ESAIM: COCV 15 (2009) 295–321
DOI: 10.1051/cocv:2008026

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

RELAXATION OF ISOTROPIC FUNCTIONALS WITH LINEAR GROWTH DEFINED ON MANIFOLD CONSTRAINED SOBOLEV MAPPINGS

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Abstract. In this paper we study the lower semicontinuous envelope with respect to the L^1-topology of a class of isotropic functionals with linear growth defined on mappings from the n-dimensional ball into $\mathbb{R}^N$ that are constrained to take values into a smooth submanifold $\mathcal{Y}$ of $\mathbb{R}^N$.

Mathematics Subject Classification. 49J45, 49Q20.

Received April 19, 2007.
Published online March 28, 2008.

Introduction

Let B^n be the unit ball in $\mathbb{R}^n$ and $\mathcal{Y}$ a smooth Riemannian manifold of dimension $M \geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary.

In this paper we shall be concerned with manifold constrained energy relaxation problems, and we consider variational functionals $\mathcal{F} : L^1(B^n, \mathcal{Y}) \to [0, +\infty]$ of the type

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{B^n} f(x, u, Du)\, dx & \text{if} \quad u \in W^{1,1}(B^n, \mathcal{Y}) \\ +\infty & \text{otherwise} \end{cases} \tag{0.1}$$

for a suitable class of integrands f, where, for $X = C^1$, L^1, BV, $W^{1,1}$, we denote

$$X(B^n, \mathcal{Y}) := \{u \in X(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n\}.$$

We shall study the lower semicontinuous envelope with respect to the L^1-topology of the variational functional (0.1), *i.e.*, the relaxed functional $\overline{\mathcal{F}} : L^1(B^n, \mathcal{Y}) \to [0, +\infty]$ defined for every function $u \in L^1(B^n, \mathcal{Y})$ by

$$\overline{\mathcal{F}}(u) := \inf\left\{ \liminf_{k\to\infty} \mathcal{F}(u_k) \mid \{u_k\} \subset W^{1,1}(B^n, \mathcal{Y}), \quad u_k \to u \text{ strongly in } L^1(B^n, \mathbb{R}^N) \right\}. \tag{0.2}$$

Motivations for the analysis of non-convex manifold constrained energy relaxation problems are originated by questions of equilibria for liquid crystals, where $n = 3$ and $\mathcal{Y} = \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. The study of minimizers of the energy of non-linear elastic complex bodies has recently been studied in [17], where the morphology of their substructures is represented by elements of some general differentiable manifold $\mathcal{Y}$.

Keywords and phrases. Relaxation, manifold constrain, BV functions.

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Among the wide literature about relaxation problems for unconstrained mappings, for future use, we only cite Fonseca and Müller [8], who studied the analogous problem for functionals with linear growth but defined for standard Sobolev mappings $u \in W^{1,1}(B^n, \mathbb{R}^N)$. As to manifold constrained mappings, Dacorogna *et al.* [5] studied the relaxation problem in topologies stronger than the L^1-topology, namely, with respect to the weak $W^{1,p}$-topology, for $p \geq 1$. Dealing with the L^1-topology, Alicandro, Corbo Esposito and Leone [2] takled the problem in the case of the target manifold $\mathcal{Y}$ equal to $\mathbb{S}^{N-1}$, the unit $(N-1)$-sphere in $\mathbb{R}^N$.

An essentially different manifold constrained relaxation problem is the one when the variational functional (0.1) is supposed to be finite only on smooth $W^{1,1}$-maps in $C^1(B^n, \mathcal{Y})$ rather than on the whole class of Sobolev maps $W^{1,1}(B^n, \mathcal{Y})$. In this setting, as to functional with linear growth, the case $\mathcal{Y} = \mathbb{S}^1$ was studied by Demengel and Hadiji [6] in the case of dimension $n = 2$, and by Giaquinta, Modica and Souček [15] in the case of higher dimension $n \geq 2$. Dealing with more general target manifolds $\mathcal{Y}$, Giaquinta and Mucci [11] studied the relaxation problem in the case of the total variation integrand $f = |Du|$, and more recently [14] in the case of integrands satisfying a suitable isotropy condition of the type $f = f(x, u, |Du^1|, \dots, |Du^N|)$.

In this paper we shall extend to the case of general target manifolds $\mathcal{Y}$ the integral representation in $BV(B^n, \mathcal{Y})$ of the relaxed functional (0.2) obtained in [2] for the case $\mathcal{Y} = \mathbb{S}^{N-1}$. More precisely, we shall assume that f satisfies the same assumptions as in [2], see (H1)–(H5) in Section 1 below. In addition, we shall assume that the *recession* function f^∞ satisfies the *isotropy condition* considered by Fonseca and Ribka [9], see property (H6) below.

1. Notation and statements

In this section we collect a few known facts that are relevant for the sequel. We then state our representation result of the relaxed functional (0.2), see Theorem 1.4.

Vector valued BV-functions. Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a function in $BV(\Omega, \mathbb{R}^N)$, *i.e.*, $u = (u^1, \dots, u^N)$ with all components $u^j \in BV(\Omega)$. The *Jump set* of u is the countably $\mathcal{H}^{n-1}$-rectifiable set J_u in Ω given by the union of the complements of the Lebesgue sets of the u^j's. Let $\nu_u = \nu_u(x)$ be a unit vector in $\mathbb{R}^n$ orthogonal to J_u at $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$. Let $u^\pm(x)$ denote the one-sided approximate limits of u on J_u, so that for $\mathcal{H}^{n-1}$-a.e. point $x \in J_u$

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^\pm(x)} |u(x) - u^\pm(x)| \, \mathrm{d}x = 0,$$

where $B_\rho^\pm(x) := \{y \in B_\rho(x) : \pm(y - x) \cdot \nu_u(x) \geq 0\}$. Note that a change of sign of ν_u induces a permutation of u^+ and u^- and that only for scalar functions there is a canonical choice of the sign of ν_u which ensures that $u^+(x) > u^-(x)$. The distributional derivative of u is the sum of a "gradient" measure, which is absolutely continuous with respect to the Lebesgue measure, of a "jump" measure, concentrated on a set that is σ-finite with respect to the $\mathcal{H}^{n-1}$-measure, and of a "Cantor-type" measure. More precisely,

$$Du = D^a u + D^J u + D^C u,$$

where

$$D^a u = \nabla u \, \mathcal{L}^n, \qquad D^J u = (u^+(x) - u^-(x)) \otimes \nu_u(x) \, \mathcal{H}^{n-1} \llcorner J_u,$$

$\nabla u := (\nabla_1 u, \dots, \nabla_n u)$ being the approximate gradient of u, compare *e.g.* [3] or [16], Volume I. We also recall that $\{u_k\}$ is said to converge to u *weakly in the BV-sense*, $u_k \rightharpoonup u$, if $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^N)$ and $Du_k \rightharpoonup Du$ weakly in the sense of (vector-valued) measures.

Tangential quasi-convexity. Let $M(N, n)$ be the space of real $(N \times n)$-matrices. We shall denote by $T_u\mathcal{Y}$ the tangent space to $\mathcal{Y}$ at $u \in \mathcal{Y}$. Moreover, writing $\xi \in M(N, n)$ as $\xi = (\xi^1, \dots, \xi^n)$, where $\xi^i \in \mathbb{R}^N$

is the i-th column of ξ, we set

$$[T_u\mathcal{Y}]^n := \{\xi \in M(N,n) \mid \xi^i \in T_u\mathcal{Y} \quad \forall\, i = 1,\dots,n\}.$$

Dealing with manifold constrained mappings, the following definition was introduced in [5], see also [1].

Definition 1.1. *Let $g : \mathcal{Y} \times M(N,n) \to [0,+\infty)$ be a continuous function. We define the* tangential quasi-convexification *of g relative to $u \in \mathcal{Y}$ at $\xi \in [T_u\mathcal{Y}]^n$ by*

$$Q_T g(u,\xi) := \inf\Big\{ \int_{(0,1)^n} g(u, \xi + D\varphi(x))\, \mathrm{d}x \mid \varphi \in W_0^{1,\infty}((0,1)^n, T_u\mathcal{Y}) \Big\}.$$

Moreover, g is said to be tangentially quasi-convex *if for every $u \in \mathcal{Y}$ and $\xi \in [T_u\mathcal{Y}]^n$*

$$g(u,\xi) = Q_T g(u,\xi).$$

Let $P_u : \mathbb{R}^N \to T_u\mathcal{Y}$ denote the orthogonal projection, and let $\overline{g} : \mathcal{Y} \times M(N,n) \to \mathbb{R}$ be given by

$$\overline{g}(u,\xi) := g(u, P_u\xi), \qquad P_u\xi := (P_u\xi^1, \dots, P_u\xi^n). \tag{1.1}$$

In [5] it was proved that for every $u \in \mathcal{Y}$ and $\xi \in [T_u\mathcal{Y}]^n$

$$Q_T g(u,\xi) = Q\overline{g}(u,\xi),$$

where $Q\overline{g}$ is the standard quasi-convex envelope of $\overline{g}$, *i.e.*,

$$Q\overline{g}(u,\xi) := \inf\Big\{ \int_{(0,1)^n} \overline{g}(u, \xi + D\varphi(x))\, \mathrm{d}x \mid \varphi \in W_0^{1,\infty}((0,1)^n, \mathbb{R}^N) \Big\}.$$

This yields that $\overline{g}$ is quasi-convex if g is tangentially quasi-convex. Moreover, we may and do identify a tangentially quasi-convex function g with the restriction of a quasi-convex function $\overline{g}$ to the subset $T(\mathcal{Y})$ of $\mathcal{Y} \times M(N,n)$ given by

$$T(\mathcal{Y}) := \{(u,\xi) \in \mathcal{Y} \times M(N,n) \mid u \in \mathcal{Y},\ \xi \in [T_u\mathcal{Y}]^n\}. \tag{1.2}$$

HYPOTHESES ON f. We shall consider integrands $f : B^n \times \mathbb{R}^N \times M(N,n) \to [0,+\infty)$ satisfying the following hypotheses:

(H1) f is continuous.

(H2) For every $x \in B^n$ the function $f(x,\cdot,\cdot) : \mathcal{Y} \times M(N,n) \to [0,+\infty)$ is tangentially quasi-convex, Definition 1.1.

(H3) There exist two absolute constants $c_i > 0$ such that

$$c_1\,|\xi| \le f(x,u,\xi) \le c_2\,(1 + |\xi|)$$

for every $x \in B^n$ and $(u,\xi) \in T(\mathcal{Y})$.

(H4) For every compact set $K \subset B^n$, there exists a non-negative continuous real function ω, with $\omega(0) = 0$, such that

$$|f(x,u,\xi) - f(x_0,u_0,\xi)| \le \omega(|x - x_0| + |u - u_0|) \cdot (1 + |\xi|)$$

for every (x,u), $(x_0,u_0) \in K \times \mathcal{Y}$ and $\xi \in M(N,n)$.

(H5) There exists two absolute constants $C > 0$ and $0 < m < 1$ such that

$$|f^\infty(x,u,\xi) - f(x,u,\xi)| \le C\,(1 + |\xi|^{1-m})$$

for every $x \in B^n$ and $(u,\xi) \in T(\mathcal{Y})$.

(H6) The *recession function* f^∞ satisfies the *isotropy condition*, *i.e.*, for every $(x,u,\xi)\in B^n\times\mathbb{R}^N\times M(N,n)$ and $\nu\in\mathbb{S}^{n-1}$
$$f^\infty(x,u,\xi)\geq f^\infty(x,u,\xi\cdot\nu\otimes\nu).$$

Remark 1.2. The hypotheses (H2), (H3), and (H5) deal with the restriction of f to $B^n\times T(\mathcal{Y})$, compare (1.2), and go back to [2]. The isotropy condition (H6) was studied by Fonseca and Rybka [9]. It is clearly satisfied if $f(x,u,\xi)=h(x,u,|\xi|)$ for some function h.

THE RECESSION FUNCTION. We recall that the *recession function* $f^\infty: B^n\times\mathbb{R}^N\times M(N,n)\to[0,+\infty)$ of f is well-defined by

$$f^\infty(x,u,\xi):=\limsup_{t\to+\infty}\frac{f(x,u,t\xi)}{t}\qquad\forall\,(x,u,\xi)\in B^n\times\mathbb{R}^N\times M(N,n).$$

If f satisfies (H2), (H3), and (H4), it turns out that:

(H2') For every $x\in B^n$ the function $f^\infty(x,\cdot,\cdot):\mathcal{Y}\times M(N,n)\to[0,+\infty]$ is tangentially quasi-convex.

(H3') $c_1\,|\xi|\leq f^\infty(x,u,\xi)\leq c_2\,|\xi|$ for every $x\in B^n$ and $(u,\xi)\in T(\mathcal{Y})$.

(H4') For every compact set $K\subset B^n$, there exists a non-negative continuous real function ω, with $\omega(0)=0$, such that
$$|f^\infty(x,u,\xi)-f^\infty(x_0,u_0,\xi)|\leq\omega(|x-x_0|+|u-u_0|)\cdot|\xi|$$
for every (x,u), $(x_0,u_0)\in K\times\mathcal{Y}$ and $\xi\in M(N,n)$.

THE SURFACE ENERGY DENSITY. Following [2,8,9], for every $\nu\in\mathbb{S}^{n-1}$ we denote

$$Q_\nu:=\{x\in\mathbb{R}^n\mid|x\cdot\nu|<1/2,\ |x\cdot\nu_i|<1/2\quad\forall\,i=1,\dots,n-1\}$$

where $\{\nu_i\}_{i=1}^{n-1}\subset\mathbb{S}^{n-1}$ are chosen so that $\{\nu_1,\dots,\nu_{n-1},\nu\}$ yields an orthonormal basis of $\mathbb{R}^n$. A function $\varphi:Q_\nu\to\mathbb{R}^N$ is said to be 1-*periodic* in the ν_i-direction if

$$\varphi(x+k\nu_i)=\varphi(z)\qquad\forall\,k\in\mathbb{Z},\quad x\in Q_\nu.$$

For every $a^-,a^+\in\mathcal{Y}$ we let

$$\begin{aligned}\mathcal{P}(a^-,a^+,\nu)\quad:=\quad&\{\varphi\in W^{1,1}(Q_\nu,\mathcal{Y})\mid\varphi(x)=a^-\text{ if }x\cdot\nu=-1/2,\ \varphi(x)=a^+\text{ if }x\cdot\nu=1/2,\\&\ \varphi\text{ is 1-periodic in the }\nu_i\text{-direction, for }i=1,\dots,n-1\}.\end{aligned}$$

Moreover, we let $K: B^n\times\mathcal{Y}\times\mathcal{Y}\times\mathbb{S}^{n-1}\to[0,+\infty)$ be defined by

$$K(x_0,a^-,a^+,\nu):=\inf\Big\{\int_{Q_\nu}f^\infty(x_0,\varphi(x),D\varphi(x))\,\mathrm{d}x\mid\varphi\in\mathcal{P}(a^-,a^+,\nu)\Big\}.\tag{1.3}$$

Arguing as in [9], it readily follows that if f^∞ satisfies the isotropy condition (H6), then

$$K(x_0,a^-,a^+,\nu)=\inf\Big\{\int_{-1/2}^{1/2}f^\infty(x_0,\gamma(t),\gamma'(t)\otimes\nu)\,\mathrm{d}t\mid\gamma\in W^{1,1}((-1/2,1/2),\mathcal{Y}),\ \gamma(\pm1/2)=a^\pm\Big\}.\tag{1.4}$$

Remark 1.3. In the case of the total variation integrand, *i.e.*, $f(x,u,\xi)=|\xi|$, we have $f^\infty(x,u,\xi)=|\xi|$ and hence
$$K(x_0,a^-,a^+,\nu)=\inf\Big\{\int_{-1/2}^{1/2}|\gamma'(t)|\,\mathrm{d}t\mid\gamma\in W^{1,1}((-1/2,1/2),\mathcal{Y}),\ \gamma(\pm1/2)=a^\pm\Big\},$$
i.e., $K(x_0,a^-,a^+,\nu)$ agrees with the *geodesic distance* $\mathrm{d}_{\mathcal{Y}}(a^-,a^+)$ between $a^-,a^+\in\mathcal{Y}$.

Main result. In this paper we will prove the following representation result of the relaxed functional.

Theorem 1.4. *Let $\mathcal{F} : L^1(B^n,\mathcal{Y}) \to [0,+\infty]$ be the variational functional (0.1), where $f : B^n \times \mathbb{R}^N \times M(N,n) \to [0,+\infty)$ satisfies the hypotheses (H1)–(H6) above, and let $\overline{\mathcal{F}} : L^1(B^n,\mathcal{Y}) \to [0,+\infty]$ be the lower semicontinuous envelope of $\mathcal{F}$ in the L^1-topology, see (0.2). Then $\overline{\mathcal{F}}(u) < +\infty$ if and only if $u \in BV(B^n,\mathcal{Y})$. Moreover, for every $u \in BV(B^n,\mathcal{Y})$ we have*

$$\overline{\mathcal{F}}(u) = \int_{B^n} f(x,u,\nabla u)\,\mathrm{d}x + \int_{B^n} f^\infty(x,u,\mathrm{d}D^C u) + \int_{J_u} K(x,u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

where the surface density term K is given by (1.3) and

$$f^\infty(x,u,\mathrm{d}D^C u) = f^\infty\left(x,\widetilde{u},\frac{\mathrm{d}D^C u}{|\mathrm{d}D^C u|}\right)\mathrm{d}|D^C u|$$

$\widetilde{u}$ being a good representative of u.

We recall that Theorem 1.4 was proved in [2] in the case $\mathcal{Y} = \mathbb{S}^{N-1}$, without assuming the isotropy condition (H6), and by [8] in the unconstrained case, $\mathcal{Y} = \mathbb{R}^N$. It remains an open problem to prove Theorem 1.4 without assuming the isotropy condition (H6).

The rest of the paper is dedicated to the proof of Theorem 1.4. By Remark 1.3, the growth condition (H3), in conjunction with the smoothness and compactness of $\mathcal{Y}$, yields that $\overline{\mathcal{F}}(u)$ is finite if and only if $u \in BV(B^n,\mathcal{Y})$. We now define for every Borel set $B \subset B^n$ and $u \in BV(B^n,\mathcal{Y})$

$$G(u,B) := \int_B f(x,u,\nabla u)\,\mathrm{d}x + \int_B f^\infty(x,u,\mathrm{d}D^C u) + \int_{J_u\cap B} K(x,u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{1.5}$$

and we let

$$G(u) := G(u,B^n).$$

In Section 2, using the same argument as in [2], that goes back to [8], we will show that

$$\overline{\mathcal{F}}(u) \geq G(u) \qquad \forall\, u \in BV(B^n,\mathcal{Y}). \tag{1.6}$$

A density result. In order to obtain the equality in (1.6), it suffices to show that *for every $u \in BV(B^n,\mathcal{Y})$ there exists a sequence of Sobolev maps $\{u_k\} \subset W^{1,1}(B^n,\mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and*

$$\lim_{k\to\infty} \int_{B^n} f(x,u_k,Du_k)\,\mathrm{d}x = G(u).$$

To this purpose, in Section 3 we will first prove:

Theorem 1.5. *For every $u \in BV(B^n,\mathcal{Y})$ there exists a sequence of maps $\{u_k\} \subset BV(B^n,\mathcal{Y})$, satisfying $|D^C u_k|(B^n) = 0$ for every k, such that u_k weakly converges to u in the BV-sense and*

$$\lim_{k\to\infty} G(u_k) = G(u).$$

In Section 4 we will then prove

Theorem 1.6. *Let $u \in BV(B^n,\mathcal{Y})$ be such that $|D^C u|(B^n) = 0$. There exists a sequence of Sobolev maps $\{u_k\} \subset W^{1,1}(B^n,\mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in the BV-sense and*

$$\lim_{k\to\infty} \int_{B^n} f(x,u_k,Du_k)\,\mathrm{d}x = G(u).$$

By a diagonal argument we then clearly obtain our density result, and hence the equality in (1.6), that concludes the proof of Theorem 1.4.

2. Estimate from below

In this section we prove the inequality (1.6). To this purpose, for every $u \in BV(B^n, \mathcal{Y})$ and every sequence $\{u_k\} \subset W^{1,1}(B^n, \mathcal{Y})$ such that $u_k \to u$ in $L^1(B^n, \mathbb{R}^N)$ and $\liminf_k \mathcal{F}(u_k) < \infty$, it suffices to show that

$$\liminf_{k\to\infty} \mathcal{F}(u_k) \geq G(u). \tag{2.1}$$

Following [8], possibly passing to a subsequence, we may and do assume that

$$f(\cdot, u_k(\cdot), Du_k(\cdot))\, \mathcal{L}^n \llcorner B^n \rightharpoonup \mu$$

weakly in the sense of the measures to some non-negative and finite Radon measure μ on B^n, that decomposes into the sum of four mutually singular measures

$$\mu = \mu_a\, \mathcal{L}^n + \mu_C |D^C u| + \mu_J |u^+ - u^-|\, \mathcal{H}^{n-1} \llcorner J_u + \mu_0.$$

Therefore, (2.1) holds true if we show that

$$\mu_a(x_0) \geq f(x_0, u(x_0), \nabla u(x_0)) \tag{2.2}$$

for $\mathcal{L}^n$-a.e. $x_0 \in B^n$,

$$\mu_C(x_0) \geq f^\infty\left(x_0, \widetilde{u}(x_0), \frac{\mathrm{d}D^C u}{|\mathrm{d}D^C u|}(x_0)\right) \tag{2.3}$$

for $|D^C u|$-a.e. $x_0 \in B^n$, and

$$\mu_J(x_0) \geq \frac{1}{|u^+(x_0) - u^-(x_0)|}\, K(x_0, u^-(x_0), u^+(x_0), \nu_u(x_0)) \tag{2.4}$$

for $|D^J u|$-a.e. $x_0 \in B^n$.

Remark 2.1. For future use, we denote by

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \mathrm{dist}(y, \mathcal{Y}) \leq \varepsilon\}$$

the ε-neighborhood of $\mathcal{Y}$ and we observe that, since $\mathcal{Y}$ is smooth and compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the *nearest point projection* Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. Therefore, the distance function $d(\cdot, \mathcal{Y})$ to $\mathcal{Y}$ is well-defined on $\mathcal{Y}_{\varepsilon_0}$ and

$$d(y, \mathcal{Y}) = |y - \Pi_{\varepsilon_0}(y)| \qquad \forall\, y \in \mathcal{Y}_{\varepsilon_0}.$$

Proof of (2.2) *and* (2.3). Let $\varphi : [0, +\infty) \to [0, 1]$ be a Lipschitz function such that $\varphi \equiv 1$ on $[0, \varepsilon_0/2]$ and $\varphi \equiv 0$ on $[\varepsilon_0, +\infty)$, and consider the function $\widetilde{f} : B^n \times \mathbb{R}^N \times M(N, n) \to [0, +\infty)$ defined for every $x \in B^n$ and $\xi \in M(N, n)$ by

$$\widetilde{f}(x, y, \xi) := \varphi(d(y, \mathcal{Y})) \cdot f(x, u, P_u(\xi)), \qquad u := \Pi_{\varepsilon_0}(y)$$

if $y \in \mathcal{Y}_{\varepsilon_0}$, see Remark 2.1, and $\widetilde{f}(x, y, \xi) := 0$ if $y \in \mathbb{R}^N \setminus \mathcal{Y}_{\varepsilon_0}$, where $P_u(\xi)$ is given by (1.1). It turns out that $\widetilde{f}$ is an extension of the restriction of f to $B^n \times T(\mathcal{Y})$, whereas the hypotheses (H1)–(H5) yield that the function

$$f_\varepsilon(x, y, \xi) := \widetilde{f}(x, y, \xi) + \varepsilon\, |\xi|$$

satisfies the hypotheses (F1)–(F5) of Theorem 2.8 in [2], *i.e.*, of Theorem 2.16 in [8]. The only non-trivial hypothesis to be checked is the following one:

(F4) For every compact set $K \subset\subset B^n \times \mathbb{R}^N$ there exists a continuous function ω, with $\omega(0) = 0$, such that

$$|f_\varepsilon(x, y, \xi) - f_\varepsilon(x_0, y_0, \xi)| \le \omega(|x - x_0| + |y - y_0|) \cdot (1 + |\xi|)$$

for all (x, y, ξ), $(x_0, y_0, \xi) \in K \times \mathbb{R}^N \times M(N, n)$.

To prove (F4), we observe that if $y, y' \in \mathcal{Y}_\varepsilon$, setting $u := \Pi_{\varepsilon_0}(y)$ and $u_0 := \Pi_{\varepsilon_0}(y_0)$, we have

$$\begin{array}{rcl} |f_\varepsilon(x, y, \xi) - f_\varepsilon(x_0, y_0, \xi)| & = & |\varphi(d(y, \mathcal{Y})) \cdot f(x, u, P_u(\xi)) - \varphi(d(y_0, \mathcal{Y})) \cdot f(x, u_0, P_{u_0}(\xi))| \\ & \le & |\varphi(d(y, \mathcal{Y})) - \varphi(d(y_0, \mathcal{Y}))| \cdot |f(x, u, P_u(\xi))| \\ & & + |f(x, u, P_u(\xi)) - f(x, u_0, P_{u_0}(\xi))|. \end{array}$$

Moreover,

$$\begin{array}{rcl} |\varphi(d(y, \mathcal{Y})) - \varphi(d(y_0, \mathcal{Y}))| & = & |\varphi(|y - u|) - \varphi(|y_0 - u_0|)| \\ & \le & \mathrm{Lip}\ \varphi \cdot \big||y - \Pi_{\varepsilon_0}(y)| - |y_0 - \Pi_{\varepsilon_0}(y_0)|\big| \\ & \le & \mathrm{Lip}\ \varphi \cdot (1 + \mathrm{Lip}\ \Pi_{\varepsilon_0}) \cdot |y - y_0|. \end{array}$$

Property (F4) then follows from (H4).

In conclusion, arguing as in Section 5.1 of [2], we infer that (2.2) and (2.3) hold true. □

Proof of (2.4). We follow the lines of the proof in Section 5.2 of [2]. More precisely, using the blow-up argument from [8], for $\mathcal{H}^{n-1}$-a.e. $x_0 \in J_u$ we find a sequence $\{v_k\} \subset W^{1,1}(Q_\nu, \mathcal{Y})$, where $\nu = \nu_u(x_0)$, such that $v_k \to u_0$ in $L^1(Q_\nu, \mathbb{R}^N)$ and

$$|u^+(x_0) - u^-(x_0)|\, \mu_J(x_0) \ge \lim_{k\to\infty} \int_{Q_\nu} f^\infty(x_0, v_k(x), Dv_k(x))\, \mathrm{d}x,$$

where $u_0 \in BV(Q_\nu, \mathcal{Y})$ is given by

$$u_0(x) := \begin{cases} u^+(x_0) & \text{if } \ x \cdot \nu_u(x_0) \ge 0 \\ u^-(x_0) & \text{if } \ x \cdot \nu_u(x_0) < 0. \end{cases}$$

Now, using the isotropy condition (H6), we prove the following:

Lemma 2.2. *Under the previous hypotheses, there exists a sequence* $\{w_k\} \subset \mathcal{P}(u^-(x_0), u^+(x_0), \nu)$, *where* $\nu = \nu_u(x_0)$, *such that*

$$\lim_{k\to\infty} \int_{Q_\nu} f^\infty(x_0, v_k(x), Dv_k(x))\, \mathrm{d}x \ge \limsup_{k\to\infty} \int_{Q_\nu} f^\infty(x_0, w_k(x), Dw_k(x))\, \mathrm{d}x.$$

On account of (1.3), Lemma 2.2 yields (2.4).

Proof of Lemma 2.2. Arguing as in [9], Proposition 2.6, by Fubini's theorem and (H6), for every $\varepsilon > 0$ and every k we find a Sobolev function $\gamma_k \in W^{1,1}((-1/2, 1/2), \mathcal{Y})$ such that

$$\int_{Q_\nu} f^\infty(x_0, v_k(x), Dv_k(x))\, \mathrm{d}x \ge \int_{-1/2}^{1/2} f^\infty(x_0, \gamma_k(t), \gamma'(t) \otimes \nu)\, \mathrm{d}t - \varepsilon, \tag{2.5}$$

where γ_k strongly converges in $L^1((-1/2, 1/2), \mathbb{R}^N)$ to the function

$$\gamma(t) := \begin{cases} u^+(x_0) & \text{if } \ 0 < t \le 1/2 \\ u^-(x_0) & \text{if } \ -1/2 \le t < 0. \end{cases}$$

By the growth condition (H3) and by (2.5), we infer that

$$\sup_k \int_{-1/2}^{1/2} |\gamma_k'(t)|\,\mathrm{d}t \le C < \infty \tag{2.6}$$

for some absolute constant $C>0$. Let $m\in\mathbb{N}^+$ to be fixed below, and let

$$I_i^+ := [(i-1)/2m, i/2m], \quad I_i^- := [-1/2m, -(i-1)/2m], \qquad i=1,\dots,m.$$

By (2.6) we infer that for every k we can find two indices $i_k^\pm \in \{1,\dots,m\}$ such that

$$\int_I |\gamma_k'(s)|\,\mathrm{d}s \le \frac{C}{m} \qquad \text{if} \quad I = I^+_{i_k^+} \quad \text{or} \quad I = I^-_{i_k^-}. \tag{2.7}$$

Now, by a straightforward adaptation of the argument from [8], Lemma 3.1, we may and do define for every k two cut-off functions

$$\varphi_k^+ : [0,1/2] \to [0,1], \qquad \varphi_k^- : [-1/2,0] \to [0,1]$$

such that $\varphi_k^\pm(t)=0$ if $0\le \pm t \le (i_k^\pm -1)/2m$, $\varphi_k^\pm(t)=1$ if $i_k^\pm/2m \le \pm t \le 1/2$, and such that, setting

$$\widetilde{w}_k^\pm(t) := \varphi_k^\pm(t)\gamma_k(t) + (1-\varphi_k^\pm(t))\,u^\pm(x_0), \qquad 0\le \pm t \le 1/2$$

and $\widetilde{w}_k : [-1/2,1/2] \to \mathbb{R}^N$ by

$$\widetilde{w}_k(t) := \begin{cases} \widetilde{w}_k^+(t) & \text{if} \quad 0\le t\le 1/2 \\ \widetilde{w}_k^-(t) & \text{if} \quad -1/2 \le t \le 0 \end{cases}$$

we have

$$\int_{-1/2}^{1/2} f^\infty(x_0, \widetilde{w}_k(t), \widetilde{w}_k'(t)\otimes\nu)\,\mathrm{d}t \le \int_{-1/2}^{1/2} f^\infty(x_0,\gamma_k(t),\gamma'(t)\otimes\nu)\,\mathrm{d}t + \frac{\widetilde{C}}{k} \tag{2.8}$$

and, by the growth condition (H3),

$$\int_{\{t\,|\,0<\varphi_k^\pm(t)<1\}} |\widetilde{w}_k'(t)\otimes\nu|\,\mathrm{d}t \le \frac{\widetilde{C}}{k} \tag{2.9}$$

for some absolute constant $\widetilde{C}>0$.

We now show that if $m\in\mathbb{N}$ is chosen sufficiently large, for k large enough

$$\mathrm{dist}(\widetilde{w}_k(t), \mathcal{Y}) \le \varepsilon_0 \qquad \forall\, t\in[-1/2,1/2]. \tag{2.10}$$

Property (2.10) is clearly satisfied if $t \notin I^\pm_{i_k^\pm}$. Now, by the L^1-convergence $\gamma_k \to \gamma$, for k sufficiently large we may and do find a number $t_k^\pm \in I^\pm_{i_k^\pm}$ such that

$$|\gamma_k(t_k^\pm) - u^\pm(x_0)| < \varepsilon_0/2. \tag{2.11}$$

Moreover, for every $t\in I^\pm_{i_k^\pm}$ we have

$$|\widetilde{w}_k^\pm(t) - u^\pm(x_0)| \le |\varphi_k^\pm(t)|\cdot|\gamma_k(t)-u^\pm(x_0)| \le |\gamma_k(t)-\gamma_k(t_k^\pm)| + |\gamma_k(t_k^\pm)-u^\pm(x_0)|,$$

whereas

$$|\gamma_k(t) - \gamma_k(t_k^\pm)| \le \int_I |\gamma_k'(s)|\,\mathrm{d}s, \qquad \text{where} \quad I = I^\pm_{i_k^\pm}.$$

Therefore, choosing m large so that $C/m < \varepsilon_0/2$ in (2.7), by (2.11) we obtain (2.10).

We finally define $w_k : [-1/2, 1/2] \to \mathcal{Y}$ by

$$w_k(t) := \Pi_{\varepsilon_0} \circ \widetilde{w}_k(t),$$

where Π_{ε_0} is the projection given by Remark 2.1. Since $w_k(t) = \widetilde{w}_k(t)$ if $t \notin I^\pm_{i_k^\pm}$, using (2.8), (2.9), and the growth condition (H3), it turns out that $\{w_k\} \subset W^{1,1}((-1/2,1/2),\mathcal{Y})$ satisfies $w_k(\pm 1/2) = u^\pm(x_0)$ and the energy estimate

$$\int_{-1/2}^{1/2} f^\infty(x_0, w_k(t), w_k'(t) \otimes \nu)\,\mathrm{d}t \le \int_{-1/2}^{1/2} f^\infty(x_0, \gamma_k(t), \gamma'(t) \otimes \nu)\,\mathrm{d}t + \mathrm{Lip}\,\Pi_{\varepsilon_0} \cdot \frac{\widehat{C}}{k},$$

where $\widehat{C} > 0$ is an absolute constant. On account of (2.5) we then obtain

$$\lim_{k\to\infty} \int_{Q_\nu} f^\infty(x_0, v_k(x), Dv_k(x))\,\mathrm{d}x \ge \limsup_{k\to\infty} \int_{-1/2}^{1/2} f^\infty(x_0, w_k(t), w_k'(t) \otimes \nu)\,\mathrm{d}t - \varepsilon$$

and finally the assertion, by letting $\varepsilon \searrow 0$. □

3. The density result, part I

In this section we prove Theorem 1.5. We shall first consider the case of the total variation integrand, $f(x,u,\xi) = |\xi|$. Using a continuity theorem by Reshetnyak, Theorem 3.4, we shall then prove Theorem 1.5 for more general integrands f as in Theorem 1.4.

The case $f(x,u,\xi) = |\xi|$. By Remark 1.3, if we consider the total variation integrand $f(x,u,\xi) = |\xi|$, we infer that $G(u,B)$ agrees with the *total variation energy*

$$\mathcal{E}_{TV}(u,B) := \int_B |\nabla u|\,\mathrm{d}x + |D^C u|(B) + \int_{J_u} \mathrm{d}_{\mathcal{Y}}(u^-(x), u^+(x))\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.1}$$

compare [11], Section 6. In the proof of Theorem 1.5 we use arguments from [11], Section 4 and [13]. For the reader's convenience we give a complete proof, that will be divided in four steps. In the case of dimension $n = 1$, the proof is a straightforward adaptation of results from [11], Section 1.

Proof of Theorem 1.5. We make use of an inductive argument on the dimension $n \ge 2$. More precisely, we will assume that Theorems 1.5 and 1.6, and hence the strong density of $W^{1,1}$-maps, hold true in dimension $n-1$, for $f(x,u,\xi) = |\xi|$.

For every point $x_0 \in B^n$ and for a.e. radius $r \in (0, r_0)$, where $2r_0 := \mathrm{dist}(x_0, \partial B^n)$, the restriction $u_{(r,x_0)} := u_{|\partial B_r(x_0)}$ of u to the boundary $\partial B_r(x_0)$ is a function in $BV(\partial B_r(x_0), \mathcal{Y})$ with jump set satisfying $J_{u_{(r,x_0)}} = J_u \cap \partial B_r(x_0)$ in the $\mathcal{H}^{n-2}$-a.e. sense. In this case we say that r is a *good radius* for u at x_0, and set

$$\begin{aligned} \mathcal{E}_{TV}(u_{(r,x_0)}, \partial B_r(x_0)) \;:=\; & \int_{\partial B_r(x_0)} |\nabla_\tau u_{(r,x_0)}|\,\mathrm{d}\mathcal{H}^{n-1} + |D^C_\tau u|(\partial B_r(x_0)) \\ & + \int_{J_u \cap \partial B_r(x_0)} \mathrm{d}_{\mathcal{Y}}(u^-(x), u^+(x))\,\mathrm{d}\mathcal{H}^{n-2}(x), \end{aligned} \tag{3.2}$$

D_τ and ∇_τ being the distributional derivative and the approximate gradient w.r.t. an orthonormal frame τ tangential to $\partial B_r(x_0)$, respectively.

Step 1: Definition of the fine cover $\mathcal{F}_m$. We define for every $m \in \mathbb{N}$ a suitable *fine cover* $\mathcal{F}_m$ of $B^n \setminus J_u$ consisting of closed balls of radius smaller than $1/m$. To this aim, let μ_d and μ_J be the mutually singular Radon measures on B^n given for every Borel set $B \subset B^n$ by

$$\mu_d(B) := \int_B |\nabla u|\, \mathrm{d}x + |D^C u|(B), \qquad \mu_J(B) := \int_{J_u \cap B} \mathrm{dist}_{\mathcal{Y}}(u^-(x), u^+(x))\, \mathrm{d}\mathcal{H}^{n-1} \tag{3.3}$$

so that by (3.1) we have the decomposition into the "diffuse" and "jump" part

$$\mathcal{E}_{TV}(u, B) = \mu_d(B) + \mu_J(B).$$

By the decomposition of the derivative Du, compare [3], Proposition 3.92, we infer that for any point x_0 in $B^n \setminus J_u$ we have

$$\liminf_{r\to 0} \frac{\mathcal{E}_{TV}(u, B_r(x_0))}{r^{n-1}} = \liminf_{r\to 0} \frac{|Du|(B_r(x_0))}{r^{n-1}} = 0. \tag{3.4}$$

Moreover, since $\mu_J = \mu_J \llcorner J_u$, where J_u is a countably $\mathcal{H}^{n-1}$-rectifiable set, and $\mathcal{E}_{TV}(u, J_u) < \infty$, for every $m \in \mathbb{N}$ we find a closed subset $J_m \subset J_u$ such that

$$J_m \subset J_{m+1} \qquad \text{and} \qquad \mathcal{E}_{TV}(u, J_u \setminus J_m) = \mu_J(J_u \setminus J_m) < \frac{1}{m} \qquad \forall\, m.$$

Setting now

$$\Omega := B^n \setminus J_u,$$

J_m being closed, for every $x_0 \in \Omega$ there exists a positive radius $r = r(x_0, m)$, smaller than the distance of x_0 to the boundary ∂B^n, such that for every $0 < R < r(x_0, m)$

$$\overline{B}_R(x_0) \cap J_m = \emptyset.$$

Finally, by (3.2), if $x_0 \in \Omega$, for every $0 < R < r(x_0, m)$ we find a good radius $r \in (R/2, R)$ such that

$$\mathcal{E}_{TV}(u_{(r,x_0)}, \partial B_r(x_0)) \le \frac{2}{R}\, \mathcal{E}_{TV}(u, \overline{B}_R(x_0)).$$

We then denote by $\mathcal{F}_m$ the union of all the closed balls centered at points $x_0 \in \Omega$ and with good radii $0 < r < \min\{r(x_0, m)/2, 1/m\}$ such that

$$\mathcal{E}_{TV}(u_{(r,x_0)}, \partial B_r(x_0)) \le \frac{2}{r}\, \mathcal{E}_{TV}(u, \overline{B}_{2r}(x_0)) \tag{3.5}$$

and, according to (3.4),

$$\frac{1}{(2r)^{n-1}}\, \mathcal{E}_{TV}(u, \overline{B}_{2r}(x_0)) \le \frac{1}{m}\cdot \tag{3.6}$$

The above construction yields that $\mathcal{F}_m$ is a *fine cover* of Ω such that

$$\bigcup \mathcal{F}_m \subset B^n \setminus J_m.$$

Step 2: Covering argument. We apply the following extension of the classical Vitali-Besicovitch covering theorem, see [11], Theorem 4.1, and *e.g.* [3], Theorem 2.19, with respect to the positive Radon measure

$$\mu := \mathcal{L}^n + \mu_d + \mu_J,$$

where $\mathcal{L}^n$ is the Lebesgue measure and μ_d, μ_J are given by (3.3). In the sequel, for any closed ball B we will denote by $\widetilde{B}$ the closed ball centered as B and with radius twice the radius of B, *i.e.*,

$$\widetilde{B} := \overline{B}_{2r}(x_0) \qquad \text{if} \qquad B = \overline{B}_r(x_0).$$

Theorem 3.1 (Vitali-Besicovitch)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel set, and let $\mathcal{F}$ be a fine cover of Ω made of closed balls. For every positive Radon measure μ in $\mathbb{R}^n$ there is a disjoint countable family $\mathcal{F}'$ of $\mathcal{F}$ such that $\mu\Big(\Omega \setminus \bigcup \mathcal{F}'\Big) = 0$. Moreover, we have*

$$\sum_{B \in \mathcal{F}'} \mu(\widetilde{B}) \leq C \cdot \mu(\Omega),$$

where $C = C(n) > 0$ is an absolute constant, only depending on the dimension n.

By Theorem 3.1 we obtain for every m a suitable denumerable disjoint family $\mathcal{F}'_m$ of closed balls contained in $B^n \setminus J_m$ and with radii smaller than $1/m$. We finally label

$$\mathcal{F}'_m = \{B_j\}_{j=1}^{\infty}, \qquad \Omega_m := \bigcup_{j=1}^{\infty} B_j$$

and notice that

$$\mu_J(\Omega_m) \leq \mu_J(B^n \setminus J_m) < \frac{1}{m} \qquad \text{and} \qquad \mu_d(B^n \setminus \Omega_m) = 0. \tag{3.7}$$

Step 3: Projecting the boundary data. Let $n \geq 3$. For any $\rho > 0$, we set $Q^n_\rho := [-\rho, \rho]^n \subset \mathbb{R}^n$ and denote by Σ^i_ρ the i-dimensional skeleton of Q^n_ρ, so that $\bigcup \Sigma^{n-1}_\rho = \partial Q^n_\rho$. Also, let $\|x\| := \max\{|x_1|, \dots, |x_n|\}$. In the sequel, we say that the i-dimensional restriction $u_{|\Sigma^i_r}$ of u to Σ^i_r belongs to $BV(\Sigma^i_r, \mathcal{Y})$ if for any i-face F of Σ^i_r its restriction $u_{|F}$ belongs to $BV(F, \mathcal{Y})$ and, for any i-faces F_1 and F_2 of Σ^i_r, the traces of $u_{|F_i}$ agree on the common $(i-1)$-face I of $F_1 \cap F_2$. In this case, moreover, we denote by $\mathcal{E}_{TV}(u, \Sigma^i_r)$ the sum of the $\mathcal{E}_{TV}$-energies $\mathcal{E}_{TV}(u, F)$ of the restrictions $u_{|F}$ of u to all the i-faces F of Σ^i_r, where

$$\mathcal{E}_{TV}(u, F) := \int_F |\nabla u_{|F}| \,\mathrm{d}\mathcal{H}^i + |D^C u_{|F}|(F) + \int_{J_{u_{|F}}} \mathrm{d}_{\mathcal{Y}}(u^-_{|F}(x), u^+_{|F}(x)) \,\mathrm{d}\mathcal{H}^{i-1}.$$

We recall that $\mathcal{Y} \subset \mathbb{R}^N$, and for $y \in \mathcal{Y}$ and $0 < \varepsilon < \varepsilon_0$ we denote by

$$B_{\mathcal{Y}}(y, \varepsilon) := \overline{B}^N(y, \varepsilon) \cap \mathcal{Y}$$

the intersection of $\mathcal{Y}$ with the closed N-ball of radius ε centered at y, so that $\Pi_\varepsilon(\overline{B}^N(y, \varepsilon)) = B_{\mathcal{Y}}(y, \varepsilon)$, where $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \to \mathcal{Y}$ is the projection map given by Remark 2.1. Moreover, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^N \to B_{\mathcal{Y}}(y, \varepsilon)$ be the retraction map given by $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \overline{B}^N(y, \varepsilon) \\ \varepsilon \, \dfrac{z - y}{|z - y|} & \text{if} \quad z \in \mathbb{R}^N \setminus \overline{B}^N(y, \varepsilon) \end{cases} \tag{3.8}$$

so that $\Psi_{(y,\varepsilon)}$ is a Lipschitz continuous function with $\operatorname{Lip} \Psi_{(y,\varepsilon)} = \operatorname{Lip} \Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

Let $B_j = \overline{B}_r(x_0) \in \mathcal{F}'_m$. By means of a deformation and slicing argument, we may and do define a bilipschitz homeomorphism $\psi_j : \overline{B}_r(x_0) \to Q^n_r$ such that $\|D\psi_j\|_\infty \leq K$, $\|D\psi_j^{-1}\|_\infty \leq K$ for some absolute constant $K > 0$, only depending on n, and

$$\psi_j(\overline{B}_\rho(x_0)) = Q^n_\rho \qquad \forall\, \rho \in (r/2, r). \tag{3.9}$$

Letting $u_j := u \circ \psi_j^{-1}$, we also may and do define ψ_j in such a way that the restriction $u_{j|\Sigma_r^i}$ belongs to $BV(\Sigma_r^i, \mathcal{Y})$ for every $i \geq 1$ and satisfies the energy estimate

$$\mathcal{E}_{TV}(u_j, \Sigma_r^i) \leq C \cdot \frac{1}{r} \cdot \mathcal{E}_{TV}(u_j, \Sigma_r^{i+1}) \qquad \forall\, i = 1, \ldots, n-2,$$

where $C > 0$ is an absolute constant, not depending on u_j. By (3.5) and (3.6) we infer that on one hand

$$\mathcal{E}_{TV}(u_j, \Sigma_r^i) \leq \widetilde{C}\, r^{i-n}\, \mathcal{E}_{TV}(u, \overline{B}_{2r}(x_0)) \qquad \forall\, i = 1, \ldots, n-1 \tag{3.10}$$

and on the other hand

$$\frac{1}{r^{i-1}} \mathcal{E}_{TV}(u_j, \Sigma_r^i) \leq \widetilde{C}\, \frac{1}{m} \qquad \forall\, i = 1, \ldots, n, \tag{3.11}$$

where $\widetilde{C} > 0$ is an absolute constant.

Remark 3.2. Setting $\varepsilon_m := 1/\sqrt{m}$, for $m \in \mathbb{N}$ sufficiently large, the inequality (3.11), with $i = 1$, yields that the image $u_j(\Sigma_r^1)$ is contained in a small geodesic ball $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$ centered at some given point $y_j \in \mathcal{Y}$.

Let $q \in \mathbb{N}^+$. Following an argument by Bethuel [4], if S_h is one of the $(n-1)$-faces of Σ_r^{n-1}, where $h = 1, \ldots, 2n$, we may and do define a partition of S_h into $(q+1)^{n-1}$ small $(n-1)$-dimensional "cubes" $C_{l,h}$ in such a way that the following facts hold:

(i) If $[C_{l,h}]_i$ denotes the i-dimensional skeleton of the boundary of $C_{l,h}$, the restriction of u_j to $[C_{l,h}]_i$ is a function in $BV([C_{l,h}]_i, \mathcal{Y})$ for every $i = 1, \ldots, n-2$.

(ii) If $n = 3$, we have

$$\sum_{l=1}^{(q+1)^2} \mathcal{E}_{TV}(u_j, \partial C_{l,h}) \leq K \left(\mathcal{E}_{TV}(u_j, \partial S_h) + \frac{q}{r}\, \mathcal{E}_{TV}(u_j, S_h) \right), \tag{3.12}$$

where $K > 0$ is an absolute constant.

(iii) If $n \geq 4$, and $[S_h]_i$ denotes the i-dimensional skeleton of S_h, for every $i = 1, \ldots, n-2$ we have

$$\sum_{l=1}^{(q+1)^{n-1}} \mathcal{E}_{TV}(u_j, [C_{l,h}]_i) \leq K \cdot \sum_{t=i}^{n-1} \left(\frac{q}{r} \right)^{t-i} \cdot \mathcal{E}_{TV}(u_j, [S_h]_t), \tag{3.13}$$

where $K > 0$ is an absolute constant.

(iv) All the $C_{l,h}$'s are bilipschitz homeomorphic to the $(n-1)$-cube $[-r/q, r/q]^{n-1}$ by linear maps $f_{l,h}$ such that $\|Df_{l,h}\|_\infty \leq K$, $\|Df_{l,h}^{-1}\|_\infty \leq K$.

Remark 3.3. By (3.11) and (3.12), or (3.13), we infer that

$$\sum_{l=1}^{(q+1)^{n-1}} \mathcal{E}_{TV}(u_j, [C_{l,h}]_1) \leq \widehat{C}\, \frac{q^{n-2}}{m},$$

where $\widehat{C} > 0$ is an absolute constant. Moreover, the image $u_j(\Sigma_r^1)$ is contained in $B_{\mathcal{Y}}(y_j, \varepsilon_m/2)$. Therefore, in the sequel we will take

$$q := \text{integer part of } ((2\widehat{C})^{-1} \cdot \varepsilon_m \cdot m)^{1/(n-2)}. \tag{3.14}$$

Arguing as in Remark 3.2, we infer that for every l and h

$$u_j([C_{l,h}]_1) \subset B_{\mathcal{Y}}(y_j, \varepsilon_m). \tag{3.15}$$

Let $\delta := r(1-q^{-1})$ and define $\Phi_q : Q^n_r \to Q^n_\delta$ by $\Phi_q(x) := (1-q^{-1})\,x$ and $\pi_{(r,\delta)} : Q^n_r \setminus Q^n_\delta \to \partial Q^n_r$ by $\pi_{(r,\delta)}(x) := r\,x/\|x\|$. Setting

$$\mathcal{M}_{(r,\delta)} := \pi^{-1}_{(r,\delta)}\Big(\bigcup_{h=1}^{2n}\ \bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,h}\Big)$$

it turns out that the $(n-1)$-skeleton

$$\mathcal{N}_{(r,\delta)} := \mathcal{M}_{(r,\delta)} \cup \partial Q^n_\rho \cup \partial Q^n_\delta$$

is the union of the boundaries of n-dimensional "cubes" $Q_{l,h}$, satisfying $C_{l,h} \subset \partial Q_{l,h}$ for every l and h, that partition $Q^n_\rho \setminus Q^n_\delta$. Moreover, each $Q_{l,h}$ is bilipschitz homeomorphic to the n-cube $[-r/q, r/q]^n$ by linear maps $\widetilde{f}_{l,h}$ such that $\|D\widetilde{f}_{l,h}\|_\infty \le K$, $\|D\widetilde{f}^{-1}_{l,h}\|_\infty \le K$, where $K>0$ is an absolute constant. Finally, set

$$\widetilde{\Sigma}^i_r := \Big(\bigcup_{h=1}^{2n}\ \bigcup_{l=1}^{(q+1)^{n-1}} [C_{l,h}]_i\Big). \tag{3.16}$$

We now define a BV-map $\widehat{u}_j \in BV(Q^n_\rho \setminus Q^n_\delta, \mathcal{Y})$ and a BV-map $\widetilde{u}_j \in BV(\partial Q^n_\delta, \mathcal{Y})$ such that the following properties hold:

(a) $\widehat{u}_j$ has small $\mathcal{E}_{TV}$-energy, see (3.18) below;
(b) $\widetilde{u}_j$ maps ∂Q^n_δ into $B_{\mathcal{Y}}(y_j, \varepsilon_m)$ and its $\mathcal{E}_{TV}$-energy is comparable to the $\mathcal{E}_{TV}$-energy of $u_{j|\partial Q^n_r}$;
(c) the trace of $\widehat{u}_j$ agrees with $u_{j|\partial Q^n_r}$ on ∂Q^n_r;
(d) the trace of $\widehat{u}_j$ agrees with $\widetilde{u}_j$ on ∂Q^n_δ.

To this purpose we first define a BV-function $u^{(2)}_j$ on the 2-skeleton of $\mathcal{N}_{(r,\delta)}$ by setting

$$u^{(2)}_j := \begin{cases} u_{j|\widetilde{\Sigma}^2_r} & \text{on } \widetilde{\Sigma}^2_r \\ \Psi_{(y_j,\varepsilon_m)} \circ u_j \circ \Phi_q{}^{-1} & \text{on } \Phi_q(\widetilde{\Sigma}^2_r) \\ u_j \circ \pi_{(r,\delta)} & \text{on } \pi^{-1}_{(r,\delta)}(\widetilde{\Sigma}^1_r). \end{cases}$$

By (3.15) and (3.16) we infer that $u^{(2)}_j$ is well-defined in the 2-skeleton of $\mathcal{N}_{(r,\delta)}$.

The case $n=3$. We then define $\widehat{u}_j$ on $Q^3_r \setminus Q^3_\delta$ by means of a radial extension on each cube $Q_{l,h}$, *i.e.*, by setting for every l and h

$$\widehat{u}_j(x) := u^{(2)}_j\Big(\widetilde{f}^{-1}_{l,h}\Big(\frac{r}{q}\cdot\frac{\widetilde{f}_{l,h}(x)}{\|\widetilde{f}_{l,h}(x)\|}\Big)\Big), \qquad x \in Q_{l,h}, \tag{3.17}$$

so that $\widehat{u}_j$ actually belongs to $BV(Q^3_r \setminus Q^3_\delta, \mathcal{Y})$. Moreover, it is readily checked that $\widehat{u}_j$ satisfies the energy estimate

$$\mathcal{E}_{TV}(\widehat{u}_j, Q_{l,h}) \le C\,\frac{r}{q}\,\mathcal{E}_{TV}(u^{(2)}_j, \partial Q_{l,h}),$$

whereas by the definition of $u^{(2)}_j$ we obtain

$$\mathcal{E}_{TV}(u^{(2)}_j, \partial Q_{l,h}) \le C\left(\mathcal{E}_{TV}(u_j, C_{l,h}) + \frac{r}{q}\,\mathcal{E}_{TV}(u_j, \partial C_{l,h})\right).$$

Therefore, by (3.12), and by summing on l and h, we estimate

$$\mathcal{E}_{TV}(\widehat{u}_j, Q^3_r \setminus Q^3_\delta) \le C\left(\frac{r}{q}\,\mathcal{E}_{TV}(u_j, \Sigma^2_r) + \left(\frac{r}{q}\right)^2 \mathcal{E}_{TV}(u_j, \Sigma^1_r)\right).$$

In conclusion, for m large, and $n=3$, by (3.14) and (3.10) we obtain the energy estimate

$$\mathcal{E}_{TV}(\widehat{u}_j, Q^n_r \setminus Q^n_\delta) \le C\,(\varepsilon_m \cdot m)^{1/(2-n)}\,\mathcal{E}_{TV}(u, \overline{B}_{2r}(x_0)) \tag{3.18}$$

where, we recall, $(\varepsilon_m \cdot m)^{1/(2-n)} \to 0$ as $m \to +\infty$, since $\varepsilon_m \cdot m = \sqrt{m}$.

The case $n \ge 4$. We define a BV-function $u_j^{(i)}$ on the i-skeleton of $\mathcal{N}_{(r,\delta)}$, arguing by iteration on the dimension $i = 3, \dots, n$. More precisely, if F is any i-dimensional face of $[Q_{l,h}]_i$, we distinguish two cases. If F is contained in ∂Q^n_r we set $u_j^{(i)} = u_j$ on F. Otherwise, we define $u_j^{(i)}$ on F by means of a "radial" extension of the boundary datum $u^{(i-1)}_{j|\partial F}$ similar to the one in (3.17), so that

$$\mathcal{E}_{TV}(u_j^{(i)}, F) \le C\,\frac{r}{q}\,\mathcal{E}_{TV}(u_j^{(i-1)}, \partial F).$$

Setting then $\widehat{u}_j = u_j^{(n)}$, by the construction, and for (3.13), we readily infer that

$$\mathcal{E}_{TV}(\widehat{u}_j, Q^n_\rho \setminus Q^n_\delta) \le C \sum_{i=1}^{n-1} \left(\frac{r}{q}\right)^{n-i} \mathcal{E}_{TV}(u_j, \Sigma^i_r),$$

so that by (3.14) and (3.10) we obtain again (3.18), for m large. The above properties (a)–(d) follow from the construction, as required.

In conclusion, for any $n \ge 3$, setting

$$w_j := \widehat{u}_j \circ \psi_j : \overline{B}_r(x_0) \setminus \overline{B}_\delta(x_0) \to \mathcal{Y},$$

on account of (3.9) we infer that w_j belongs to $BV(\overline{B}_r(x_0) \setminus \overline{B}_\delta(x_0), \mathcal{Y})$, and by (3.18) it satisfies the energy estimate

$$\mathcal{E}_{TV}(w_j, \overline{B}_r(x_0) \setminus \overline{B}_\delta(x_0)) \le C\,(\varepsilon_m \cdot m)^{1/(2-n)}\,\mathcal{E}_{TV}(u, \overline{B}_{2r}(x_0)). \tag{3.19}$$

Finally, by the properties (c)-(d) we infer that the trace of w_j on $\partial B_r(x_0)$ is equal to $u_{(r,x_0)}$ and the trace of w_j on $\partial B_\delta(x_0)$ is equal to v_j, where $v_j \in BV(\partial B_\delta(x_0), \mathcal{Y})$ satisfies

$$v_j(\partial B_\delta(x_0)) \subset B_{\mathcal{Y}}(y_j, \varepsilon_m) \tag{3.20}$$

and the energy estimate

$$\mathcal{E}_{TV}(v_j, \partial B_\delta(x_0)) \le C \cdot \mathcal{E}_{TV}(u_{(r,x_0)}, \partial B_r(x_0)).$$

In the case of dimension $n=2$ we simply take $\delta = r$ and $v_j := u_{(r,x_0)}$. In this case, in fact, the energy bounds (3.5) and (3.6) yield that (3.20) holds true, see Remark 3.2.

Step 4: Approximation on the balls of $\mathcal{F}'_m$. We set $\widehat{B}_j := \overline{B}_\delta(x_0)$, and we now apply the above mentioned inductive hypothesis to the BV-map $v_j \in BV(\partial \widehat{B}_j, \mathcal{Y})$ defined in Step 3. Therefore, we find a sequence of Sobolev maps $\{v_h^{(j)}\} \subset W^{1,1}(\partial \widehat{B}_j, \mathcal{Y})$ such that $\|v_h^{(j)} - v_j\|_{L^1(\partial \widehat{B}_j)} \to 0$ and

$$\int_{\partial \widehat{B}_j} |D_\tau v_h^{(j)}|\, \mathrm{d}\mathcal{H}^{n-1} \le \mathcal{E}_{TV}(v_j, \partial \widehat{B}_j \times \mathcal{Y}) \cdot (1 + 2^{-h})$$

for every h. Now, since v_j satisfies the property (3.20), by the proof of Theorem 1.5 and of Theorem 1.6 below we infer that we may and do assume that the approximating sequence satisfies

$$v_h^{(j)}(\partial \widehat{B}_j) \subset B_{\mathcal{Y}}(y_j, \varepsilon_m) \qquad \forall\, h. \tag{3.21}$$

Taking k sufficiently large, and using the argument by Gagliardo [10], we then define a map $W_k^{(j)} \in W^{1,1}(A^\delta_{\rho_k},\mathbb{R}^N)$, where $0<\rho_k<\delta$ and $A^R_\rho = A^R_\rho(x_0)$ denotes the annulus

$$A^R_\rho := \overline{B}_R(x_0)\setminus B_\rho(x_0), \qquad 0<\rho<R,$$

in such a way that $W^{(j)}_{k|\partial B_\delta(x_0)} = v_{j|\partial B_\delta(x_0)}$ in the sense of traces,

$$W_k^{(j)}\left(x_0+\rho_k\,\frac{x-x_0}{|x-x_0|}\right) = v_k^{(j)}\left(x_0+\delta\,\frac{x-x_0}{|x-x_0|}\right)$$

and the energy $\int_{A^\delta_{\rho_k}} |DW_k^{(j)}|\,\mathrm{d}x$ is arbitrarily small, if $\rho_k\nearrow\delta$ sufficiently rapidly. Since $W_k^{(j)}$ is built up by means of affine interpolations between $v_h^{(j)}$ and $v_{h+1}^{(j)}$, for $h\ge k$, by condition (3.21) we infer that

$$\operatorname{dist}(W_k^{(j)}(x),\mathcal{Y})<\varepsilon_0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x\in A^\delta_{\rho_k} \tag{3.22}$$

for m large enough, hence we may and do define $w_k^{(j)} := \Pi_{\varepsilon_0}\circ W_k^{(j)}$ on $A^\delta_{\rho_k}$, where Π_{ε_0} is the Lipschitz projection on $\mathcal{Y}$ given by Remark 2.1, so that $w_k^{(j)}(A^\delta_{\rho_k})\subset B_{\mathcal{Y}}(y_j,\varepsilon_m)$.

We now extend $w_k^{(j)}$ to the whole ball $\widehat{B}_j$ by the map $\widetilde{w}_k^{(j)} : \overline{B}_{\rho_k}(x_0)\to B_{\mathcal{Y}}(y_j,\varepsilon_m)$ given by

$$\widetilde{w}_k^{(j)}(x) := \begin{cases} w_k^{(j)}\circ\psi_{(\delta,\sigma)}(x) & \text{if} \quad x\in A^{\delta-\sigma}_{\delta-2\sigma} \\ \Psi_{(y_j,\varepsilon_m)}\circ u\circ\phi_{(\delta,\sigma,r)}(x) & \text{if} \quad x\in B_{\delta-2\sigma}(x_0), \end{cases} \tag{3.23}$$

where $\sigma:=\delta-\rho_k$, $\psi_{(\delta,\sigma)} : A^{\delta-\sigma}_{\delta-2\sigma}\to A^\delta_{\delta-\sigma}$ is the reflection map

$$\psi_{(\delta,\sigma)}(x) := \big(-|x-x_0|+2\,(\delta-\sigma)\big)\,\frac{x-x_0}{|x-x_0|}$$

and $\phi_{(\delta,\sigma,r)} : B_{\delta-2\sigma}(x_0)\to B_r(x_0)$ is the homothetic map

$$\phi_{(\delta,\sigma)}(x) := x_0+\frac{r}{\delta-2\sigma}\,(x-x_0).$$

Set now $\rho:=\rho_k=\delta-\sigma$. Since the image of $B_\rho(x_0)$ by $\widetilde{w}_k^{(j)}$ is contained in the geodesic ball $B_{\mathcal{Y}}(y_j,\varepsilon_m)$, by means of a convolution argument we can approximate $\widetilde{w}_k^{(j)}$ on $B_\rho(x_0)$ by a smooth sequence $v_\varepsilon^{(j)} : B_\rho(x_0)\to \overline{B}^N(y_j,\varepsilon_m)$ that converges in the L^1-sense to $\widetilde{w}^{(j)}_{k|B_\rho(x_0)}$ and with total variation converging to the total variation $|D\widetilde{w}_k^{(j)}|(B_\rho(x_0))$. We then set $w_\varepsilon^{(j)} := \Pi_{\varepsilon_m}\circ v_\varepsilon^{(j)} : B_\rho(x_0)\to B_{\mathcal{Y}}(y_j,\varepsilon_m)$, so that clearly $w_\varepsilon^{(j)}\rightharpoonup \widetilde{w}_k^{(j)}$ weakly in the BV-sense, whereas

$$|Dw_\varepsilon^{(j)}|(B_\rho(x_0)) \le (\operatorname{Lip}\Pi_{\varepsilon_m})\cdot|Dv_\varepsilon^{(j)}|(B_\rho(x_0)).$$

Therefore, the energy of $\widetilde{w}_k^{(j)}$ being small on $A^{\delta-\sigma}_{\delta-2\sigma}$, we may and do assume that

$$\limsup_{\varepsilon\to 0}\int_{B_\rho(x_0)} |Dw_\varepsilon^{(j)}|\,\mathrm{d}x \le (\operatorname{Lip}\Pi_{\varepsilon_m})^2\cdot|Du|(B_r(x_0))+\frac{2^{-j}}{k}\cdot \tag{3.24}$$

Moreover, by suitably defining the convolution kernel, we may and do assume that the traces are equal, so that $w^{(j)}_{\varepsilon|\partial B_\rho(x_0)} = v^{(j)}_{\varepsilon|\partial B_\rho(x_0)} = \widetilde{w}^{(j)}_{k|\partial B_\rho(x_0)}$. We finally define $u^{(j)}_k \in BV(\overline{B}_r(x_0), \mathcal{Y})$ by

$$u^{(j)}_k(x) := \begin{cases} w_j(x) & \text{if} \quad x \in A^r_\delta \\ w^{(j)}_k(x) & \text{if} \quad x \in A^\delta_{\rho_k} \\ w^{(j)}_{\varepsilon_k}(x) & \text{if} \quad x \in B_{\rho_k}(x_0) \end{cases}$$

where $\varepsilon_k \searrow 0$ along a sequence and $\rho_k \nearrow \delta$ sufficiently rapidly so that

$$\mathcal{E}_{TV}(u^{(j)}_k, A^\delta_{\rho_k}) = \int_{A^\delta_{\rho_k}} |Du^{(j)}_k| \, \mathrm{d}x \le \frac{2^{-j}}{k} \cdot \tag{3.25}$$

Step 5: Approximating maps on the whole domain. For any $n \ge 2$ we now define $u^{(m)}_k : B^n \to \mathcal{Y}$ by

$$u^{(m)}_k(x) := \begin{cases} u^{(j)}_k(x) & \text{if} \quad x \in B_j, \quad j \in \mathbb{N} \\ u(x) & \text{if} \quad x \in B^n \setminus \Omega_m, \end{cases} \qquad \Omega_m := \bigcup_{j=1}^{\infty} B_j. \tag{3.26}$$

By Step 4 we know that $u^{(j)}_k \in BV(B_j, \mathcal{Y})$ and $u^{(j)}_{k\,|\widehat{B}_j} \in W^{1,1}(\widehat{B}_j, \mathcal{Y})$ for every j and k. Moreover, since $u^{(j)}_k = u$ on ∂B_j for every j, we infer that $u^{(m)}_k$ is for every k a function in $BV(B^n, \mathcal{Y})$.

As to the energy estimates of $u^{(m)}_k$, if $n \ge 3$, by (3.19) we infer that

$$\sum_{j=1}^{\infty} \mathcal{E}_{TV}(u^{(m)}_k, B_j \setminus \widehat{B}_j) \le C \, (\varepsilon_m \cdot m)^{1/(2-n)} \sum_{j=1}^{\infty} \mathcal{E}_{TV}(u, \widetilde{B}_j)$$

where, we recall,

$$\widetilde{B}_j = \overline{B}_{2r}(x_0), \quad B_j = \overline{B}_r(x_0), \quad \widehat{B}_j = \overline{B}_\delta(x_0),$$

whereas by Theorem 3.1, on account of (3.3), we obtain

$$\sum_{j=1}^{\infty} \mathcal{E}_{TV}(u, \widetilde{B}_j) \le C \cdot \Big(\mathcal{E}_{TV}(u, B^n) + \mathcal{L}^n(B^n) \Big) < \infty, \tag{3.27}$$

and $(\varepsilon_m \cdot m)^{1/(2-n)} \to 0$ as $m \to \infty$. On the other hand, by (3.24) and (3.25) we estimate

$$\sum_{j=1}^{\infty} \mathcal{E}_{TV}(u^{(m)}_k, \widehat{B}_j) \le (\mathrm{Lip}\,\Pi_{\varepsilon_m})^2 \cdot |Du|(\Omega_m) + \frac{2}{k} \cdot$$

Now, (3.7) yields

$$|Du|(\Omega_m) \le \mu_d(\Omega_m) + \frac{1}{m} \cdot$$

Therefore, by a diagonal argument, setting $u_m := u^{(m)}_{k_m}$ for a suitable sequence $k_m \to \infty$ as $m \to \infty$, we have

$$\limsup_{m \to \infty} \sum_{j=1}^{\infty} \mathcal{E}_{TV}(u_m, B_j) \le \mu_d(B^n),$$

that clearly holds even in the case $n=2$. Since moreover $u_m = u$ on $B^n \setminus \Omega_m$, by (3.7) we then infer that

$$\limsup_{m\to\infty} \mathcal{E}_{TV}(u_m, B^n) \le \mu_d(B^n) + \mu_J(B^n) = \mathcal{E}_{TV}(u, B^n). \tag{3.28}$$

To prove the L^1-convergence of u_m to u as $m \to \infty$, and hence weakly in the BV-sense, we recall that the radii of the balls B_j in $\mathcal{F}'_m$ are smaller than $1/m$, whereas $u_k^{(m)} = u$ on ∂B_j and outside Ω_m. Therefore, since by the above energy estimates we may assume that $|Du_m|(B_j) \le 2|Du|(B_j)$ for every j, if m is sufficiently large, the Poincaré inequality yields

$$\int_{B^n} |u_m - u| \,\mathrm{d}x = \sum_{j=1}^{\infty} \int_{B_j} |u_{k_m}^{(m)} - u| \,\mathrm{d}x \le \sum_{j=1}^{\infty} C_n \cdot \frac{1}{m} \cdot |Du|(B_j) \le C_n \cdot \frac{1}{m} \cdot |Du|(B^n),$$

where $C_n > 0$ is an absolute constant, whence $u_m \to u$ in $L^1(B^n, \mathbb{R}^N)$. The lower semicontinuity of $u \mapsto \mathcal{E}_{TV}(u, B^n)$, see (2.1), in conjunction with (3.28), yields the convergence $\mathcal{E}_{TV}(u_m, B^n) \to \mathcal{E}_{TV}(u, B^n)$.

Finally, we observe that the Cantor part of Du_m is non-zero only possibly in the annuli $B_r(x_0) \setminus B_\delta(x_0)$. However, due to the energy estimates (3.18)-(3.27), by summing on j, we may and do assume that for m large enough

$$|D^C u_m|(B^n) \le \frac{1}{2} |D^C u|(B^n).$$

Therefore, using an iteration argument on the approximating sequences $\{u_m\}$, similar *e.g.* to the one giving Theorem 1.6 from Proposition 4.1 in Section 4 below, we find the approximating sequence $\{u_k\}$ such that $|D^C u_k|(B^n) = 0$ for every k, as required. □

THE CASE OF GENERAL INTEGRANDS f. To prove Theorem 1.5 for general integrands f, arguing as in [14], we shall make use of a continuity property from [16], Volume II. This property relies on the following continuity theorem due to Reshetnyak [18], compare Theorem 1 in Section 1.3.4 of [16], Volume II.

We first notice that the sequence $\{u_m\}$ obtained in the proof of Theorem 1.5, in the case $f(x,u,\xi) = |\xi|$, actually satisfies

$$\lim_{m\to\infty} |Du_m|(B^n) = |Du|(B^n). \tag{3.29}$$

Moreover, denoting by $\widetilde{D}u := \nabla u \,\mathcal{L}^n + D^C u$ the "diffuse" part of Du, we also have

$$\lim_{m\to\infty} |\widetilde{D}u_m|(B^n) = |\widetilde{D}u|(B^n). \tag{3.30}$$

In fact, $u_m = u$ on $B^n \setminus \Omega_m$, where $\Omega_m \subset B^n \setminus J_m$. By (3.7) and the growth condition (H3) we obtain

$$|D^J u_m|(B^n \setminus J_m) \le C \cdot \mu_J(B^n \setminus J_m) \le \frac{C}{m}, \tag{3.31}$$

that clearly yields both (3.29) and (3.30). We now let

$$\mathcal{F}_f(u) := \int_{B^n} f(x, u, \nabla u) \,\mathrm{d}x + \int_{B^n} f^\infty(x, u, \mathrm{d}D^C u)$$

so that if $f(x,u,\xi) = |\xi|$, we have $\mathcal{F}_f(u) = |\widetilde{D}u|(B^n)$. Using (3.29), (3.30) and Theorem 3.4 below we will prove that

$$\lim_{m\to\infty} \mathcal{F}_f(u_m) = \mathcal{F}_f(u). \tag{3.32}$$

Now, the first two terms in $G(u)$, corresponding to the "diffuse" part $\widetilde{D}u$, agree with $\mathcal{F}_f(u)$, see (1.5). Moreover, since $\Omega_m \subset B^n \setminus J_m$, by property (H3), and by the compactness and smoothness of $\mathcal{Y}$, we obtain

$$\begin{aligned}\int_{J_u\cap\Omega_m} K(x,u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} &\le \int_{J_u\setminus J_m} K(x,u^-,u^+,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le C\cdot \mu_J(B^n\setminus J_m) \le C\cdot |D^J u|(B^n\setminus J_m).\end{aligned}$$

By (3.31) and (3.32), and since $u_m = u$ on $B^n \setminus \Omega_m$, we readily conclude that $G(u_m) \to G(u)$.

In order to prove (3.32), for any $\mathbb{R}^m$-valued Radon measure μ defined on an open set $U \subset \mathbb{R}^{n+N}$, we will denote by $\overrightarrow{\mu}$ its Radon Nikodym derivative with respect to the total variation $|\mu|$, and by $\mu_k \rightharpoonup \mu$ the weak convergence in the sense of the measures.

Theorem 3.4. (Reshetnyak). *Let $G(z,p)$ be a non-negative continuous function defined in $U\times\mathbb{R}^m$ satisfying the following properties:*

(i) *$G(z,\cdot)$ is positively homogeneous of degree one for every z.*
(ii) *$G(\cdot,p)$ is uniformly bounded as $p\in\mathbb{S}^{m-1}$.*
(iii) *$G(z,\cdot)$ is* essentially convex *for every z,* i.e.,

$$G(z,p+q)\le G(z,p)+G(z,q) \qquad \forall\, p,q\in\mathbb{R}^m,$$

where the equality holds if and only if $q=\lambda p$ for some $\lambda\ge 0$.

Let $F(z,p)$ be a non-negative continuous function that is homogeneous of degree one in p for every z and that satisfies

$$0\le F(z,p)\le c_1\,G(z,p)+c_2 \qquad \forall\,(z,p)\in U\times\mathbb{R}^m$$

for some absolute constants $c_i>0$. Then we have

$$\lim_{k\to\infty}\int_U F(z,\overrightarrow{\mu}_k(z))\,\mathrm{d}|\mu_k| = \int_U F(z,\overrightarrow{\mu}(z))\,\mathrm{d}|\mu|$$

provided that μ_k, μ are $\mathbb{R}^m$-valued Radon measures on U satisfying

$$\mu_k\rightharpoonup\mu, \qquad \int_U G(z,\overrightarrow{\mu}_k(z))\,\mathrm{d}|\mu_k| \to \int_U G(z,\overrightarrow{\mu}(z))\,\mathrm{d}|\mu| \qquad \textit{as } k\to\infty.$$

We now take $U=B^n\times\mathbb{R}^N$, $z=(x,u)$, $m=1+nN$, and we identify $v\in BV(B^n,\mathcal{Y})$ with the vector valued measure $\mu_v:=(\mu_v^{(0)},\mu_v^{(1)})$, where

$$\mu_v^{(0)}(\phi) := \int_{B^n}\phi(x,v(x))\,\mathrm{d}x \qquad \forall\phi=\phi(x,y)\in C_c^0(B^n\times\mathbb{R}^N)$$

and $\mu_v^{(1)} := ((\mu_v)_i^j)$, for $i=1,\ldots,n$, $j=1,\ldots,N$, is defined in components by

$$(\mu_v)_i^j(\phi) := \int_{B^n}\nabla_i v^j(x)\,\phi(x,v(x))\,\mathrm{d}x + \int_{B^n}\phi(x,u(x))\,\mathrm{d}(D^C v^j)_i.$$

Following [16], Volume II, [12,14], it turns out that

$$|\widetilde{D}v|(B^n) = \int_U G(z,\overrightarrow{\mu}(v)(z))\,\mathrm{d}|\mu(v)|, \qquad \mathcal{F}_f(v) = \int_U F_f(z,\overrightarrow{\mu}(v)(z))\,\mathrm{d}|\mu(v)|,$$

where G and F_f are the *parametric polyconvex l.s.c. envelope* of the total variation integrand and of f, respectively. Moreover, we have:

Lemma 3.5. *The weak convergence $u_m \rightharpoonup u$ in the BV-sense, in conjunction with (3.30), yields the weak convergence $\mu_{u_m} \rightharpoonup \mu_u$ in the sense of the measures.*

Therefore, by the growth condition (H3), by (3.29), and by Lemma 3.5, we readily check the hypotheses of Theorem 3.4, for G and $F = F_f$ as above, that clearly yields (3.32), as required.

Proof of Lemma 3.5. Setting $U := B^n \times \mathbb{R}^N$, by (3.30) and by Hahn-Banach theorem, since

$$|\mu_v|(U) = \int_{B^n} \sqrt{1 + |\nabla v|^2}\, \mathrm{d}x + |D^C v|(B^n), \qquad |\mu_v^{(1)}|(U) = |\widetilde{D}v|(B^n),$$

possibly passing to a subsequence (μ_{u_m}) weakly converges to some vector-valued measure $\widetilde{\mu}$ with finite total variation. Writing $\widetilde{\mu} = (\widetilde{\mu}^{(0)}, \widetilde{\mu}^{(1)})$ as above, the L^1-convergence $u_m \to u$ and the Lebesgue theorem clearly yield that the first component

$$\widetilde{\mu}^{(0)}(\phi) = \int_{B^n} \phi(x, u(x))\, \mathrm{d}x = \mu_u^{(0)}(\phi) \qquad \forall \phi \in C_c^0(B^n \times \mathbb{R}^N).$$

As to the second component $\widetilde{\mu}^{(1)} := ((\widetilde{\mu})_i^j)$, the weak convergence $u_m \rightharpoonup u$ in the BV-sense yields that for every i and j we may decompose $(\widetilde{\mu})_i^j$ into two mutually singular measures

$$(\widetilde{\mu})_i^j = (\mu_u)_i^j + (\widehat{\mu})_i^j, \qquad (\mu_u)_i^j \perp (\widehat{\mu})_i^j,$$

the first one being the corresponding component of $\mu_u^{(1)}$, so that

$$|(\widetilde{\mu})_i^j|(U) = |(\mu_u)_i^j|(U) + |(\widehat{\mu})_i^j|(U).$$

By lower semicontinuity, using (3.30) we obtain

$$|(\mu_u)_i^j|(U) + |(\widehat{\mu})_i^j|(U) \le \liminf_{m\to\infty} |(\mu_{u_m})_i^j|(U) = |(\mu_u)_i^j|(U)$$

which yields $(\widehat{\mu})_i^j = 0$ for every i and j and hence $\widetilde{\mu} = \mu_u$. The assertion readily follows, as the limit $\widetilde{\mu}$ does not depend on the chosen subsequence. □

4. The density result, part II

In this section we prove Theorem 1.6 in any dimension $n \ge 2$, the case $n = 1$ being an immediate adaptation of results from [11]. In the sequel, for every map $v \in BV(B^n, \mathcal{Y})$ we will denote by $\mu_{a,v}$ and $\mu_{J,v}$ the Radon measures on B^n given for every Borel set $B \subset B^n$ respectively by

$$\mu_{a,v}(B) := \int_B f(x, v, \nabla v)\, \mathrm{d}x, \qquad \mu_{J,v}(B) := \int_{J_v \cap B} K(x, v^-, v^+, \nu_v)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.1}$$

so that if $|D^C v|(B^n) = 0$ we have, compare (1.5),

$$G(v, B) = \mu_{a,v}(B) + \mu_{J,v}(B).$$

The proof of Theorem 1.6 is based on the following:

Proposition 4.1. *Let* $\widetilde{u} \in BV(B^n, \mathcal{Y})$ *be such that* $|D^C \widetilde{u}|(B^n) = 0$*. Let* $\varepsilon \in (0, 1/2)$ *and* $k \in \mathbb{N}$*. We can find a function* $\widehat{u} \in BV(B^n, \mathcal{Y})$ *such that*

$$\begin{aligned} &G(\widehat{u}, B^n) \leq G(\widetilde{u}, B^n) + \varepsilon^k, &&\|\widehat{u} - \widetilde{u}\|_{L^1(B^n)} \leq \varepsilon^k, \\ &\mu_{J,\widehat{u}}(B^n) \leq \frac{1}{2} \cdot \mu_{J,\widetilde{u}}(B^n) &&\textit{and} \quad |D^C \widehat{u}|(B^n) = 0. \end{aligned} \tag{4.2}$$

In fact, for any $\varepsilon \in (0, 1/2)$ we apply iteratively Proposition 4.1 as follows. Letting $u_0^\varepsilon := u$, at the k^{th} step, in correspondence of $\widetilde{u} := u_{k-1}^\varepsilon$ we find $\widehat{u} := u_k^\varepsilon$ such that (4.2) holds true. By induction on $k \in \mathbb{N}$, we define $u^\varepsilon := u_\infty^\varepsilon \in BV(B^n, \mathcal{Y})$ such that

$$G(u^\varepsilon, B^n) \leq G(u, B^n) + \sum_{k=1}^{\infty} \varepsilon^k \leq G(u, B^n) + 2\varepsilon$$

and $|D^C u^\varepsilon|(B^n) = 0$. Moreover, since for every k

$$\mu_{J,u_k^\varepsilon}(B^n) \leq 2^{-k} \cdot \mu_{J,u}(B^n),$$

letting $k \to \infty$ we obtain that $\mu_{J,u^\varepsilon}(B^n) = 0$. Finally, since

$$\|u^\varepsilon - u\|_{L^1(B^n)} \leq \sum_{k=1}^{\infty} \|u_k^\varepsilon - u_{k-1}^\varepsilon\|_{L^1(B^n)} \leq \sum_{k=1}^{\infty} \varepsilon^k \leq 2\varepsilon,$$

letting $u_k := u^{\varepsilon_k}$ for some sequence $\varepsilon_k \searrow 0$, the sequence $\{u_k\} \subset BV(B^n, \mathcal{Y})$ weakly converges to u, with $G(u_k) \to G(u)$ as $k \to \infty$, by lower semicontinuity. Since $|D^C u_k|(B^n) = 0$ and $\mu_{J,u_k}(B^n) = 0$ for every k, by the growth condition (H3) we infer that $|D^J u_k|(B^n) = 0$, whence $\{u_k\} \subset W^{1,1}(B^n, \mathcal{Y})$, as required.

Proof of Proposition 4.1. We set $\widetilde{u} = u$, for simplicity, and divide the proof in five steps, where we will use arguments from [11,13].

Step 1: Blow-up argument. We apply an argument by Federer [7], Section 4.2.19, to the rectifiable measure

$$\mu_{J,u} := \mathcal{L}_u \, \mathcal{H}^{n-1} \llcorner J_u, \qquad \mathcal{L}_u(x) := K(x, u^-, u^+, \nu_u),$$

the density $\mathcal{L}_u(x)$ being a non-negative $\mathcal{H}^{n-1} \llcorner J_u$-summable function on J_u. Therefore, by [7], Section 3.2.29, there exists a countable family $\mathcal{G}$ of $(n-1)$-dimensional C^1-submanifolds $\mathcal{M}_j$ of B^n such that $\mu_{J,u}$-almost all of B^n is covered by $\mathcal{G}$. Moreover, since $\mu_{J,u}(B^n) < \infty$, we can find a positive number $\theta > 0$ so that the subset

$$\widetilde{J} := \{x \in J_u \mid \mathcal{L}_u(x) > \theta\}$$

satisfies the following properties:

$$\mathcal{H}^{n-1}(\widetilde{J}) < \infty \qquad \text{and} \qquad \mu_{J,u}(B^n \setminus \widetilde{J}) < \frac{1}{4} \cdot \mu_{J,u}(B^n). \tag{4.3}$$

Moreover, by the smoothness and compactness of $\mathcal{Y}$, and by the growth condition (H3), we infer that the function $x \mapsto \mathcal{L}_u(x)$ is uniformly bounded on $\widetilde{J}$, *i.e.*,

$$\mathcal{L}_u(x) \leq C(\widetilde{J}) < \infty \qquad \forall\, x \in \widetilde{J}. \tag{4.4}$$

Let $0 < \sigma < 1$ to be fixed. By [7], Section 2.10.19, and by the Vitali-Besicovitch theorem, Theorem 3.1, we can find a number $t_\sigma \in (1/2, 1)$, a countable disjoint family of closed balls B_j, contained in B^n and centered at points in $\widetilde{J}$, and a bilipschitz homeomorphism ψ_σ from B^n onto itself satisfying the properties listed below, where $c > 0$ is an absolute constant, possibly varying from line to line, which is independent of σ and of the radii r_j of the balls B_j.

(i) $\mu_{J,u}(B^n \setminus \bigcup_j B_j) = 0$.

(ii) If $B_j := \overline{B}(p_j, r_j)$, for every j there is a manifold $\mathcal{M}_j$ of $\mathcal{G}$ such that $p_j \in \mathcal{M}_j$, and the radius $r_j \in (0, \sigma)$.

(iii) Since $\mathcal{H}^{n-1}(\widetilde{J}) < \infty$, then

$$\sum_{j=1}^{\infty} {r_j}^{n-1} \leq c \cdot \mathcal{H}^{n-1}(\widetilde{J}) < \infty. \tag{4.5}$$

(iv) Letting $C_j := B(p_j, t_\sigma r_j) \cap \mathcal{M}_j$, we have

$$\mu_{J,u}(B(p_j, r_j) \setminus C_j) \leq \sigma \cdot \mu_{J,u}(B(p_j, r_j)) \qquad \forall\, j. \tag{4.6}$$

(v) The one-sided approximate limits $u^{\pm}(p_j)$ of u at p_j are well-defined.

(vi) The "blow-up" map $u_j : \mathbb{R}^n \to \mathcal{Y}$ of u at p_j

$$u_j(x) := \begin{cases} u^+(p_j) & \text{if } (x - p_j) \cdot \nu_u(p_j) > 0 \\ u^-(p_j) & \text{if } (x - p_j) \cdot \nu_u(p_j) < 0 \end{cases}$$

has small L^1-distance from u on B_j, *i.e.*

$$\|u_j - u\|_{L^1(B_j)} \leq c \cdot \sigma \cdot {r_j}^{n-1}. \tag{4.7}$$

(vii) By a slicing argument, we may and will assume that for some $R \in (r_j, 2r_j)$ the sliced map $u_{|\partial B(p_j, tr_j)}$ satisfies

$$|Du_{|\partial B(p_j, tr_j)}|(\partial B(p_j, tr_j) \setminus C_j) \leq \frac{c}{r_j} \cdot |Du|(B(p_j, R) \setminus C_j).$$

Moreover, by the construction we may assume that both properties (4.6) and

$$\mu_{J,u}(B(p_j, \rho)) \leq c\, \mathcal{L}_u(p_j)\, \rho^{n-1}$$

hold true for any $0 < \rho < 2r_j$. Therefore, taking $\sigma > 0$ small so that $\sqrt{\sigma} \leq 1/C(\widetilde{J})$, by (4.4) and the growth condition (H3) we obtain that

$$|Du_{|\partial B(p_j, tr_j)}|(\partial B(p_j, tr_j) \setminus C_j) \leq c \cdot \sqrt{\sigma}\, {r_j}^{n-2}. \tag{4.8}$$

(viii) The ball $B(p_j, tr_j)$ is divided by $\mathcal{M}_j$ into two open connected "half" balls, denoted by $\Omega_j^{\pm}$, and by a slicing argument we have

$$\int_{\partial B(p_j, tr_j) \cap \Omega_j^{\pm}} |u(x) - u^{\pm}(p_j)| \leq c \cdot \sigma\, {r_j}^{n-2}. \tag{4.9}$$

(ix) By the hypothesis (H4) on f, for every $x \in B_j$ we have

$$|f(x, u, \xi) - f(p_j, u, \xi)| \leq \sigma\, (1 + |\xi|) \qquad \forall\, u \in \mathcal{Y},\ \forall\, \xi \in M(N, n). \tag{4.10}$$

(x) Since $\mathcal{L}_u(p_j)$ is the $(n-1)$-dimensional density of $\mu_{J,u}$ at p_j, we have

$$|\mu_{J,u}(B_j) - \mathcal{L}_u(p_j) \cdot \omega_{n-1}\, {r_j}^{n-1}| \le \sigma \cdot \omega_{n-1}\, {r_j}^{n-1}. \tag{4.11}$$

(xi) $\operatorname{Lip}\psi_\sigma \le 2$ and $\operatorname{Lip}\psi_\sigma^{-1} \le 2$. Moreover, ψ_σ maps bijectively B_j onto B_j, with $\psi_{\sigma|\partial B_j} = Id_{|\partial B_j}$ and $\psi_\sigma(p_j) = p_j$ for all j, and ψ_σ is equal to the identity outside the union of the balls B_j.

(xii) $\psi_\sigma(C_j) = B(p_j,\rho_j) \cap (p_j + \operatorname{Tan}(\mathcal{M}_j, p_j))$ for every j, where $\operatorname{Tan}(\mathcal{M}_j, p_j)$ is the $(n-1)$-dimensional tangent space to $\mathcal{M}_j$ at p_j and $\rho_j \in (r_j/2, r_j)$.

As a consequence, we define $u_j^\sigma \in BV(\operatorname{int}(B_j), \mathcal{Y})$ by $u_j^\sigma := (u \circ \psi_\sigma^{-1})_{|\operatorname{int}(B_j)}$, and observe that

$$\mu_{J,u_j^\sigma} = \psi_{\sigma\#}(\mu_{J,u} \llcorner \operatorname{int}(B_j)).$$

Step 2: Projecting the boundary data. Set

$$x = (\widetilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Without loss of generality we may and will assume that $p_j = 0$, $\nu_u(p_j) = \nu_{u_j^\sigma}(0) = e_n$, and

$$B_j = \overline{B}_d^n, \qquad B(p_j,\rho_j) = B_R^n, \qquad 0 < R < d,$$

where $B_\rho^n := B^n(0,\rho)$, so that $d = r_j$ and $R = \rho_j$, and

$$B(p_j,\rho_j) \cap (p_j + \operatorname{Tan}(\mathcal{M}_j, p_j)) = D_R \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R},$$

where $D_\rho := B^{n-1}(0_{\mathbb{R}^{n-1}}, \rho)$. We also set

$$B_\rho^\pm := \{x \in B_\rho^n \mid \pm x_n > 0\}, \qquad \partial B_\rho^\pm := \{x \in \partial B_\rho^n \mid \pm x_n > 0\}.$$

By (4.8) and (4.9) we readily infer

$$|Du^\sigma_{j|\partial B_R^n}|(\partial B_R^\pm) \le c \cdot \sqrt{\sigma}\, R^{n-2}, \qquad \int_{\partial B_R^\pm} |u_j^\sigma(x) - z_j^\pm| \le c \cdot \sigma\, R^{n-2}, \tag{4.12}$$

where $z_j^\pm$ are the one-sided approximate limits of u_j^σ at the point $0 \in J_{u_j^\sigma}$, *i.e.*, of u at p_j. Therefore, for $n \ge 3$, using an argument very similar to the one in Step 3 of Section 3, see also (3.23), but this time working separately on the half-balls $B_r^\pm$, and taking σ instead of $1/m$, due to (4.12) we are able to define two BV-maps $w_j^\pm : B_R^\pm \setminus B_r^\pm \to \mathcal{Y}$ such that the following properties hold:

(a) We have $R - r = R/q$, where $q := c/\sigma^{1/2(n-2)}$.

(b) $w_j^\pm(rx) = \Psi_{(z_j^\pm,\varepsilon_\sigma)} \circ u_j^\sigma(Rx)$ if $|x| = 1$, where $\Psi_{(z_j^\pm,\varepsilon_\sigma)}(z) := \Pi_{\varepsilon_\sigma} \circ \xi_{(z_j^\pm,\varepsilon_\sigma)}$, see (3.8), and $\varepsilon_\sigma := c \cdot \sqrt{\sigma}$.

(c) We have $|Dw_j^\pm|(B_R^\pm \setminus B_r^\pm) \le c \cdot \dfrac{R}{q} \cdot |Du^\sigma_{j|\partial B_R^n}|(\partial B_R^\pm)$, so that by (4.12)

$$|Dw_j^\pm|(B_R^\pm \setminus B_r^\pm) \le c \cdot \frac{\sqrt{\sigma}}{q} \cdot R^{n-1} = c \cdot \sigma^{(n-1)/2(n-2)} \cdot R^{n-1}.$$

(d) By (4.12), since $0 < R < d < 1$, we also have

$$\|w_j^\pm - z_j^\pm\|_{L^1(B_R^\pm \setminus B_r^\pm)} \le c \cdot \frac{\sigma}{q} \cdot R^{n-1} \le c \cdot \sigma \cdot d^{n-1}. \tag{4.13}$$

(e) Using the properties $|D^C u_j^\sigma|(B_R^n) = 0$ and $|D^J u_j^\sigma|(B_R^n \setminus (D_R \times \{0\})) \leq c\,\sigma |D^J u_j^\sigma|(B_d^n)$, that follows from (4.6), from the growth condition (H3), and from the smoothness and compactness of $\mathcal{Y}$, we infer that we may and do define $w_j^\pm$ in such a way that the function $w_j : B_R^n \setminus B_r^n \to \mathcal{Y}$, given by $w_j(x) = w_j^\pm(x)$ if $\pm x_n > 0$, belongs to $BV(B_R^n \setminus B_r^n, \mathcal{Y})$ and satisfies

$$|D^J w_j|((B_R^n \setminus B_r^n) \cap (D_R \times \{0\})) \leq c \cdot \frac{R}{q} \cdot |D u^\sigma_{j|\partial B_R^n}|(\partial B_R^+ \cup \partial B_R^-),$$

so that again by (4.12) and property (c) we deduce that

$$|D w_j|(B_R^n \setminus B_r^n) \leq c \cdot \sigma^{(n-1)/2(n-2)} \cdot R^{n-1}. \tag{4.14}$$

Step 3: Approximation on the balls B_j. Using property (b) in Step 2, we now define $\widetilde{u}_j^\sigma : B_r^n \to \mathcal{Y}$ by setting

$$\widetilde{u}_j^\sigma(x) := \begin{cases} \Psi_{(z_j^+, \varepsilon_\sigma)} \circ u_j^\sigma(Rx/r) & \text{if} \quad x_n > 0 \\ \Psi_{(z_j^-, \varepsilon_\sigma)} \circ u_j^\sigma(Rx/r) & \text{if} \quad x_n < 0. \end{cases} \tag{4.15}$$

Remark 4.2. In the case $n = 2$, by (4.12) we infer that $u_j^\sigma(\partial B_R^\pm) \subset B_{\mathcal{Y}}(z_j^\pm, \varepsilon_\sigma)$, and hence we simply take $r = R$ in (4.15).

We apply for every j a "dipole construction" to approximate almost all the Jump part of the energy of $\widetilde{u}_j^\sigma$. Let $y(\widetilde{x}) := (r - |\widetilde{x}|)$ denote the distance of $\widetilde{x}$ from the boundary of the $(n-1)$-disk D_r. For $\delta > 0$ small, let

$$\phi_\delta(x) := (\widetilde{x}, \varphi_\delta(y(\widetilde{x}))x_n), \quad x \in D_r \times [-1, 1], \quad \varphi_\delta(y) := \min\{y, \delta\}.$$

Let $\Omega_\delta := \phi_\delta(D_r \times [-1, 1])$ be the "neighborhood" of $D_r \times \{0\}$ given by

$$\Omega_\delta = \{(\widetilde{x}, x_n) \mid \widetilde{x} \in D_r, \quad |x_n| \leq \varphi_\delta(y(\widetilde{x}))\},$$

and let

$$\widetilde{\Omega}_\delta := \phi_\delta(D_r \times [-1/2, 1/2]) = \{(\widetilde{x}, x_n) \mid \widetilde{x} \in D_r,\ |x_n| \leq \varphi_\delta(y(\widetilde{x}))/2\}.$$

Also, set

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0\}).$$

Let $v_j^\sigma : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathcal{Y}$ be given by $v_j^\sigma(x) := \widetilde{u}_j^\sigma \circ \psi_j^\sigma(x)$, where $\psi_j^\sigma : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \Omega_{(r,\delta)}$ is the bijective map

$$\psi_j^\sigma(\widetilde{x}, x_n) := \left(\widetilde{x}, \left(2 - \frac{\varphi_\delta(y(\widetilde{x}))}{|x_n|}\right) x_n\right).$$

Since we have

$$|\nabla v_j^\sigma(x)| \leq c\, |\nabla \widetilde{u}_j^\sigma(\widetilde{x}, (2 - \varphi_\delta(y(\widetilde{x}))/|x_n|)\, x_n)| \cdot (1 + \varphi_\delta(y(\widetilde{x}))/|x_n|),$$

and $\varphi_\delta(y(\widetilde{x}))/|x_n| \in]1/2, 1]$, we infer that $v_j^\sigma \in BV(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathcal{Y})$, with

$$\int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla v_j^\sigma| \, dx \leq c \int_{\Omega_\delta} |\nabla \widetilde{u}_j^\sigma| \, dx. \tag{4.16}$$

Therefore, by absolute continuity and by the growth condition (H3) in Section 1, we infer that for δ small $\mu_{a, v_j^\sigma}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)$ is small. Moreover, we have

$$\mu_{J, v_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \leq \mu_{J, \widetilde{u}_j^\sigma}(\mathrm{int}(\Omega_{(r,\delta)})),$$

so that by (4.6) and the definition of $\widetilde{u}_j^\sigma$ we obtain

$$\mu_{J,v_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \leq c\,\sigma\, \mu_{J,u_j^\sigma}(B_R^n). \tag{4.17}$$

We now define $w_j^\sigma : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathbb{R}^N$ by

$$w_j^\sigma(x) := \left(\frac{2\,|x_n|}{\varphi_\delta(y(\widetilde{x}))} - 1 \right) \cdot v_j^\sigma(\widetilde{x}, x_n) + \left(2 - \frac{2\,|x_n|}{\varphi_\delta(y(\widetilde{x}))} \right) \cdot z_j^\pm ,$$

where $\pm$ is the sign of x_n.

If $r - \delta \leq |\widetilde{x}| \leq r$ and $(r - |\widetilde{x}|)/2 < |x_n| < (r - |\widetilde{x}|)$, then

$$|\nabla w_j^\sigma|(x) \leq \frac{c}{r - |\widetilde{x}|}\,|v_j^\sigma(x) - z_j^\pm| + c\,|\nabla v_j^\sigma(x)|,$$

whereas if $|\widetilde{x}| \leq r - \delta$ and $\delta/2 < |x_n| < \delta$, we estimate

$$|\nabla w_j^\sigma|(x) \leq \frac{c}{\delta}\,|v_j^\sigma(x) - z_j^\pm| + c\,|\nabla v_j^\sigma(x)|.$$

Moreover, by the definitions of $\widetilde{u}_j^\sigma$ and of v_j^σ, we infer that for every $x \in \Omega_\delta \setminus \widetilde{\Omega}_\delta$

$$v_j^\sigma(x) \in B_{\mathcal{Y}}(z_j^\pm, \varepsilon_\sigma) \qquad \text{if} \quad \pm\, x_n > 0,$$

see (3.8), where $\varepsilon_\sigma := c \cdot \sqrt{\sigma}$. As a consequence, on account of (4.16) we obtain

$$\int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla w_j^\sigma|\,\mathrm{d}x \leq c\,\sqrt{\sigma}\,\mathcal{L}^n(\Omega_\delta \setminus \widetilde{\Omega}_\delta) + c \int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla v_j^\sigma|\,\mathrm{d}x \tag{4.18}$$

which is small if δ and σ are small, by the absolute continuity. Also, since the oscillation of w_j^σ is smaller than $c\sqrt{\sigma}$ on both the sets $\{x \in \Omega_\delta \setminus \widetilde{\Omega}_\delta \mid \pm x_n > 0\}$, by projecting w_j^σ into the manifold $\mathcal{Y}$, see Remark 2.1, we may and will assume that w_j^σ is a function in $BV(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathcal{Y})$. We then clearly have

$$\mu_{J,w_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \leq \mu_{J,v_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)), \tag{4.19}$$

whereas by the construction

$$w_j^\sigma(\widetilde{x}, \pm\varphi_\delta(y(\widetilde{x}))/2) = z_j^\pm \qquad \forall\, \widetilde{x} \in D_r .$$

By (4.17), (4.18), (4.19), and by the growth condition (H3), taking δ small, we infer that w_j^σ satisfies the energy estimate

$$\begin{aligned} G(w_j^\sigma, \mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) &\leq c \int_{\Omega_\delta \setminus \widetilde{\Omega}_\delta} |\nabla w_j^\sigma|\,\mathrm{d}x + \mu_{J,w_j^\sigma}(\mathrm{int}(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \\ &\leq c\,\sigma\,R^{n-1} + c\,\sigma \mu_{J,u_j^\sigma}(B_R^n). \end{aligned} \tag{4.20}$$

In conclusion, defining $\widehat{u}_j^\sigma : (B_d^n \setminus \widetilde{\Omega}_\delta) \to \mathcal{Y}$ by

$$\widehat{u}_j^\sigma(x) := \begin{cases} u_j^\sigma(x) & \text{if } x \in B_d^n \setminus B_R^n \\ w_j(x) & \text{if } x \in B_R^n \setminus B_r^n \\ \widetilde{u}_j^\sigma(x) & \text{if } x \in B_r^n \setminus \Omega_\delta \\ w_j^\sigma(x) & \text{if } x \in \Omega_\delta \setminus \widetilde{\Omega}_\delta \end{cases} \tag{4.21}$$

it turns out that $\widehat{u}_j^\sigma$ belongs to $BV((B_d^n \setminus \widetilde{\Omega}_\delta), \mathcal{Y})$, satisfies the boundary condition

$$\widehat{u}_j^\sigma = z_j^+ \quad \text{on} \quad \partial\widetilde{\Omega}_\delta \cap B_r^+, \qquad \widehat{u}_j^\sigma = z_j^- \quad \text{on} \quad \partial\widetilde{\Omega}_\delta \cap B_r^-, \tag{4.22}$$

where $z_j^\pm = u^\pm(p_j)$. Moreover, (4.6), (4.14), the growth condition (H3), and (4.20) yield the energy estimate

$$G(\widehat{u}_j^\sigma, B_d^n \setminus \widetilde{\Omega}_\delta) \le \mu_{a,u_j^\sigma}(B_d^n) + c\,\sigma^{\alpha(n)}\, d^{n-1} + c\,\sigma\mu_{J,u_j^\sigma}(B_d^n), \tag{4.23}$$

where $0 < \alpha(n) \le 1$ is an absolute positive constant, only depending on the dimension $n \ge 2$.

Step 4: The dipole construction. In order to extend $\widehat{u}_j^\sigma$ to a function in $BV(B_j, \mathcal{Y})$, we use a dipole-type argument. To this purpose, we recall that $B_j = \overline{B}_d^n$, $J_{u_j^\sigma} = \psi_\sigma(J_u \cap \mathrm{int}(B_j))$, $p_j = 0$, $\nu_u(p_j) = \nu_{u_j^\sigma}(0) = e_n$, and $z_j^\pm = u^\pm(p_j)$. Let $\gamma_j \in W^{1,1}((-1/2, 1/2), \mathcal{Y})$ be such that $\gamma_j(\pm 1/2) = z_j^\pm$ and

$$\mathcal{L}(\gamma_j) := \int_{-1/2}^{1/2} f^\infty(p_j, \gamma_j(t), \gamma_j'(t) \otimes e_n)\,\mathrm{d}t \le K(p_j, z_j^-, z_j^+, e_n) + \varepsilon_j, \tag{4.24}$$

with $\varepsilon_j > 0$ arbitrarily small, compare (1.4). Using the homogeneity of $f^\infty(x, u, \cdot)$, we observe that the function $\varphi_j : D_r \times [-\rho/2, \rho/2] \to \mathcal{Y}$ given by $\varphi_j(\widetilde{x}, x_n) := \gamma_j(x_n/\rho)$ satisfies

$$\int_{D_r \times [-\rho/2, \rho/2]} f^\infty(p_j, \varphi_j(x), D\varphi_j(x))\,\mathrm{d}x = \mathcal{H}^{n-1}(D_r) \cdot \mathcal{L}(\gamma_j)$$

for every $\rho > 0$. Therefore, setting $\widehat{u}_j^\sigma : \widetilde{\Omega}_\delta \to \mathcal{Y}$ by

$$\widehat{u}_j^\sigma(x) := \gamma_j\Big(\frac{x_n}{\varphi_\delta(y(\widetilde{x}))}\Big), \qquad \widetilde{x} \in D_r, \quad |x_n| \le \varphi_\delta(y(\widetilde{x}))/2,$$

since

$$\widehat{u}_j^\sigma(x) := (v_j \circ \phi_\delta^{-1})(x), \qquad x \in \phi_\delta(D_r \times [-1/2, 1/2]),$$

where $v_j : D_r \times [-1/2, 1/2] \to \mathcal{Y}$ is given by $v_j(\widetilde{x}, x_n) := \gamma_j(x_n)$, we readily estimate

$$\begin{aligned}
\int_{\widetilde{\Omega}_\delta} f^\infty(p_j, \widehat{u}_j^\sigma(x), D\widehat{u}_j^\sigma(x))\,\mathrm{d}x \;&\le\; \mathcal{L}(\gamma_j) \cdot (\mathcal{H}^{n-1}(D_{r-\delta}) + c\,\mathcal{H}^{n-1}(D_r \setminus D_{r-\delta})) \\
&\le\; \sigma\, r^{n-1} + \mathcal{H}^{n-1}(D_r) \cdot \mathcal{L}(\gamma_j) \\
&\le\; \sigma\, r^{n-1} + \mathcal{H}^{n-1}(D_r) \cdot \mathcal{L}_u(p_j)
\end{aligned} \tag{4.25}$$

if $\delta > 0$ is small, where in the last inequality we have used (4.24) and the fact that

$$K(p_j, z_j^-, z_j^+, e_n) = \mathcal{L}_u(p_j).$$

On the other hand, by (4.10) and by the growth condition (H3) we obtain

$$\begin{aligned}
&\left| \int_{\widetilde{\Omega}_\delta} f(x, \widehat{u}_j^\sigma(x), D\widehat{u}_j^\sigma(x))\,\mathrm{d}x - \int_{\widetilde{\Omega}_\delta} f(p_j, \widehat{u}_j^\sigma(x), D\widehat{u}_j^\sigma(x))\,\mathrm{d}x \right| \le \\
&\qquad \sigma \cdot \int_{\widetilde{\Omega}_\delta} (1 + |D\widehat{u}_j^\sigma(x)|)\,\mathrm{d}x \le \sigma \left(|\widetilde{\Omega}_\delta| + c \cdot \int_{\widetilde{\Omega}_\delta} f^\infty(p_j, \widehat{u}_j^\sigma(x), D\widehat{u}_j^\sigma(x))\,\mathrm{d}x \right)
\end{aligned}$$

where $c>0$ is an absolute constant. As a consequence, by (4.11) and (4.25) we infer that if $\delta>0$ is small $\widehat{u}_j^\sigma$ satisfies the energy estimate

$$\int_{\widetilde{\Omega}_\delta} f(x,\widehat{u}_j^\sigma, D\widehat{u}_j^\sigma)\,\mathrm{d}x \le c\,\sigma\,d^{n-1} + (1+c\,\sigma)\,\mu_{J,u_j^\sigma}(B_j). \tag{4.26}$$

In conclusion, by (4.21) and (4.22) we deduce that $\widehat{u}_j^\sigma \in W^{1,1}(B_j,\mathcal{Y})$, whereas by (4.23) and (4.26)

$$G(\widehat{u}_j^\sigma, B_d^n) \le G(u_j^\sigma, B_d^n) + c\,\sigma^{\alpha(n)}\,d^{n-1} + c\,\sigma\mu_{J,u_j^\sigma}(B_d^n). \tag{4.27}$$

Step 5: Approximation on the whole domain. Setting again $B_j=\overline{B}_d^n$, we now show that for δ small enough

$$\|\widehat{u}_j^\sigma - u_j^\sigma\|_{L^1(B_d^n)} \le c\,\sigma\left(d^{n-1}+\mu_{J,u}(B_d^n)\right). \tag{4.28}$$

In fact, $\widehat{u}_j^\sigma$ agrees with u_j^σ outside B_R^n, whereas by (4.7) and the definition (4.21) of $\widehat{u}_j^\sigma$ we readily infer that

$$\|u_j - u_j^\sigma\|_{L^1(B_d^n\setminus\widetilde{\Omega}_\delta)} \le c\,\sigma\,d^{n-1}.$$

On the other hand, by (4.21), (4.22), the Poincaré inequality, and the growth condition (H3) we have

$$\|u_j - \widehat{u}_j^\sigma\|_{L^1(\widetilde{\Omega}_\delta)} \le c\,r\int_{\widetilde{\Omega}_\delta}|\nabla\widehat{u}_j^\sigma|\,\mathrm{d}x \le c\,r\int_{\widetilde{\Omega}_\delta} f(x,\widehat{u}_j^\sigma,\nabla\widehat{u}_j^\sigma)\,\mathrm{d}x.$$

Therefore, by (4.26) and (4.4), and since $0<r<d<\sigma<1$, we obtain

$$\|u_j - \widehat{u}_j^\sigma\|_{L^1(\widetilde{\Omega}_\delta)} \le c\,\sigma\left(d^{n-1}+\mu_{J,u}(B_d^n)\right)$$

and hence (4.28).

Setting then $U_j^\sigma \in W^{1,1}(B_j,\mathcal{Y})$ by $U_j^\sigma := (\widehat{u}_j^\sigma\circ\psi_\sigma)_{|B_j}$, since $d=r_j$, by (4.27) we infer that for every j

$$G(U_j^\sigma, B_j) \le \mu_{a,u}(B_j) + (1+c\,\sigma)\,\mu_{J,u}(B_j) + c\,\sigma\,{r_j}^{n-1}, \tag{4.29}$$

whereas by (4.28) we get

$$\|U_j^\sigma - u\|_{L^1(B_j)} \le c\,\sigma\,({r_j}^{n-1}+\mu_{J,u}(B_j)). \tag{4.30}$$

Let now $u^\sigma\in W^{1,1}(B^n,\mathcal{Y})$ be given by

$$u^\sigma(x) := \begin{cases} U_j^\sigma(x) & \text{if } x\in B_j \\ u(x) & \text{if } x\in B^n\setminus\Omega_m, \end{cases} \qquad \Omega_m = \bigcup_{j=1}^\infty B_j.$$

By (4.29) and (4.5) we obtain

$$G(u^\sigma, B^n) \le \mu_{a,u}(B^n) + (1+c\,\sigma)\,\mu_{J,u}(B^n) + c\,\sigma\,\mathcal{H}^{n-1}(\widetilde{J}),$$

so that if $\sigma=\sigma(\varepsilon,k,\widetilde{J},\mu_{J,u})>0$ is small, we have

$$G(u^\sigma, B^n) \le G(u,B^n) + \varepsilon^k.$$

Moreover, by (4.3) and (4.6), taking σ small, the above construction yields that

$$\begin{aligned}\mu_{J,u^\sigma}(B^n) &\leq c\sum_{j=1}^{\infty}\mu_{J,u}(B_j\setminus C_j)+\mu_{J,u}(B^n\setminus\widetilde{J})\\ &\leq c\,\sigma\,\mu_{J,u}(B^n)+\frac{1}{4}\,\mu_{J,u}(B^n)<\frac{1}{2}\cdot\mu_{J,u}(B^n).\end{aligned}$$

Finally, by (4.30) and (4.5), the balls B_j being pairwise disjoint, we have

$$\|u^\sigma-u\|_{L^1(B^n)}\leq\sum_{j=1}^{\infty}\|U_j^\sigma-u\|_{L^1(B_j)}\leq c\,\sigma\sum_{j=1}^{\infty}{r_j}^{n-1}+c\,\sigma\mu_{J,u}(B^n)<\varepsilon^k$$

if $\sigma=\sigma(\varepsilon,k,\widetilde{J},\mu_{J,u})>0$ is small. Since Du^σ has no Cantor part, the proof is complete. □

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