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UNIQUE CONTINUATION PROPERTY NEAR A CORNER AND ITS FLUID-STRUCTURE CONTROLLABILITY CONSEQUENCES*

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Abstract. We study a non standard unique continuation property for the biharmonic spectral problem $\Delta^2 w = -\lambda \Delta w$ in a 2D corner with homogeneous Dirichlet boundary conditions and a supplementary third order boundary condition on one side of the corner. We prove that if the corner has an angle $0 < \theta_0 < 2\pi$, $\theta_0 \neq \pi$ and $\theta_0 \neq 3\pi/2$, a unique continuation property holds. Approximate controllability of a 2-D linear fluid-structure problem follows from this property, with a control acting on the elastic side of a corner in a domain containing a Stokes fluid. The proof of the main result is based in a power series expansion of the eigenfunctions near the corner, the resolution of a coupled infinite set of finite dimensional linear systems, and a result of Kozlov, Kondratiev and Mazya, concerning the absence of strong zeros for the biharmonic operator [*Math. USSR Izvestiya* **34** (1990) 337–353]. We also show how the same methodology used here can be adapted to exclude domains with corners to have a local version of the Schiffer property for the Laplace operator.

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1. Introduction and main results

Let us consider a circular sector of $\mathbb{R}^2$ described in polar coordinates

$$G = \{(r, \theta),\ 0 < r < r_0,\ 0 < \theta < \theta_0\} \tag{1.1}$$

and centered at the origin. Let Ω be a Lipschitz bounded subset of $\mathbb{R}^2$ with a straight corner of angle θ_0 at the origin, that is by definition: if B_{r_0} is the open ball of radius r_0 centered at the origin, then

$$\Omega \cap B_{r_0} = G. \tag{1.2}$$

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Let n be the outward normal vector to Ω and let us take the notations

$$\Gamma_0 = \{(r,0),\ 0<r<r_0\}, \quad \Gamma_1 = \{(r,\theta_0),\ 0<r<r_0\}. \tag{1.3}$$

We are interested in the following local unique continuation property: prove (or disprove) that every weak solution $w \in H^2(\Omega)$ of the overdetermined spectral problem

$$\Delta^2 w = -\lambda \Delta w \quad \text{in } \Omega \tag{1.4}$$
$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \tag{1.5}$$
$$\frac{\partial \Delta w}{\partial n} = 0 \quad \text{on } \Gamma_0 \tag{1.6}$$

necessarily vanishes in Ω. Our main result is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded subset with a straight corner of angle $0 < \theta_0 < 2\pi$ at the origin and assume that*

$$\theta_0 \neq \pi, \quad \theta_0 \neq \frac{3\pi}{2}, \tag{1.7}$$

then any weak solution $w \in H^2(\Omega)$ of the problem (1.4)–(1.6) vanishes in Ω.

We have become interested in this kind of problems because they are related to the approximate controllability of some simplified fluid-structure models (see for instance [5] or [8]). Indeed, a non vanishing solution of (1.4)–(1.6) is the stream function of an eigenfunction of the Stokes operator with Dirichlet boundary conditions on $\Gamma_0 \cup \Gamma_1$ and having *constant pressure* on Γ_0. Theorem 1.1 says that such eigenfunction does not exist under condition (1.7). From this fact, a fluid-structure approximate controllability result can be obtained *in the presence of a corner*. In order to state the result, we consider the boundary of our domain Ω splitted into an elastic and a rigid part, that is

$$\partial\Omega = \overline{\Gamma_E} \cup \overline{\Gamma_R}, \quad \Gamma_E \cap \Gamma_R = \emptyset \tag{1.8}$$

and we introduce the Sobolev spaces

$$H = \{z \in L^2(\Omega)^2 \mid \operatorname{div} z = 0 \text{ in } \Omega,\ z\cdot n = 0 \text{ on } \Gamma_R\}$$
$$L_0^2(\Gamma_E) = \{\xi \in L^2(\Gamma_E) \mid \int_{\Gamma_E} \xi \, \mathrm{d}\sigma = 0\}.$$

With these definitions the result reads as follows:

Theorem 1.2. *Let Ω be as in Theorem 1.1. We suppose that Γ_E and Γ_R are such that $\Gamma_0 = \Gamma_E \cap B_{r_0}$, $\Gamma_1 = \Gamma_R \cap B_{r_0}$. Given $T > 0$, $u_0 \in L^2(\Omega)$, $\eta_0 \in H_0^2(\Gamma_E) \cap L_0^2(\Gamma_E)$ and $\eta_1 \in H_0^1(\Gamma_E) \cap L_0^2(\Gamma_E)$, let u, η be* *solutions of the fluid-structure system*

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = 0 \quad \textit{in } \Omega \times (0,T) \tag{1.9}$$
$$\operatorname{div} u = 0 \quad \textit{in } \Omega \times (0,T) \tag{1.10}$$
$$u = 0 \quad \textit{on } \Gamma_R \times (0,T) \tag{1.11}$$
$$u = \frac{\partial \eta}{\partial t} n \quad \textit{on } \Gamma_E \times (0,T) \tag{1.12}$$
$$u(0) = u_0 \quad \textit{in } \Omega \tag{1.13}$$
$$\frac{\partial^2 \eta}{\partial t^2} + \mathcal{B}\eta = -\sigma(u,p)n \cdot n + h \quad \textit{on } \Gamma_E \times (0,T) \tag{1.14}$$
$$\eta(t) \in H_0^2(\Gamma_E) \cap L_0^2(\Gamma_E) \quad \textit{a.e. in } (0,T) \tag{1.15}$$
$$\eta(0) = \eta_0, \ \frac{\partial \eta}{\partial t}(0) = \eta_1 \quad \textit{on } \Gamma_E, \tag{1.16}$$

where $\mathcal{B}$ is a selfadjoint differential operator which is uniformly elliptic in $H_0^2(\Gamma_E)$. Then, for any $u^T \in L^2(\Omega)$, $\eta_0^T \in H_0^2(\Gamma_E) \cap L_0^2(\Gamma_E)$ and $\eta_1^T \in H_0^1(\Gamma_E) \cap L_0^2(\Gamma_E)$ and $\varepsilon > 0$, there exists a function $h \in L^2(\Gamma_E)$ such that

$$\|u(T) - u^T\|_{0,\Omega} + \|\eta(T) - \eta_0^T\|_{2,\Omega} + \|\eta_t(T) - \eta_1^T\|_{1,\Omega} < \varepsilon.$$

Remark 1.1. In [8] we have shown the well posedness of problem (1.9)–(1.16) by a transposition method.

The presence of the corner is important to obtain the results above. In fact, the local unique continuation property given in Theorem 1.1 is not true in a ball (see [5]).

The result of Theorem 1.1 was already known for right corners ($\theta_0 = \pi/2$), firstly for rectangular domains where a direct spectral solution of (1.4)–(1.6) can be obtained (see [7]) and secondly for general domains using the fact that for $\theta_0 = \pi/2$ the solution of (1.4)–(1.5) is analytic in G (see [8]). In this paper, we consider general corners except for $\theta_0 = 3\pi/2$ (and $\theta_0 = \pi$ which does not correspond to a corner of course) by considering that w is not necessarily analytic but regular enough. We first prove in Section 2 that w is C^∞ in G for r_0 small enough using regularity results for the biharmonic operator near boundary corners. Then, in Sections 3 and 4, by using a non trivial generalization of the technique used in [8], we show that all the derivatives of w vanish at the origin, *i.e.*, w has a zero of infinite order. In Section 5 using a result of Kozlov, Kondratiev and Mazya [3], we deduce that this is only possible if w vanishes.

The same method developed in the proof of Theorem 1.1 can be used to obtain some local unique continuation properties for the Laplace operator in domains with corners, which correspond in fact to a kind of *local Schiffer's conjecture* (see for instance [1]).

More precisely, let Ω be an open subset of $\mathbb{R}^N$ with unit exterior normal n and suppose that there exists an eigenvalue λ and an eigenfunction $w \in H^1(\Omega)$ satisfying

$$-\Delta w = \lambda w \quad \text{in } \Omega \tag{1.17}$$
$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.18}$$
$$w = \text{constant} \neq 0 \quad \text{on } \Gamma_0 \subset \partial\Omega, \tag{1.19}$$

for some Γ_0 *strictly* included in $\partial\Omega$. In this case we say that the domain Ω satisfies a local Schiffer property of Neumann type. The conjecture can be stated as:

Conjecture. The only simply connected domain satisfying a local Schiffer property of Neumann type is a ball.

The analogous local Schiffer conjecture of Dirichlet type is obtained by interchanging the role of Neumann and Dirichlet boundary conditions in (1.18)–(1.19).

If $\Gamma_0 = \partial\Omega$ or $\partial\Omega$ is analytic, then condition (1.19) holds on the whole boundary and we are in the case of classical Schiffer conjectures. In this context, it is well known that the classical Schiffer conjecture of Neumann type is equivalent to say that the ball is the only simply connected domain having the Pompeiu property [9].

The general case $\Gamma_0 \neq \partial\Omega$ is different and, to our knowledge, a relationship between the local Schiffer property of Neumann type and some kind of restricted Pompeiu property is not known. But, there is another characterization that admits a local version. Indeed, in the case of simple eigenvalues, we can rewrite the local Schiffer property of Neumann type as an extremal Neumann eigenvalue problem under volume constraint as in [1]. That is, if λ_n is a simple Neumann eigenvalue of (1.17)–(1.18) associated to a normalized eigenvector w_n in $L^2(\Omega)$, then

$$\mathrm{d}\lambda(\Omega; v) = \int_{\Gamma_0} (v \cdot n) \left(\lambda_n\, w_n^2 - \left| \frac{\partial w_n}{\partial n} \right|^2 \right) \mathrm{d}\sigma$$

gives its derivative with respect to the domain for a regular deformation field v which *vanishes* in $\partial\Omega \setminus \Gamma_0$. The characterization is that (λ_n, w_n) is a solution of (1.17)–(1.19) iff there exists a constant c such that

$$\mathrm{d}\lambda_n(\Omega; v) = c \int_{\Gamma_0} (v \cdot n)\, \mathrm{d}\sigma, \qquad \text{for all deformation } v \text{ with } v|_{\partial\Omega\setminus\Gamma_0} = 0.$$

The proof of this equivalence is exactly the same as in [1], Proposition 2.2, after replacing $\partial\Omega$ by Γ_0.

Here we prove a partial answer to the previous local Schiffer conjecture of Neumann type in the case $N = 2$:

Theorem 1.3. *Let Ω be as in Theorem 1.1. If there exists a pair $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$ satisfying (1.17), (1.18) and (1.19), then w necessarily vanishes in Ω.*

The proof of this result is given in Section 6.

In the case $\Gamma_0 = \partial\Omega$ or $\partial\Omega$ analytic, Theorem 1.3 becomes a particular case of a more general result due to Williams [10], saying that a Lipschitz domain with a boundary which is not analytic everywhere does not have the Pompeiu property.

2. Local regularity at the origin

We recall the following H^{m+3}-regularity result for the biharmonic operator near a corner (see Grisvard [2], Th. 7.2.2.3 and Rem. 7.2.2.4):

Theorem 2.1 (Grisvard [2])**.** *Given a corner $G_\infty = \{(r, \theta),\ r > 0,\ 0 < \theta < \theta_0\}$, $f \in H^m(G_\infty)$, $m \geq 0$, if w is a bounded support solution of*

$$\Delta^2 w = f \quad \text{in} \quad G_\infty, \quad w \in H_0^2(G_\infty) \tag{2.1}$$

then

$$w = w_r + w_s \eta \tag{2.2}$$

where $w_r \in H^{m+3}(G_\infty)$, η is a $C^\infty(G_\infty)$ cut-off function equal to 1 near the origin and independent of m and

$$\begin{aligned} w_s = & \sum_{-(m+1) \leq \Im p_k < 0} r^{1+ip_k}\, a_k\, u_{p_k}(\theta) + \\ & \sum_{-(m+1) \leq \Im q_\ell < 0} r^{1+iq_\ell} \Big(b_\ell\, u_{q_\ell}(\theta) + c_\ell \Big(v_{q_\ell}(\theta) + i(\ln r) u_{q_\ell}(\theta) \Big) \Big), \end{aligned} \tag{2.3}$$

where a_k, b_ℓ, c_ℓ are complex constants and p_k, q_ℓ are respectively the simple and double roots τ of

$$\sinh^2(\tau\theta_0) = \tau^2 \sin^2(\theta_0) \tag{2.4}$$

with imaginary part in $[-(m+1),0)$ *and excluding the root* $-i$ *if* $\tan\theta_0 \neq \theta_0$. *The functions* u_{p_k}, u_{q_ℓ}, v_{q_ℓ} *can be chosen uniquely prescribed as solutions of the linear fourth order ordinary differential equations*

$$u_\tau^{(iv)} + 2(1-\tau^2)\,u_\tau'' + (1+\tau^2)^2\,u_\tau = 0 \quad \text{in } (0,\theta_0) \tag{2.5}$$
$$u_\tau(0) = u_\tau(\theta_0) = u_\tau'(0) = u_\tau'(\theta_0) = 0,\ \|u_\tau\|_{L^2(0,\theta_0)} = 1, \tag{2.6}$$

corresponding both to $\tau = p_k$ *and* $\tau = q_\ell$ *and*

$$v_\tau^{(iv)} + 2(1-\tau^2)\,v_\tau'' + (1+\tau^2)^2\,v_\tau = 4\tau u_\tau'' - 4\tau(1+\tau^2)\,u_\tau \quad \text{in } (0,\theta_0) \tag{2.7}$$
$$v_\tau(0) = v_\tau(\theta_0) = v_\tau'(0) = v_\tau'(\theta_0) = 0,\ (u_\tau,\overline{v}_\tau)_{L^2(0,\theta_0)} = 0, \tag{2.8}$$

corresponding only to $\tau = q_\ell$.

Remark 2.1. A classical H^{m+4} regularity result holds for a more restrictive set of angles θ_0. More precisely, if in the previous theorem, all p_k and q_l have imaginary part different from $-(m+2)$ then $w_r \in H^{m+4}(G_\infty)$ and an analogous decomposition as (2.3) holds with sums over imaginary part in the range $[-(m+2),0)$. It is important to use here the H^{m+3} regularity result to avoid unnecessary restrictions on θ_0.

First we will prove the following regularity result:

Theorem 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a Lipschitz bounded domain with a straight corner of angle* $0 < \theta_0 < 2\pi$ *at the origin, then any weak solution* $w \in H^2(\Omega)$ *of the problem* (1.4)–(1.6) *is* C^∞ *at the origin.*

In order to prove Theorem 2.2 we will need some extra auxiliary lemmas. The first one is obtained by localization in an standard way as in [2].

Lemma 2.1. *If* $w \in H^2(\Omega)$ *is solution of* (1.4)–(1.5) *then* $w \in C^\infty(G_\infty \setminus B_\delta)$, $\forall\delta > 0$.

The second one concerns some particular properties of the singular part of the solution:

Lemma 2.2. (i) *The functions* r^{1+ip_k}, r^{1+iq_ℓ}, $r^{1+iq_\ell}\ln r$ *appearing in* (2.3) *are linearly independent.*
(ii) *The singular component* w_s *given by the expansion* (2.3) *is a biharmonic function.*

Proof. (i) It is clear, by definition, that the roots p_k, q_ℓ of the characteristic equations are all different so the above functions are clearly independent. (ii) The fact that $\Delta^2 w_s = 0$ is implicitly given by construction in [2], Section 7.2. Nevertheless, we give here an explicit proof in order to be self-contained. Let $r = \mathrm{e}^t$ and $z_s(t,\theta) = \mathrm{e}^{-t} w_s(\mathrm{e}^t,\theta)$. We have

$$\Delta^2 w_s = \mathrm{e}^{-3t} P(D_t, D_\theta) z_s$$

where D_t, D_θ stand for the derivatives with respect to t and θ respectively and the differential operator P is given by

$$P(D_t, D_\theta) = (D_t^2 - 1)^2 + 2(D_t^2+1)D_\theta^2 + D_\theta^4.$$

From (2.3), it follows that z_s is a linear combination of the functions

$$\mathrm{e}^{\mathrm{i}\tau t} u_\tau \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\tau t}(v_\tau + itu_\tau),$$

where τ is a simple or double root of (2.4) with imaginary part in $[-(m+1),0)$, excluding the root $-i$ if $\tan\theta_0 \neq \theta_0$. The functions u_τ, v_τ are the unique solutions of (2.5), (2.6) and (2.7), (2.8) respectively. If

$$p(\tau, D_\theta) = P(i\tau, D_\theta) = (\tau^2+1)^2 + 2(1-\tau^2)D_\theta^2 + D_\theta^4$$

it is easy to verify that

$$P(D_t, D_\theta)\mathrm{e}^{i\tau t}u_\tau = \mathrm{e}^{i\tau t}p(\tau, D_\theta)u_\tau = 0$$
$$P(D_t, D_\theta)\mathrm{e}^{i\tau t}(v_\tau + itu_\tau) = \mathrm{e}^{i\tau t}\left(it\,p(\tau, D_\theta)u_\tau + p(\tau, D_\theta)v_\tau + \tfrac{\partial}{\partial\tau}p(\tau, D_\theta)u_\tau\right) = 0,$$

where we have used (2.5) and (2.7). We conclude that $\Delta^2 w_s = 0$. □

Lemma 2.3. *Let τ be a simple or double root of (2.4) with imaginary part in $[-(m+1), 0)$, excluding the root $-i$ if $\tan\theta_0 \neq \theta_0$. Let u_τ be the corresponding solution of (2.5) with boundary conditions (2.6). If $u_\tau'''(0) = 0$ then u_τ is zero.*

Proof. The cases $\tau = 0$, $\tau = +i$ are not allowed since the imaginary part of τ is strictly negative. So, the only cases to consider are: (i) $\tau \neq \pm i$, $\tau \neq 0$ if $\tan\theta_0 \neq \theta_0$ or (ii) $\tau = -i$ if $\tan\theta_0 = \theta_0$.

In case (i), the characteristic values of (2.5) are $\tau + i$, $\tau - i$, $-\tau - i$, $-\tau + i$, so the solution is of the form $u_\tau = A\exp((\tau + i)\theta) + B\exp((\tau - i)\theta) + C\exp((-\tau - i)\theta) + D\exp((-\tau + i)\theta)$. In case (ii), the characteristic values of (2.5) are 0 (double) and $2i$, $-2i$, so the solution is of the form $u_\tau = A + B\theta + C\exp(2i\theta) + D\exp(-2i\theta)$.

In both cases, and for the boundary conditions (2.6), the corresponding linear homogeneous system associated to the coefficients A, B, C and D has vanishing determinant, since of course τ verifies (2.4) to have subspaces of nontrivial solutions. Indeed, in case (i) the determinant is $\Delta_1 = 4(\sinh(\tau\theta_0) - \tau^2\sin^2(\theta_0))$ which vanishes due to (2.4) and in case (ii), it is $\Delta_2 = 16i\sin(\theta_0)(\theta_0\cos\theta_0 - \sin\theta_0)$ which vanishes since $\tan\theta_0 = \theta_0$.

Now, if we add the overdetermined condition $u_\tau'''(0) = 0$ and we replace the first row of the corresponding matrices associated to the previous linear homogeneous system with this condition, we obtain the non vanishing determinants $\widetilde{\Delta}_1 = 16\tau(1+\tau^2)(\sinh(2\tau\theta_0) + \tau\sin(2\theta_0))$ or using (2.4) $\widetilde{\Delta}_1 = 32\tau(1+\tau^2)\sinh(\tau\theta_0)(\cosh(\tau\theta_0) \pm \cos\theta_0)$ in the first case and $\widetilde{\Delta}_2 = -4\sin(\theta_0)$ in the second case.

In both cases we obtain the trivial solution, *i.e.* $A = B = C = D = 0$. □

The other two lemmas are related to regularity properties.

Lemma 2.4. *Let s be a non-negative real number, $\alpha \in \mathbb{C}$, $\beta \in \{0, 1\}$ and $\varphi \in C^\infty([0, \theta_0])$, and let η be a C^∞ cut-off function equal to one near the origin. Let $D = G_\infty$ (in this case we set $N = 2$) or $D = \partial G_\infty \cap \{\theta = 0\}$ (in this case we set $N = 1$). Then*

$$r^\alpha(\ln r)^\beta\varphi(\theta)\eta|_D \in H^s(D) \Leftrightarrow \left(\left(\Re\alpha > s - \frac{N}{2}\right)\ \textit{or}\ (\alpha \in \mathbb{N}\ \textit{and}\ \beta = 0)\ \textit{or}\ \varphi|_D = 0\right).$$

Proof. Let V be a neighborhood of the origin where $\eta = 1$ in such a way to consider the H^s regularity in $D \cap V$ in which case we can drop off the function η. If $s = k$ is a non-negative integer, then $D^\gamma(r^\alpha(\ln r)^\beta\varphi(\theta)|_{D\cap V})$ for $|\gamma| \leq k$, is a linear combination of functions of the form $r^{\alpha-m}(\ln r)^\beta$ and $r^{\alpha-m}$ for $0 \leq m \leq k$ when $N = 1$ or $r^{\alpha-m}(\ln r)^\beta\psi(\theta)$ and $r^{\alpha-m}\phi(\theta)$, where $0 \leq m \leq k$ and ψ and ϕ are $C^\infty([0, \theta_0])$ when $N = 2$. This occurs except for the case $\alpha \in \mathbb{N}$ and $\beta = 0$ when r^α is a polynomial in r or in the trivial case where $\varphi|_{D\cap V} = 0$. Assuming we are not in these cases, the function is in $H^s(D \cap V)$ if and only if $2(\Re\alpha - k) + N > 0$, which gives the characterization of the lemma.

For non-integer s this follows as in [2], Theorem 1.4.5.3, by an embedding method due to Babuška. Take a non-negative integer $k \leq s$ and $p \geq 2$ such that $k - \frac{N}{p} = s - \frac{N}{2}$. Since $\Re\alpha \leq k - \frac{N}{p}$ then the functions are not in $W_p^k(D \cap V)$, except if $\alpha \in \mathbb{N}$ and $\beta = 0$ or $\varphi|_{D\cap V} = 0$, and the Sobolev imbedding of $H^s(D \cap V)$ into $W_p^k(D \cap V)$ gives the result. □

Lemma 2.5 (see [4])**.** *Let Ω be a Lipschitz domain. If $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ then $\frac{\partial u}{\partial n} \in H^{-1/2}(\partial\Omega)$.*

Let us now proceed with the proof of Theorem 2.2. Let Ω, r_0, G, Γ_0, Γ_1 be given by (1.1), (1.2) and (1.3). First of all, we will localize problem (1.4)–(1.5)–(1.6) near the origin. Let $\xi = \xi(r)$ be a C^∞ cut-off function such that $\xi = 1$ for $0 < r < \frac{1}{3}r_0$ and $\xi = 0$ for $r > \frac{2}{3}r_0$. By defining $\widetilde{w} = \xi w$ if $(r,\theta) \in G$ and $\widetilde{w} = 0$ if $(r,\theta) \in G_\infty \setminus G$, it is easy to see that it satisfies

$$\widetilde{w} \in H_0^2(G_\infty) \quad \text{and} \quad \Delta^2 \widetilde{w} = -\lambda \Delta \widetilde{w} + h,$$

where

$$h = \lambda \Delta\xi w + 2\lambda \nabla\xi \cdot \nabla w + 2\nabla\xi \cdot \nabla\Delta w + 2\Delta\xi\Delta w + 2\Delta(\nabla\xi \cdot \nabla w) + 2\nabla\Delta\xi \cdot \nabla w + \Delta^2\xi\, w.$$

Notice that $h \in C^\infty(G_\infty)$ and vanishes for $r < r_0/3$ since $\nabla\xi$ vanishes in this domain and w is C^∞ out of this region (Lem. 2.1). Therefore, we can apply Theorem 2.1 with $m = 0$ to the function $\widetilde{w}$ since it is a compactly supported $H_0^2(G_\infty)$ solution of

$$\Delta^2 \widetilde{w} = -\lambda \Delta \widetilde{w} + h = f \in L^2(G_\infty).$$

Therefore, we have the decomposition $\widetilde{w} = w_r + w_s\eta$, where $w_r \in H^3(G_\infty)$, η is a $C^\infty(G_\infty)$ cut-off function equal to 1 near the origin (say, for $r < r_1 < r_0/3$) and

$$w_s = \sum_{-1 \le \Im p_k < 0} r^{1+ip_k} a_k u_{p_k}(\theta) + \sum_{-1 \le \Im q_\ell < 0} r^{1+iq_\ell} \left(b_l\, u_{q_\ell}(\theta) + c_l \left(v_{q_\ell}(\theta) + i(\ln r) u_{q_\ell}(\theta)\right)\right). \tag{2.9}$$

The idea is to prove that all the complex coefficients in the previous expansion vanish, except possibly for those a_k, b_ℓ associated to roots p_k, q_ℓ such that $1 + ip_k$ or $1 + iq_\ell$ is a non-negative integer. In fact, we will exclude the $H^3(G_\infty)$ terms associated to these roots from w_s by supposing they are already included in the regular part w_r.

Notice that $\Delta^2 w_s = 0$ (see Lem. 2.2) and therefore $\Delta^2(w_s\,\eta) = 0$ for $r < r_1$. Then $\Delta^2 w_r = \Delta^2\widetilde{w} - \Delta^2(w_s\eta) = f \in L^2(G_\infty \cap B_{r_1})$. Also $\Delta w_r \in H^1(G_\infty \cap B_{r_1})$ since $w_r \in H^3(G_\infty \cap B_{r_1})$, so by Lemma 2.5 we have

$$\frac{\partial \Delta w_r}{\partial n} \in H^{-1/2}((\overline{\Gamma}_0 \cup \overline{\Gamma}_1) \cap B_{r_1}).$$

The space $H^{-1/2}((\overline{\Gamma}_0 \cup \overline{\Gamma}_1) \cap B_{r_1})$ can not be restricted to $H^{-1/2}(\Gamma_0 \cap B_{r_1})$. We are only interested in the regularity of $\frac{\partial \Delta w_r}{\partial n}$ near the origin. In order to avoid the other part of the boundary of $\Gamma_0 \cap B_{r_1}$ we use another C^∞ cut-off function η_1 equal to one for $r < r_1/3$ and vanishing for $r > 2r_1/3$. We claim that

$$\eta_1 \frac{\partial \Delta w_r}{\partial n} \in \left(H_{00}^{1/2}(\Gamma_0 \cap B_{r_1})\right)'. \tag{2.10}$$

Indeed, the functions of $H_{00}^{1/2}(\Gamma_0 \cap B_{r_1})$ can be extended by zero continuously to $H^{1/2}((\overline{\Gamma}_0 \cup \overline{\Gamma}_1) \cap B_{r_1})$. For the definition of the trace space $H_{00}^{1/2}$ see [4]. One characterization of this space is

$$f \in H_{00}^{1/2}(\Gamma_0 \cap B_{r_1}) \Leftrightarrow f \in H^{1/2}(\Gamma_0 \cap B_{r_1}) \text{ and } r^{-1/2}(r - r_1)^{-1/2} f \in L^2(\Gamma_0 \cap B_{r_1}).$$

Using functions with behavior like $\frac{1}{\ln r}$ near $r = 0$, which belong to $H_{00}^{1/2}$, we can show that

$$r^\alpha \notin \left(H_{00}^{1/2}(\Gamma_0 \cap B_{r_1})\right)' \Leftrightarrow \Re\alpha \le -1. \tag{2.11}$$

Since $\widetilde{w} = w$ for $r < r_1$, the overdetermined boundary condition (1.6) also holds for $\widetilde{w}$ if $r < r_1$ and therefore

$$\eta_1 \frac{\partial \Delta w_s}{\partial n} = \eta_1 \frac{\partial \Delta \widetilde{w}}{\partial n} - \eta_1 \frac{\partial \Delta w_r}{\partial n} = -\eta_1 \frac{\partial \Delta w_r}{\partial n} \in \left(H_{00}^{1/2}(\Gamma_0 \cap B_{r_1})\right)'.$$

Using the formula for the Laplacian in polar coordinates we compute

$$\begin{aligned}\frac{\partial\Delta(f(r)g(\theta))}{\partial n} &= \frac{1}{r}\frac{\partial}{\partial\theta}\left(\left(\frac{1}{r}f'+f''\right)g+\frac{1}{r^2}fg''\right)\\ &= \frac{1}{r}\left(\frac{1}{r}f'+f''\right)g'+\frac{1}{r^3}fg'''\end{aligned}$$

and if $g'(0)=0$ we obtain

$$\left.\frac{\partial\Delta(f(r)g(\theta))}{\partial n}\right|_{\theta=0}=\frac{1}{r^3}fg'''(0).$$

Using this in (2.9) and recalling the boundary conditions (2.6) and (2.8) we obtain on Γ_0 for $r<r_1/3$

$$\frac{\partial\Delta w_s}{\partial n}(r)=\sum_{-1\le\Im p_k<0} r^{-2+ip_k}a_k u'''_{p_k}(0)+\sum_{-1\le\Im q_\ell<0} r^{-2+iq_\ell}\left(b_l\,u'''_{q_\ell}(0)+c_l\left(v'''_{q_\ell}(0)+i(\ln r)u'''_{q_\ell}(0)\right)\right).$$

From the previous expansion and the characterization (2.11) we verify that

$$\eta_1\frac{\partial\Delta w_s}{\partial n}\notin\left(H^{1/2}_{00}(\Gamma_0\cap B_{r_1})\right)',$$

since the real part $\Re\alpha$ of the powers of r satisfies

$$\Re\alpha\le -2+1=-1.$$

From (2.10) the only possibility is

$$\left.\eta_1\frac{\partial\Delta w_s}{\partial n}\right|_{\Gamma_0}=0.$$

We recall that the functions

$$r^{-2+ip_k},\ r^{-2+iq_\ell}, r^{-2+iq_\ell}\ln r$$

are linearly independent, therefore we get

$$\begin{aligned}a_k\,u'''_{p_k}(0) &= 0 \quad \forall p_k\\ c_l u'''_{q_\ell}(0) &= 0 \quad \forall q_\ell\\ b_l u'''_{q_\ell}(0)+c_l v'''_{q_\ell}(0) &= 0 \quad \forall q_\ell.\end{aligned}$$

From the first equation, if $a_k\neq 0$ for some k then $u'''_{p_k}(0)=0$. This is an overdetermined condition for (2.5)–(2.6), which would imply $u_{p_k}=0$. But this is a contradiction due to the normalized condition appearing in (2.6). Then $a_k=0$ for all k. Now, from the second equation, if $c_l\neq 0$ for some l then $u'''_{q_\ell}(0)=0$. This is also an overdetermined condition for (2.5)–(2.6) and would imply $u'''_{q_\ell}=0$, which gives a contradiction. Therefore $c_l=0$ for all l. Finally, from the third equation, if $b_l\neq 0$ then $u'''_{q_\ell}(0)=0$ and this gives again a contradiction. Notice that the function v_{q_ℓ}, which is solution of a non homogeneous equation (2.7)–(2.8), does not intervene in this analysis.

Consequently

$$w_s=0$$

and therefore

$$\widetilde{w}=w_r\in H^3(G\cap B_{r_1}).$$

The next steps are easier. Let us suppose now that $\widetilde{w} \in H^{m+2}(G_\infty)$ for some $m \geq 1$. Then we apply Theorem 2.1 to obtain that $\widetilde{w} = w_r + w_s\eta$ with $w_r \in H^{m+3}(G_\infty)$ and w_s of the form (2.3). With the same argument as before, it is easy to verify that

$$\frac{\partial \Delta w_s}{\partial n} = \frac{\partial \Delta \widetilde{w}}{\partial n} - \frac{\partial \Delta w_r}{\partial n} = -\frac{\partial \Delta w_r}{\partial n} \in H^{m-1/2}(\Gamma_0 \cap B_{r_1}).$$

But

$$r^\alpha (\ln r)^\beta \eta \;\notin H^{m-1/2}(\Gamma_0 \cap B_{r_1}) \quad \text{if} \quad \Re\alpha \leq m-1,\ \beta = 0 \text{ or } 1,$$

by Lemma 2.4, except if α is a non-negative integer and $\beta = 0$ in which case the functions are H^{m+3} and they are already included in w_r. In the present case, the real part of the powers of r in $\frac{\partial \Delta w_s}{\partial n}$ are given by $\Re\alpha = \Re(1 + i\tau - 3) = -2 - \Im\tau$, $\tau = p_k$ or $\tau = q_\ell$ and they satisfy

$$1 < \Re\alpha \leq m - 1$$

since $\Im\tau \in [-(m+1), 0)$. Therefore

$$\frac{\partial \Delta w_s}{\partial n} = 0 \quad \text{on } \Gamma_0 \cap B_{r_1}$$

and the same argument of linear independence as before shows that this is possible only if w_s vanishes. Then

$$\widetilde{w} = w_r \in H^{m+3}(G_\infty).$$

So we obtain that w is in $H^{m+3}(G_\infty)$ for all $m \geq 0$ and this gives the C^∞ regularity at the origin. This finishes the proof of Theorem 2.2.

3. Power series expansion

We choose Cartesian axes (x, y) centered in $r = 0$, $\theta = 0$ where the edge Γ_0 of G coincides with the horizontal axis. Since $w \in C^\infty(G \cap B_\rho)$ for some $\rho > 0$, for (x, y) in a neighborhood of $(0, 0)$, we can write for each $k \geq 0$

$$w(x, y) = \sum_{\substack{i,j \geq 0 \\ i+j \leq k+4}} a_{ij}\, x^i y^j + o(x^{k+4} + y^{k+4}). \tag{3.1}$$

First derivatives give:

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \sum_{\substack{i \geq 1, j \geq 0 \\ i+j \leq k+4}} i\, a_{ij}\, x^{i-1}\, y^j + o(x^{k+3} + y^{k+3}) \\
\frac{\partial w}{\partial y} &= \sum_{\substack{i \geq 0, j \geq 1 \\ i+j \leq k+4}} j\, a_{ij}\, x^i\, y^{j-1} + o(x^{k+3} + y^{k+3}) \\
\Delta w &= \sum_{\substack{i \geq 2, j \geq 0 \\ i+j \leq k+4}} i(i-1)\, a_{ij}\, x^{i-2}\, y^j + \sum_{\substack{i \geq 0, j \geq 2 \\ i+j \leq k+4}} j(j-1)\, a_{ij}\, x^i\, y^{j-2} + o(x^{k+2} + y^{k+2}),
\end{aligned}$$

from where

$$\frac{\partial w}{\partial x} = \sum_{\substack{i,j\geq 0\\ i+j\leq k+3}} (i+1)\, a_{i+1,j}\, x^i\, y^j + o(x^{k+3}+y^{k+3}) \tag{3.2}$$

$$\frac{\partial w}{\partial y} = \sum_{\substack{i,j\geq 1\\ i+j\leq k+3}} (j+1)\, a_{i,j+1}\, x^i\, y^j + o(x^{k+3}+y^{k+3}) \tag{3.3}$$

$$\Delta w = \sum_{\substack{i,j\geq 0\\ i+j\leq k+2}} \Big((i+2)(i+1)\, a_{i+2,\,j} + (j+2)(j+1) a_{i,j+2}\Big) x^i\, y^j + o(x^{k+2}+y^{k+2}) \tag{3.4}$$

$$\Delta^2 w = \sum_{\substack{i,j\geq 0\\ i+j\leq k}} \Big((i+4)(i+3)(i+2)(i+1)\, a_{i+4,j} + 2(i+2)(i+1)(j+2)(j+1) a_{i+2,j+2} + (j+4)(j+3)(j+2)(j+1) a_{i,j+4}\Big) x^i\, y^j + o(x^k+y^k). \tag{3.5}$$

Note that the expansions (3.1) to (3.5) are valid for all $k\geq 0$. We can then rewrite equation (1.4) by identifying coefficients in (3.4)–(3.5) as

$$(i+4)!j!\, a_{i+4,j} + 2(i+2)!(j+2)!\, a_{i+2,j+2} + i!(j+4)!\, a_{i,j+4} = -\lambda\Big((i+2)!j!\, a_{i+2,j} + i!(j+2)!\, a_{i,j+2}\Big),\ \forall i,j\geq 0. \tag{3.6}$$

Boundary conditions (1.5)–(1.6) on Γ_0 are also equivalent to

$$a_{i,0}=0, \quad a_{i,1}=0, \quad a_{i,3}=0, \quad \forall i\geq 0. \tag{3.7}$$

Now we introduce a parameter $\alpha=\tan\theta_0$ such that

$$(x,y)\in\Gamma_1 \quad \text{implies} \quad \alpha=\tan\theta_0. \tag{3.8}$$

Note that, from hypothesis (1.7) and since the case $\theta_0=\pi/2$ was specially treated in [8], we can assume that

$$\alpha\neq 0,\ \alpha\neq\pm\infty.$$

The first boundary condition (1.5) on Γ_1 becomes

$$\begin{aligned} w(x,\alpha x) &= \sum_{\substack{i,j\geq 0\\ i+j\leq k+4}} a_{ij}\, \alpha^j\, x^{i+j} + o(x^{k+4})\\ &= \sum_{\ell=0}^{k+4}\Bigg(\sum_{\substack{i,j\geq 0\\ i+j=\ell}} a_{ij}\,\alpha^j\Bigg) x^\ell + o(x^{k+4})\\ &= 0 \end{aligned}$$

and then

$$\sum_{\substack{i,j\geq 0\\ i+j=k}} a_{ij}\,\alpha^j = 0 \qquad \forall k\geq 0. \tag{3.9}$$

Since on Γ_1 the normal is $n = (\alpha, -1)/\sqrt{1+\alpha^2}$ and the tangent $\tau = (1, \alpha)/\sqrt{1+\alpha^2}$ then

$$\frac{\partial w}{\partial n} = \frac{1}{\sqrt{1+\alpha^2}} \left(\alpha \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \right) \tag{3.10}$$

$$\frac{\partial w}{\partial \tau} = \frac{1}{\sqrt{1+\alpha^2}} \left(\frac{\partial w}{\partial x} + \alpha \frac{\partial w}{\partial y} \right) = 0, \tag{3.11}$$

and the other boundary condition (1.5) on Γ_1 can be replaced by

$$\begin{aligned} \frac{\partial w}{\partial x}(x, \alpha x) &= \sum_{\substack{i,j \geq 0 \\ i+j \leq k+3}} (i+1)\, a_{i+1,j}\, \alpha^j\, x^{i+j} + o(x^{k+3}) \\ &= \sum_{\ell=0}^{k+3} \left(\sum_{\substack{i,j \geq 0 \\ i+j = \ell}} (i+1)\, a_{i+1,j}\, \alpha^j \right) x^\ell + o(x^{k+3}) \\ &= 0 \end{aligned}$$

that is

$$\sum_{\substack{i,j \geq 0 \\ i+j = k}} (i+1)\, a_{i+1,j}\, \alpha^j = 0 \qquad \forall k \geq 0. \tag{3.12}$$

Now we have replaced equations (1.4)–(1.6) by (3.6)–(3.12).

The idea is first to show that

$$a_{i,2k+1} = 0 \qquad \forall k \geq 0,\ \forall i \geq 0. \tag{3.13}$$

This is easily shown by induction since from (3.7) $a_{i,1} = a_{i,3} = 0$, $\forall i \geq 0$ and if we suppose that $a_{i,2k-3} = a_{i,2k-1} = 0$ equation (3.6) for $j = 2k-3$ gives

$$i!(2k+1)!\, a_{i,2k+1} = 0$$

from which we deduce (3.13).

The hardest technical part is to show that

$$a_{i,2k} = 0 \qquad \forall k \geq 0,\ \forall i \geq 0, \tag{3.14}$$

and this is given in the next section.

4. An infinite set of finite dimensional systems

Let us show (3.14) by induction. From (3.7) we know that $a_{i,0} = 0$, $\forall i \geq 0$ and from (3.13) we know that $a_{i,2k+1} = 0$, $\forall k \geq 0$. Using (3.6) with both $i + j = k = 2, 3$ we obtain

$$a_{0,2}\alpha^2 + a_{1,1}\alpha + a_{2,0} = 0 \quad \Rightarrow \quad a_{0,2} = 0 \tag{4.1}$$

$$a_{0,3}\alpha^3 + a_{1,2}\alpha^2 + a_{2,1}\alpha + a_{3,0} = 0 \quad \Rightarrow \quad a_{1,2} = 0. \tag{4.2}$$

Now, suppose that

$$a_{2k+2,0} = a_{2k,2} = \ldots = a_{2,2k} = a_{0,2k+2} = 0 \quad \text{(index sum } 2k+2) \tag{4.3}$$

$$a_{2k+1,0} = a_{2k-1,2} = \ldots = a_{3,2k-2} = a_{1,2k} = 0 \quad \text{(index sum } 2k+1) \tag{4.4}$$

let us prove that

$$
\begin{aligned}
&a_{2k+4,0} = a_{2k+2,2} = \ldots = a_{2,2k+2} = a_{0,2k+4} = 0 \quad \text{(index sum } 2k+4) && (4.5)\\
&a_{2k+3,0} = a_{2k+1,2} = \ldots = a_{3,2k} = a_{1,2k+2} = 0 \quad \text{(index sum } 2k+3). && (4.6)
\end{aligned}
$$

Taking $i = 2k$, $j = 0$, $i = 2k-2$, $j = 2$, …, $i = 2$, $j = 2k-2$, $i = 0$, $j = 2k$ in order to have a zero right hand side in (3.6) (index sum $2k+2$), if we also take into account (3.7) we obtain the following system of $k+1$ equations which do not depend on λ explicitly:

$$
\begin{aligned}
2(2k+2)!2!a_{2k+2,2} + (2k)!4!a_{2k,4} &= 0 \qquad (4.7)\\
(2k+2)!2!a_{2k+2,2} + 2(2k)!4!a_{2k,4} + (2k-2)!6!a_{2k-2,6} &= 0 \qquad (4.8)\\
&\vdots\\
4!2k!a_{4,2k} + 2\cdot 2!(2k+2)!a_{2,2k+2} + 0!(2k+4)!a_{0,2k+4} &= 0. \qquad (4.9)
\end{aligned}
$$

This system has $k+2$ unknowns, *i.e.* the vector:

$$A_{k+2} = (a_{2k+2,2}, a_{2k,4}, \ldots, a_{2,2k+2}, a_{0,2k+4}).$$

In fact the system is overdetermined by the additional two conditions (3.9) and (3.12). More precisely, if we take $i+j = 2k+4$ in (3.9) and $i+j = 2k+3$ in (3.12) we respectively obtain

$$
\begin{aligned}
\alpha^2 a_{2k+2,2} + \alpha^4 a_{2k,4} + \ldots + \alpha^{2k+2} a_{2,2k+2} + \alpha^{2k+4} a_{0,2k+4} &= 0 \qquad (4.10)\\
(2k+2)\alpha^2 a_{2k+2,2} + (2k)\alpha^4 a_{2k,4} + \ldots + 2\alpha^{2k+2} a_{2,2k+2} &= 0. \qquad (4.11)
\end{aligned}
$$

If we introduce the variable

$$\beta = \frac{1}{\alpha} \qquad -\infty < \beta < +\infty$$

it is clear that A_{k+2} satisfies simultaneously the systems

$$M_{k+2}A_{k+2} = 0, \qquad N_{k+2}A_{k+2} = 0, \tag{4.12}$$

where M_{k+2} and N_{k+2} are both square $k+2 \times k+2$ matrices, M_{k+2} including the coefficients of (4.7) and (4.10) and N_{k+2} including the coefficients of (4.7) and (4.11), *i.e.*

$$M_{k+2} = \begin{pmatrix}
2(2k+2)!2! & (2k)!4! & 0 & \ldots & \ldots & 0 & 0\\
(2k+2)!2! & 2(2k)!4! & (2k-2)! & 0 & \ldots & 0 & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & \ldots & 0 & 4!(2k)! & 2\cdot 2!(2k+2)! & (2k+4)!\\
\beta^{2k+2} & \beta^{2k} & \ldots & \ldots & \beta^4 & \beta^2 & 1
\end{pmatrix}$$

and

$$N_{k+2} = \begin{pmatrix}
2(2k+2)!2! & (2k)!4! & 0 & \ldots & \ldots & 0 & 0\\
(2k+2)!2! & 2(2k)!4! & (2k-2)! & 0 & \ldots & 0 & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & \ldots & 0 & 4!(2k)! & 2\cdot 2!(2k+2)! & (2k+4)!\\
(2k+2)\beta^{2k+2} & (2k)\beta^{2k} & \ldots & \ldots & 4\beta^4 & 2\beta^2 & 1
\end{pmatrix}.$$

We have to prove that

$$D_{k+2}(\beta) = \frac{\det M_{k+2}}{(2k+2)!2!(2k)!4!\dots 2!(2k+2)!0!(2k+4)!}, \tag{4.13}$$

$$E_{k+2}(\beta) = \frac{\det N_{k+2}}{(2k+2)!2!(2k)!4!\dots 2!(2k+2)!0!(2k+4)!} \tag{4.14}$$

can not vanish simultaneously. It is clear that

$$D_{k+2} = \begin{vmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & 1 & \\ & & & & 1 & 2 & 1 \\ \frac{\beta^{2k+2}}{(2k+2)!2!} & \frac{\beta^{2k}}{(2k)!4!} & \cdots & \cdots & \cdots & \frac{\beta^2}{2!(2k+2)!} & \frac{1}{0!(2k+4)!} \end{vmatrix}$$

and

$$E_{k+2} = \begin{vmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & 1 & \\ & & & & 1 & 2 & 1 \\ \frac{(2k+2)\beta^{2k+2}}{(2k+2)!2!} & \frac{(2k)\beta^{2k}}{(2k)!4!} & \cdots & \cdots & \cdots & \frac{2\beta^2}{2!(2k+2)!} & \frac{0}{0!(2k+4)!} \end{vmatrix}.$$

From the formula for the derivative of a determinant it is clear that

$$E_{k+2} = \beta\frac{\partial}{\partial\beta}D_{k+2},$$

so assuming that for some parameter β_0 we have

$$D_{k+2}(\beta_0) = 0, \quad E_{k+2}(\beta_0) = 0$$

this implies that

$$D_{k+2}(\beta_0) = 0, \quad \frac{\partial}{\partial\beta}D_{k+2}(\beta_0) = 0,$$

that is to say, β_0 is a double root of D_{k+2}.

In fact D_{k+2} can be calculated explicitly by developing the determinant (4) with respect to the last column, giving the following recursive formula

$$D_{k+2} = \frac{1}{0!(2k+4)!}\Delta_{k+1} - \beta^2 D_{k+1},$$

where

$$\Delta_{k+1} = \begin{vmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{vmatrix} = k+2.$$

From the above recurrence, it is possible to deduce that

$$\begin{aligned} D_{k+2} &= \sum_{j=0}^{k+1} \frac{(-1)^j \beta^{2j}(k+2-j)}{(2j)!(2(k-j)+4)!} \\ &= \frac{1}{2(2k+3)!} Re(1+i\beta)^{2k+3}. \end{aligned} \tag{4.15}$$

Notice that $D_{k+2}(\beta)$ is a polynomial of degree $2k+2$ in β. If

$$\omega = \arg(1+i\beta), \quad \omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

then the number of different roots of D_{k+2} corresponds to the number of different arguments in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solutions of the equation

$$(2k+3)\omega = \frac{\pi}{2} + \ell\pi, \quad \ell \in Z.$$

These different solutions are

$$\left\{ w_\ell = \frac{\pi}{2(2k+3)}(1+2\ell), \quad -(k+1) \leq \ell \leq k \right\}, \tag{4.16}$$

that is, exactly $2k+2$ different values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, all the roots of D_{k+2} are distinct and they are not double roots.

From the above analysis, the only solution of systems (4.12) is the trivial one:

$$A_{k+2} = (a_{2k+2,2}, a_{2k,4}, \ldots, a_{2,2k+2}, a_{0,2k+4}) = (0, 0, \ldots, 0, 0).$$

With an analogous technique, it is possible to show that

$$B_{k+2} = (a_{2k+1,2}, a_{2k-1,4}, \ldots, a_{3,2k+2}, a_{1,2k+4}) = (0, 0, \ldots, 0, 0).$$

This shows that all the coefficients of the Taylor expansion of w solution of (1.4)–(1.6) near the origin are zero, so that the origin is a zero of infinite order of w.

5. A zero of infinite order

We use the following result of Kozlov, Kondratiev and Mazya about the zeros of infinite order for the biharmonic operator. First we introduce the space $V_n^k(G)$ as the space of functions defined in G for which

$$r^{(-k+|\alpha|+n)} D^\alpha w \in L^2(G), \quad |\alpha| \leq k.$$

Theorem 5.1 ([3]). *Suppose $\theta_0 \neq \pi$ and $\theta_0 \neq 2\pi$ and $w \in V_0^4(G)$ is solution of the differential inequality*

$$|\Delta^2 w| \leq \frac{C}{r^2}\left(|\Delta w| + \frac{1}{r}|w|\right) \quad \text{for } r < r_0,\ 0 < \theta < \theta_0 \tag{5.1}$$

$$w(r,0) = w(r,\theta_0) = \tfrac{\partial w}{\partial \theta}(r,0) = \tfrac{\partial w}{\partial \theta}(r,\theta_0) = 0 \quad \text{for } r < r_0 \tag{5.2}$$

and suppose also that

$$w \in V_n^4(G), \quad \forall n \leq -1 \tag{5.3}$$

then $w = 0$ in $G \cap B_{r_0}$.

It is clear that if (λ, w) is solution of (1.4)–(1.5) and λ is fixed, then w satisfies (5.1)–(5.2) for r_0 small enough. From the previous sections we know that $w \in C^\infty(G \cap B_\rho)$ for some $\rho > 0$ and that all the derivatives of w vanish at the origin, which implies that (5.3) holds. We then necessarily have

$$w = 0 \quad \text{in } G \cap B_\rho,$$

for some $\rho > 0$ sufficiently small. Finally, by standard unique continuation we deduce that w vanishes in G. This ends the proof of Theorem 1.1.

6. Non standard unique continuation for the Laplace operator in a corner

This section gives the proof of Theorem 1.3 which essentially says that domains with corners do not have the local Schiffer property of Neumann type defined in the Introduction. For this aim, we prove a non standard local unique continuation for solutions of (1.17)–(1.19) in a corner G of the form (1.1) forming an angle θ_0 and verifying (1.7).

The proof follows the same steps of the proof of Theorem 1.1, but it is simpler. First, from classical results of elliptic regularity near corners (see [2,6]), it is possible to prove that the solution w of (1.17)–(1.19) is in fact C^∞ at the origin. This is done as in Section 2 after proving that the eventually singular part of the Neumann eigenfunctions vanishes thanks to the overdetermined condition (1.19). We have to use a $H^{m+2-\varepsilon_m}$, regularity result with $\varepsilon_m > 0$ chosen in order to avoid corner's angle restrictions and after a bootstrap argument to obtain the C^∞ regularity at the origin. Then we expand w in finite series near the origin of the corner for each $k \geq 0$

$$w(x,y) = \sum_{\substack{i,j \geq 0 \\ i+j \leq k+2}} a_{ij}\, x^i y^j + o(x^{k+2} + y^{k+2}). \tag{6.1}$$

As done in Section 3, and introducing the slope $\alpha = \tan\theta_0$ of

$$\Gamma_1 = (\partial\Omega \setminus \Gamma_0) \cap B_{r_0},$$

(B_{r_0} from Th. 1.3) equations (1.17)–(1.19) can be rewritten as

$$(i+2)(i+1)\, a_{i+2,j} + (j+2)(j+1)\, a_{i,j+2} = -\lambda\, a_{ij} \quad \forall i,j \geq 0 \tag{6.2}$$

$$\textstyle\sum_{\substack{i,j \geq 0 \\ i+j=k}} \alpha^j \Big(\alpha(i+1)\, a_{i+1,j} - (j+1)\, a_{i,j+1}\Big) = 0 \qquad \forall k \geq 0, \tag{6.3}$$

$$a_{i,1} = 0 \quad \forall i \geq 0 \tag{6.4}$$

$$a_{0,0} = c, \quad a_{i,0} = 0 \quad \forall i \geq 1 \tag{6.5}$$

where $c \neq 0$ is the constant appearing in condition (1.19). Condition (6.2) comes from formula (3.4) and condition (6.3) is deduced from the expression for the normal derivative on Γ_1 given by (3.10) and partial derivatives (3.2)–(3.3). It is easy to see that from the above conditions we necessarily have $c = 0$. Indeed, from (6.2), (6.4) and (6.5) it is easy to see that all coefficients a_{ij} vanishes, except eventually for

$$a_{0,0} = c \quad \text{and} \quad a_{0,2k} \quad \forall k \geq 1.$$

But from (6.2) with $i = j = 0$ and from (6.2) with $k = 2$ we respectively have

$$2a_{0,2} = -\lambda\, c \quad \text{and} \quad 2\alpha\, a_{0,2} = 0.$$

Since $\lambda \neq 0$ and $\alpha \neq 0$, we would have $c = 0$ and this would be possible only if $w = 0$.

Remark 6.1. The case of a local Dirichlet-Schiffer property is completely similar.

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