show that f_b can be expressed in terms of the functions

$$\begin{aligned}f_b^D(x) &:= \max\{|g_t(x)-b_t|,\ t\in D;\ g_t(x)-b_t,\ t\in T\setminus D\} \\ &= \max\left\{\max_{t\in T}(g_t(x)-b_t);\ b_s-g_s(x),\ s\in D\right\},\end{aligned} \tag{3.1}$$

for certain finite subsets of indices $D \subset T$ with $|D| = n$. At this moment we announce that each f_b^D is a d.c. function (see Sect. 4.2), which entails a notable advantage when we are looking for an operative expression of the Lipschitz modulus (see again Sect. 4.2).

Theorem 3.1. *Assume that* ENC *is satisfied at* $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\in \operatorname{gph}(\mathcal{F}^*)$. *Then, a certain* $\delta_0 > 0$ *exists such that for all* $\delta \in]0,\delta_0]$ *there exist some neighborhoods* U *and* V *of* $\overline{x}$ *and* $\overline{b}$, *respectively, verifying*

$$f_b(x) = \min_{D\in\mathcal{T}_b^\delta(\overline{x})} f_b^D(x),\ \textit{for all } x\in U \textit{ and all } b\in V. \tag{3.2}$$

(The minimum above is attained.)

Proof. Take $\delta_0 > 0$, the scalar provided by Theorem 2.1(v); *i.e.*, such that if $\delta \in]0,\delta_0]$, there exist neighborhoods U_δ and V_δ of $\overline{x}$ and $\overline{b}$, respectively, satisfying

$$\emptyset \neq \mathcal{T}_b(x) \subset \mathcal{T}_b^\delta(\overline{x}) \subset \mathcal{D}(x),\ \text{for all } x\in U_\delta \text{ and all } b\in V_\delta. \tag{3.3}$$

Fix any $\delta \in]0, \delta_0]$ and associated neighborhoods $\widetilde{U}$ $(=U_\delta)$ and $\widetilde{V}$ $(=V_\delta)$ verifying (3.3). As a consequence of Theorem 2.1(i), we may assume without loss of generality that ENC is satisfied at any $((\overline{c}, b), x) \in \text{gph}(\mathcal{F}^*)$ such that $x \in \widetilde{U}$ and $b \in \widetilde{V}$. Now the proof is organized in four steps.

Step 1. First we prove that

$$f_b(x) \leq \inf_{D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_b^D(x), \text{ for all } x \in \widetilde{U} \text{ and all } b \in \widetilde{V}. \tag{3.4}$$

Consider arbitrarily fixed elements $D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$, $x \in \widetilde{U}$ and $b \in \widetilde{V}$, and let us see that $f_b(x) \leq f_b^D(x)$. Reasoning by contradiction, assume

$$f_b(x) > \gamma > f_b^D(x),$$

for a certain γ (> 0). Now we construct $\widetilde{b} \in C(T, \mathbb{R})$ such that $\widetilde{b} \in \widetilde{\mathcal{G}}(x)$ and $\left\|b - \widetilde{b}\right\|_\infty < \gamma$. In this way we would attain the contradiction $f_b(x) \leq \left\|b - \widetilde{b}\right\|_\infty < \gamma$.

Define

$$\widetilde{b}_t := \varphi(t) g_t(x) + (1 - \varphi(t)) b_t, \text{ for } t \in T,$$

where $\varphi : T \to [0, 1]$ is a continuous function, whose existence follows from Urysohn's lemma, satisfying

$$\varphi(t) := \begin{cases} 1, \text{ if } t \in D \text{ or } g_t(x) \geq b_t, \\ 0, \text{ if } g_t(x) \leq b_t - \gamma. \end{cases}$$

(If $\{t \in T : g_t(x) \leq b_t - \gamma\} = \varnothing$, then we take $\varphi \equiv 1$.) This implies, for every $t \in T$,

$$g_t(x) - \widetilde{b}_t = (1 - \varphi(t))(g_t(x) - b_t) \leq 0,$$

which states the feasibility of x with respect to $\widetilde{b}$; *i.e.*, $x \in \mathcal{F}\left(\widetilde{b}\right)$. Moreover, since $\widetilde{b}_t = g_t(x)$, for $t \in D$, and $D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x}) \subset \mathcal{D}(x)$ (by (3.3)), KKT optimality conditions yield $\widetilde{b} \in \widetilde{\mathcal{G}}(x)$. Now in order to prove $\left\|b - \widetilde{b}\right\|_\infty < \gamma$, we write

$$\left\|b - \widetilde{b}\right\|_\infty = \max_{t \in T} \varphi(t) \left|g_t(x) - b_t\right|,$$

and observe that, in the non-trivial case $\varphi(t) > 0$ we have

$$-\gamma < g_t(x) - b_t \leq f_b^D(x) < \gamma.$$

So,

$$\left\|b - \widetilde{b}\right\|_\infty = \max_{\varphi(t) > 0} \varphi(t) \left|g_t(x) - b_t\right| < \gamma$$

(observe that the maximum is attained). This finishes the proof of Step 1.

Step 2. Now we establish the existence of neighborhoods $U \subset \widetilde{U}$ and $V \subset \widetilde{V}$ of $\overline{x}$ and $\overline{b}$, respectively, such that

$$d\left(b, \widetilde{\mathcal{G}}(x) \cap \widetilde{V}\right) = d\left(b, \widetilde{\mathcal{G}}(x)\right) \text{ for all } x \in U \text{ and } b \in V. \tag{3.5}$$

Let $\varepsilon > 0$ be such that $\widetilde{V} \supset B_{3\varepsilon}\left(\overline{b}\right)$, and take

$$V := B_\varepsilon\left(\overline{b}\right).$$

Now, according to Theorem 2.1(iv), we consider a neighborhood of $\overline{x}$, $U \subset \widetilde{U}$ such that

$$\widetilde{\mathcal{G}}(x) \cap V \neq \emptyset, \text{ for all } x \in U.$$

In this way, if $x \in U$, $b \in V$, and $b^1 \in \widetilde{\mathcal{G}}(x) \cap V$, we have,

$$d\left(b, \widetilde{\mathcal{G}}(x) \cap \widetilde{V}\right) \leq d\left(b, \widetilde{\mathcal{G}}(x) \cap V\right) \leq d\left(b, b^1\right) < 2\varepsilon$$

and, if $b^2 \in \widetilde{\mathcal{G}}(x) \setminus \widetilde{V}$, then,

$$d\left(b, b^2\right) \geq d\left(b^2, \overline{b}\right) - d\left(\overline{b}, b\right) > 3\varepsilon - \varepsilon > d\left(b, \widetilde{\mathcal{G}}(x) \cap \widetilde{V}\right),$$

leading us to

$$d\left(b, \widetilde{\mathcal{G}}(x) \cap \widetilde{V}\right) = d\left(b, \widetilde{\mathcal{G}}(x)\right),$$

as we wanted to prove.

Step 3. The following equality holds

$$f_b(x) = \inf_{D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_b^D(x), \text{ for all } x \in U \text{ and } b \in V, \tag{3.6}$$

where U and V are as in Step 2.

We only need to prove the inequality '$\geq$' (see (3.4)). Reasoning by contradiction, assume that there exist $x^0 \in U$ and $b^0 \in V$ such that $f_{b^0}(x^0) < \inf_{D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_{b^0}^D(x^0)$. From (3.5) we have

$$f_{b^0}(x^0) = d\left(b^0, \widetilde{\mathcal{G}}(x^0)\right) = d\left(b^0, \widetilde{\mathcal{G}}(x^0) \cap \widetilde{V}\right),$$

entailing the existence of $\widetilde{b}^0 \in \widetilde{\mathcal{G}}(x^0) \cap \widetilde{V}$ such that

$$\left\| b^0 - \widetilde{b}^0 \right\|_\infty < \inf_{D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_{b^0}^D(x^0). \tag{3.7}$$

Our choice of $\widetilde{V}$ entails that $\sigma\left(\widetilde{b}^0\right)$ satisfies SCQ. Hence, there must exist $D_0 \in \mathcal{T}_{\widetilde{b}^0}\left(x^0\right)$ providing KKT optimality conditions. But then, (3.3) entails $D_0 \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$. From the facts that $x^0 \in \mathcal{F}\left(\widetilde{b}^0\right)$, $g_t\left(x^0\right) = \widetilde{b}_t^0$ for $t \in D_0$, and taking the definition of $f_{b^0}^{D_0}(x^0)$ into account, we obtain

$$f_{b^0}^{D_0}(x^0) \leq \left\| b^0 - \widetilde{b}^0 \right\|_\infty,$$

and so, (3.7) yields a contradiction.

Step 4. Finally we show that the infimum in (3.6) is attained.

For fixed $b \in V$ and $x \in U$, let $D^r := \{t_1^r, ..., t_n^r\} \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$, $r = 1, 2, ...$, be such that

$$\inf_{D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_b^D(x) = \lim_{r \to \infty} f_b^{D^r}(x), \tag{3.8}$$

and write, for each $r \in \mathbb{N}$,

$$u^r + \bar{c} = \sum_{i=1}^{n} \lambda_i^r u_i^r,$$

where, for all $r \in \mathbb{N}$ and all $i \in \{1, ..., n\}$,

$$u^r \in \partial f(\bar{x}), \; u_i^r \in -\partial g_{t_i^r}(\bar{x}) \text{ for some } t_i^r \in T_{\overline{b}}^{\delta}(\bar{x}), \text{ and } \lambda_i^r \geq 0.$$

Then, standard arguments yield the existence of some subsequence of r's, denoted as the original sequence for the sake of simplicity, such that

$$u^r \to u,\ u_i^r \to u_i,\ t_i^r \to t_i,\ \text{and } \lambda_i^r \to \lambda_i,\ i = 1, ..., n,$$

for certain

$$u \in \partial f\left(\overline{x}\right),\ u_i \in -\partial g_{t_i}\left(\overline{x}\right),\ t_i \in T_{\overline{b}}^{\delta}\left(\bar{x}\right),\ \text{and } \lambda_i \geq 0,$$

verifying

$$u + \overline{c} = \sum_{i=1}^{n} \lambda_i u_i.$$

(Specifically, [23], Thm. 24.5, ensures the convergence of subgradients, and SCQ entails the boundedness of the sequence $\{\sum_{i=1}^{n} \lambda_i^r\}$ which yields the existence of convergent subsequences of $\{\lambda_i^r\}_{r\in\mathbb{N}}$, for $i = 1, ..., n$.)

Now, ENC at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ entails that $\{u_1, ..., u_n\}$ is a basis of $\mathbb{R}^n$ (otherwise, Carathéodory's theorem would allow us to remove some term in $\sum_{i=1}^{n} \lambda_i u_i$, which contradicts ENC), and $\lambda_i > 0$ for all $i = 1, ..., n$. Therefore, the index set $\widetilde{D} := \{t_1, ..., t_n\}$ satisfies $\widetilde{D} \subset T_{\overline{b}}^{\delta}\left(\bar{x}\right)$, $\left|\widetilde{D}\right| = n$, and

$$\left(\partial f\left(\overline{x}\right) + \bar{c}\right) \cap \operatorname{cone}\left(\bigcup_{t\in\widetilde{D}} \left(-\partial g_t(\bar{x})\right)\right) \neq \emptyset;$$

in other words $\widetilde{D} \in \mathcal{T}_{\overline{b}}^{\delta}\left(\bar{x}\right)$. Consequently, from (3.1) we easily conclude that $f_b^{D^r}(x) \to f_b^{\widetilde{D}}(x)$ as $r \to \infty$; *i.e.*, (3.8) reads

$$\inf_{D\in\mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_b^{D}(x) = f_b^{\widetilde{D}}(x). \qquad \square$$

As a consequence of Theorem 3.1 we will establish the Lipschitz continuity around $\overline{x}$ of functions f_b for b close enough to $\overline{b}$. We need the following lemma.

Lemma 3.1 ([23], Thm. 10.6)**.** *For every compact neighborhood of $\overline{x}$, U, the functions g_t, $t \in T$, are equi-Lipschitzian on U;* i.e. *there exists $L > 0$ such that*

$$\sup_{t\in T} |g_t(z) - g_t(z')| \leq L \|z - z'\|, \ \textit{for all } z, z' \in U. \tag{3.9}$$

Proposition 3.1. *Let $L > 0$ be such that (3.9) holds for a certain neighborhood of $\overline{x}$, U. If* ENC *is satisfied at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}\left(\mathcal{F}^*\right)$, then there exist neighborhoods U_0 and V_0 of $\overline{x}$ and $\overline{b}$, respectively, such that*

$$|f_b(z) - f_b(z')| \leq L \|z - z'\|, \ \textit{for all } z, z' \in U_0, \ \textit{and } b \in V_0.$$

Proof. According to Theorem 3.1, take $\delta > 0$ and neighborhoods of $\overline{x}$ and $\overline{b}$, U_0 and V_0, respectively, such that

$$f_b(x) = \min_{D\in\mathcal{T}_{\overline{b}}^{\delta}(\overline{x})} f_b^{D}(x), \ \text{for all } x \in U_0 \text{ and } b \in V_0. \tag{3.10}$$

We may assume, without loss of generality, that $U_0 \subset U$. So, in addition we have

$$\sup_{t\in T} |g_t(z) - g_t(z')| \leq L \|z - z'\|, \ \text{for all } z, z' \in U_0.$$

Thus, for each $b \in V_0$, $D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$, and $z, z' \in U_0$, we have:

$$\begin{aligned} f_b^D(z) &= \max\{b_t - g_t(z),\ t \in D;\ g_t(z) - b_t,\ t \in T\} \\ &\leq \max\{b_t - g_t(z'),\ t \in D;\ g_t(z') - b_t,\ t \in T\} + \sup_{t \in T} |g_t(z) - g_t(z')| \\ &\leq f_b^D(z') + L \|z - z'\|, \end{aligned}$$

which leads us, by (3.10), to $f_b(z) \leq f_b(z') + L\|z - z'\|$. (Note that, for all $z, z' \in U_0$, $f_b(z) - L\|z - z'\| \leq f_b(z')$.) By symmetry, we finally obtain

$$|f_b(z) - f_b(z')| \leq L \|z - z'\|. \qquad \square$$

4. Lipschitz modulus of $\mathcal{F}^*$ in terms of the problem's data

This section makes use of the representation of f_b given in Theorem 3.1 for providing lower and upper bounds, as well as the exact value when T is finite, for $\operatorname{lip}\mathcal{F}^*(\overline{c}, \overline{b})$. These lower and upper bounds have the virtue of relying on the problem's data. Specifically, the lower bound comes from the Lipschitz constant appearing in Proposition 3.1, while the upper bound comes from applying Theorem 2.2.

4.1. **Lower bound on the Lipschitz modulus**

Proposition 4.1. *Let $L > 0$ be such that (3.9) holds for some neighborhood of $\overline{x}$, U. If* ENC *is satisfied at* $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$, *then*

$$\operatorname{lip}\mathcal{F}^*(\overline{c}, \overline{b}) \geq L^{-1}.$$

Proof. Recall that $\operatorname{lip}\mathcal{F}^*(\overline{c}, \overline{b})$ is the infimum of all (Lipschitz) constants $\kappa \geq 0$ verifying (1.4) for some associated neighborhoods. Take any of these constants, say $\kappa \geq 0$, and consider associated neighborhoods U of $\overline{x}$ and W of $\left(\overline{c}, \overline{b}\right)$, such that

$$d\left(x, \mathcal{F}^*(c, b)\right) \leq \kappa d\left((c, b), (\mathcal{F}^*)^{-1}(x)\right), \tag{4.1}$$

for all $x \in U$ and all $(c, b) \in W$. We have to prove that $\kappa \geq L^{-1}$.

According to Proposition 3.1 let U_0 and V_0 neighborhoods of $\overline{x}$ and $\overline{b}$, respectively, such that

$$|f_b(z) - f_b(z')| \leq L \|z - z'\|, \text{ for all } z, z' \in U_0, \text{ and } b \in V_0.$$

Assume, without loss of generality, that $U_0 \subset U$ and $\{\overline{c}\} \times V_0 \subset W$. Then, applying (4.1) in the particular case $(c, b) = \left(\overline{c}, \overline{b}\right)$, we obtain, for each $z \in U_0$,

$$\begin{aligned} d(z, \overline{x}) &= d(z, \mathcal{F}^*(\overline{c}, \overline{b})) \leq \kappa d((\overline{c}, \overline{b}), (\mathcal{F}^*)^{-1}(z)) \leq \kappa d((\overline{c}, \overline{b}), \{\overline{c}\} \times \widetilde{\mathcal{G}}(z)) \\ &= \kappa d(\overline{b}, \widetilde{\mathcal{G}}(z)) = \kappa f_{\overline{b}}(z) = \kappa (f_{\overline{b}}(z) - f_{\overline{b}}(\overline{x})) \leq \kappa L d(z, \bar{x}). \end{aligned}$$

We have taken into account that $\mathcal{F}^*(\overline{c}, \overline{b}) = \{\overline{x}\}$ and $f_{\overline{b}}(\overline{x}) = 0$. Hence, obviously $\kappa L \geq 1$, which finishes the proof. $\square$

Corollary 4.1. *When confined to the linear case, say $g_t(x) := \langle a_t, x \rangle$ with $a_t \in \mathbb{R}^n$, for all $t \in T$, then*

$$\operatorname{lip}\mathcal{F}^*\left(\overline{x} \mid (\overline{c}, \overline{b})\right) \geq \left(\sup_{t \in T} \|a_t\|_*\right)^{-1}.$$

This inequality may be strict, in general. See Example 4.1 (at the end of this section), where

$$\operatorname{lip}\mathcal{F}^*\left(\overline{x} \mid (\overline{c}, \overline{b})\right) = \sqrt{5} > \frac{1}{\sqrt{2}} = \left(\sup_{t \in T} \|a_t\|_*\right)^{-1}.$$

4.2. Upper bound on the Lipschitz modulus

The aimed upper bound on $\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b})$ follows from the relationship between the regular subdifferentials of the functions f_b and f_b^D, with $D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$ (recall Thm. 3.1). First, we provide an operative expression for $\widehat{\partial} f_b^D(z)$, with z and b near $\overline{x}$ and $\overline{b}$, respectively, and $D \in \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$. To do this, let us observe that each f_b^D is a d.c. function since

$$f_b^D = f_{1,b}^D - f_{2,b}^D,$$

where

$$f_{1,b}^D(z) := \max\left\{ \begin{array}{l} (\max_{t\in T}(g_t(z)-b_t)) + \sum_{s\in D}(g_s(z)-b_s); \\ \sum_{s\in D\setminus\{t\}}(g_s(z)-b_s),\ t\in D \end{array} \right\}$$

and

$$f_{2,b}^D(z) := \sum_{t\in D}(g_t(z)-b_t).$$

Now, appealing to (2.5) and applying Valadier's formula (see [26] or, for instance, [12], Thm. VI.4.4.2) for calculating $\partial f_{1,b}^D(z)$ and $\partial f_{2,b}^D(z)$, with $z\in\mathbb{R}^n$, the following lemma is easily derived.

Lemma 4.1. *For any $b \in C(T,\mathbb{R})$, $D \subset T$ with $|D| = n$, and $z \in \mathbb{R}^n$, we have*

$$\widehat{\partial} f_b^D(z) = \partial f_{1,b}^D(z) \div \partial f_{2,b}^D(z),$$

with

$$\partial f_{1,b}^D(z) = \operatorname{co}\left\{ \partial g_t(z) + \sum_{s\in D}\partial g_s(z),\ t\in I_b^D(z);\ \sum_{s\in D\setminus\{t\}}\partial g_s(z),\ t\in J_b^D(z) \right\},$$

and

$$\partial f_{2,b}^D(z) = \sum_{t\in D}\partial g_t(z),$$

where

$$I_b^D(z) := \left\{ t\in T \;\middle|\; f_{1,b}^D(z) = g_t(z) - b_t + \sum_{s\in D}(g_s(z)-b_s) \right\},$$

and

$$J_b^D(z) := \left\{ t\in D \;\middle|\; f_{1,b}^D(z) = \sum_{s\in D\setminus\{t\}}(g_s(z)-b_s) \right\}.$$

Lemma 4.2. *Assume that* ENC *is satisfied at $\left((\overline{c},\overline{b}),\overline{x}\right) \in \operatorname{gph}(\mathcal{F}^*)$. Then, a certain $\delta_0 > 0$ exists such that for all $\delta \in \left]0,\delta_0\right]$ there exist some neighborhoods U and V of $\overline{x}$ and $\overline{b}$, respectively, satisfying*

$$\widehat{\partial} f_b(z) \subset \bigcap_{D\in\mathcal{S}(z,b,\delta)} \widehat{\partial} f_b^D(z),\ \text{for all } z\in U \text{ and } b\in V,$$

where

$$\mathcal{S}(z,b,\delta) := \left\{ D\in\mathcal{T}_{\overline{b}}^{\delta}(\overline{x}) \mid f_b(z) = f_b^D(z) \right\}.$$

Proof. The proof is a straightforward consequence of (2.4) and Theorem 3.1. □

The following theorem provides the aimed upper bound for the Lipschitz modulus of $\mathcal{F}^*$, whose main feature is that it only relies on the problem's data. Further, this upper bound equals the exact modulus when T is finite.

Theorem 4.1. *Assuming that* ENC *is satisfied at* $\left((\overline{c},\overline{b}),\overline{x}\right)\in \operatorname{gph}\left(\mathcal{F}^*\right)$, *one has*

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) \leq \limsup_{\substack{(z,b,\delta)\to(\overline{x},\overline{b},0)\\ f_b(z)>0}} \inf_{D\in\mathcal{S}(z,b,\delta)} \left(d_*\left(0_n, \partial f_{1,b}^D(z) \div \partial f_{2,b}^D(z)\right)\right)^{-1}. \tag{4.2}$$

(Formulae for $\partial f_{1,b}^D(z)$ *and* $\partial f_{2,b}^D(z)$ *in* (4.2) *are given in Lem.* 4.1.*)*

In the particular case when T *is finite we get*

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n, \partial f_{1,b}^D(z) \div \partial f_{2,b}^D(z)\right)\right)^{-1}. \tag{4.3}$$

Proof. Write, according to Theorem 2.2,

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \lim_{r\to\infty}\left(d_*\left(0_n, \widehat{\partial} f_{b^r}(z^r)\right)\right)^{-1}, \tag{4.4}$$

where $b^r\to\overline{b}$, $z^r\to\overline{x}$ and $f_{b^r}(z^r)>0$ for all r. If we take any sequence $\delta_k\searrow 0$, according to Lemma 4.2 (and keeping the notation introduced there), for k large enough there must exist neighborhoods U_k and V_k of $\overline{x}$ and $\overline{b}$, respectively, such that

$$\widehat{\partial} f_b(z) \subset \bigcap_{D\in\mathcal{S}(z,b,\delta^k)} \widehat{\partial} f_b^D(z), \text{ for all } z\in U_k \text{ and all } b\in V_k. \tag{4.5}$$

Now, take subsequences $\{z^{r_k}\}$ and $\{b^{r_k}\}$ of $\{z^r\}$ and $\{b^r\}$, respectively, such that $z^{r_k}\in U_k$ and $b^{r_k}\in V_k$ for all k. Then, as a consequence of (4.5) particularized at $b=b^{r_k}$ and $z=z^{r_k}$, we conclude, for each k,

$$d_*\left(0_n, \widehat{\partial} f_{b^{r_k}}(z^{r_k})\right) \geq \sup_{D\in\mathcal{S}(z^{r_k},b^{r_k},\delta^k)} d_*\left(0_n, \widehat{\partial} f_{b^{r_k}}^D(z^{r_k})\right).$$

Thus (4.4) yields

$$\begin{aligned}\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) &\leq \limsup_{k\to\infty} \inf_{D\in\mathcal{S}(z^{r_k},b^{r_k},\delta^k)} \left(d_*\left(0_n, \widehat{\partial} f_{b^{r_k}}^D(z^{r_k})\right)\right)^{-1}\\ &\leq \limsup_{\substack{(z,b,\delta)\to(\overline{x},\overline{b},0)\\ f_b(z)>0}} \inf_{D\in\mathcal{S}(z,b,\delta)} \left(d_*\left(0_n, \widehat{\partial} f_b^D(z)\right)\right)^{-1}.\end{aligned}$$

Hence, (4.2) comes from Lemma 4.1.

From now on we suppose that T is finite, which entails $\mathcal{T}_{\overline{b}}^{\delta}(\overline{x})=\mathcal{T}_{\overline{b}}(\overline{x})$ for $\delta>0$ small enough. Then inequality '$\leq$' of (4.3) comes straightforwardly from (4.2). So, let us see that

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) \geq \limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n, \widehat{\partial} f_b^D(z)\right)\right)^{-1},$$

which yields (4.3), by taking again Lemma 4.1 into account.

From Theorem 2.2 we have

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}} \left(\left|\nabla f_b\right|(z)\right)^{-1} < +\infty, \tag{4.6}$$

where the finiteness comes from the fact that ENC implies strong Lipschitz stability of $\mathcal{F}^*$ (and then $\mathrm{lip}\mathcal{F}^*(\bar{c},\bar{b}) < +\infty$). Next, take any sequence $\{(z^r,b^r)\}$ converging to $(\overline{x},\overline{b})$ such that $f_{b^r}(z^r) > 0$ for all r, and let us prove that

$$\mathrm{lip}\mathcal{F}^*(\bar{c},\bar{b}) \geq \limsup_{r\to\infty} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_{b^r}(z^r)=f_{b^r}^D(z^r)}} \left(d_*\left(0_n,\widehat{\partial} f_{b^r}^D(z^r)\right)\right)^{-1}.$$

Expression (4.6) ensures $|\nabla f_{b^r}|(z^r) > 0$ for r large enough (w.l.o.g. for all r), and so we can write

$$|\nabla f_{b^r}|(z^r) = \lim_k \frac{f_{b^r}(z^r) - f_{b^r}(z^{r,k})}{\|z^r - z^{r,k}\|},$$

for a certain $\{z^{r,k}\}_{k\in\mathbb{N}}$ converging to z^r. Take $\delta_0 > 0$ according to Theorem 3.1 and such that $\mathcal{T}_{\overline{b}}^{\delta}(\overline{x}) = \mathcal{T}_{\overline{b}}(\overline{x})$ for all $\delta \leq \delta_0$. Now fix any $\delta \leq \delta_0$ and consider associated open neighborhoods U and V of $\overline{x}$ and $\overline{b}$, respectively, verifying

$$f_b(x) = \min_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} f_b^D(x), \text{ for } x \in U \text{ and } b \in V. \tag{4.7}$$

For r large enough, w.l.o.g. for all r, $x^r \in U$ and $b^r \in V$.

Consider an arbitrarily fixed $r \in \mathbb{N}$. Let $k(r) \in \mathbb{N}$ be such that $z^{r,k} \in U$ for all $k \geq k(r)$, which entails, taking (4.7) into account,

$$f_{b^r}(z^{r,k}) = f_{b^r}^{D^{r,k}}(z^{r,k})$$

for a certain $D^{r,k} \in \mathcal{T}_{\overline{b}}(\overline{x})$. The finiteness of T allows us to assume (by considering a suitable subsequence if necessary) $D^{r,k} = D^r$ for all $k \geq k(r)$. Then, because of the continuity of each f_{b^r} and $f_{b^r}^{D^r}$, we have

$$f_{b^r}(z^r) = \lim_{k\geq k(r)} f_{b^r}(z^{r,k}) = \lim_{k\geq k(r)} f_{b^r}^{D^r}(z^{r,k}) = f_{b^r}^{D^r}(z^r).$$

So,

$$|\nabla f_{b^r}|(z^r) = \lim_{k\geq k(r)} \frac{f_{b^r}^{D^r}(z^r) - f_{b^r}^{D^r}(z^{r,k})}{\|z^r - z^{r,k}\|} \leq \left|\nabla f_{b^r}^{D^r}\right|(z^r) \leq d_*\left(0_n,\widehat{\partial} f_{b^r}^{D^r}(z^r)\right),$$

where the last inequality comes from (2.2). Therefore, (4.6) entails

$$\begin{aligned}\mathrm{lip}\mathcal{F}^*(\bar{c},\bar{b}) &\geq \limsup_{r\to\infty}\left(|\nabla f_{b^r}|(z^r)\right)^{-1} \geq \limsup_{r\to\infty}\left(d_*\left(0_n,\widehat{\partial} f_{b^r}^{D^r}(z^r)\right)\right)^{-1}\\ &\geq \limsup_{r\to\infty} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_{b^r}(z^r)=f_{b^r}^D(z^r)}} \left(d_*\left(0_n,\widehat{\partial} f_{b^r}^D(z^r)\right)\right)^{-1},\end{aligned}$$

as we aimed to prove. □

When confined to the case of linear semi-infinite problems in the form

$$\begin{aligned} P(c,b):\quad & \mathrm{Inf}\,\langle c,x\rangle \\ & \text{s.t. } \langle a_t,x\rangle \leq b_t,\ \ t\in T, \end{aligned} \tag{4.8}$$

Theorem 4.1 adopts the following simplified form:

Corollary 4.2. *Assuming that $\mathcal{F}^*$, associated with problem (4.8), is strongly Lipschitz stable at $(\overline{x},(\bar{c},\overline{b})) \in \mathrm{gph}(\mathcal{F}^*)$, one has*

$$\mathrm{lip}\mathcal{F}^*(\bar{c},\bar{b}) \leq \limsup_{\substack{(z,b,\delta)\to(\overline{x},\overline{b},0)\\ f_b(z)>0}} \inf_{\substack{D\in\mathcal{T}_{\overline{b}}^{\delta}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n,C_b^D(z)\right)\right)^{-1}, \tag{4.9}$$

where

$$\begin{aligned}
C_b^D(z) &:= \mathrm{co}\{a_t,\ t \in I_b^D(z);\ \ -a_t,\ t \in J_b^D(z)\},\\
I_b^D(z) &:= \{t \in T \mid \langle a_t, z\rangle - b_t = f_b^D(z)\},\\
J_b^D(z) &:= \{t \in D \mid b_t - \langle a_t, z\rangle = f_b^D(z)\},\\
f_b^D(z) &:= \max\{|\langle a_t, z\rangle - b_t|\,,\ t \in D;\ \langle a_t, z\rangle - b_t,\ t \in T \setminus D\}.
\end{aligned}$$

In the particular case when T is finite, one gets

$$\mathrm{lip}\mathcal{F}^*\left(\overline{x} \mid (\overline{c}, \overline{b})\right) = \limsup_{\substack{(z,b)\to(\overline{x},\overline{b})\\ f_b(z)>0}} \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n, C_b^D(z)\right)\right)^{-1}. \tag{4.10}$$

Proof. First, the strong Lipschitz stability assumption in this linear context is equivalent to ENC (see [3], Thm. 16). So, Theorem 4.1 applies and the reader can easily check that $\partial f_{1,b}^D(z) \div \partial f_{2,b}^D(z) = C_b^D(z)$ for all $z \in \mathbb{R}^n$ and $b \in C(T, \mathbb{R})$ (recall Lem. 4.1 and (2.5)). □

We finish this section with an example devoted to illustrate the ingredients of Corollary 4.2 and, in general, the main ingredients of this subsection.

Example 4.1. Consider the linear problem, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lrl}
P(\overline{c}, b): & \mathrm{Inf} & 2x_1 + x_2\\
\quad s.\,t. & & -x_1 - x_2 \le b_1,\\
 & & -x_1 + x_2 \le b_2,\\
 & & -x_1 \le b_3,
\end{array}$$

with $\overline{b} = 0_3$ and $\overline{x} = 0_2$.

In this case $\mathcal{T}_{\overline{b}}(\overline{x}) = \{\{1,2\},\{1,3\}\}$. For (z,b) close enough to $\left(\overline{x}, \overline{b}\right)$ we have

$$f_b(z) = \min\left\{f_b^{\{1,2\}}(z), f_b^{\{1,3\}}(z)\right\},$$

where

$$\begin{aligned}
f_b^{\{1,2\}}(z) &= \max\{|-z_1 - z_2 - b_1|, |-z_1 + z_2 - b_2|, -z_1 - b_3\},\\
f_b^{\{1,3\}}(z) &= \max\{|-z_1 - z_2 - b_1|, -z_1 + z_2 - b_2, |-z_1 - b_3|\}.
\end{aligned}$$

Next we apply the previous corollary to obtain the regularity modulus. Observe that, for z and b close enough to $\left(\overline{x}, \overline{b}\right)$, the expression

$$\rho(z,b) := \min_{\substack{D\in\mathcal{T}_{\overline{b}}(\overline{x})\\ f_b(z)=f_b^D(z)}} \left(d_*\left(0_n, C_b^D(z)\right)\right)^{-1},$$

only takes finitely many different values, specifically,

$$\rho(z,b) \in \left\{\tfrac{1}{\sqrt{2}}, 1, \sqrt{5}\right\}.$$

For the sake of simplicity we do not enumerate all possible combinations of $\{I_{b^r}^D(z^r), J_{b^r}^D(z^r)\}$ with $(z^r, b^r) \to (\overline{x}, \overline{b})$, $D \in \mathcal{T}_{\overline{b}}(\overline{x})$, $f_{b^r}^D(z^r) = f_{b^r}(z^r)$, but only the three cases of the following table, which lead to the three only possible values of $\rho(z, b)$:

z^r, b^r	$f_{b^r}^{\{1,2\}}(z^r)$, $f_{b^r}^{\{1,3\}}(z^r)$	$D\in\mathcal{T}_{\overline{b}}(\overline{x})$: $f_{b^r}^D(z^r)=f_{b^r}(z^r)$	$I_{b^r}^D(z^r), J_{b^r}^D(z^r)$ ($D\in\mathcal{T}_{\overline{b}}(\overline{x})$, with $f_{b^r}^D(z^r)=f_{b^r}(z^r)$)	$\rho(z^r,b^r)$
$\left(\frac{-1}{r}, \frac{-1}{r}\right), 0_3$	$\frac{2}{r}, \frac{2}{r}$	$\{1,2\}, \{1,3\}$	$\{1\}, \varnothing$	$\frac{1}{\sqrt{2}}$
$\left(\frac{1}{r}, 0\right), \left(0, 0, \frac{1}{r}\right)$	$\frac{1}{r}, \frac{2}{r}$	$\{1,3\}$	$\varnothing, \{3\}$	1
$\left(0, \frac{1}{r}\right), \left(0, 0, \frac{-1}{r}\right)$	$\frac{1}{r}, \frac{1}{r}$	$\{1,2\}, \{1,3\}$	$\{2,3\}, \{1\}$	$\sqrt{5}$

In particular, it is clear that
$$\mathrm{lip}\mathcal{F}^*\left(\overline{x} \mid (\overline{c}, \overline{b})\right) = \sqrt{5}.$$

5. Final comments and further research

5.1. Tchebycheff approximation

A typical application of semi-infinite optimization can be found in functional approximation (see, for instance, [11], Sect. 1.2). Specifically, the uniform approximation of a given continuous function $h : [a, b] \to \mathbb{R}$ by a function of the n-dimensional subspace $\mathcal{V} \subset C([a, b], \mathbb{R})$ spanned by linearly independent functions $f_1, ..., f_n \in C([a, b], \mathbb{R})$, gives rise to the linear semi-infinite programming problem with variables $x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$,

$$\begin{aligned} P(h): \quad & \text{Inf } x_{n+1} \\ & \text{s.t. } \pm(f_1(t)x_1 + ... + f_n(t)x_n - h(t)) \le x_{n+1}, \ \forall\, t \in [a, b]. \end{aligned} \tag{5.1}$$

$P(h)$ can be rewritten into our standard format with $T := [a, b] \times \{1, 2\}$, $g_{(t,i)}(x) = (-1)^i (f_1(t)x_1 + ...+ f_n(t)x_n) - x_{n+1}$, $\overline{c} = (0_n, 1)$, and $b_{(t,i)} = (-1)^i h(t)$ for $(t, i) \in T$. We shall confine ourselves to those parameters $b \in C(T, \mathbb{R})$ which preserve the structure of (5.1), *i.e.*, $b_{(t,1)} = -b_{(t,2)}$ for all $t \in [a, b]$. In such a way, we regard $h \in C([a, b], \mathbb{R})$ as the parameter, and write everywhere h instead of b in the notation followed in this paper. It is easy to see that $\mathcal{F}^*(\overline{c}, h) \neq \varnothing$ for any $h \in C([a, b], \mathbb{R})$ ([11], Sect. 1.2.2). Moreover, SCQ holds for all $h \in C([a, b], \mathbb{R})$; in fact $(0_n, \rho + 1)$, with $\rho := \max_{t\in[a,b]} |h(t)|$, is a Slater point for $P(h)$.

In uniform functional approximation, the following well-known condition characterizes the unicity of the optimal solution of $P(h)$, whichever $h \in C([a, b], \mathbb{R})$ we take (see, for instance, [18], Thm. 3.4.6):

Definition 5.1. The subspace $\mathcal{V}$ verifies the *Haar condition* if every function $g \in \mathcal{V}$ has no more than $n - 1$ zeros in $[a, b]$. Equivalently, it is said that the functions generating $\mathcal{V}$, *i.e.*, $f_1, ..., f_n$, form a *Tchebycheff system*.

It is obvious that $f_i(t) := t^{i-1}$, $i = 1, ..., n$, form a Tchebycheff system whichever interval $[a, b]$ we consider. It is well-known (see, for instance, the proof of Prop. 3.4.2 in [18]) that $f_1, ..., f_n$ form a Tchebycheff system if and only if, for every set of distinct points $t_1, ..., t_n \in [a, b]$, the determinant of the matrix whose (i, j)-entry is $f_i(t_j)$ is different from zero (this condition is trivially satisfied in the case of polynomial approximation, since we get there a Van der Monde matrix). Provided that $f_1, ..., f_n$ form a Tchebycheff system, the Tchebycheff alternancy theorem (see, for instance, [18], Thm. 3.5.2) characterizes the best approximation of $h \in C([a, b], \mathbb{R}) \setminus \mathcal{V}$ in $\mathcal{V}$, say $v = x_1 f_1 + ... + x_n f_n$ (*i.e.*, $\mathcal{F}^*(\overline{c}, h) = \{(x_1, ..., x_n, x_{n+1})\}$, with $x_{n+1} = \|v - h\|_\infty$) in the following terms: there exist $t_1 < t_2 < ... < t_{n+1}$ in $[a, b]$ such that $|v(t_i) - h(t_i)| = \|v - h\|_\infty$ for $i = 1, ..., n + 1$, and $v(t_i) - h(t_i) = -(v(t_{i+1}) - h(t_{i+1}))$ for $i = 1, ..., n$.

Proposition 5.1. *Given the parametrized problem* (5.1), *if* $f_1, ..., f_n$ *form a Tchebycheff system, then ENC holds at* $\left(\left(\overline{c}, \overline{h}\right), \overline{x}\right)$ *for every* $\overline{h} \in C\left([a,b], \mathbb{R}\right) \setminus \mathcal{V}$, $\overline{x}$ *being the only point of* $\mathcal{F}^*\left(\overline{c}, \overline{h}\right)$.

Proof. First, note that $h \in \mathcal{V}$ if and only if $P\left(h\right)$ has a feasible solution such that $x_{n+1} = 0$ (such a solution is obviously optimal). So, if $\overline{h} \in C\left([a,b], \mathbb{R}\right) \setminus \mathcal{V}$ and $\overline{x} \in \mathcal{F}^*\left(\overline{c}, \overline{h}\right)$, we have $\overline{x}_{n+1} > 0$. Reasoning by contradiction, assume that we may write, for some $D = (S_1 \times \{1\}) \cup (S_2 \times \{2\}) \subset T_{\overline{h}}\left(\overline{x}\right)$, $S_1, S_2 \subset [a,b]$, $|S_1| + |S_2| < n+1$,

$$(0_n, 1) = \sum_{t \in S_1} \lambda_{(t,1)} \left(f_1\left(t\right), ..., f_n\left(t\right), 1\right) + \sum_{t \in S_2} \lambda_{(t,2)} \left(-f_1\left(t\right), ..., -f_n\left(t\right), 1\right). \tag{5.2}$$

In particular, $0_n = \sum_{(t,i) \in D} \left(-1\right)^{i+1} \lambda_{(t,i)} \left(f_1\left(t\right), ..., f_n\left(t\right)\right)$. Moreover, the fact that $D \subset T_{\overline{h}}\left(\overline{x}\right)$, together with $\overline{x}_{n+1} > 0$, entails $S_1 \cap S_2 = \varnothing$. Completing with zeros, if needed, the $\left(-1\right)^{i+1} \lambda_{(t,i)}$'s are solutions of an homogeneous system of n $(\geq |S_1| + |S_2|)$ linear equations and n variables associated with n different points $t_1, ..., t_n \in [a,b]$ such that $S_1 \cup S_2 \subset \{t_1, ..., t_n\}$. Since we have assumed the Haar condition, the coefficient matrix of this homogeneous system is non-singular and, so, the unique possible solution is the identically zero vector. The last statement contradicts $\sum_{t \in S_1} \lambda_{(t,1)} + \sum_{t \in S_2} \lambda_{(t,2)} = 1$, which comes from the last coordinate in (5.2). □

Remark 5.1. For the parametrized problem (5.1), the previous proposition points out that ENC at $\left(\left(\overline{c}, \overline{h}\right), \overline{x}\right)$ is weaker than the Haar condition (for $\mathcal{V}$) and entails, not only the unicity of the uniform best approximation of h $(\notin \mathcal{V})$ in $\mathcal{V}$, but also the Lipschitzian dependence of this best approximation on h.

5.2. More about ENC

As pointed out in [3], Remark 12, ENC is in general rather strong in relation to the strong Lipschitz stability of minimizers, although it has some remarkable virtues. One of them is that ENC is formulated in terms of the nominal data $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, not involving parameters and points in a neighborhood. Moreover, it entails a nice behavior not only for $\mathcal{F}^*$, but also for its inverse, which turns out to be lower semicontinuous (see [5], Thm. 4.3). Roughly speaking, points near $\overline{x}$ are optimal for parameters near $\left(\overline{c}, \overline{b}\right)$. In the following example $\mathcal{F}^*$ is strongly Lipschitz stable at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, ENC fails at this point, and $\left(\mathcal{F}^*\right)^{-1}$ is not lower semicontinuous at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right)$. In particular, $\widetilde{\mathcal{G}}$ is not lower semicontinuous at $\left(\overline{x}, \overline{b}\right)$ (see Thm. 2.1(iv)). (Recall that Thm. 2.1(iv) is one of the consequences of ENC used in the proof of Thm. 3.1, and the latter is a key result in the present paper.)

Example 5.1 ([5], Ex. 4.1)**.** Consider the problem in $\mathbb{R}^2$,

$$P\left(c,b\right) : \text{Inf}\ \{c_1 x_1 + c_2 x_2 \mid |x_1| - x_2 \leq b_1, \ \ -x_1 \leq b_2\}.$$

Take $\overline{c} = \left(\frac{1}{2}, 1\right), \overline{b} = 0_2$, $\overline{x} = 0_2$. One can easily check that $\mathcal{F}^*\left(c,b\right) = \{((-b_2)^+, \ -b_1 + (-b_2)^+)\}$, for all c such that $|c_1| - c_2 < 0$ and all $b \in \mathbb{R}^2$, where $(-b_2)^+ := \max\{-b_2, 0\}$. Thus, $\mathcal{F}^*$ is strongly Lipschitz stable at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, although ENC is not satisfied at this point since $\overline{c} \in \text{cone}\left(-\partial g_1\left(\overline{x}\right)\right)$ (here $f \equiv 0$). Nevertheless, the point $\left(\frac{-1}{r}, \frac{1}{r}\right)$, $r \in \mathbb{N}$, is not optimal for any b such that $\left(\frac{-1}{r}, \frac{1}{r}\right) \in \mathcal{F}(b)$ and any c such that $|c_1| - c_2 < 0$.

Due to the crucial role of ENC along the paper, a natural question is how typical this property is. In relation to this, Theorem 2.1(i) guarantees that the set of $((c,b), x)$ for which ENC holds is open in the topology relative to gph$(\mathcal{F}^*)$, but this open set may be quite small as the following example shows.

Example 5.2. Consider the problem in $\mathbb{R}^2$,

$$P\left(c,b\right) : \text{Inf}\ \left\{c_1 x_1 + c_2 x_2 \mid x_1 + x_2^2 \leq b_1, \ \ -x_1 + x_2^2 \leq b_2\right\}.$$

One can easily check that for any pair of positive numbers $b=(b_1,b_2)$ the set of (c,x) such that ENC holds at $((c,b),x)\in \mathrm{gph}(\mathcal{F}^*)$ is exactly $\left(U_1\times\{z^1\}\right)\cup\left(U_2\times\{z^2\}\right)$, where, for $i=1,2$,

$$\begin{aligned} z^i &= \left(\frac{b_1-b_2}{2},(-1)^{i-1}\sqrt{\frac{b_1+b_2}{2}}\right), \ \textit{and} \\ U_i &= \left\{c\in\mathbb{R}^2 \mid (-1)^{i-1}c_2 < -\sqrt{2(b_1+b_2)}\,|c_1|\right\}. \end{aligned}$$

Observe that sets U_1 and U_2 shrink as far as b_1+b_2 increase.

The reader is addressed to [3,5] for additional details about ENC at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ and its implications. Nevertheless, for completeness purposes, here we gather some of these implications:

- [3], Proposition 14 and Remark 15. *Strong uniqueness* of $\overline{x}$ as a minimizer of $P\left(\overline{c},\overline{b}\right)$, which constitutes a first order growth condition on f at $\overline{x}$ with respect to $\sigma(\overline{b})$, and it can be formalized in terms of the existence of a positive scalar α such that

$$f(y)+\overline{c}'y\geq f(\overline{x})+\overline{c}'\overline{x}+\alpha\left\|y-\overline{x}\right\|, \text{ for all } y\in\mathcal{F}\left(\overline{b}\right). \tag{5.3}$$

  Indeed, ENC at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ implies that $P(c,b)$ has a strongly unique minimizer (also called *sharp minima*) for (c,b) close to $\left(\overline{c},\overline{b}\right)$. The reader is addressed, for instance, to [1,25] for further information, in particular, for the analysis of the so called *weak sharp minima* (coming from removing the uniqueness).
- Observe that at least n constraints have to be active at $\overline{x}$. In the case of finite optimization problems with twice differentiable data, certain second-order growth conditions (typically held with less than n active constraints) are sufficient and necessary for the strong Lipschitz stability of $\mathcal{F}^*$ (see, *e.g.*, [15], Chap. 8, and [16]). The generalization of such results to our current framework remains as an open problem. Note that the requirement of having at least n active constraints yields a notable difference with respect to the classical optimization theory with twice differentiable data (see, for instance, [10]).
- [5], Theorem 4.1. From a geometric point of view, the following implication goes beyond the condition of Theorem 2.1(ii): *For all* $D:=\{t_1,...,t_n\}\in\mathcal{T}_{\overline{b}}(\overline{x})$ and *every* $u_i\in-\partial g_{t_i}(\overline{x})$, $i=1,...,n$, one has

$$\partial f(\overline{x})+\overline{c}\subset \mathrm{int}(\mathrm{cone}(\{u_1,...,u_n\})). \tag{5.4}$$

  In fact, one can easily find examples verifying the condition of Theorem 2.1(ii), but not this stronger one.

Example 5.3 (Ex. 5.1 revisited. [3], Ex. 6)**.** Consider the problem:

$$P(c,b): \mathrm{Inf}\ \{c_1x_1+c_2x_2 \mid -x_1-x_2\leq b_1,\ -x_1+x_2\leq b_2,\ -x_1\leq b_3\}.$$

Let $\overline{c}:=(1,0)$, $\overline{b}=0_3$, and $\overline{x}=0_2$. Obviously, $\overline{c}\in \mathrm{int}(\mathrm{cone}(\{(1,1),(1,-1),(1,0)\}))$, but it is not in the interior of the cone associated with every possible choice of indices $\{t_1,t_2\}\in\mathcal{T}_{\overline{b}}(\overline{x})$. For instance, $\overline{c}\notin \mathrm{int}(\mathrm{cone}(\{(1,1),(1,0)\}))$.

- Since ENC is preserved in a neighborhood of $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$, under ENC, the previous condition (5.4) (and then Thm. 2.1(ii)) also holds in this neighborhood. However, the condition of Theorem 2.1(ii) itself does not guarantee its fulfillment around $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$. Just take $\{((\overline{c},b^r),x^r)\}$ converging to $\left\{\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)\right\}$ in Example 5.3, with $x^r=(1/r,0)$ and

$$P(\overline{c},b^r): \mathrm{Inf}\ \{x_1 \mid -x_1-x_2\leq 0, -x_1+x_2\leq 0, -x_1\leq 1/r\}.$$

  Observe that the condition of Theorem 2.1(ii) is not satisfied at $\{((\overline{c},b^r),x^r)\}$, while it is at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$.

Moreover, as commented in Section 4, when confined to the case of linear semi-infinite optimization problems, ENC at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ turns out to be an equivalent condition to the strong Lipschitz stability of $\mathcal{F}^*$ at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$. In the convex case, ENC is still a sufficient condition for the latter but no longer necessary ([3], Rem. 11), and the problem of finding intermediate conditions remains open. More specifically, the question of whether or not the strong uniqueness of minimizers for (c,b) near $\left(\overline{c},\overline{b}\right)$ is such an intermediate condition remains as an open problem (see [3], Sect. 5).

5.3. Simplified expressions for the Lipschitz modulus in particular cases

A serious difficulty in order to apply (4.3) in practice is given by the problem of finding those $D \in \mathcal{T}_{\overline{b}}(\overline{x})$ such that $f_b(z) = f_b^D(z)$ for (z,b) close to $\left(\overline{x},\overline{b}\right)$. In the finite case ($T$ finite), [6] provides a strategy for avoiding such a problem. Specifically, Theorem 2 in [6] establishes, under ENC at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ and T being finite, that

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \operatorname{lip}\mathcal{F}_D^*(\overline{c},\overline{b}_D), \tag{5.5}$$

where the subscript D in $\mathcal{F}_D^*$ and $\overline{b}_D$ means that only those constraints with indices in D are considered in the model. In this case, our f_b^D is reduced to $h_\beta^D(z) := \max\left\{|\langle a_i, z\rangle - \beta_i| : i \in D\right\}$, for $z \in \mathbb{R}^n$, where $\beta \in \mathbb{R}^D$ denotes the new parameter. Combining this result with (4.3), we obtain

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \limsup_{\substack{(z,\beta)\to(\overline{x},\overline{b}_D)\\ h_\beta^D(z)>0}} \left(d_*\left(0_n, \widehat{\partial} h_\beta^D(z)\right)\right)^{-1}. \tag{5.6}$$

A remaining difficulty in the previous expression is the fact that it appeals to parameters and points near the nominal $\overline{b}$ and $\overline{x}$, respectively. This difficulty can be overcome in the finite linear case, *i.e.*, (4.8) with T finite. In this case, it is well-known in the literature (see *e.g.* Cor. 1 in [6] and references therein) that $\operatorname{lip}\mathcal{F}_D^*(\overline{c},\overline{b}_D) = \left\|A_D^{-1}\right\|$, where A_D is the matrix whose rows are a_t, $t \in D$. In this way, (5.5) reads as

$$\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \max_{D\in\mathcal{T}_{\overline{b}}(\overline{x})} \left\|A_D^{-1}\right\|.$$

The last expression was already obtained in [2]. In fact, that paper shows that $\sup_{D\in\mathcal{T}_{\overline{b}}(\overline{x})}\left\|A_D^{-1}\right\|$ is always a lower bound on the modulus in the semi-infinite linear case, and we have the equality under a certain additional hypothesis which is always satisfied for dimensions $n \leq 3$. The question or whether or not $\operatorname{lip}\mathcal{F}^*(\overline{c},\overline{b}) = \sup_{D\in\mathcal{T}_{\overline{b}}(\overline{x})}\left\|A_D^{-1}\right\|$ fulfils under weaker (or none) additional assumptions remains as an open problem.

In summary, the state of the art concerning the Lipschitz modulus of the argmin mapping is as follows:

- A desirable goal would be to obtain an exact expression for the Lipschitz modulus in the convex semi-infinite case in terms of the nominal data, not involving parameter and points in a neighborhood. At the moment, this goal has been completely attained in the finite linear case, and partially attained in the linear semi-infinite case.
- In the finite convex case (4.3) and (5.6) provide exact formulae for the modulus still involving elements in a neighborhood. In relation to this points, the main contribution of the present paper is the fact of providing an expression, (4.3), in terms of the functions describing the model, namely, f and g_t, $t \in T$.
- In the convex semi-infinite case, we have an upper estimation, (4.2), in the same circumstances as in the previous paragraph.

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