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MINIMIZING MOVEMENTS FOR DISLOCATION DYNAMICS WITH A MEAN CURVATURE TERM

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Abstract. We prove existence of minimizing movements for the dislocation dynamics evolution law of a propagating front, in which the normal velocity of the front is the sum of a non-local term and a mean curvature term. We prove that any such minimizing movement is a weak solution of this evolution law, in a sense related to viscosity solutions of the corresponding level-set equation. We also prove the consistency of this approach, by showing that any minimizing movement coincides with the smooth evolution as long as the latter exists. In relation with this, we finally prove short time existence and uniqueness of a smooth front evolving according to our law, provided the initial shape is smooth enough.

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1. Introduction

In this paper, we investigate the existence of minimizing movements (see Almgren, Taylor and Wang [1], Ambrosio [5], and the book by Ambrosio, Gigli and Savaré [6]) for a non-local geometric law governing the movement of a family $\{K(t)\}_{0\leq t\leq T}$ of compact subsets of $\mathbb{R}^N$:

$$V_{x,t} = H_{x,t} + c_0(\cdot,t)\star \mathbf{1}_{K(t)}(x) + c_1(x,t), \tag{1.1}$$

where $V_{x,t}$ denotes the normal velocity at time t of a point x of $\partial K(t)$, $H_{x,t}$ the mean curvature of $\partial K(t)$ at x (with negative sign for convex sets), $\star$ is the convolution in space, $\mathbf{1}_{K(t)}$ is the indicator function of the set $K(t)$ and $c_0, c_1 : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ are given functions.

The non-local dependence $c_0(\cdot,t)\star\mathbf{1}_{K(t)}$ in the expression of $V_{x,t}$ is typical of models for dislocation dynamics (see Alvarez *et al.* [4]). Moreover we think of the term c_1 as a prescribed driving force. Equation (1.1) with only these two terms (and without a mean curvature term) is currently also a center of interest: in the context

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of viscosity solutions, its level-set formulation, namely

$$u_t(x,t) = [c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)]|Du(x,t)|, \tag{1.2}$$

was first investigated by Alvarez *et al.* [4], who proved short time existence and uniqueness of a viscosity solution to (1.2), and then by Alvarez, Cardaliaguet and Monneau [2], and by Barles and Ley [9], who proved, by different methods, long time existence and uniqueness under suitable monotonicity assumptions. In (1.2) and throughout the paper, u_t denotes the time derivative of u, Du denotes the space gradient of u, and $|\cdot|$ is the standard Euclidean norm. The mean curvature term in (1.1) corresponds to an additional line tension term in the elastic energy of the dislocation which better approximates what happens near the dislocation (see the introduction of [18] for a discussion on the model). The level-set formulation of the geometric law (1.1),

$$u_t(x,t) = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right)(x,t) + c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right]|Du(x,t)|, \tag{1.3}$$

was studied by the first author in [18]. He proved short time existence and uniqueness of a viscosity solution to (1.3).

In both cases, the source of major difficulties is the non-local dependence in the expression of the velocity, $c_0(\cdot,t)\star\mathbf{1}_{K(t)}$, which prevents comparison principle to hold. Indeed, c_0 is not necessarily non-negative, and physical models show that this situation can not be avoided. The problem of existence and uniqueness of a viscosity solution to the level-set equations (1.2) and (1.3) for general kernels c_0 is therefore still open. For example, the long time existence and uniqueness results mentioned above were obtained under the assumption that $c_0(\cdot,t)\star\mathbf{1}_E+c_1(x,t)\geq 0$ for any set E, which guarantees that the dislocation is expanding, and a regularity assumption on the initial shape $K(0)$. The short time existence and uniqueness for (1.3) was obtained in the case where the initial shape is a graph or a Lipschitz curve, without assumption on the sign of the non-local term. It is worth mentioning however that this equation benefits from the regularizing effect of the mean curvature term.

To overcome this difficulty, Barles *et al.* defined in [7] a notion of weak solution for (1.2), and proved existence of such weak solutions under general assumptions on c_0 and c_1. A similar concept of solution already appears in [28] for FitzHugh-Nagumo systems. In this work, we wish to provide such weak solutions for (1.1). We will work with set-valued mappings $E:[0,T]\to\mathcal{P}(\mathbb{R}^N)$ with uniformly bounded images which are continuous in the L^1 topology, that is to say, $t\mapsto\mathbf{1}_{E(t)}$ belongs to $C^0([0,T],L^1(\mathbb{R}^N))$. We assume that c_0 and c_1 satisfy some regularity assumptions which guarantee that $(x,t)\mapsto c_0(\cdot,t)\star\mathbf{1}_{E(t)}(x)+c_1(x,t)$ is smooth enough for such a mapping E. Let us now explain what we call a weak solution of (1.1):

Definition 1.1 (weak solutions)**.** Assume that $c_0\in Lip\left([0,T],L^1(\mathbb{R}^N)\right)$, $c_1\in Lip\left([0,T],L^\infty(\mathbb{R}^N)\right)$, that c_0 and c_1 are continuous on $\mathbb{R}^N\times[0,T]$ and Lipschitz continuous in space. Let $E:[0,T]\to\mathcal{P}(\mathbb{R}^N)$ be a set-valued mapping with uniformly bounded images such that $t\mapsto\mathbf{1}_{E(t)}$ belongs to $C^0([0,T],L^1(\mathbb{R}^N))$.

Let u be the unique uniformly continuous viscosity solution of

$$\begin{cases} u_t(x,t) = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right)(x,t) + c_0(\cdot,t)\star\mathbf{1}_{E(t)}(x)+c_1(x,t)\right]|Du(x,t)| & \text{for } (x,t)\in\mathbb{R}^N\times(0,T)\\ u(x,0)=u_0(x) \quad \text{for } x\in\mathbb{R}^N, \end{cases} \tag{1.4}$$

where u_0 is a uniformly continuous function such that $\overline{E_0}=\{u_0\geq 0\}$, $\overset{\circ}{E_0}=\{u_0>0\}$.

We say that E is a weak solution of (1.1) if we have, for all $t\in[0,T]$, and almost everywhere in $\mathbb{R}^N$,

$$\{u(\cdot,t)>0\}\subset E(t)\subset\{u(\cdot,t)\geq 0\}.$$

The goal of this paper is to construct a weak solution to the geometric law (1.1). To do this, we wish to adapt the approach of Almgren, Taylor and Wang [1] (also discovered independently by Luckhaus and Sturzenhecker [22]) – initially proposed for the mean curvature motion – to the geometric law (1.1) with its additional non-local term and driving force. The idea of minimizing movements is, for a given initial set E_0, to select a sequence of sets $E_h(k)$ associated with time-steps of size h by minimizing a suitable functional, so that the corresponding Euler equation is a discretization of our evolution law. A compactness result for sets of finite perimeter guarantees the existence of a subsequence (h_n) and a set-valued mapping $E : [0,T] \mapsto \mathcal{P}(\mathbb{R}^N)$ such that $E_{h_n}([t/h_n])$ converges to $E(t)$ in $L^1(\mathbb{R}^N)$ for all t, where $[\cdot]$ denotes the integer part. Such a E is called a minimizing movement (or generalized minimizing movement) associated to the geometric law. Moreover, we prove *a priori* estimates for the discrete evolution E_h, which imply the Hölder continuity of the limit E in the appropriate metric. This guarantees that the sets $E(t)$ cannot vary in a wildly discontinuous way.

Let us now explain the interest of this approach in the perspective of proving existence of weak solutions. For any sequence (h_n) going to 0 and such that $E_{h_n}([\cdot/h_n])$ converges to a minimizing movement E, we are able, thanks to the Euler equation corresponding to our minimization procedure, to compute the velocity (in the viscosity sense) of the upper and lower limit of the $E_{h_n}(k)$'s as $n \to \infty$, E^* and E_*, in function of E. This enables us to compare E_* and E^* with the 0 level set of the viscosity solution u appearing in Definition 1.1. Since $E_* \subset E \subset E^*$, we will deduce that E is a weak solution of (1.1). In case no fattening occurs for u, we remark that u is a viscosity solution of (1.3).

Of course it is a natural request that this construction be consistent with smooth flows if they exist. To verify this, we further show that if ∂E_0 is a smooth hypersurface, then there is a unique smooth solution for small times of the evolution law (1.1), and that any minimizing movement E coincides with this smooth evolution as long as the latter exists. This uses the notions of lower/upper limits mentioned above and of sub/super pairs of solutions of Cardaliaguet and Pasquignon [14].

To state our results in more details below, we first need to fix some notation and assumptions that will be used throughout the paper.

Notation

• For $k \in \mathbb{N}$, $B^k_r(x)$ (resp. $\overline{B}^k_r(x)$) denotes the open (resp. closed) ball of radius r centered at $x \in \mathbb{R}^k$, and $\mathcal{L}^k$ is the Lebesgue measure on $\mathbb{R}^k$. If k is not specified, we mean that $k = N$. We set $\omega_k = \mathcal{L}^k(B^k_1(0))$. The Hausdorff measure of dimension k on $\mathbb{R}^N$ is denoted by $\mathcal{H}^k$.

• The notation Sym_N represents the set of real square symmetric matrices of size N.

• We say that a sequence (E_n) of subsets of $\mathbb{R}^N$ converges to E in $L^1(\mathbb{R}^N)$ if $\mathbf{1}_{E_n} \to \mathbf{1}_E$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

• Let $\mathcal{P}$ be the set of all bounded subsets of $\mathbb{R}^N$ having finite perimeter (see [15] for the definition and properties of sets of finite perimeter). We denote by $P(E)$ the perimeter of $E \in \mathcal{P}$, by $P(E,U)$ the perimeter of E in U subset of $\mathbb{R}^N$, and we endow $\mathcal{P}$ with the metric

$$\delta(E,F) = \|\mathbf{1}_E - \mathbf{1}_F\|_{L^1(\mathbb{R}^N)} = \mathcal{L}^N(E\Delta F),$$

where $E\Delta F$ is the symmetric difference of E and F, *i.e.*, $E\Delta F = (E \cup F) \setminus (E \cap F)$.

In particular we call *equivalent* two sets E and F such that $\delta(E,F) = 0$, and we also say that $E = F$ almost everywhere (a.e.). Similarly, we say that $E \subset F$ almost everywhere if $\mathcal{L}^N(E \setminus F) = 0$.

Moreover $\partial^* E$ denotes the reduced boundary of $E \in \mathcal{P}$. We also define a notion of boundary for $E \in \mathcal{P}$ that is invariant in the class of E formed by the sets that are equivalent to E:

$$\partial E = \{x \in \mathbb{R}^N;\, 0 < \mathcal{L}^N(E \cap B_r(x)) < \mathcal{L}^N(B_r(x)) \text{ for all } r > 0\}.$$

Then ∂E is closed, and in fact $\partial E = \overline{\partial^* E}$.

Definitions of tubes (see [12])

• For any subset E of $\mathbb{R}^N \times [0,T]$, we set $E(t) = \{x \in \mathbb{R}^N; (x,t) \in E\}$. Conversely a mapping $t \in [0,T] \mapsto E(t) \in \mathcal{P}(\mathbb{R}^N)$ can be seen as a subset of $\mathbb{R}^N \times [0,T]$ by identifying E with its graph $\cup_{t\in[0,T]} E(t) \times \{t\}$.

• We call *tube* a bounded subset E of $\mathbb{R}^N \times [0,T]$. We call *regular tube* a tube E with C^1 boundary in $\mathbb{R}^N \times [0,T]$ such that for any regular point $(x,t) \in \partial E$, the unit outer normal (ν_x, ν_t) to E at (x,t) satisfies $\nu_x \neq 0$. In this case, the *normal velocity* of E at (x,t) is $-\nu_t/|\nu_x|$.

• Finally a mapping $t \in [0,T] \mapsto E_r(t)$ is said to be a smooth evolution with $C^{3+\alpha}$ boundary if E_r is a compact regular tube such that $E_r(t)$ has $C^{3+\alpha}$ boundary for all $t \in [0,T]$.

Assumptions on c_0 and c_1

Throughout the paper, c_0 and c_1 are assumed to satisfy the following regularity assumption:

(A) $c_0 \in Lip\left([0,T], L^1(\mathbb{R}^N)\right), \quad c_1 \in Lip\left([0,T], L^\infty(\mathbb{R}^N)\right)$.

In particular, we set $K_0 = Lip(c_0)$, and $K_1 = Lip(c_1)$, so that for all $t, s \in [0,T]$,

$$\|c_0(\cdot,t) - c_0(\cdot,s)\|_1 \leq K_0|t-s| \quad \text{and} \quad \|c_1(\cdot,t) - c_1(\cdot,s)\|_\infty \leq K_1|t-s|.$$

We finally set

$$L_0 = \|c_0\|_{L^\infty([0,T],L^1(\mathbb{R}^N))},\ L_1 = \|c_1\|_{L^\infty([0,T],L^\infty(\mathbb{R}^N))} \text{ and } L = L_0 + L_1. \tag{1.5}$$

We will sometimes need additional regularity for c_0 and c_1. When this happens, we will specify which assumptions are made in each of the statements of theorems. In particular we will sometimes need to require that c_0 be symmetric, so that the gradient flow of our functional is, at least formally, a solution of (1.1):

(Symmetry of c_0) We say that c_0 is symmetric if $c_0(-(\cdot),t) = c_0(\cdot,t)$ for all $t \in [0,T]$.

Main results

For $h > 0$ (the time step), $k \in \mathbb{N}$ such that $kh \leq T$, E and F in $\mathcal{P}$, we define, following the original idea of Almgren, Taylor and Wang [1], the functional

$$\mathcal{F}(h,k,E,F) = P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}(x)\,\mathrm{d}x - \int_E \left(\frac{1}{2}c_0(\cdot,kh) \star \mathbf{1}_E(x) + c_1(x,kh)\right)\mathrm{d}x, \tag{1.6}$$

where d_C is the distance function to a closed set C.

Let us now define a minimizing movement:

Definition 1.2 (minimizing movement [1])**.** Let $T > 0$ and $E_0 \in \mathcal{P}$. We say that $E : [0,T] \to \mathcal{P}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition E_0 if there exist a sequence (h_n), $h_n \to 0^+$, and sets $E_{h_n}(k) \in \mathcal{P}$ for all $k \in \mathbb{N}$ verifying $kh_n \leq T$, such that:

(1) $E_{h_n}(0) = E_0$.
(2) For any $k, n \in \mathbb{N}$ with $(k+1)h_n \leq T$,

$$E_{h_n}(k+1) \text{ minimizes the functional } E \to \mathcal{F}(h_n, k+1, E, E_{h_n}(k)) \tag{1.7}$$

among all $E's$ in $\mathcal{P}$.
(3) For any $t \in [0,T]$, $E_{h_n}([t/h_n]) \to E(t)$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$, where $[\cdot]$ denotes the integer part.

The first result of the paper is the existence of minimizing movements associated to our functional $\mathcal{F}$:

Theorem 1.3 (existence of minimizing movements)**.** *Assume that c_0 and c_1 satisfy* **(A)***. Let $E_0 \in \mathcal{P}$ with $\mathcal{L}^N(\partial E_0) = 0$. Then, there exists a minimizing movement E associated to $\mathcal{F}$ with initial condition E_0 such that for all t, s verifying $t \leq T$ and $0 \leq s \leq t < s+1$, we have*

$$\delta(E(t), E(s)) \leq \gamma\,(t-s)^{\frac{1}{N+1}}, \tag{1.8}$$

where $\gamma = \gamma(N, T, E_0, K_0, K_1, L_0, L_1)$ is a constant.

We then prove that any such minimizing movement is a weak solution of (1.1):

Theorem 1.4 (minimizing movements are weak solutions)**.** *Assume that c_0 is symmetric, that c_0 and c_1 satisfy* **(A)***, are continuous on $\mathbb{R}^N \times [0,T]$ and Lipschitz continuous in space. Let $E_0 \in \mathcal{P}$ with $\mathcal{L}^N(\partial E_0) = 0$. Let E be any minimizing movement associated to $\mathcal{F}$ with initial condition E_0.*

Then E is a weak solution of (1.1) in the sense of Definition 1.1. In particular if no fattening occurs, i.e. *if the corresponding solution u of (1.4) is such that $\{u(\cdot,t) = 0\}$ has zero $\mathcal{L}^N$ measure, then u is a viscosity solution of (1.3) with initial datum u_0.*

Let us already point out that even in the absence of fattening (a favorable situation which is not known to be generic), uniqueness for (1.3) is, to our knowledge, an open problem. The approach we use here provides one particular solution.

Our third result states that any minimizing movement E coincides with the smooth evolution E_r as long as the latter exists:

Theorem 1.5 (agreement with the smooth flow)**.** *Assume that c_0 is symmetric, that c_0 and c_1 satisfy* **(A)***, are continuous on $\mathbb{R}^N \times [0,T]$ and Lipschitz continuous in space. Let E_0 be a compact subset of $\mathbb{R}^N$ with uniformly $C^{3+\alpha}$ boundary. Let E_r be a smooth evolution with $C^{3+\alpha}$ boundary defined on $[0,T]$, starting from E_0, with normal velocity given by*

$$V_{x,t} = H_{x,t} + c_0(\cdot,t) \star \mathbf{1}_{E_r(t)}(x) + c_1(x,t), \tag{1.9}$$

where $H_{x,t}$ is the mean curvature of $\partial E_r(t)$ at x.

Then any minimizing movement E associated to $\mathcal{F}$ with initial condition E_0 verifies $E(t) = E_r(t)$ almost everywhere, for all $t \in [0,T]$.

In relation with this, we finally prove short time existence and uniqueness of a smooth solution E_r to (1.1), when E_0 is sufficiently smooth. The regularity assumptions on c_0 and c_1 are the following ones:

$$c_0 \in L^\infty([0,T], W^{2,\infty}(\mathbb{R}^N)) \cap W^{1,\infty}([0,T], L^\infty(\mathbb{R}^N)) \tag{1.10}$$

and

$$c_1 \in W^{2,1;\infty}(\mathbb{R}^N \times [0,T]), \tag{1.11}$$

where $f \in W^{1,\infty}([0,T], L^\infty(\mathbb{R}^N))$ means that f is Lipschitz continuous with respect to $t \in [0,T]$, uniformly with respect to $x \in \mathbb{R}^N$, and for $n \in \mathbb{N}^*$,

$$W^{n,1;\infty}(\mathbb{R}^N \times (0,T)) = \left\{ f \in L^\infty(\mathbb{R}^N \times (0,T)) \;\middle|\; \begin{array}{l} f_t,\ \frac{\partial^\alpha f}{\partial x^\alpha} \in L^\infty(\mathbb{R}^N \times (0,T)) \\ \text{for } \alpha \in \mathbb{N}^N \ s.t.\ \sum_{i=0}^N \alpha_i \leq n \end{array} \right\}.$$

Theorem 1.6 (existence and uniqueness of a smooth solution)**.** *Assume the regularity (1.10)–(1.11). Let E_0 be a compact subset of $\mathbb{R}^N$ with uniformly $C^{3+\alpha}$ boundary. Then there exists a small time $t_0 > 0$ and a unique smooth evolution E_r with $C^{3+\alpha}$ boundary defined on $[0,t_0]$, starting from E_0, with normal velocity given by (1.9).*

Let us now explain how this paper is organized. First, in Section 2, we prove the existence of minimizing movements and the Hölder estimate Theorem 1.3. Section 3 is devoted to proving a regularity result for $\mathcal{F}$-minimizers that we use in Section 4 to prepare the proofs of Theorems 1.4 and 1.5, respectively given in Sections 5 and 6. Finally, in Section 7, we prove Theorem 1.6.

2. Existence of minimizing movements

This section is concerned with the existence of minimizing movements associated to $\mathcal{F}$ (Th. 1.3). Let us start with existence and basic properties of $\mathcal{F}$-minimizers.

2.1. $\mathcal{F}$-minimizers

The first point to check is the existence of $\mathcal{F}$-minimizers:

Proposition 2.1 (existence of $\mathcal{F}$-minimizers)**.** *For all $h > 0$, $k \in \mathbb{N}$ with $kh \leq T$, and $F \in \mathcal{P}$, there exists a minimizer of $E \mapsto \mathcal{F}(h, k, E, F)$ on $\mathcal{P}$. Moreover, if L is defined by (1.5), then*

$$F \subset B_R(0) \text{ a.e.} \Rightarrow E \subset B_{R+Lh}(0) \text{ a.e.}$$

whenever E is a minimizer.

Proof. Let us fix $F \in \mathcal{P}$ with $F \subset B_R(0)$ a.e., and set $B = B_{R+Lh}(0)$. Let (E_n) be a minimizing sequence for $\mathcal{F}(h, k, \cdot, F)$. We want to prove that for all $n \in \mathbb{N}$,

$$\mathcal{F}(h, k, E_n \cap B, F) \leq \mathcal{F}(h, k, E_n, F). \tag{2.1}$$

First, since B is open and convex, we know that

$$P(E_n \cap B) \leq P(E_n). \tag{2.2}$$

Let us compare $\int_{E_n} c_0(\cdot, kh) \star \mathbf{1}_{E_n}(x)\,\mathrm{d}x$ and $\int_{E_n \cap B} c_0(\cdot, kh) \star \mathbf{1}_{E_n \cap B}(x)\,\mathrm{d}x$: for all $x \in \mathbb{R}^N$,

$$\begin{aligned} c_0(\cdot, kh) \star \mathbf{1}_{E_n}(x) &= \int_{E_n} c_0(x - y, kh)\,\mathrm{d}y \\ &= c_0(\cdot, kh) \star \mathbf{1}_{E_n \cap B}(x) + \int_{E_n \setminus B} c_0(x - y, kh)\,\mathrm{d}y. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{E_n} c_0(\cdot, kh) \star \mathbf{1}_{E_n}(x)\,\mathrm{d}x = &\int_{E_n \cap B} c_0(\cdot, kh) \star \mathbf{1}_{E_n \cap B}(x)\,\mathrm{d}x \\ &+ \int_{E_n \setminus B} c_0(\cdot, kh) \star \mathbf{1}_{E_n \cap B}(x)\,\mathrm{d}x + \int_{E_n} \int_{E_n \setminus B} c_0(x - y, kh)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Since $\|c_0(\cdot, kh) \star \mathbf{1}_A\|_{L^\infty(\mathbb{R}^N)} \leq L_0$ for any measurable set A, it follows that

$$\begin{aligned} &\int_{E_n} \left(\frac{1}{2} c_0(\cdot, kh) \star \mathbf{1}_{E_n}(x) + c_1(x, kh)\right)\,\mathrm{d}x \geq \\ &\qquad \int_{E_n \cap B} \left(\frac{1}{2} c_0(\cdot, kh) \star \mathbf{1}_{E_n \cap B}(x) + c_1(x, kh)\right)\,\mathrm{d}x - L\mathcal{L}^N(E_n \setminus B) \end{aligned} \tag{2.3}$$

thanks to the definition of L (see (1.5)). Moreover $F \subset B$, so that

$$E_n \Delta F = (E_n \cap B)\Delta F \ \cup \ (E_n \setminus B),$$

whence

$$\begin{aligned} \frac{1}{h}\int_{E_n\Delta F} d_{\partial F}(x)\,\mathrm{d}x &= \frac{1}{h}\int_{(E_n\cap B)\Delta F} d_{\partial F}(x)\,\mathrm{d}x + \frac{1}{h}\int_{E_n\setminus B} d_{\partial F}(x)\,\mathrm{d}x \\ &\geq \frac{1}{h}\int_{(E_n\cap B)\Delta F} d_{\partial F}(x)\,\mathrm{d}x + L\mathcal{L}^N(E_n\setminus B), \end{aligned} \tag{2.4}$$

since $d_{\partial F}(x) \geq Lh$ for all $x \in E_n \setminus B$ by definition of B. Putting (2.2), (2.3) and (2.4) together proves (2.1). Therefore we can replace (E_n) by $(E_n \cap B)$ as a minimizing sequence, and in particular we can assume that $E_n \subset B$ for all n. Then

$$\begin{aligned} \mathcal{F}(h,k,E_n,F) &\geq -\int_{E_n}\left(\frac{1}{2}c_0(\cdot,kh)\star \mathbf{1}_{E_n}(x) + c_1(x,kh)\right)\mathrm{d}x \\ &\geq -\left(\frac{1}{2}L_0 + L_1\right)\mathcal{L}^N(B), \end{aligned}$$

so that $\inf_{E\in\mathcal{P}} \mathcal{F}(h,k,E,F) > -\infty$. Besides, for n large enough,

$$\mathcal{F}(h,k,E_n,F) \leq \inf_{E\in\mathcal{P}} \mathcal{F}(h,k,E,F) + 1.$$

This implies that

$$P(E_n) \leq \inf_{E\in\mathcal{P}} \mathcal{F}(h,k,E,F) + 1 + \left(\frac{1}{2}L_0 + L_1\right)\mathcal{L}^N(B),$$

and gives a uniform bound on the perimeter of the E_n's. Since they are also uniformly bounded by B, it follows from the compactness theorem for sets of finite perimeter [15], Section 5.2.3, that we can extract a converging subsequence (E_{n_k}) of (E_n) in the sense that there exists $E_\infty \in \mathcal{P}$, $E_\infty \subset B$, such that $E_{n_k} \to E_\infty$ in $L^1(\mathbb{R}^N)$. Therefore

$$\mathcal{F}(h,k,E_\infty,F) \leq \liminf_{k\to\infty} \mathcal{F}(h,k,E_{n_k},F) = \inf_{E\in\mathcal{P}} \mathcal{F}(h,k,E,F),$$

because all terms in the expression of $\mathcal{F}$ are at least lower semi-continuous in the E variable for the L^1 topology. Thus E_∞ is a minimizer of $E \mapsto \mathcal{F}(h,k,E,F)$ on $\mathcal{P}$. Finally, if E is any other minimizer, then the previous comparisons show that $P(E\cap B) = P(E)$, whence $E \subset B$ almost everywhere (see the comparison theorem [5], p. 216). $\square$

Remark 2.2. This proposition shows that the $E_h(k)$'s are uniformly bounded for all h and k, if $E_0 \in \mathcal{P}$: more precisely, if $E_0 \subset B_R(0)$ a.e., then since $kh \leq T$, we can choose $E_h(k) \subset B_{R+LT}(0)$ independently of h, k. Therefore we can choose $\Omega = B_{R+LT+1}(0)$ so that $E_h(k) \Subset \Omega$ for all k, h. We will always do so in the sequel, and set $D = R + LT + 1$.

Remark 2.2 gives a uniform bound Ω for $E_h(k)$, independently of h, k, provided that E_0 is bounded. In order to have compactness in $\mathcal{P}$, so as to construct our minimizing movement, we also want a uniform bound on the perimeter of $E_h(k)$.

Proposition 2.3 (uniform bound on the perimeter)**.** *Let $E_0 \in \mathcal{P}$ with $E_0 \subset B_R(0)$. Then, there exists a constant $c = c(T, E_0, D, K_0, K_1, L_0, L_1) > 0$ independent of h and k such that if E_h is defined by the procedure (1.7), we have*

$$P(E_h(k)) \leq c \quad \forall\, h,k \text{ such that } kh \leq T.$$

Proof. By definition of E_h, we have for all j such that $jh \le T$,

$$\mathcal{F}(h,j,E_h(j),E_h(j-1)) \le \mathcal{F}(h,j,E_h(j-1),E_h(j-1)),$$

and in particular,

$$P(E_h(j)) - \int_{E_h(j)} \left(\frac{1}{2}c_0(\cdot,jh)\star \mathbf{1}_{E_h(j)}(x) + c_1(x,jh)\right)\,\mathrm{d}x \le$$
$$P(E_h(j-1)) - \int_{E_h(j-1)} \left(\frac{1}{2}c_0(\cdot,jh)\star \mathbf{1}_{E_h(j-1)}(x) + c_1(x,jh)\right)\,\mathrm{d}x.$$

Adding these inequalities for $j = 1,\dots,k$ with $kh \le T$, we find:

$$\begin{aligned} P(E_h(k)) - P(E_0) &\le \sum_{j=1}^{k} J_h(j,j) - J_h(j-1,j) \\ &= \sum_{j=1}^{k} \int_\Omega c_1(\cdot,jh)\mathbf{1}_{E_h(j)} - c_1(\cdot,jh)\mathbf{1}_{E_h(j-1)} \\ &\quad + \frac{1}{2}\sum_{j=1}^{k}\int_\Omega (c_0(\cdot,jh)\star\mathbf{1}_{E_h(j)})\mathbf{1}_{E_h(j)} - (c_0(\cdot,jh)\star\mathbf{1}_{E_h(j-1)})\mathbf{1}_{E_h(j-1)} \end{aligned} \tag{2.5}$$

where we have set

$$J_h(i,j) = \int_{E_h(i)} \left(\frac{1}{2}c_0(\cdot,jh)\star \mathbf{1}_{E_h(i)}(x) + c_1(x,jh)\right)\,\mathrm{d}x. \tag{2.6}$$

Doing an Abel transformation on the first sum of the last member of (2.5) yields

$$\begin{aligned} \sum_{j=1}^{k} \int_\Omega c_1(\cdot,jh)\mathbf{1}_{E_h(j)} - c_1(\cdot,jh)\mathbf{1}_{E_h(j-1)} &= \int_\Omega c_1(\cdot,kh)\mathbf{1}_{E_h(k)} - \int_\Omega c_1(\cdot,h)\mathbf{1}_{E_0} \\ &\quad + \sum_{j=1}^{k-1}\int_\Omega [c_1(\cdot,jh) - c_1(\cdot,(j+1)h)]\mathbf{1}_{E_h(j)} \\ &\le\ 2L_1\,\mathcal{L}^N(\Omega) + (k-1)K_1\,h\,\mathcal{L}^N(\Omega) \\ &\le\ (2L_1 + K_1\,T)\,\mathcal{L}^N(\Omega). \end{aligned}$$

The same manipulation with the second sum gives

$$\sum_{j=1}^{k}\int_\Omega (c_0(\cdot,jh)\star\mathbf{1}_{E_h(j)})\mathbf{1}_{E_h(j)} - (c_0(\cdot,jh)\star\mathbf{1}_{E_h(j-1)})\mathbf{1}_{E_h(j-1)} \le (2L_0 + K_0\,T)\,\mathcal{L}^N(\Omega).$$

This proves that for all k such that $kh \le T$,

$$\sum_{j=1}^{k} J_h(j,j) - J_h(j-1,j) \le \left(L_0 + 2L_1 + \frac{1}{2}K_0\,T + K_1\,T\right)\,\mathcal{L}^N(\Omega) \tag{2.7}$$

and gives the result, with $c = P(E_0) + (L_0 + 2L_1 + \frac{1}{2}K_0\,T + K_1\,T)\,\mathcal{L}^N(\Omega)$. □

2.2. Minimizing movements

We are now ready to address the problem of existence of minimizing movements. Proofs in this section closely follow the ideas of Almgren, Taylor and Wang [1], and are adaptations of Ambrosio's simplified presentation of these ideas (see [5]).

The main result in the perspective of the proof of existence of minimizing movements is the following theorem on the behaviour of the solutions of procedure (1.7):

Theorem 2.4 (discrete Hölder estimate). *Let $E_0 \in \mathcal{P}$ with $E_0 \subset B_R(0)$. There exists a constant $\gamma = \gamma(N, D) > 0$ (where D is defined in Rem. 2.2) and $h_0 > 0$ such that for all $h \in (0, h_0)$, for all $m, l \in \mathbb{N}$ verifying $mh \leq T$ and $0 < l < m < l + \frac{1}{h}$, we have:*

$$\delta(E_h(m), E_h(l)) \leq \gamma\, c\,[h(m-l)]^{\frac{1}{N+1}}, \tag{2.8}$$

where c is the uniform bound on $P(E_h(k))$ given by Proposition 2.3.

Theorem 1.3 is a corollary of this result, as proved in [5], pp. 231–232. However the arguments of [1], Theorem 4.4, or [5], Theorem 3.3, for the proof of Theorem 2.4 in the mean curvature motion case need a few adaptations due to the particular form of $\mathcal{F}$. This is what the rest of this section is devoted to. We begin by giving some preliminary results which will be necessary in the proof of Theorem 2.4.

Lower density bound for $\mathcal{F}$-minimizers

Theorem 2.5 (density bound for $\mathcal{F}$-minimizers). *There exist two positive constants α and β (depending only on N) and $h_0 > 0$ such that if $E \in \mathcal{P}$ is a minimizer of $\mathcal{F}(h, k, \cdot, F)$ with $F \in \mathcal{P}$, $E \cup F \subset B_{D-1}(0)$, and $h \in (0, h_0)$, then*

$$\forall x \in \partial E,\ \forall \rho \in \left(0, \frac{\alpha h}{D}\right), \quad P(E, B_\rho(x)) \geq \beta \rho^{N-1}. \tag{2.9}$$

Proof. The proof relies on the following lemma relating the perimeter of $E \in \mathcal{P}$ and the perimeter of E replaced by a cone in a small ball:

Lemma 2.6 ([5], Lem. 3.5).
Let $E \in \mathcal{P}$, $x \in \mathbb{R}^N$ and $f(\rho) = P(E, B_\rho(x))$. Set

$$E_\rho = (E \cap (\mathbb{R}^N \setminus B_\rho(x))) \cup \left\{ y \in B_\rho(x);\ x + \rho \frac{y-x}{|y-x|} \in E \right\}.$$

Then for almost all $\rho > 0$ (all ρ such that f is differentiable at ρ), we have

$$P(E_\rho, \overline{B}_\rho(x)) \leq \rho \frac{f'(\rho)}{N-1}\cdot$$

Let us now prove Theorem 2.5. Fix $x \in \partial^* E$ and $\rho > 0$ such that f is differentiable at ρ. By definition of E, we know that $\mathcal{F}(h, k, E, F) \leq \mathcal{F}(h, k, E_\rho, F)$, that is to say

$$\begin{aligned} P(E) \leq{}& P(E_\rho) + \frac{1}{h}\left\{ \int_{E_\rho \Delta F} d_{\partial F}(y)\,\mathrm{d}y - \int_{E \Delta F} \mathrm{d}_{\partial F}(y)\,\mathrm{d}y \right\} \\ &+ \int_E \left(\frac{1}{2} c_0(\cdot, kh) \star \mathbf{1}_E(y) + c_1(y, kh) \right) \mathrm{d}y - \int_{E_\rho} \left(\frac{1}{2} c_0(\cdot, kh) \star \mathbf{1}_{E_\rho}(y) + c_1(y, kh) \right) \mathrm{d}y. \end{aligned} \tag{2.10}$$

But since E coincides with E_ρ in $\mathbb{R}^N \setminus \overline{B}_\rho$, we have

$$P(E, \mathbb{R}^N \setminus \overline{B}_\rho(x)) = P(E_\rho, \mathbb{R}^N \setminus \overline{B}_\rho(x)).$$

Moreover f is continuous at ρ, which together with (2.10) implies that

$$P(E, B_\rho(x)) = P(E, \overline{B}_\rho(x)) \leq P(E_\rho, \overline{B}_\rho(x)) + \frac{2D}{h}\omega_N \rho^N + 2L\,\omega_N \rho^N,$$

due to the fact that $d_{\partial F}(y) \leq 2D$ for all $y \in B_\rho(x)$, provided $\rho < 1$. Now Lemma 2.6 implies that for almost all $\rho \in (0,1)$,

$$f(\rho) \leq \rho \frac{f'(\rho)}{N-1} + \left(\frac{2D}{h} + 2L\right) \omega_N \rho^N. \tag{2.11}$$

Therefore, the function

$$g : \rho \mapsto \frac{f(\rho)}{\rho^{N-1}} + \left(\frac{2D}{h} + 2L\right)(N-1)\,\omega_N\,\rho$$

is nondecreasing on $(0,1)$. In particular if $x \in \partial^* E$ and $\rho \in (0,1)$,

$$g(\rho) \geq \liminf_{\overline{\rho} \to 0^+} g(\overline{\rho}) \geq \omega_{N-1} \tag{2.12}$$

because of [15], Corollary 1 (ii) p. 203. As a consequence, for all $\rho \in (0,1)$,

$$f(\rho) \geq \omega_{N-1}\rho^{N-1} - \left(\frac{2D}{h} + 2L\right)(N-1)\,\omega_N\,\rho^N. \tag{2.13}$$

Let us set $\alpha = \frac{\omega_{N-1}}{8(N-1)\omega_N}$ and $\beta = \frac{\omega_{N-1}}{2}$. Then, provided $h < \min\{\frac{D}{L}, \frac{D}{\alpha}\} =: h_0$, we deduce from (2.13) that for all $\rho \in (0, \frac{\alpha h}{D})$,

$$P(E, B_\rho(x)) = f(\rho) \geq \beta\,\rho^{N-1}.$$

Since $\partial^* E$ is dense in ∂E, this also holds for all $x \in \partial E$. □

Corollary 2.7 ([5], Cor. 3.6)**.** *Let $E \in \mathcal{P}$ be a minimizer of $\mathcal{F}(h,k,\cdot,F)$ with $F \in \mathcal{P}$ and $h \in (0,h_0)$. Then*

$$\mathcal{H}^{N-1}(\partial E \setminus \partial^* E) = 0.$$

Distance-volume comparison

We recall here a general result which makes it possible to compare $\mathcal{L}^N(A \setminus C)$ and $\int_A d_{\partial C}$ under conditions of density of C similar to (2.9). Such comparison will be essential to prove Theorem 2.4.

Theorem 2.8 (distance-volume comparison, [5] p. 230)**.** *Let C be a compact subset of $\mathbb{R}^N$ such that there exist $\beta > 0$, $\tau > 0$ with*

$$\mathcal{H}^{N-1}(C \cap B_\rho(x)) \geq \beta\rho^{N-1} \quad \forall\, x \in \partial C,\ \forall\, \rho \in (0,\tau).$$

Then there exists a constant $\Gamma = \Gamma(N) > 0$ such that for all $R > \tau$, for all Borel set $A \subset \mathbb{R}^N$, we have

$$\mathcal{L}^N(A \setminus C) \leq \left[2\Gamma\left(\frac{R}{\tau}\right)^{N-1}\mathcal{H}^{N-1}(C)\right]^{\frac{1}{2}} \left[\int_A d_C(x)\,\mathrm{d}x\right]^{\frac{1}{2}} + \frac{1}{R}\int_A d_C(x)\,\mathrm{d}x. \tag{2.14}$$

We are now able to prove Theorem 2.4.

Proof of Theorem 2.4. Let us fix $h \in (0,h_0)$, where h_0 is given by Theorem 2.5. By definition of E_h, we have for all j such that $jh \leq T$,

$$\mathcal{F}(h,j,E_h(j),E_h(j-1)) \leq \mathcal{F}(h,j,E_h(j-1),E_h(j-1)),$$

that is to say,

$$\int_{E_h(j)\Delta E_h(j-1)} d_{\partial E_h(j-1)}(x)\,\mathrm{d}x \leq \ h\,[P(E_h(j-1)) - P(E_h(j))] + \ h\,[J_h(j,j) - J_h(j-1,j)],$$

where $J_h(i,j)$ is defined by (2.6). Let us set

$$I_h(j) = \{[P(E_h(j-1)) - P(E_h(j))] + [J_h(j,j) - J_h(j-1,j)]\}^{\frac{1}{2}}.$$

We now use Theorem 2.8 with $C = \partial E_h(j-1)$, $A = E_h(j)\Delta E_h(j-1)$, $\tau = \frac{\alpha h}{D}$, which is justified for $j \geq 2$ because of the density estimate (2.9). Thanks to Corollary 2.7, we know that $\mathcal{L}^N(C) = 0$, so that for all $R > \frac{\alpha h}{D}$,

$$\mathcal{L}^N(E_h(j)\Delta E_h(j-1)) \leq \left[2\Gamma\left(\frac{R}{\tau}\right)^{N-1}\mathcal{H}^{N-1}(\partial E_h(j-1))\right]^{\frac{1}{2}}\sqrt{h}\,I_h(j) + \frac{1}{R}\,h\,I_h(j)^2. \tag{2.15}$$

Recall that Proposition 2.3 gives a uniform bound c on the perimeter of $\mathcal{F}$-minimizers, so that $\mathcal{H}^{N-1}(\partial E_h(j-1)) \leq c$.

Let $m, l \in \mathbb{N}$ verify $mh \leq T$ and $0 < l < m < l + \frac{1}{h}$. We choose

$$R = \frac{\alpha h}{D}\,[h(m-l)]^{\frac{-1}{N+1}} > \frac{\alpha h}{D},$$

and add up inequalities (2.15) for $j = l+1, \ldots, m$. Recall that (2.5) and (2.7) show that

$$\begin{aligned}\sum_{j=l+1}^{m} I_h(j)^2 &\leq P(E_h(l)) + \sum_{j=l+1}^{m} J_h(j,j) - J_h(j-1,j)\\ &\leq P(E_0) + \sum_{j=1}^{m} J_h(j,j) - J_h(j-1,j) \leq c.\end{aligned}$$

Moreover, the Cauchy-Schwarz inequality shows that

$$\sum_{j=l+1}^{m} I_h(j) \leq \sqrt{m-l}\left\{\sum_{j=l+1}^{m} I_h(j)^2\right\}^{\frac{1}{2}} \leq \sqrt{m-l}\,\sqrt{c}.$$

Finally, we find that

$$\begin{aligned}\mathcal{L}^N(E_h(m)\Delta E_h(l)) &\leq \left[2\Gamma[h(m-l)]^{-\frac{N-1}{N+1}}c\right]^{\frac{1}{2}}\sqrt{h(m-l)}\,\sqrt{c} + \frac{D}{\alpha h}\,[h(m-l)]^{\frac{1}{N+1}}\,h\,c\\ &= \left(\sqrt{2\Gamma} + \frac{D}{\alpha}\right)c\,[h(m-l)]^{\frac{1}{N+1}},\end{aligned}$$

which concludes the proof. □

3. Regularity for $\mathcal{F}$-minimizers

One of the main interests of the variational approach used in [1] is that it enables to use the regularity theory for area-minimizing currents described for instance in [11,17,25,27]. This is the idea we follow in this section. We use the notation of [1]. In particular, the notation **M** and **S** stand respectively for the mass and size of an integral current: if T is a k integral current associated to a k rectifiable set $S \subset \mathbb{R}^N$ and a density function θ, then $\mathbf{M}(T) = \int_S \theta \, \mathrm{d}\mathcal{H}^k$, while $\mathbf{S}(T) = \mathcal{H}^k(S)$ (see [1], Sect. 3.1.3). Besides, if $E \in \mathcal{P}$, $[E]$ denotes the solid associated to E, *i.e.* the canonical N-dimensional Euclidean current restricted to E. We use the notation $T \llcorner C$ for the restriction of a current T to a set C.

3.1. **Existence of tangent cones**

A fundamental notion in regularity theory is that of tangent cones defined as follows:

Definition 3.1. Let $f_{p,R} : x \mapsto R(x-p)$, for $p \in \mathbb{R}^N$, $R > 0$. A locally integral current $[J]$ is called a tangent current to ∂E at $p \in \partial E$ if there exists a sequence $(R_i) \to +\infty$ such that if we set $E(R) = f_{p,R}(E)$, then $[E(R_i)] \to [J]$ locally as $i \to +\infty$, in the sense that $\mathcal{L}^N((J \Delta E(R_i)) \cap B_r(q)) \to 0$ for each $q \in \mathbb{R}^N$ and $r > 0$.

Lemma 3.2 (existence of tangent cones)**.** *Let $F \in \mathcal{P}$ and let E be a minimizer of $\mathcal{F}(h,k,\cdot,F)$ on $\mathcal{P}$. For each $p \in \partial E$, there exists a tangent current $[J]$ to ∂E at p. Each such tangent current $[J]$ is a cone and locally minimizes the perimeter P. Moreover $0 \in \partial J$.*

Proof. The proof is inspired by that of [1], Theorem 3.9. We easily check that for all $R > 0$,

$$\begin{gathered}
P(E(R)) = R^{N-1} P(E), \\
\frac{1}{h} \int_{E(R) \Delta F(R)} d_{\partial F(R)}(y) \, \mathrm{d}y = R^{N+1} \frac{1}{h} \int_{E \Delta F} d_{\partial F}(y) \, \mathrm{d}y, \\
\int_{E(R)} \frac{1}{2} c_0(\cdot, kh) \star \mathbf{1}_{E(R)}(y) \, \mathrm{d}y = R^{2N} \int_E \frac{1}{2} c_0(R(\cdot), kh) \star \mathbf{1}_E(y) \, \mathrm{d}y \\
\int_{E(R)} c_1(y, kh) \, \mathrm{d}y = R^N \int_E c_1(R(y-p), kh) \, \mathrm{d}y.
\end{gathered}$$

By definition of E we find that $E(R)$ minimizes

$$E \mapsto P(E) + \frac{1}{R^2 h} \int_{E \Delta F(R)} d_{\partial F(R)}(y) \, \mathrm{d}y - \frac{1}{R^{N+1}} \int_E \frac{1}{2} c_0^R(\cdot, kh) \star \mathbf{1}_E(y) \, \mathrm{d}y - \frac{1}{R} \int_E c_1^R(y, kh) \, \mathrm{d}y, \tag{3.1}$$

where we have set $c_0^R(y,t) = c_0(y/R, t)$, $c_1^R(y,t) = c_1(p + y/R, t)$. Let us compare $E(R)$ and $E(R) \setminus B_r(q)$ for fixed $q \in \mathbb{R}^N$ and $r > 0$, with respect to this last functional. It follows from manipulations similar to those in the proof of Proposition 2.1 that for almost all $r > 0$,

$$P(E(R), B_r(q)) \leq P(B_r(q)) + \frac{1}{R^2 h} \int_{B_r(q)} d_{\partial F(R)}(x) \, \mathrm{d}x + \frac{L}{R} \mathcal{L}^N(B_r(q)),$$

where L is defined by (1.5). But $\operatorname{diam} F(R) = R \operatorname{diam} F$, so that

$$R \mapsto \frac{1}{R^2 h} \int_{B_r(q)} d_{\partial F(R)}(x) \, \mathrm{d}x$$

is bounded as a function of R, and even converges to 0 as R goes to infinity. This provides the sufficient bound on the perimeter of $E(R)$ in balls to infer the existence of a tangent current $[J]$ (using the compactness result [26], Th. 1.1 p. 225).

Let us prove that $[J]$ locally minimizes the perimeter. This means that for all $q \in \mathbb{R}^N$, all $r > 0$, and all $(N-1)$ integral current X with $\partial X = 0$ and having support in $C = \overline{B}_r(q)$, then $\mathbf{M}(\partial[J] \llcorner C) \leq \mathbf{M}(\partial[J] \llcorner C + X)$.

We first recall from [1], Section 3.1.6, that there exists an N integral current Q with compact support in C such that $\partial Q = X$ and

$$\mathbf{S}(Q) \leq \mathbf{M}(Q) \leq \frac{r}{N}\mathbf{M}(X). \tag{3.2}$$

Then according to [17], Section 4.5.17, we can write Q as

$$Q = \sum_{i=0}^{+\infty}[Q_i] - \sum_{i=0}^{+\infty}[P_i],$$

where $Q_i, P_i \in \mathcal{P}$ and (Q_i), (P_i) are nested families such that $P_1 \cup Q_1 \subset \mathrm{Supp}(Q)$ and $P_1 \cap Q_1 = \emptyset$. Let us set $K = (E(R) \setminus P_1) \cup Q_1$ and compare $E(R)$ and K with respect to the functional defined by (3.1). We obtain that

$$\begin{aligned} P(E(R)) &\leq P(K) + \frac{1}{R^2 h}\int_{P_1 \cup Q_1} d_{\partial F(R)}(x)\,\mathrm{d}x + \frac{L}{R}\mathcal{L}^N(P_1 \cup Q_1) \\ &\leq \mathbf{M}(\partial[E(R)] + \partial Q) + \frac{1}{R^2 h}\int_C d_{\partial F(R)}(x)\,\mathrm{d}x + \frac{L}{R}\mathbf{S}(Q). \end{aligned}$$

Since Q and $\partial Q = X$ have compact support in C, and since $P(E(R), C) = \mathbf{M}(\partial[E(R)] \llcorner C)$, we deduce that

$$\mathbf{M}(\partial[E(R)] \llcorner C) \leq \mathbf{M}(\partial[E(R)] \llcorner C + X) + \frac{1}{R^2 h}\int_C d_{\partial F(R)}(x)\,\mathrm{d}x + \frac{L}{R}\mathcal{L}^N(C).$$

Knowing this, we can adapt [27], Theorem 34.5, to show that $[J]$ locally minimizes the perimeter and also that if R_i is such that $[E(R_i)] \to [J]$ as $i \to +\infty$, then for all $x \in \mathbb{R}^N$ and almost all $\rho > 0$, $P(E(R_i), B_\rho(x)) \to P(J, B_\rho(x))$ as $i \to +\infty$.

Finally we check that $[J]$ is a cone, *i.e.* that J is invariant under all homothetic expansions $z \mapsto \lambda z$ for $\lambda > 0$. To see this we recall from (2.11) and (2.12) that for all $x \in \partial E$, the function

$$g : \rho \mapsto \frac{P(E, B_\rho(x))}{\rho^{N-1}} + c\rho$$

is nondecreasing on $(0, 1)$, where c is a constant, and that for all $\rho \in (0, 1)$,

$$\frac{P(E, B_\rho(x))}{\rho^{N-1}} + c\rho \geq \omega_{N-1}.$$

It follows that ∂E has a density $\theta(\partial E, x)$ at x with $\theta(\partial E, x) \geq 1$. For all $\rho > 0$,

$$\frac{P(E(R), B_\rho(0))}{\rho^{N-1}} = \frac{P(E, B_{\rho/R}(p))}{(\rho/R)^{N-1}} \underset{R \to +\infty}{\longrightarrow} \theta(\partial E, p)\,\omega_{N-1}.$$

Moreover for almost all $\rho > 0$,

$$\frac{P(E(R_i), B_\rho(0))}{\rho^{N-1}} \underset{i \to +\infty}{\longrightarrow} \frac{P(J, B_\rho(0))}{\rho^{N-1}}.$$

This shows that the ratio $\rho^{1-N}P(J, B_\rho(0))$ is independent of ρ, which is known to imply that J is a cone (see [21], proof of Th. 9.3). Moreover $\rho^{1-N}P(J, B_\rho(0)) = \theta(\partial E, p)\,\omega_{N-1} > 0$, so that $0 \in \partial J$. We finally observe that $\theta(\partial J, 0) = \theta(\partial E, p)$. $\square$

3.2. The regularity results

The existence of tangent cones enables us to prove regularity results for $\mathcal{F}$-minimizers, as in [1], Sections 3.5 and 3.7.

Theorem 3.3 (C^1-regularity for $\mathcal{F}$-minimizers)**.** *Let $F \in \mathcal{P}$, and let E be a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Then ∂E is a C^1-hypersurface, except for a set of Hausdorff dimension less than $N-8$ (empty if $N \le 7$).*

Proof. We verify that E is an almost minimal current in the sense of Bombieri, that is, for some $\delta > 0$, for all $(N-1)$ integral current X with $\partial X = 0$ and having compact support in C with $\mathrm{diam}(C) = r \le \delta$, then

$$\mathbf{M}(\partial[E] \llcorner C) \le (1 + \omega(r))\, \mathbf{M}(\partial[E] \llcorner C + X) \tag{3.3}$$

for some function ω such that $\omega(r) \to 0$ as $r \to 0^+$. To do so we proceed as in the previous proof, write $X = \partial Q$ with

$$Q = \sum_{i=0}^{+\infty} [Q_i] - \sum_{i=0}^{+\infty} [P_i],$$

set $K = (E \setminus P_1) \cup Q_1$ and compare E and K with respect to $\mathcal{F}$:

$$P(E, C) \le P(K) + \frac{1}{h} \int_{P_1 \cup Q_1} d_{\partial F}(y)\, \mathrm{d}y + L\mathcal{L}^N(P_1 \cup Q_1).$$

Let $D > 0$ be such that $E \cup F \subset B_{D-1}(0)$. If $B_{D-1}(0) \cap C \neq \emptyset$ (otherwise (3.3) is obvious), and $\delta \le 1$, the previous comparison yields

$$\begin{aligned}
\mathbf{M}(\partial[E] \llcorner C) &\le \mathbf{M}(\partial[E] \llcorner C + \partial Q) + \left(\frac{2D}{h} + L\right) \mathbf{S}(Q) \\
&\le \mathbf{M}(\partial[E] \llcorner C + X) + \left(\frac{2D}{h} + L\right) \frac{r}{N} \mathbf{M}(X) \qquad \text{(using (3.2))} \\
&\le \mathbf{M}(\partial[E] \llcorner C + X) + \left(\frac{2D}{h} + L\right) \frac{r}{N} (\mathbf{M}(\partial[E] \llcorner C + X) + \mathbf{M}(\partial[E] \llcorner C)).
\end{aligned}$$

This easily implies the result with $\omega(r) = 3\left(\frac{2D}{h} + L\right)\frac{r}{N}$ and $\delta = \frac{N}{3}\left(\frac{2D}{h} + L\right)^{-1}$.

In addition, at any point p of ∂E there exists a tangent cone $[J]$ which minimizes the perimeter (Lem. 3.2). Such a cone must be a hyperplane for $N \le 7$ ([27], Appendix B), so that in particular $\theta(E, p) = \theta(J, 0) = 1$. We then deduce the result from the final remark in [11]. In case $N \ge 8$, we use the dimension reduction argument of Federer ([21], Th. 11.8). □

Now, we prove that minimizers are smooth at contact points with smooth hypersurfaces:

Theorem 3.4. *Let $F \in \mathcal{P}$, and let E be a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Assume that there exists $K \subset \mathbb{R}^N$ closed such that ∂K is a C^1 hypersurface and $\partial E \cap K = \{p\}$. Then ∂E is a C^1 hypersurface near p.*

Proof. Let $[J]$ be any tangent cone to ∂E at p. The assumption that $\partial E \cap K = \{p\}$ guarantees that ∂J is contained in the closed half-space orthogonal to the outer unit normal $\mathbf{n}$ to K at p and containing $\mathbf{n}$. Since $0 \in \partial J$, [21], Theorem 15.5, p. 174, implies that ∂J is regular at 0, and therefore is a hyperplane. The result follows as in the proof of Theorem 3.3. □

Actually, we can deduce higher regularity for $\mathcal{F}$-minimizers at each point where they are C^1 hypersurfaces:

Theorem 3.5 (higher regularity for $\mathcal{F}$-minimizers)**.** *Assume that c_0 is symmetric, that c_0 and c_1 satisfy* **(A)** *and are Lipschitz continuous in space. Let $F \in \mathcal{P}$, and let E be a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$. Set $g(p) = \pm d_{\partial F}(p)$, where we take the $-$ sign if $p \in F$, and the $+$ sign otherwise.*

Let $p \in \partial E$ be such that ∂E is a C^1 hypersurface near p: there exist $R > 0$, $M > 0$ and a C^1 function $f : B_R^{N-1}(p) \to (-M, M)$ such that, possibly rotating and relabelling, we have

$$E \cap (B_R^{N-1}(p) \times (-M, M)) = \{(x, y);\, x \in B_R^{N-1}(p), -M < y < f(x)\}.$$

Then f is of class $C^{2,\alpha}$ in $B_R^{N-1}(p)$ for some $0 < \alpha < 1$ and satisfies

$$\frac{1}{h} g((x, f(x))) = \Delta f(x) + c_0(\cdot, kh) \star \mathbf{1}_E((x, f(x))) + c_1((x, f(x)), kh). \tag{3.4}$$

Therefore the mean curvature H_p of ∂E at p verifies

$$\frac{1}{h} g(p) = H_p + c_0(\cdot, kh) \star \mathbf{1}_E(p) + c_1(p, kh). \tag{3.5}$$

Proof. We begin by verifying that f satisfies (3.4) in the sense of distributions. This is simply the Euler-Lagrange equation for $\mathcal{F}$, and the proof is the same as that of Ambrosio ([5], after statement of Th. 3.3), with the additional observation that the first variation of

$$K \mapsto \frac{1}{2} \int_K c_0(\cdot, kh) \star \mathbf{1}_K(x)\, \mathrm{d}x, \quad K \mapsto \int_K c_1(x, kh)\, \mathrm{d}x$$

in the direction of a C^2 vector field Φ is respectively

$$K \mapsto \int_{\partial K} c_0(\cdot, kh) \star \mathbf{1}_K(x) \langle \Phi(x), \nu_x \rangle\, \mathrm{d}\mathcal{H}^{N-1}(x), \quad K \mapsto \int_{\partial K} c_1(x, kh) \langle \Phi(x), \nu_x \rangle\, \mathrm{d}\mathcal{H}^{N-1}(x),$$

where ν_x is the outer unit normal to K at $x \in \partial K$. The symmetry of c_0 is used here, along with the continuity of c_1 and $c_0 \star \mathbf{1}_K$ in space.

Knowing this, we apply [19], Theorem 1.2, p. 219, to f and to each of the $\frac{\partial f}{\partial x_i}$, to conclude that f is $C^{2,\alpha}$ in $B_R^{N-1}(p)$. This last assertion uses the Lipschitz continuity of c_1 and $c_0 \star \mathbf{1}_K$ in space. Both conclusions immediately follow. □

4. The upper and lower limits

In this section we are going to prepare the proofs of Theorems 1.4 and 1.5. Let E be a minimizing movement with initial condition E_0 and let (h_n) be a sequence such that $E_{h_n}([t/h_n])$ converges to $E(t)$ in $L^1(\mathbb{R}^N)$ for all $t \in [0, T]$ as n goes to infinity. We define the upper and lower limits of the sets $E_{h_n}(k)$ for $n \to \infty$ and $k \in \mathbb{N}$ as follows:

$$E^*(t) = \{x \in \mathbb{R}^N; \exists (h_{n'}) \subset (h_n), k_{n'} \to +\infty \text{ and } x_{n'} \in E_{h_{n'}}(k_{n'}) \text{ with } k_{n'}h_{n'} \to t \text{ and } x_{n'} \to x\},$$
$$E_*(t) = \mathbb{R}^N \setminus \{x \in \mathbb{R}^N; \exists (h_{n'}) \subset (h_n), k_{n'} \to +\infty \text{ and } x_{n'} \notin E_{h_{n'}}(k_{n'}) \text{ with } k_{n'}h_{n'} \to t \text{ and } x_{n'} \to x\}.$$

By construction, E^* is closed while E_* is open, and $E_*(t) \subset E(t) \subset E^*(t)$ for all $t \in [0, T]$ and almost everywhere in $\mathbb{R}^N$. Indeed $E_*(t)$ and $E^*(t)$ were defined respectively as the sets of cluster points of sets $E_{h_n}(k)$ and $\mathbb{R}^N \setminus E_{h_n}(k)$ for *all* $k \to +\infty$ such that $kh_n \to t$, and, up to a subsequence and a set of zero $\mathcal{L}^N$ measure, our minimizing movement at time t, $E(t)$, was constructed as the pointwise limit of sets $E_{h_n}(k)$ for *some* such $k = [t/h_n]$.

We will use the regularity result Theorem 3.5 to compute the normal velocity of the evolutions $t \mapsto E^*(t)$ and $t \mapsto E_*(t)$ in function of E. Then we will prove a regularity result for E^* and E_*, and compare the initial sets $E_*(0)$, $E^*(0)$ and E_0.

In order that our minimizing procedure be consistent with the evolution law (1.1) as ensured by Theorem 3.5, we will assume in particular throughout this section that c_0 is symmetric.

4.1. Velocity of E^* and E_*

Here we are going to prove a rigorous version of the heuristic fact that E^* moves with velocity

$$V_{x,t} \leq H_{x,t} + c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t),$$

while E_* moves with velocity

$$V_{x,t} \geq H_{x,t} + c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t),$$

where $H_{x,t}$ respectively denotes the mean curvature of ∂E^* and ∂E_*. Following Cardaliaguet [12], we formulate this statement in terms of test functions: let us first define the classical mean curvature operator

$$h(p,X) = Trace(X) - \frac{\langle Xp,p\rangle}{|p|^2},$$

for $X \in Sym_N$ and $p \in \mathbb{R}^N \setminus \{0\}$, and let us define, for any subset A of $\mathbb{R}^N$, $\widehat{A} = \overline{\mathbb{R}^N \setminus A}$, and for any subset B of $\mathbb{R}^N \times [0,T]$, $\widehat{B} = \overline{(\mathbb{R}^N \times [0,T]) \setminus B}$.

Proposition 4.1. *Under the assumptions of Theorem 1.4, we have:*

1. *For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local maximum on E^* at some point $(x,t) \in \partial E^*$, with $D\phi(x,t) \neq 0$, then*

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$

2. *For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local minimum on $\widehat{E_*}$ at some point $(x,t) \in \partial\widehat{E_*}$, with $D\phi(x,t) \neq 0$, then*

$$\phi_t(x,t) \leq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|.$$

Proof. We only prove the first point, the proof of the second being similar. Let $t \in (0,T)$ and ϕ of class C^2 have a local maximum on E^* at some point $(x,t) \in \partial E^*$, with $D\phi(x,t) \neq 0$. We can assume without loss of generality that it is a strict maximum. By definition of E^*, there exist $k_n \to +\infty$ and $x_n \in \partial E_{h_n}(k_n)$ with $k_n h_n \to t$ and $x_n \to x$, such that ϕ has a local maximum (that we can assume to be strict) on $E_{h_n} = \cup_k E_{h_n}(k) \times \{kh_n\}$ at $(x_n, k_n h_n)$, with $D\phi(x_n, k_n h_n) \neq 0$. It follows that $\Gamma_{h_n}(k_n) = \{x \in \mathbb{R}^N; \phi(x, k_n h_n) = \phi(x_n, k_n h_n)\}$ is a smooth exterior contact surface to $E_{h_n}(k_n)$ at x_n, and therefore Theorems 3.4 and 3.5 imply that $\partial E_{h_n}(k_n)$ is a $C^{2,\alpha}$ hypersurface near x_n. We now infer from the local relative position of Γ and $\partial E_{h_n}(k_n)$ that the curvature of $\partial E_{h_n}(k_n)$ at x_n, $H^n_{x_n}$, is less than the curvature of Γ at x_n:

$$H^n_{x_n} \leq -\frac{1}{|D\phi(x_n, k_n h_n)|} h(D\phi(x_n, k_n h_n), D^2\phi(x_n, k_n h_n)).$$

Now (3.5) implies, if $k_n \geq 1$, that

$$\pm \frac{1}{h_n} d_{\partial E_{h_n}(k_n-1)}(x_n) = H^n_{x_n} + c_0(\cdot, k_n h_n) \star \mathbf{1}_{E_{h_n}(k_n)}(x_n) + c_1(x_n, k_n h_n),$$

where we take the $-$ sign if $x_n \in E_{h_n}(k_n - 1)$, and the $+$ sign otherwise. With this convention,

$$\pm \frac{1}{h_n} d_{\partial E_{h_n}(k_n-1)}(x_n) \geq \pm \frac{1}{h_n} d_{\Gamma_{h_n}(k_n-1)}(x_n) = -\frac{\phi_t(x_n, k_n h_n)}{|D\phi(x_n, k_n h_n)|} + o(1).$$

Putting together the last three equations yields

$$\begin{aligned} \phi_t(x_n, k_n h_n) + o(1) \geq h(D\phi(x_n, k_n h_n), D^2\phi(x_n, k_n h_n)) \\ - \left[c_0(\cdot, k_n h_n) \star \mathbf{1}_{E_{h_n}(k_n)}(x_n) + c_1(x_n, k_n h_n)\right] |D\phi(x_n, k_n h_n)|. \end{aligned} \tag{4.1}$$

Thanks to the discrete Hölder estimate, Theorem 2.4, we know, since $k_n h_n \to t$, that $E_{h_n}(k_n) \to E(t)$ in $L^1(\mathbb{R}^N)$. Up to a subsequence, we can assume that $E_{h_n}(k_n) \to E(t)$ almost everywhere. As a consequence, sending n to $+\infty$, we get the result, namely:

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) - \left[c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|. \qquad \square$$

4.2. Regularity of E^* and E_*

Now we are going to prove a regularity result for the tubes E^* and E_* which allows in particular to treat the degenerate case $D\phi(x,t) = 0$ in Proposition 4.1:

Proposition 4.2. *For all x in $\mathbb{R}^N$, the maps $t \mapsto d_{E^*(t)}(x)$ and $t \mapsto d_{\widehat{E_*}(t)}(x)$ are left-continuous on $(0,T]$.*

To prove this we first need to estimate in a finer way than what we have done in Section 2 how $E_h(k)$ can expand or shrink at most at each iteration. This is the equivalent of [1], Theorem 5.4. Let us first define for simplicity of forthcoming estimates the scaled ball $W_R = \overline{B}_{R/(\omega_N)^{1/N}}(0) = \overline{B}_{R/\omega_*}(0)$, so that $\mathcal{L}^N(W_R) = R^N$. Then W_R minimizes the perimeter among all sets $E \in \mathcal{P}$ such that $\mathcal{L}^N(E) = R^N$. This property will provide the necessary estimates.

Let us also define, for any subsets A and B of $\mathbb{R}^N$, $A - B = \mathbb{R}^N \setminus ((\mathbb{R}^N \setminus A) + B)$.

Lemma 4.3. *Let $F \in \mathcal{P}$ and let E be a minimizer of $\mathcal{F}(h,k,\cdot,F)$ on $\mathcal{P}$. Let L be defined as in (1.5). Let $R(h) = 2L\omega_* h + 2\sqrt{L^2\omega_*^2 h^2 + 2\omega_* h P(W_1)}$. Then*

$$F - W_{R(h)} \subset E \subset F + W_{R(h)} \quad a.e.$$

Proof. We begin by proving the left-hand side inclusion, and we will see that the other inclusion immediately follows. We adapt the proofs of [1], Section 5. $\square$

Step 1. Let us first prove that if $0 < R < S$, $W_S \subset F$ and $0 < 2\mathcal{L}^N(W_R \setminus E) \leq R^N$, then

$$\frac{S-R}{\omega_* h} R - 2LR \leq \frac{N-1}{N} P(W_1) + \frac{2^{1/N}(N-1)}{N^2} P(W_1) \frac{\mathcal{L}^N(W_R \setminus E)}{R^N}.$$

We compare E and $E \cup W_R$ with respect to the functional $\mathcal{F}(h,k,\cdot,F)$:

$$\begin{aligned} P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}(x)\,\mathrm{d}x - \int_E \left(\frac{1}{2}c_0(\cdot,kh) \star \mathbf{1}_E(x) + c_1(x,kh)\right) \mathrm{d}x \leq \\ P(E \cup W_R) + \frac{1}{h}\int_{(E\cup W_R)\Delta F} d_{\partial F}(x)\,\mathrm{d}x - \int_{E\cup W_R} \left(\frac{1}{2}c_0(\cdot,kh) \star \mathbf{1}_{E\cup W_R}(x) + c_1(x,kh)\right) \mathrm{d}x. \end{aligned}$$

Since $W_R \subset F$, we check that $E\Delta F = ((E \cup W_R)\Delta F) \cup (W_R \setminus E)$. This, together with manipulations similar to those of previous proofs, implies that

$$\begin{aligned} P(E \cup W_R) - P(E) &\geq \frac{1}{h}\int_{W_R \setminus E} d_{\partial F}(x)\,\mathrm{d}x - 2L\mathcal{L}^N(W_R \setminus E) \\ &\geq \left(\frac{S-R}{\omega_* h} - 2L\right) \mathcal{L}^N(W_R \setminus E), \end{aligned} \tag{4.2}$$

since the inclusion $W_S \subset F$ implies that $d_{\partial F}(x) \geq (S-R)/\omega_*$ for each $x \in W_R$. But conclusion (4) of [1], Proposition 5, implies that

$$P(E \cup W_R) - P(E) \leq$$
$$R^{N-1}P(W_1)\left\{\frac{N-1}{N}\frac{\mathcal{L}^N(W_R \setminus E)}{R^N} + \frac{2^{1/N}(N-1)}{N^2}\left(\frac{\mathcal{L}^N(W_R \setminus E)}{R^N}\right)^2\right\},$$

and the result follows from the last two inequalities.

Step 2. Now let us assume that the conclusion of the lemma does not hold, *i.e.* that if we set $A = (F - W_{R(h)}) \setminus E$, then $\mathcal{L}^N(A) > 0$. There must exist $p \in A$ such that for any $r > 0$, $\mathcal{L}^N(A \cap B_r(p)) > 0$. We can assume, possibly applying a translation, that $p = 0$. Therefore $W_{R(h)} \subset F$ and $\mathcal{L}^N(W_{R(h)/2} \setminus E) > 0$. Moreover we also have

$$2\mathcal{L}^N(W_{R(h)/2} \setminus E) \leq \left(\frac{R(h)}{2}\right)^N,$$

otherwise we would obtain as in Step 1 with $S = R(h)$ and $R = R(h)/2$ that

$$P(E \cup W_{R(h)/2}) - P(E) \geq \left(\frac{R(h)}{2\omega_* h} - 2L\right)\mathcal{L}^N(W_{R(h)/2} \setminus E) > \left(\frac{R(h)}{2\omega_* h} - 2L\right)\frac{1}{2}\left(\frac{R(h)}{2}\right)^N,$$

because $\frac{R(h)}{2\omega_* h} - 2L > 0$. But $P(E \cup W_{R(h)/2}) \leq P(E) + P(W_{R(h)/2})$, whence

$$\left(\frac{R(h)}{2\omega_* h} - 2L\right)\frac{1}{2}\left(\frac{R(h)}{2}\right)^N < P(W_{R(h)/2}) = \left(\frac{R(h)}{2}\right)^{N-1}P(W_1),$$

or equivalently

$$\frac{1}{\omega_* h}\left(\frac{R(h)}{2}\right)^2 - LR(h) < 2P(W_1),$$

which is contradictory with the choice of $R(h)$, since equality should hold instead of the last inequality. Then we can apply Step 1 with $S = R(h)$ and $R = R(h)/2$ to infer that

$$\frac{1}{\omega_* h}\left(\frac{R(h)}{2}\right)^2 - LR(h) \leq \frac{N-1}{N}P(W_1) + \frac{2^{1/N}(N-1)}{N^2}P(W_1)\frac{\mathcal{L}^N(W_{R(h)/2} \setminus E)}{(R(h)/2)^N},$$

or thanks to the choice of $R(h)$:

$$2 \leq \frac{N-1}{N} + \frac{2^{1/N}(N-1)}{N^2}\frac{1}{2},$$

which is false. This proves the left-hand side inclusion of Lemma 4.3.

Step 3. Let us now explain why the left-hand side inclusion is sufficient to deduce the right-hand side one. Let $B = B_D(0)$ be a large ball. It is easy to check that if $F \in \mathcal{P}$ with $F \subset B_{D-1}(0)$, and if $E \in \mathcal{P}$ with $E \subset B_{D-1}(0)$ is a minimizer of $\mathcal{F}(h, k, \cdot, F)$ on $\mathcal{P}$, then $B \setminus E$ is a minimizer of

$$E \mapsto P(E) + \frac{1}{h}\int_{E \Delta (B \setminus F)} d_{\partial F}(x)\,\mathrm{d}x - \int_E \left(\frac{1}{2}c_0(\cdot, kh) \star \mathbf{1}_E(x) + \overline{c_1}(x, kh)\right)\mathrm{d}x$$

among all sets in $\mathcal{P}$ and included in B, where $\overline{c_1}(x, kh) = -c_1(x, kh) + c_0(\cdot, kh) \star \mathbf{1}_B(x)$. Therefore, taking h small enough so that $R < 1$, the arguments on E and F in the previous steps transform into the same arguments for $B \setminus E$ and $B \setminus F$, since in particular the term $2L$ appearing in (4.2) was taken so large (with the *a priori* useless factor 2) as to get the lower bound there also with $\overline{c_1}$ in place of c_1. The conclusion $F - W_R \subset E$ transforms into $(B \setminus F) - W_R \subset B \setminus E$, that is exactly $E \subset F + W_R$.

The last lemma provides a bound on the growth of $\mathcal{F}$-minimizers at each iteration equal to $2L\omega_* h + 2\sqrt{L^2\omega_*^2 h^2 + 2\omega_* h P(W_1)}$, and of the order of $\sqrt{h}$. This is not fine enough to conclude the left continuity, mainly because if $kh \to t$, then $k\sqrt{h} \to +\infty$ and the bound is lost in the limit movement. The following lemma refines the bound to the order h.

Lemma 4.4. *Let us set $\delta = 2\frac{N-1}{N}P(W_1)$ and $R(h) = 2L\omega_* h + 2\sqrt{L^2\omega_*^2 h^2 + 2\omega_* h P(W_1)}$.*

1. *Assume that $p + W_S \subset E_h(k)$ a.e. for some $p \in \mathbb{R}^N$ and k, h such that $kh \le T$. If h and j are small enough so that $R(h) < \frac{S}{4}$ and $jh \le \min\{\frac{S^2}{4\omega_*(\delta+2LS)}, T - kh\}$, then*

$$p + W_{S-\omega_*(\frac{\delta}{S}+2L)jh} \subset E_h(k+j) \quad a.e.$$

2. *Assume that $p + W_S \subset \mathbb{R}^N \setminus E_h(k)$ a.e. for some $p \in \mathbb{R}^N$ and k, h such that $kh \le T$. If h and j are small enough so that $R(h) < \frac{S}{4}$ and $jh \le \min\{\frac{S^2}{4\omega_*(\delta+2LS)}, T - kh\}$, then*

$$p + W_{S-\omega_*(\frac{\delta}{S}+2L)jh} \subset \mathbb{R}^N \setminus E_h(k+j) \quad a.e.$$

Proof. Let us prove the first assertion. For simplicity we assume without loss of generality that $p = 0$. We prove the result by induction on j. The result for $j = 0$ is the assumption. Let us assume that the result holds for some j such that $(j+1)h \le \min\{\frac{S^2}{4\omega_*(\delta+2LS)}, T - kh\}$. We know thanks to Lemma 4.3 that

$$E_h(k+j) - W_{R(h)} \subset E_h(k+j+1) \quad a.e. \tag{4.3}$$

Since the induction assumption states that $W_{S-\omega_*(\frac{\delta}{S}+2L)jh} \subset E_h(k+j)$, and since the assumptions on j and h imply that

$$R(h) < \frac{S}{2} - \omega_*\left(\frac{\delta}{S} + 2L\right) jh,$$

we deduce from (4.3) that $W_{S/2} \subset E_h(k+j+1)$ almost everywhere. Let us set

$$r_{max} = \sup\{r; W_r \subset E_h(k+j+1)\ a.e.\} \ge \frac{S}{2}\cdot$$

Step 1 of Lemma 4.3 shows, by sending R to r_{max}^+, that

$$\frac{1}{\omega_* h}\left(\left\{S - \omega_*\left(\frac{\delta}{S} + 2L\right) jh\right\} - r_{max}\right) r_{max} - 2L r_{max} \le \frac{N-1}{N} P(W_1) = \frac{\delta}{2},$$

from which we infer that

$$\left\{S - \omega_*\left(\frac{\delta}{S} + 2L\right) jh\right\} - r_{max} \le \left(\frac{\delta}{2r_{max}} + 2L\right)\omega_* h \le \omega_*\left(\frac{\delta}{S} + 2L\right) h,$$

and the result for $E_h(k+j+1)$ follows, so that the proof by induction is complete. The proof of the second point is entirely identical, according to the remark in Step 3 of the proof of Lemma 4.3. □

We are now ready to prove Proposition 4.2. This proof is inspired by the proof of [13], Lemma 4.7.

Proof of Proposition 4.2. Let us start with E^*. Assume on the contrary of our claim that there exist $x \in \mathbb{R}^N$ and $t \in (0, T]$ such that $s \mapsto d_{E^*(s)}(x)$ is not left continuous at t. Since this map is lower semi-continuous thanks to the closeness of E^*, we deduce that there exist $\varepsilon > 0$ and a sequence (t_p) converging to t^- such that for all $p \in \mathbb{N}$,

$$d_{E^*(t_p)}(x) > d_{E^*(t)}(x) + \varepsilon.$$

Let $S = \varepsilon\omega_*$, so that W_S is the closed ball of radius ε centered at 0. By considering a projection of x on $E^*(t)$, we can assume that $x \in E^*(t)$ and for all $p \in \mathbb{N}$,

$$d_{E^*(t_p)}(x) > \varepsilon.$$

Set for a fixed p, $k_n = [t_p/h_n]$, so that $k_n h_n \to t_p^-$ as $n \to +\infty$. By definition of $E^*(t_p)$, there exists n_0 large enough depending on p so that for all $n \geq n_0$, $d_{E_{h_n}(k_n)}(x) > \varepsilon$. Let us set

$$M = \left(\frac{\delta}{S} + 2L\right).$$

Then we can apply assertion 2 of Lemma 4.4 to deduce that for all $n \geq n_0$ such that $R(h_n) < \frac{S}{4}$ and for all j such that $jh_n \leq \min\{\frac{S^2}{4\omega_*(\delta+2LS)}, T - k_n h_n\}$

$$d_{E_{h_n}(k_n+j)}(x) \geq \varepsilon - Mjh_n. \tag{4.4}$$

Indeed we have $W_{S-\omega_*(\frac{\delta}{S}+2L)jh_n}(x) \subset \mathbb{R}^N \setminus E_{h_n}(k_n + j)$. Let us set

$$\tau = \min\left\{\frac{\varepsilon}{2M}, \frac{S^2}{4\omega_*(\delta + 2LS)}\right\}$$

and fix $s \in (0, \tau)$ with $s \leq T - t_p$. We set $j_n = [s/h_n]$ so that $j_n h_n \to s^-$ as $n \to +\infty$. Then $j_n h_n \leq \min\{\frac{S^2}{4\omega_*(\delta+2LS)}, T - k_n h_n\}$ for n large enough, so that sending n to $+\infty$ in (4.4) yields, by definition of $E^*(t_p+s)$,

$$d_{E^*(t_p+s)}(x) \geq \varepsilon - Ms \geq \frac{\varepsilon}{2}.$$

Taking $s = t - t_p$ for p big enough so that $0 < s < \tau$, we get $d_{E^*(t)}(x) \geq \frac{\varepsilon}{2}$, which contradicts the fact that $x \in E^*(t)$.

The proof for $d_{\widehat{E_*}}$ is obtained in the same way by using assertion 1 of Lemma 4.4. □

4.3. Comparison at initial time

We finish by giving a consequence of previous growth results on the comparison of the initial sets $E_*(0)$ and $E^*(0)$ with E_0. This result will be essential for comparison at later times:

Proposition 4.5. *We have* $\overset{\circ}{E_0} \subset E_*(0) \subset E^*(0) \subset \overline{E_0}$.

Proof. We only prove that $E^*(0) \subset \overline{E_0}$, the left-hand side inclusion is obtained by similar arguments. Suppose on the contrary that there exists $x \in E^*(0) \setminus \overline{E_0}$. Then we can find some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{R}^N \setminus \overline{E_0}$. By definition of $E^*(0)$, there exist sequences $k_n \to +\infty$ and $x_n \to x$ with $k_n h_n \to 0$ and $x_n \in E_{h_n}(k_n)$. Thanks to Lemma 4.4 and the facts that $E_{h_n}(0) = E_0$ and $k_n h_n \to 0$, we know that there exists $M > 0$ depending only on ε, L and N such that if n is large enough, then

$$B_{\varepsilon - Mk_n h_n}(x) \subset \mathbb{R}^N \setminus E_{h_n}(k_n).$$

But $x_n \to x$ and $\varepsilon - Mk_n h_n \to \varepsilon$, so that $x_n \in B_{\varepsilon - Mk_n h_n}(x)$ for n large enough. This is a contradiction since $x_n \in E_{h_n}(k_n)$, and this proves the proposition. □

5. Minimizing movements and weak solutions

With the tools of Section 4, we are now ready to prove Theorem 1.4. Since $E_*(t) \subset E(t) \subset E^*(t)$ a.e. for all $t \in [0,T]$, it suffices to prove that for all $t \in [0,T]$,

$$\{u(\cdot,t) > 0\} \subset E_*(t) \quad \text{and} \quad E^*(t) \subset \{u(\cdot,t) \geq 0\}.$$

To this end, we will use a comparison principle for discontinuous viscosity solutions. We therefore start by giving equations satisfied by $\mathbf{1}_{E_*}$ and $\mathbf{1}_{E^*}$ in the viscosity sense, in relation with Theorem 4.1:

Theorem 5.1. *Under the assumptions of Theorem 1.4, we have:*

1. *For any $(x,t) \in \mathbb{R}^N \times (0,T)$, if a test function ϕ of class C^2 is such that $\mathbf{1}_{E^*} - \phi$ has a local maximum at (x,t), then:*

- *if $D\phi(x,t) \neq 0$, we have*

$$\phi_t(x,t) \leq h(D\phi(x,t), D^2\phi(x,t)) + \left[c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|;$$

- *if $D\phi(x,t) = 0$ and $D^2\phi(x,t) = 0$, we have*

$$\phi_t(x,t) \leq 0.$$

2. *For any $(x,t) \in \mathbb{R}^N \times (0,T)$, if a test function ϕ of class C^2 is such that $\mathbf{1}_{E_*} - \phi$ has a local minimum at (x,t), then:*

- *if $D\phi(x,t) \neq 0$, we have*

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) + \left[c_0(\cdot,t) \star \mathbf{1}_{E(t)}(x) + c_1(x,t)\right] |D\phi(x,t)|;$$

- *if $D\phi(x,t) = 0$ and $D^2\phi(x,t) = 0$, we have*

$$\phi_t(x,t) \geq 0.$$

Proof. We only prove the first point, since the second point uses the same arguments. We only need to consider the case where $(x,t) \in \partial E^*$, since otherwise all derivatives of ϕ at (x,t) vanish and the equation is obviously satisfied.

First case. $D\phi(x,t) \neq 0$. In this case it is straightforward to check that $-\phi$ has a local maximum on E^* at (x,t). Therefore, the first point of Proposition 4.1 gives the result.

Second case. $D\phi(x,t) = 0$ and $D^2\phi(x,t) = 0$. We can always assume that our maximum is equal to 0, *i.e.* $\phi(x,t) = \mathbf{1}_{E^*}(x,t) = 1$. Let us also assume that $\phi_t(x,t) > 0$. Then a Taylor expansion of ϕ at (x,t) shows that there exist $\delta > 0$ and $k > 0$ such that for all (y,s) verifying $s \in (t-\delta,t)$ and $|y-x| < 2k(t-s)^{1/3}$, $\mathbf{1}_{E^*}(y,s) \leq \phi(y,s) < \phi(x,t) = 1$, whence $y \notin E^*(s)$. As a consequence for all $s \in (t-\delta,t)$,

$$d_{E^*(s)}(x) > k(t-s)^{1/3}.$$

Now we can proceed as in the proof of Proposition 4.2, using the growth control given by Lemma 4.4, to prove that there are positive constants k_1 and k_2 such that for all $s < t$ close enough to t,

$$d_{E^*(t)}(x) > k(t-s)^{1/3} - \left(\frac{k_1}{(t-s)^{1/3}} + k_2\right)(t-s) > 0,$$

which contradicts the fact that $x \in E^*(t)$. Therefore $\phi_t(x,t) \leq 0$. $\square$

Proof of Theorem 1.4. The previous theorem shows that $\mathbf{1}_{E^*}$ is a subsolution of the level-set equation (1.4), while $\mathbf{1}_{E_*}$ is a supersolution. Indeed, an argument of Barles and Georgelin [8], Proposition 1 shows that under the conclusions of Theorem 5.1 there is no property to check when the test function satisfies $D\phi(x,t) = 0$ and $D^2\phi(x,t) \neq 0$. To conclude we use a method initiated by Barles, Soner and Souganidis [10], Theorem 2.1: let (Φ_n) be a sequence of smooth functions such that $\Phi_n \equiv 1$ on $[0,+\infty)$, $\Phi_n' \geq 0$ in $\mathbb{R}$, $\Phi_n(\mathbb{R}) \subset [0,1]$ and $\inf_n \Phi_n = 0$ on $(-\infty,0)$. Thanks to Lemma 4.5, we know that $\mathbf{1}_{E^*(0)} \leq \Phi_n(u_0)$ in $\mathbb{R}^N$. Since (1.4) is a geometric equation, $\Phi_n(u)$ is a uniformly continuous solution of this equation. The comparison principle [10], Theorem 1.3, implies that for all $t \in [0,T)$,

$$\mathbf{1}_{E^*(t)} \leq \Phi_n(u(\cdot,t)).$$

If $x \in \{u(\cdot,t) < 0\}$, we therefore have

$$\mathbf{1}_{E^*(t)}(x) \leq \inf_n \Phi_n(u(x,t)) = 0,$$

which means that $x \notin E^*(t)$. As a consequence $E^*(t) \subset \{u(\cdot,t) \geq 0\}$ for all $t \in [0,T)$, which also holds for $t = T$ by continuity of u and thanks to Proposition 4.2. The argument to prove that $\{u(\cdot,t) > 0\} \subset E_*(t)$ is similar.

In case there is no fattening, we deduce that for all $t \in [0,T]$, $E(t) = \{u(\cdot,t) \geq 0\}$ almost everywhere, and we can replace $\{u(\cdot,t) \geq 0\}$ by $E(t)$ in (1.4) to deduce that u is a viscosity solution of (1.4). This concludes the proof of Theorem 1.4. $\square$

6. Comparison with the smooth flow

Now we are going to show that our construction is consistent with smooth flows if they exist: we turn to the proof of Theorem 1.5. Following Cardaliaguet and Pasquignon [14], we define a sub/super pair of solutions for our non-local motion. Roughly speaking, it is a pair $(\mathcal{K}_1,\mathcal{K}_2)$ of tubes, where $\mathcal{K}_1$ moves with velocity

$$V_{x,t} \leq H_{x,t} + \inf_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)} \{c_0(\cdot,t)\star\mathbf{1}_K(x)\} + c_1(x,t),$$

while $\mathcal{K}_2$ moves with velocity

$$V_{x,t} \geq H_{x,t} + \sup_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)} \{c_0(\cdot,t)\star\mathbf{1}_K(x)\} + c_1(x,t).$$

As we did at the beginning of Section 4.1, we formulate this in terms of test functions:

Definition 6.1 ([14], Def. 2.5)**.** Let K_1 and K_2 be compact subsets of $\mathbb{R}^N$ such that $K_1 \subset \overset{\circ}{K}_2$. A sub/super pair of solutions with initial data (K_1,K_2) is a pair $(\mathcal{K}_1,\mathcal{K}_2)$ of tubes such that
1. $\mathcal{K}_1 \subset \mathcal{K}_2$.
2. $\mathcal{K}_1(0) = K_1$ and $\widehat{\mathcal{K}_2}(0) \subset \widehat{K_2}$.
3. For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local maximum on $\mathcal{K}_1$ at some point $(x,t) \in \partial\mathcal{K}_1$, then

$$\phi_t(x,t) \geq h(D\phi(x,t), D^2\phi(x,t)) - \left[\inf_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)} \{c_0(\cdot,t)\star\mathbf{1}_K(x)\} + c_1(x,t)\right]|D\phi(x,t)|.$$

4. For any $t \in (0,T)$, if a test function ϕ of class C^2 has a local minimum on $\widehat{\mathcal{K}_2}$ at some point $(x,t) \in \partial\widehat{\mathcal{K}_2}$, then

$$\phi_t(x,t) \leq h(D\phi(x,t), D^2\phi(x,t)) - \left[\sup_{\mathcal{K}_1(t)\subset K\subset\mathcal{K}_2(t)} \{c_0(\cdot,t)\star\mathbf{1}_K(x)\} + c_1(x,t)\right]|D\phi(x,t)|.$$

Such sub/super pairs of solutions exist and we can define, following Cardaliaguet and Pasquignon, *extremal* sub/super pairs of solutions $(\mathcal{K}_1^\varepsilon, \mathcal{K}_2^\varepsilon)$ with initial data $(E_0 - \varepsilon B_1(0), E_0 + \varepsilon \overline{B}_1(0))$. The extremality holds with respect to the inclusion. Moreover, if E_0 is compact with uniformly $C^{3+\alpha}$ boundary, and if E_r is a smooth evolution with $C^{3+\alpha}$ boundary, starting from E_0 with normal velocity given by (1.9), then $\mathcal{K}_1^\varepsilon \subset E_r \subset \mathcal{K}_2^\varepsilon$ and both $\mathcal{K}_1^\varepsilon$ and $\mathcal{K}_2^\varepsilon$ converge to E_r in the Hausdorff distance as $\varepsilon \to 0$, as proved by Cardaliaguet [12]. This implies in particular that a smooth evolution with $C^{3+\alpha}$ boundary is necessarily unique.

Now, owing to the respective velocities of $\mathcal{K}_1^\varepsilon$, E_*, E^* and $\mathcal{K}_2^\varepsilon$, we want to compare these sets. Going through the corresponding proofs in [14] and [12], we check that the estimation on the velocities of E^* and E_* (Prop. 4.1), their regularity property (Prop. 4.2) and their initial position relatively to E_0 (Prop. 4.5) give the following result:

Theorem 6.2 ([14], Th. 2.11)**.** *Under the assumptions of Theorem 1.5, let $(\mathcal{K}_1^\varepsilon, \mathcal{K}_2^\varepsilon)$ be an extremal sub/super pair of solutions with initial data $(E_0 - \varepsilon B_1(0), E_0 + \varepsilon \overline{B}_1(0))$. If $\mathcal{K}_1^\varepsilon(t)$ and $\mathcal{K}_2^\varepsilon(t)$ are non-empty for all $t \in [0,T]$, then*

$$\mathcal{K}_1^\varepsilon(t) \subset E_*(t) \subset E^*(t) \subset \mathcal{K}_2^\varepsilon(t) \quad \forall t \in [0,T).$$

We are finally ready to prove Theorem 1.5.

Proof of Theorem 1.5. Since $\mathcal{K}_1^\varepsilon$ and $\mathcal{K}_2^\varepsilon$ converge to the smooth evolution E_r starting from E_0 in the Hausdorff distance if the latter exists, we deduce that for all $t \in [0,T)$, $E_*(t) = E^*(t) = E_r(t)$. This also holds for $t = T$ thanks to Proposition 4.2. Moreover we know that for all $t \in [0,T]$, $E_*(t) \subset E(t) \subset E^*(t)$ a.e., so the result follows. □

7. Existence and uniqueness of a smooth solution

To conclude this work, it is natural to verify that such a smooth evolution exists (we already know that it must be unique). This is the claim of Theorem 1.6, that we prove now, using a fixed point method. We therefore begin by constructing a smooth solution for the local problem (*i.e.* with prescribed velocity).

7.1. Existence of smooth solutions for the local problem

Theorem 7.1 (existence of a smooth solution for the local problem)**.** *Assume that E_0 is a compact subset of $\mathbb{R}^N$ with uniformly $C^{3+\alpha}$ boundary and that $c \in W^{2,1;\infty}(\mathbb{R}^N \times [0,T])$. Then there exist a small time $t_0 > 0$ depending only on E_0 and on an upper bound on $\|c\|_{W^{2,1;\infty}(\mathbb{R}^N \times [0,T])}$, and a smooth evolution E_r with $C^{3+\alpha}$ boundary defined on $[0,t_0]$, starting from E_0, with normal velocity*

$$V_{x,t} = H_{x,t} + c(x,t), \tag{7.1}$$

where $H_{x,t}$ is the mean curvature of $\Gamma(t) = \partial E_r(t)$ at x.

The proof is an adaptation of the one proposed by Evans and Spruck [16] for the classical mean curvature motion (see also Giga and Goto [20] and Maekawa [24] for more general equations). For the reader's convenience, we give the steps of the proof to explain how to treat the dependence in the space variable of the velocity c.

Assume we are given the smooth hypersurface $\Gamma_0 = \partial E_0$, a time $t_0 > 0$ and a smooth evolution $t \mapsto \Gamma(t) = \partial E(t)$ of surfaces developing from Γ_0 on $[0,t_0]$ with normal velocity $V_{x,t}$. Heuristically, one can show (see [16]) that the signed distance function d to $\Gamma(t)$ defined by

$$d(x,t) = \begin{cases} -\mathrm{dist}(x,\Gamma(t)) & x \in \mathbb{R}^N \setminus E(t) \\ \mathrm{dist}(x,\Gamma(t)) & x \in E(t) \end{cases}$$

is a solution of

$$v_t = F(D^2 v, v) + c(x - v(x,t)Dv(x,t), t) \tag{7.2}$$

with

$$F(R,z)=f(\lambda_1(R),...,\lambda_n(R),z)=\sum_{i=1}^{N}\frac{\lambda_i(R)}{1-\lambda_i(R)z}, \tag{7.3}$$

where $\lambda_1(R)\le\lambda_2(R)\le...\le\lambda_N(R)$ are the eigenvalues of R. F is *a priori* defined and smooth for $|R|$ and $|z|$ small enough, but we extend it to be smooth on all of $Sym_N\times\mathbb{R}$ with $|F|$, $|DF|$ and $|D^2F|$ bounded as in [16].

The idea is to study directly the PDE (7.2). To this aim, we set $\Gamma_0=\partial E_0$ and let

$$g(x)=\begin{cases}-\mathrm{dist}(x,\Gamma_0) & x\in\mathbb{R}^N\backslash E_0\\ \mathrm{dist}(x,\Gamma_0) & x\in E_0\end{cases} \tag{7.4}$$

be the signed distance function to Γ_0. We fix δ_0 so small that g is of class $C^{3+\alpha}$ within

$$V=\{x\in\mathbb{R}^N,\ -\delta_0<g(x)<\delta_0\}$$

and we set, for $t_0>0$ to be determined,

$$Q=V\times(0,t_0),\quad \Sigma=\partial V\times[0,t_0].$$

The plan is to consider a solution to the PDE

$$\begin{cases}v_t=F(D^2v,v)+c(x-vDv,t) & \text{in} \quad Q\\ |Dv|^2=1 & \text{on}\quad \Sigma\\ v=g & \text{on}\quad V\times\{t=0\}\end{cases} \tag{7.5}$$

and prove that the zero level sets of $v(\cdot,t)$ are smooth hypersurfaces evolving with normal velocity given by (7.1).

First, we have the following existence result for this non-linear PDE (see Lunardi [23], Th. 8.5.4 and Prop. 8.5.6):

Theorem 7.2 (existence for the non-linear PDE)**.** *There exist δ_0 depending only on E_0 and $t_0>0$ depending only on E_0 and on an upper bound on $\|c\|_{W^{2,1;\infty}(\mathbb{R}^N\times[0,T])}$ such that there exists a unique solution $v\in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})$ of the PDE* (7.5)*. Moreover the first order space derivatives v_{x_k}, for $1\le k\le N$, belong to $C^{2+\alpha,\frac{2+\alpha}{2}}(V\times[0,t_0])$.*

Evolution of the zero level set of v

The rest of the proof is devoted to proving that, possibly reducing t_0, the mapping

$$t\in[0,t_0]\mapsto E_r(t)=(E_0\setminus V)\cup\{x\in V,\ v(x,t)\ge 0\}$$

is a smooth evolution with $C^{3+\alpha}$ boundary, with normal velocity given by (7.1).

Proposition 7.3 (distance property of v)**.** *Let v be the solution of* (7.5) *given by Theorem 7.2. Then we have*

$$|Dv|^2=1 \quad \text{in } \overline{Q}. \tag{7.6}$$

Proof. We adapt the proof of Evans and Spruck [16], Theorem 3.1.

Step 1. Let $w=|Dv|^2-1$. Then $w\in C^{2+\alpha,\frac{2+\alpha}{2}}(V\times[0,t_0])$. Moreover, using the PDE (7.5) and the definition of g given by (7.4), we get that

$$w=0 \quad \text{on } \Sigma\cup(V\times\{t=0\}).$$

Step 2. Differentiating (7.5), we compute (with implicit summations over i, j, k)

$$v_{tx_k} = \frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_ix_jx_k} + \frac{\partial F}{\partial z}(D^2v, v)v_{x_k} + \frac{\partial}{\partial x_k}c(x - vDv, t).$$

Therefore

$$\begin{aligned} w_t =& 2v_{x_k}v_{x_kt} \\ =& 2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_k}v_{x_kx_ix_j} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2 + 2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k} \\ =& \frac{\partial F}{\partial r_{ij}}(D^2v, v)w_{x_ix_j} - 2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_kx_i}v_{x_kx_j} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2 + 2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k}. \end{aligned} \tag{7.7}$$

Now

$$\begin{aligned} 2\frac{\partial}{\partial x_k}(c(x - vDv, t))v_{x_k} =& 2\sum_{i,k=1}^{N} \frac{\partial c}{\partial x_i}(x - vDv, t)(\delta_{ik} - v_{x_k}v_{x_i} - vv_{x_kx_i})\, v_{x_k} \\ =& -2(|Dv|^2 - 1)\sum_{i=1}^{N} \frac{\partial c}{\partial x_i}(x - vDv, t)\, v_{x_i} - \sum_{i=1}^{N} \frac{\partial c}{\partial x_i}(x - vDv, t)\, v\, w_{x_i} \\ =& -w\, l_1(x, t) - w_{x_i}\, l_{2,i}(x, t), \end{aligned}$$

where

$$l_1(t, x) = 2\sum_{i=1}^{N} \frac{\partial c}{\partial x_i}(x - vDv, t)\, v_{x_i}$$

and

$$l_{2,i}(x, t) = \frac{\partial c}{\partial x_i}(x - vDv)\, v.$$

Moreover as recalled in [16],

$$\frac{\partial F}{\partial r_{ij}}(D^2v)v_{x_kx_i}v_{x_kx_j} = \frac{\partial F}{\partial z}(D^2v, v).$$

As a consequence (7.7) becomes

$$w_t = \frac{\partial F}{\partial r_{ij}}(D^2v, v)w_{x_ix_j} + \left(2\frac{\partial F}{\partial z}(D^2v, v) - l_1(x, t)\right) w - l_{2,i}(x, t)\, w_{x_i}.$$

In view of the uniform ellipticity of F (see [16], Lem. 2.1), we get that this is a uniformly parabolic equation. Using the fact that $w = 0$ on the parabolic boundary of Q, we deduce that $w = 0$ in $\overline{Q}$. This ends the proof of the proposition. □

Now, using (7.6), we get that

$$\Gamma = \{(x, t) \in \overline{Q},\ v = 0\}$$

is a C^1 hypersurface in $\overline{Q}$ and each slice $\Gamma(t) = \{x \in V,\ v(x, t) = 0\}$ is a $C^{3+\alpha}$ hypersurface in V. Moreover we have the following equivalent of [16], Theorem 3.2:

Theorem 7.4 (existence of a classical evolution)**.** *The surfaces* $\{\Gamma(t)\}_{0\le t\le t_0}$ *comprise a classical motion starting from* Γ_0 *with normal velocity*

$$V_{x,t} = H_{x,t} + c(x, t).$$

Given that $\Gamma(t) = \partial E_r(t)$ for all $t \in [0, t_0]$, provided t_0 is small enough depending only on an upper bound on $\|c\|_{W^{2,1;\infty}(\mathbb{R}^N \times [0,T])}$, this concludes the proof of Theorem 7.1.

7.2. Existence of smooth solution for the non-local problem

With the results of the previous section, we are now ready to carry out the fixed point argument. We use the same notation as in the previous section, in particular F, Q, Σ and V, with the same δ_0 fixed, but for some t_0 to be determined. Using the same method as in Section 7.1, our goal is to construct a solution to the PDE

$$\begin{cases} v_t = F(D^2 v, v) + (c_0(\cdot, t) \star_V \mathbf{1}_{\{v(\cdot,t)\geq 0\}})(x - vDv, t) + \tilde{c}(x - vDv, t) & \text{in} \quad Q \\ |Dv|^2 = 1 & \text{on} \quad \Sigma \\ v = g & \text{on} \quad V \times \{t = 0\} \end{cases} \tag{7.8}$$

where $\star_V$ denotes the convolution restricted to V, *i.e.*

$$c_0(\cdot, t) \star_V \mathbf{1}_{\{v(\cdot,t)\geq 0\}}(x) = \int_V c_0(x - y, t)\mathbf{1}_{\{v(\cdot,t)\geq 0\}}(y)\, \mathrm{d}y$$

and

$$\tilde{c}(x, t) = \int_{E_0 \setminus V} c_0(x - y, t)\mathrm{d}y + c_1(x, t).$$

We define the set

$$E = \left\{ v \in C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q}) \;\middle|\; \begin{array}{l} ||v - g||_{C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 \\ |Dv|^2 = 1 \text{ in } Q \\ v = g \text{ on } V \times \{t = 0\} \\ v_t = h_0 \text{ on } V \times \{t = 0\} \end{array} \right\},$$

where g is defined by (7.4), R_0 is a small constant which will be precised later and

$$h_0 = F(D^2 g, g) + c_0 \star \mathbf{1}_{E_0}(x - gDg, 0) + c_1(x - gDg, 0).$$

For $w \in E$, we set

$$c_w(x, t) = c_0(\cdot, t) \star_V \mathbf{1}_{\{w(\cdot,t)\geq 0\}}(x) + \tilde{c}(x, t).$$

Under the assumptions on c_0 and c_1 it is easy to check that $c_w \in W^{2,1;\infty}(\mathbb{R}^N \times [0, T])$ (see the definition of $W^{2,1;\infty}(\mathbb{R}^N \times [0, T])$ after (1.11)). Indeed, the only difficulty is to check that c_w is Lipschitz in time. To do this, let us state the following lemma:

Lemma 7.5 (estimate on characteristic functions)**.** *There exists a constant C which does not depend on t_0, such that if u_1, $u_2 \in C^1(V)$ satisfy $Du_i \cdot Dg \geq \frac{1}{2}$ in V for $i = 1, 2$, then*

$$\|\mathbf{1}_{\{u_1\geq 0\}} - \mathbf{1}_{\{u_2\geq 0\}}\|_{L^1(\overline{V})} \leq C\|u_1 - u_2\|_{L^\infty(\overline{V})}.$$

The proof is an easy adaptation of [3], Lemma 42 (using local cards and a partition of unity), so we skip it.

For any $u \in E$, Du satisfies $Du(\cdot, 0) = Dg$ and is Hölder in time. As a consequence, for t_0 small enough depending only on an upper bound on

$$\|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})},$$

we have $Du(\cdot,t)\cdot Dg \geq 1/2$ in V for any $u \in E$ and $t \in [0,t_0]$. Therefore, using the previous lemma, we can compute

$$\begin{aligned}|c_w(x,t)-c_w(x,s)| &= |c_0(\cdot,t)\star_V \mathbf{1}_{\{w(\cdot,t)\geq 0\}}(x) - c_0(\cdot,s)\star_V \mathbf{1}_{\{w(\cdot,s)\geq 0\}}(x) + \tilde{c}(x,t)-\tilde{c}(x,s)|\\ &\leq |c_0(\cdot,t)\star_V \mathbf{1}_{\{w(\cdot,t)\geq 0\}}(x) - c_0(\cdot,t)\star_V \mathbf{1}_{\{w(\cdot,s)\geq 0\}}(x)|\\ &\quad + |c_0(\cdot,t)\star_V \mathbf{1}_{\{w(\cdot,s)\geq 0\}}(x) - c_0(\cdot,s)\star_V \mathbf{1}_{\{w(\cdot,s)\geq 0\}}(x)| + |\tilde{c}(x,t)-\tilde{c}(x,s)|\\ &\leq C_w|t-s|,\end{aligned}$$

where

$$C_w = C\|c_0\|_{L^\infty(\mathbb{R}^N\times[0,T])}\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} + 2\|c_0\|_{W^{1,\infty}([0,T];L^\infty(\mathbb{R}^N))}\mathcal{L}^N(E_0) + \|c_1\|_{W^{2,1;\infty}(\mathbb{R}^N\times(0,T))}.$$

The factor 2 appears if we assume that $\mathcal{L}^N(V\setminus E_0)\leq \mathcal{L}^N(E_0)$, which is always possible. We remark that this constant C_w can be chosen independently of w since we have $\|w\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$. This, together with similar estimates on space derivatives, implies that for any $w\in E$,

$$\|c_w\|_{W^{2,1;\infty}(\overline{Q})} \leq C(1+R_0),$$

where the constant C does not depend on t_0, R_0.

As a consequence of Theorem 7.2, for t_0 small enough (depending only on R_0), we can therefore define for any $w\in E$, $v=\Phi(w)$ as the unique solution of

$$\begin{cases} v_t = F(D^2v,v)+c_w(x-vDv,t) & \text{in} \quad Q\\ |Dv|^2=1 & \text{on} \quad \Sigma\\ v=g & \text{on} \quad V\times\{t=0\}.\end{cases}$$

Moreover the proof of Theorem 7.2 shows that provided t_0 is small enough (depending only on R_0), then $v\in E$ for any $w\in E$. Let us now prove that Φ is a contraction, for a good choice of parameters R_0 and t_0.

Let $w_1,\ w_2\in E$, $v_1=\Phi(w_1)$, $v_2=\Phi(w_2)$ and $v=v_2-v_1$. Then v is a solution of

$$\begin{cases} v_t - a_{ij}v_{x_ix_j} + f_iv_{x_i} + ev = \delta + A(D^2v,Dv,v,x,t) & \text{in} \quad Q\\ \dfrac{\partial v}{\partial \nu} = a(Dv,x,t) & \text{on} \quad \Sigma\\ v=0 & \text{on} \quad V\times\{t=0\},\end{cases}$$

where

$$a_{ij}=\frac{\partial F}{\partial r_{ij}}(D^2v_1,v_1)v_{ij},\quad f_i=\frac{\partial c}{\partial x_i}v_1,\quad e=Dc_{w_1}\cdot Dv_1-\frac{\partial F}{\partial z}(D^2v_1,v_1),$$

$$\delta = c_{w_2}(x-v_2Dv_2,t)-c_{w_1}(x-v_2Dv_2,t),$$

$$\begin{aligned}A(R,p,z,x,t) &= F(D^2v_1+R,v_1+z)-F(D^2v_1,v_1)-\frac{\partial F}{\partial z}(D^2v_1,v_1)z-\frac{\partial F}{\partial r_{ij}}(D^2v_1,v_1)r_{ij}\\ &\quad + c_{w_1}(x-(v_1+z)(Dv_1+p),t)-c_{w_1}(x-v_1Dv_1,t)\\ &\quad + (Dc_{w_1}(x-v_1Dv_1,t)\cdot Dv_1)z + \frac{\partial c_{w_1}(x-v_1Dv_1,t)}{\partial x_i}v_1p_i\end{aligned}$$

and

$$a(p,x,t)=\begin{cases}-\dfrac{1}{2}\left(2p\cdot(Dv_1(x,t)-Dg(x))+|p|^2\right) & \text{on } \{g=\delta_0\}\\ \dfrac{1}{2}\left(2p\cdot(Dv_1(x,t)-Dg(x))+|p|^2\right) & \text{on } \{g=-\delta_0\},\end{cases}$$

where we have used the fact that Dg is a unit normal to ∂V. Using the same arguments as those of Evans and Spruck [16], Lemma 5.3 (*i.e.* a Taylor expansion) and the fact that $||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}\le 2R_0$, we get that

$$||A||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})},||a||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma)}\le C_0R_0||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}, \tag{7.9}$$

where C_0 does not depend on t_0,R_0. Using [16], Lemma 2.2, we then deduce that:

$$\|v_1-v_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}=||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}\le C_1\left(\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}+\|A\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}+\|a\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Sigma)}\right),$$

where C_1 does not depend on t_0 and R_0, which together with (7.9) implies that

$$\|v\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}\le 2C_1\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} \tag{7.10}$$

as soon as $R_0\le(4C_0C_1)^{-1}$. Let us fix from now on such a R_0.

We now use the following lemma, the proof of which is postponed:

Lemma 7.6 (estimate on the velocities)**.** *With the previous notation, there exists C independent of t_0 such that if $w=w_1-w_2$, we have for t_0 small enough*

$$||\delta||_{W^{1,1;\infty}(\overline{Q})}\le C||w||_{W^{1,1;\infty}(\overline{Q})}.$$

This implies in particular, also using the Hölder regularity of w and the fact that $w_t(\cdot,0)=0=Dw(\cdot,0)$, that

$$\|\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})}\le||\delta||_{W^{1,1;\infty}(\overline{Q})}\le C||w_1-w_2||_{W^{1,1;\infty}(\overline{Q})}\le Ct_0^{\frac{\alpha}{2}}\|w_1-w_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}.$$

Using (7.10), we deduce that for t_0 small enough,

$$\|v_1-v_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}\le\frac{1}{2}\|w_1-w_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})}.$$

This implies that Φ is a contraction whence, using the Banach fixed point theorem, we deduce that there exists a unique solution v of (7.8).

Using Theorem 7.4, we finally obtain that, possibly reducing t_0,

$$t\in[0,t_0]\mapsto E_r(t)=(E_0\setminus V)\cup\{x\in V,\ v(x,t)\ge 0\}$$

defines a smooth evolution with $C^{3+\alpha}$ boundary starting from E_0 with normal velocity

$$V_{x,t}=H_{x,t}+c_0(\cdot,t)\star\mathbf{1}_{E_r(t)}(x)+c_1(x,t).$$

This concludes the proof of Theorem 1.6.

We end with the proof of Lemma 7.6:

Proof of Lemma 7.6. We begin by estimating the derivative of δ in time. Writing out the expression of $\frac{\partial\delta}{\partial t}$, we see that thanks to the regularity of c_0 and the fact that

$$\|v_2\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})},$$

the only difficult term to treat is $\frac{\partial}{\partial t}(c_{w_2} - c_{w_1})$. However we have, using Hadamard's formula:

$$\begin{aligned}\frac{\partial(c_{w_2} - c_{w_1})}{\partial t}(x,t) &= \int_V (c_0)_t(x-y,t)(\mathbf{1}_{\{w_1(\cdot,t)\geq 0\}} - \mathbf{1}_{\{w_2(\cdot,t)\geq 0\}})(y)\,\mathrm{d}y \\ &- \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y) + \int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y).\end{aligned} \tag{7.11}$$

First, using Lemma 7.5, we have that

$$\left|\int_V (c_0)_t(x-y,t)(\mathbf{1}_{\{w_1(\cdot,t)\geq 0\}} - \mathbf{1}_{\{w_2(\cdot,t)\geq 0\}})(y)\mathrm{d}y\right| \leq C\|c_0\|_{W^{1,\infty}([0,T];L^\infty(\mathbb{R}^N))}\|w\|_{L^\infty(\overline{Q})}. \tag{7.12}$$

For the second term, we write

$$\int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y) - \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y) = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 = \int_{\{w_2(\cdot,t)=0\}} (w_2)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y) - \int_{\{w_2(\cdot,t)=0\}} (w_1)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y)$$

and

$$\mathcal{I}_2 = \int_{\{w_2(\cdot,t)=0\}} (w_1)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y) - \int_{\{w_1(\cdot,t)=0\}} (w_1)_t(y,t)c_0(x-y,t)\mathrm{d}\mathcal{H}^{N-1}(y).$$

We remark that

$$|\mathcal{I}_1| \leq C\|c_0\|_{L^\infty(\overline{Q})}\|w_t\|_{L^\infty(\overline{Q})}, \tag{7.13}$$

where the constant C is a bound on the perimeter of $\{u(\cdot,t)=0\}$, uniform for $u \in E$ and $t \in [0,t_0]$.

We now treat $\mathcal{I}_2$, and to this aim we use a local parameterization. We choose local coordinates and r small enough such that if $B_r = B_r^{N-1}(0)$, then

$$\frac{\partial g}{\partial x_N} \geq \frac{3}{4} \quad \text{in } B_r \times [-r,r].$$

Now, for t_0 small enough (depending only on R_0 and g), recalling that

$$w_i(\cdot,0) = g \quad \text{and} \quad \|w_i\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q})} \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})},$$

we get that

$$\frac{\partial w_i}{\partial x_N} \geq \frac{1}{2} \quad \text{in } B_r \times [-r,r]. \tag{7.14}$$

We fix $t \leq t_0$ and we assume that $\{w_i(\cdot,t)=0\} = \{(x', f_i(x')),\ x' \in B_r\}$. Using a partition of unity, we will then recover the complete estimate. We define $\varepsilon(x') = f_2(x') - f_1(x')$. For t_0 small enough (depending only on R_0 and g) we can assume that

$$|\varepsilon(x')| \leq \frac{1}{2(R_0 + \|g\|_{C^{2+\alpha}(\overline{V})})}. \tag{7.15}$$

We then have

$$\begin{aligned} |\mathcal{I}_2| &\leq C \int_{y'\in B_r} \Big| \sqrt{1+|Df_1|^2}\, c_0(x'-y', x_N - f_1(y'), t) \\ &\qquad - \sqrt{1+|Df_1 + D\varepsilon|^2}\, c_0(x'-y', x_N - f_1(y') - \varepsilon(y'), t) \Big| \, \mathrm{d}y' \\ &\leq C\|\varepsilon\|_{W^{1,\infty}(B_r)}, \end{aligned}$$

where we have used the fact that $c_0 \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}^N))$ and where the constant C depends only on R_0, g and c_0.

Our goal now is just to estimate $\|\varepsilon\|_{W^{1,\infty}(B_r)}$ with respect to $\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}$. For simplicity of notation, we forget the dependence in time of w, w_1 and w_2. We recall that

$$\begin{aligned} w_1(x', f_1(x')) = 0 &= w_2(x', f_1(x') + \varepsilon(x')) \\ &= w_1(x', f_1(x') + \varepsilon(x')) - w(x', f_1(x') + \varepsilon(x')). \end{aligned} \tag{7.16}$$

Using a Taylor expansion, we get that

$$w_1(x', f_1(x') + \varepsilon(x')) = w_1(x', f(x')) + \frac{\partial w_1}{\partial x_N}(x', f(x')) \cdot \varepsilon(x') + o(\varepsilon), \tag{7.17}$$

where

$$\|o(\varepsilon)\|_{L^\infty} \leq \frac{1}{2}\left|\frac{\partial^2 w_1}{\partial x_N^2}\right| \|\varepsilon\|^2_{L^\infty} \leq \frac{1}{4}\|\varepsilon\|_{L^\infty},$$

thanks to (7.15) and the fact that $\left|\frac{\partial^2 w_1}{\partial x_N^2}\right| \leq R_0 + \|g\|_{C^{2+\alpha}(\overline{V})}$. We then deduce from (7.16), (7.17) and (7.14) that

$$\|\varepsilon\|_{L^\infty} \leq 4\|w\|_{L^\infty(\overline{Q})}. \tag{7.18}$$

Differentiating (7.16) with respect to x_i and using a Taylor expansion, we get as above

$$\|\varepsilon_{x_i}\|_{L^\infty} \leq C \frac{\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}}{|\frac{\partial w_2}{\partial x_N}|} \leq C\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}.$$

Combining the last inequality with (7.18), we have

$$|\mathcal{I}_2| \leq C\|w\|_{L^\infty([0,t_0],W^{1,\infty}(\overline{V}))}. \tag{7.19}$$

Using (7.12), (7.13) and (7.19), we finally obtain

$$\|\frac{\partial \delta}{\partial t}\|_{L^\infty(\overline{Q})} \leq C\|w\|_{W^{1,1,\infty}(\overline{Q})}.$$

The estimates on $\|\delta\|_{L^\infty(\overline{Q})}$ and $\|D\delta\|_{L^\infty(\overline{Q})}$ are easier (they use the regularity of c_0), so we skip their proofs. This ends the proof of the lemma. □

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