

ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS IN VARYING DOMAINS WITH BOUNDARY CONDITIONS ON VARYING SETS

CARMEN CALVO-JURADO¹, JUAN CASADO-DÍAZ² AND MANUEL LUNA-LAYNEZ²

Abstract. For a fixed bounded open set $\Omega \subset \mathbb{R}^N$, a sequence of open sets $\Omega_n \subset \Omega$ and a sequence of sets $\Gamma_n \subset \partial\Omega \cap \partial\Omega_n$, we study the asymptotic behavior of the solution of a nonlinear elliptic system posed on Ω_n , satisfying Neumann boundary conditions on Γ_n and Dirichlet boundary conditions on $\partial\Omega_n \setminus \Gamma_n$. We obtain a representation of the limit problem which is stable by homogenization and we prove that this representation depends on Ω_n and Γ_n locally.

Mathematics Subject Classification. 35B40.

Received March 15, 2007. Revised April 10, 2007 and June 26, 2007.
Published online March 6, 2008.

1. INTRODUCTION

For a given Lipschitz bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a sequence of open sets $\Omega_n \subset \Omega$ and a sequence of sets $\Gamma_n \subset \partial\Omega \cap \partial\Omega_n$, we study the asymptotic behavior of the solution u_n of the nonlinear elliptic system

$$\begin{cases} -\operatorname{div}(a(x, Du_n) - G_n) = g_n & \text{in } \Omega_n \\ u_n = 0 & \text{on } \partial\Omega_n \setminus \Gamma_n \\ (a(x, Du_n) - G_n)\nu = 0 & \text{on } \Gamma_n, \end{cases} \quad (1.1)$$

where $a : \Omega \times \mathbb{R}^{M \times N} \rightarrow \mathbb{R}^{M \times N}$, $M \geq 1$, is a Carathéodory function which satisfies standard assumptions so that the operator $v \in W_0^{1,p}(\Omega)^M \mapsto -\operatorname{div} a(x, Dv) \in W^{-1,p'}(\Omega)^M$, $p \geq 2$, defines a monotone operator in the sense of Leray and Lions [16] (see Sect. 2 for the precise assumptions on a) and ν denotes the unitary outward normal to Ω . The sequences g_n and G_n are assumed to converge in $L^{p'}(\Omega)^M$ weakly and $L^{p'}(\Omega)^{M \times N}$ strongly to some functions g and G respectively.

Assuming that $\|u_n\|_{W^{1,p}(\Omega_n)^M}$ is bounded (this holds for example if there exists $C > 0$ independent of n with $\|v\|_{W^{1,p}(\Omega_n)} \leq C \|\nabla v\|_{L^p(\Omega_n)^N}$, for every $v \in W^{1,p}(\Omega_n)$, $v = 0$ on $\partial\Omega_n \setminus \Gamma_n$) and extending u_n by zero outside Ω_n , we prove the existence of a nonnegative Borel measure μ in $\bar{\Omega}$ which does not charge sets of p -capacity zero, and a μ -Carathéodory function $F : \bar{\Omega} \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ satisfying monotonicity and continuity properties related

Keywords and phrases. Homogenization, varying domains, nonlinear problems.

¹ Dpto. de Matemáticas, Escuela Politécnica, Avenida de la Universidad s/n, 10071 Cáceres, Spain. ccalvo@unex.es

² Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Fac. de Matemáticas, C. Tarfia s/n, 41012 Sevilla, Spain. jcasadod@us.es; mllayne@us.es

to those imposed to a (see (3.20), (3.21), (3.22)), such that u_n converges weakly in $W^{1,p}(\Omega)^M$ and strongly in $W^{1,q}(\Omega)^M$, $1 \leq q < p$, to the solution u of the problem

$$\begin{cases} u \in W^{1,p}(\Omega)^M \cap L^p_{\mu}(\bar{\Omega})^M \\ \int_{\Omega} a(x, Du) : Dv \, dx + \int_{\bar{\Omega}} F(x, u)v \, d\mu = \int_{\Omega} gv \, dx + \int_{\Omega} G : Dv \, dx \\ \forall v \in W^{1,p}(\Omega)^M \cap L^p_{\mu}(\bar{\Omega})^M, \end{cases} \quad (1.2)$$

which (if μ is smooth) can be written as

$$\begin{cases} -\operatorname{div}(a(x, Du) - G) + F(x, u)\mu = g & \text{in } \Omega \\ (a(x, Du) - G)\nu + F(x, u)\mu = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

The pair (F, μ) does not depend on g_n or G_n , and it depends on Ω_n and Γ_n locally in the sense that if we consider a Lipschitz open set $\omega \subset \Omega$ and we replace in (1.1) Ω_n by $\Omega_n \cap \omega$, Γ_n by $\Gamma_n \cap \bar{\omega}$, then the previous result holds with (F, μ) replaced by $(F|_{\bar{\omega}}, \mu|_{\bar{\omega}})$.

The term $F(x, u)\mu$ in (1.2) is similar to the *strange term* which appears in the homogenization of Dirichlet problems on varying domains (see [1, 3–12, 19, 20]). In fact, if Γ_n is empty, our result follows from [5]. When Γ_n is not empty the main difference is that now μ is defined on $\bar{\Omega}$ and not only on Ω and then the term $F(x, u)\mu$ does not only appears in the equation but also in the boundary conditions of (1.3). Taking $\Omega = \Omega_n$ for every $n \in \mathbb{N}$, the above result proves that the boundary condition corresponding to the limit of a sequence of nonlinear elliptic systems with Dirichlet and Neumann conditions on varying subsets of $\partial\Omega$ is a Fourier-Robin condition. Indeed, the proof of this fact was the origin of the present work. We have preferred to present here the more general case where the open sets Ω_n are variable, in order to show that the homogenization of elliptic Dirichlet problems in varying domains (corresponding to $\Gamma_n = \emptyset$) and the homogenization of elliptic problems with Neumann and Dirichlet conditions imposed on varying sets of the boundary admit a common formulation.

As in the case of Dirichlet problems on varying domains [9], we observe that (1.1) can be written in such way that its structure is similar to (1.2). For this purpose, it is enough to define μ_n as (C_p stands for the p -capacity, see Sect. 2)

$$\mu_n(B) = \begin{cases} +\infty & \text{if } C_p(B \cap (\bar{\Omega} \setminus (\Omega_n \cup \Gamma_n))) > 0 \\ 0 & \text{if } C_p(B \cap (\bar{\Omega} \setminus (\Omega_n \cup \Gamma_n))) = 0, \end{cases} \quad \forall B \subset \bar{\Omega} \text{ Borel}, \quad (1.4)$$

and $F_n : \bar{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ as, for example, $F_n(x, s) = |s|^{p-2}s$. Then (1.1) is equivalent to

$$\begin{cases} u_n \in W^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\bar{\Omega})^M \\ \int_{\Omega} a(x, Du_n) : Dv \, dx + \int_{\bar{\Omega}} F_n(x, u_n)v \, d\mu_n = \int_{\Omega} g_nv \, dx + \int_{\Omega} G_n : Dv \, dx \\ \forall v \in W^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\bar{\Omega})^M. \end{cases} \quad (1.5)$$

Hence, we can consider (1.1) as a particular case of (1.2). For this reason, better than the homogenization of (1.1), we will study the homogenization of (1.5) for a sequence μ_n of Borel measures in $\bar{\Omega}$ (not necessarily defined from sequences Ω_n, Γ_n as above) which vanish on sets of p -capacity zero and a sequence $F_n : \bar{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ of monotone μ_n -measurable functions (see Sect. 2 for the precise hypotheses on F_n). We prove that, in this more general form, the problem is stable for homogenization, *i.e.* for every sequences μ_n and F_n there exist μ and F such that, at least for a subsequence, the limit problem of (1.5) is still given by (1.2).

Throughout the paper we just consider the case $p \geq 2$. The case $1 \leq p < 2$ can be treated in a similar way, after proper modification on the growth and coerciveness hypotheses for the functions a and F_n . The case of linear equations and μ_n concentrated on $\partial\Omega$ (which for problem (1.1) means $\Omega_n = \Omega$ for every $n \in \mathbb{N}$) has been studied in [2], see also [13] for related problems.

2. NOTATIONS AND DEFINITIONS

The minimum and the maximum of two numbers a, b are respectively denoted by $a \wedge b, a \vee b$.

The scalar product of two matrices $A, B \in \mathbb{R}^{M \times N}$ will be denoted by $A : B$.

For a Borel set $B \subset \mathbb{R}^N$ and a Borel measure μ in B , we denote by $L_\mu^q(B)$, $1 \leq q \leq +\infty$, the usual Lebesgue spaces with respect to the measure μ . If μ is the Lebesgue measure, we use the standard notation $L^q(B)$.

For every Lipschitz open set $O \subset \mathbb{R}^N$, we denote by $W^{1,q}(O)$, $1 \leq q \leq +\infty$, the usual Sobolev spaces. We recall that, since we are assuming O Lipschitz, the elements of $W^{1,q}(O)$ have a trace on ∂O and then, they are defined in \overline{O} . Moreover, $C^\infty(\overline{O})$ is dense in $W^{1,q}(O)$ if $q < +\infty$. For every subset Υ of ∂O , we define $W_\Upsilon^{1,q}(O)$ as the closure in $W^{1,q}(O)$ of the functions in $C^\infty(\overline{O})$ which vanish in a neighborhood of $\overline{\Upsilon}$. In the case $\Upsilon = \partial O$, we write $W_0^{1,q}(O)$ instead of $W_\Upsilon^{1,q}(O)$.

Along the paper we denote by p a fixed number such that $p \geq 2$. Also we consider a bounded Lipschitz open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and a bounded open set $\hat{\Omega}$, such that $\overline{\Omega} \subset \hat{\Omega}$.

We denote by $P : W^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\hat{\Omega})$ a bounded linear operator such that

$$P(u) = u \text{ in } \Omega, \quad \forall u \in W^{1,p}(\Omega). \quad (2.6)$$

This operator is also chosen bounded from $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ into $W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$ and such that it transforms nonnegative functions into nonnegative functions. The existence of this extension operator is guaranteed because Ω is Lipschitz (see *e.g.* [17]).

For a Lipschitz open set $\omega \subset \Omega$, we denote

$$\mathcal{S}_\omega = \{\varphi : \varphi \in W^{1,\infty}(\omega), \quad \varphi = 0 \text{ in a neighborhood of } \overline{\partial\omega \cap \Omega}\}.$$

Also, we define the bounded linear operators $Z_\omega : W_{\partial\omega \cap \Omega}^{1,p}(\omega) \rightarrow W^{1,p}(\Omega)$, $Q_\omega : W_{\partial\omega \cap \Omega}^{1,p}(\omega) \rightarrow W_0^{1,p}(\hat{\Omega})$ as

$$Z_\omega(u) = \begin{cases} u & \text{in } \overline{\omega} \\ 0 & \text{in } \overline{\Omega} \setminus \overline{\omega}, \end{cases} \quad Q_\omega = P \circ Z_\omega.$$

When $u = (u_1, \dots, u_M)$ is vectorial, we denote

$$\begin{aligned} P(u) &= (P(u_1), \dots, P(u_M)), & Z_\omega(u) &= (Z_\omega(u_1), \dots, Z_\omega(u_M)), \\ Q_\omega(u) &= (Q_\omega(u_1), \dots, Q_\omega(u_M)). \end{aligned}$$

For $E \subset \hat{\Omega}$ and $1 < p < +\infty$, the p -capacity of E in $\hat{\Omega}$, denoted by $C_p(E)$, is defined by

$$C_p(E) = \inf \left\{ \int_{\hat{\Omega}} |\nabla u|^p dx : u \in W_0^{1,p}(\hat{\Omega}), u \geq 1 \text{ a.e. in a neighborhood of } E \right\}.$$

This definition depends on $\hat{\Omega}$, however the sets of p -capacity zero are independent of $\hat{\Omega}$.

We say that a property $\mathcal{P}(x)$ holds quasi everywhere (abbreviated as q.e.) in a set $B \subset \hat{\Omega}$ if it holds for all $x \in B \setminus N$, with $C_p(N) = 0$.

A function $u : \hat{\Omega} \longrightarrow \mathbb{R}$ is said to be quasi continuous if for every $\varepsilon > 0$ there exists a set $B \subset \hat{\Omega}$, with $C_p(B) < \varepsilon$, such that the restriction of u to $\hat{\Omega} \setminus B$ is continuous. It is well known (see *e.g.* [14,15,21]) that every $u \in W^{1,p}(\hat{\Omega})$ has a quasi continuous representative. We shall always identify $u \in W^{1,p}(\hat{\Omega})$ with this quasi continuous representative.

A subset O of $\hat{\Omega}$ is said to be quasi open if for every $\varepsilon > 0$ there exists $B \subset \hat{\Omega}$, with $C_p(B) < \varepsilon$, such that $O \cup B$ is open.

Following [8,9], for every Borel subset B of $\hat{\Omega}$, we denote by $\mathcal{M}_0^p(B)$ the class of all non negative Borel measures μ in B which vanish on Borel sets of p -capacity zero and satisfy the following condition

$$\mu(E) = \inf\{\mu(O \cap B) : O \text{ quasi open, } E \subset O \subset \hat{\Omega}\}, \quad \forall E \subset B \text{ Borel.} \quad (2.7)$$

We will denote by $a : \Omega \times \mathbb{R}^{M \times N} \longrightarrow \mathbb{R}^{M \times N}$ a Carathéodory function such that there exist two positive constants α, γ , and $r \in L^{\frac{p}{p-2}}(\Omega)$ satisfying

$$a(x, 0) = 0 \quad \text{a.e. } x \in \Omega, \quad (2.8)$$

$$(a(x, \xi_1) - a(x, \xi_2)) : (\xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^{M \times N}, \quad \text{a.e. } x \in \Omega, \quad (2.9)$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leq (r(x) + \gamma(|\xi_1| + |\xi_2|)^{p-2}) |\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^{M \times N}, \quad \text{a.e. } x \in \Omega. \quad (2.10)$$

Observe that these hypotheses imply in particular that there exist $\beta > 0$ and $h \in L^{p'}(\Omega)$ such that

$$a(x, \xi) : \xi \geq \alpha |\xi|^p, \quad \forall \xi \in \mathbb{R}^{M \times N}, \quad \text{a.e. } x \in \Omega, \quad (2.11)$$

$$|a(x, \xi)| \leq h(x) + \beta |\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}^{M \times N}, \quad \text{a.e. } x \in \Omega. \quad (2.12)$$

For every $n \in \mathbb{N}$, we will also consider $\mu_n \in \mathcal{M}_0^p(\overline{\Omega})$ and $F_n : \overline{\Omega} \times \mathbb{R}^M \longrightarrow \mathbb{R}^M$ such that

$$F_n(\cdot, s) \text{ } \mu_n\text{-measurable, } \forall s \in \mathbb{R}^M, \quad (2.13)$$

$$F_n(x, 0) = 0, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \quad (2.14)$$

$$(F_n(x, s_1) - F_n(x, s_2))(s_1 - s_2) \geq \alpha |s_1 - s_2|^p, \quad \forall s_1, s_2 \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \quad (2.15)$$

$$|F_n(x, s_1) - F_n(x, s_2)| \leq \gamma(|s_1| + |s_2|)^{p-2} |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}. \quad (2.16)$$

Thus, for $\beta > 0$ as above, we have

$$F_n(x, s)s \geq \alpha |s|^p, \quad \forall s \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}, \quad (2.17)$$

$$|F_n(x, s)| \leq \beta |s|^{p-1}, \quad \forall s \in \mathbb{R}^M, \quad \mu_n\text{-a.e. } x \in \overline{\Omega}. \quad (2.18)$$

Remark 2.1. To simplify the exposition we have considered $p \geq 2$. The case $1 < p < 2$ can be studied similarly after proper modification on the hypotheses on a and F_n . We can also consider hypotheses less restrictive than (2.10) and (2.16), assuming that $a(x, \xi)$ and $F_n(x, s)$ are locally Hölder continuous with respect to ξ and s respectively.

We denote by C a generic constant which does not depend on n and can change from line to line.

We denote by $O_{m,n}$ and O_n generic sequences of real numbers which can change from line to line and satisfy

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} |O_{m,n}| = 0, \quad \lim_{n \rightarrow \infty} O_n = 0.$$

3. HOMOGENIZATION RESULT

In this section we state the main results of the paper, relative to the homogenization problem (1.5).

Theorem 3.1. *Let a , μ_n and F_n be in the conditions of Section 2. Then, there exist a subsequence of n , still denoted by n , a measure $\mu \in \mathcal{M}_0^p(\overline{\Omega})$ and a function $F : \overline{\Omega} \times \mathbb{R}^M \rightarrow \mathbb{R}^M$, with*

$$F(\cdot, s) \text{ } \mu\text{-measurable, } \forall s \in \mathbb{R}^M, \quad (3.19)$$

$$F(x, 0) = 0, \quad \mu\text{-a.e. } x \in \overline{\Omega}, \quad (3.20)$$

$$|F(x, s_2) - F(x, s_1)| \leq C_1(|s_1| + |s_2|)^{\frac{p(p-2)}{p-1}} |s_2 - s_1|^{\frac{1}{p-1}}, \quad \forall s_1, s_2 \in \mathbb{R}^M, \quad \mu\text{-a.e. } x \in \overline{\Omega}, \quad (3.21)$$

$$(F(x, s_2) - F(x, s_1))(s_2 - s_1) \geq C_2|s_2 - s_1|^p, \quad \forall s_1, s_2 \in \mathbb{R}^M, \quad \mu\text{-a.e. } x \in \overline{\Omega}, \quad (3.22)$$

such that the following homogenization result holds: Let $\omega \subset \Omega$ be a Lipschitz open set and consider a sequence $g_n \in L^{p'}(\omega)^M$ which converges weakly in $L^{p'}(\omega)^M$ to a function g , a sequence $G_n \in L^{p'}(\omega)^{M \times N}$ which converges strongly in $L^{p'}(\omega)^{M \times N}$ to a function G and a sequence $u_n \in W^{1,p}(\omega)^M \cap L_{\mu_n}^p(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})$ which satisfies

$$\|u_n\|_{W^{1,p}(\omega)^M \cap L_{\mu_n}^p(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})} \leq C, \quad (3.23)$$

and

$$\begin{cases} \int_{\omega} a(x, Du_n) : Dv \, dx + \int_{\overline{\omega}} F_n(x, u_n)v \, d\mu_n = \int_{\omega} g_n v \, dx + \int_{\omega} G_n : Dv \, dx \\ \forall v \in W_{\partial\omega \cap \Omega}^{1,p}(\omega)^M \cap L_{\mu_n}^p(\overline{\omega})^M. \end{cases} \quad (3.24)$$

Then, every cluster point u of u_n in the weak topology of $W^{1,p}(\omega)^M$ satisfies

$$\begin{cases} u \in W^{1,p}(\omega)^M \cap L_{\mu}^p(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M \\ \int_{\omega} a(x, Du) : Dv \, dx + \int_{\overline{\omega}} F(x, u)v \, d\mu = \int_{\omega} g v \, dx + \int_{\omega} G : Dv \, dx \\ \forall v \in W_{\partial\omega \cap \Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\overline{\omega})^M. \end{cases} \quad (3.25)$$

Moreover, the measure μ can be taken independently of a .

The proof of this result is carried on in Section 6. To do it, in Section 5 we consider a bounded open set $\hat{\Omega}$ with $\overline{\Omega} \subset \hat{\Omega}$ and then, for u_n and u as in the statement of Theorem 3.1, we estimate the difference between u_n and the corrector with limit u relative to the homogenization problem for the operator $v \rightarrow -\operatorname{div}(|\nabla v|^{p-2}\nabla v) + |v|^{p-2}v \, d\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions. The properties of this corrector will be recalled in Section 4. As a consequence we obtain some estimates for Du_n (Lems. 5.2 and 5.3) which allow us to prove (Prop. 5.4) the existence of $\mu \in \mathcal{M}_0^p(\overline{\Omega})$ and $T \in L_{\mu}^p(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ such that u belongs to $L_{\mu}^p(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ and

$$\int_{\omega} a(x, Du) : Dv \, dx + \int_{\overline{\omega}} T v \, d\mu = \int_{\omega} g v \, dx + \int_{\omega} G : Dv \, dx,$$

for every $v \in W_{\partial\omega \cap \Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\overline{\omega})^M$ (see [8,10]). The estimates obtained in Section 5 prove that T is of the form $F(x, u(x))$ (estimate (5.60)), but for a function F only defined on the set of pairs $(x, v(x))$ such that v is the limit of some sequence v_n in the conditions of the sequence u_n which appears in the statement of Theorem 3.1. We will prove in Lemma 6.1 that the set of such functions v is large enough to allow us to define F in the whole of $\overline{\Omega} \times \mathbb{R}^M$ and then to conclude Theorem 3.1.

We will also prove in Section 6 the following consequence of Theorem 3.1.

Theorem 3.2. *Under the same assumptions that Theorem 3.1, the following results hold:*

(i) *For every $\lambda > 0$, the unique solution u_n of*

$$\begin{cases} u_n \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu_n}^p(\bar{\omega})^M \\ \int_{\omega} a(x, Du_n) : Dv \, dx + \lambda \int_{\omega} |u_n|^{p-2} u_n v \, dx + \int_{\bar{\omega}} F_n(x, u_n) v \, d\mu_n = \\ \int_{\omega} g_n v \, dx + \int_{\omega} G_n : Dv \, dx \\ \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu_n}^p(\bar{\omega})^M, \end{cases} \quad (3.26)$$

converges weakly in $W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M$ and strongly in $W_{\partial\omega\cap\Omega}^{1,q}(\omega)^M$, $1 \leq q < p$, to the unique solution u of

$$\begin{cases} u \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\bar{\omega})^M \\ \int_{\omega} a(x, Du) : Dv \, dx + \lambda \int_{\omega} |u|^{p-2} uv \, dx + \int_{\bar{\omega}} F(x, u) v \, d\mu = \int_{\omega} gv \, dx + \int_{\omega} G : Dv \, dx \\ \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\bar{\omega})^M. \end{cases} \quad (3.27)$$

(ii) *Assume that there exists (a Poincaré's constant) $C_P > 0$ such that*

$$\|v\|_{L^p(\omega)} \leq C_P \left(\|\nabla v\|_{L^p(\omega)^N}^p + \|v\|_{L_{\mu_n}^p(\bar{\omega})}^p \right)^{\frac{1}{p}}, \quad \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega) \cap L_{\mu_n}^p(\bar{\omega}), \quad \forall n \in \mathbb{N}. \quad (3.28)$$

Then, the unique solution u_n of

$$\begin{cases} u_n \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu_n}^p(\bar{\omega})^M \\ \int_{\omega} a(x, Du_n) : Dv \, dx + \int_{\bar{\omega}} F_n(x, u_n) v \, d\mu_n = \int_{\omega} g_n v \, dx + \int_{\omega} G_n : Dv \, dx \\ \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu_n}^p(\bar{\omega})^M, \end{cases} \quad (3.29)$$

converges weakly in $W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M$ and strongly in $W_{\partial\omega\cap\Omega}^{1,q}(\omega)^M$, $1 \leq q < p$, to the unique solution u of

$$\begin{cases} u \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\bar{\omega})^M \\ \int_{\omega} a(x, Du) : Dv \, dx + \int_{\bar{\omega}} F(x, u) v \, d\mu = \int_{\omega} gv \, dx + \int_{\omega} G : Dv \, dx \\ \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega)^M \cap L_{\mu}^p(\bar{\omega})^M. \end{cases} \quad (3.30)$$

Remark 3.3. As we said in the Introduction, the homogenization of problem (3.29) gives in particular the homogenization of problem (1.1). For this, given a sequence of Lipschitz open sets $\Omega_n \subset \Omega$ and a sequence $\Gamma_n \subset \partial\Omega \cap \Omega_n$, we define a sequence of measures $\mu_n \in \mathcal{M}_0^p(\bar{\Omega})$ by (1.4). Then, problem (1.1), understood in the variational form

$$\begin{cases} u_n \in W^{1,p}(\Omega_n)^M, & u_n = 0 \text{ q.e. on } \partial\Omega_n \setminus \Gamma_n \\ \int_{\Omega_n} a(x, Du_n) : Dv \, dx = \int_{\Omega_n} g_n v \, dx + \int_{\Omega_n} G_n : Dv \, dx \\ \forall v \in W^{1,p}(\Omega_n)^M, & v = 0 \text{ q.e. on } \partial\Omega_n \setminus \Gamma_n, \end{cases}$$

is equivalent to problem (3.29) with $\omega = \Omega$ and (for example) $F_n(x, s) = |s|^{p-2}s$. An interesting particular case is when $\Omega_n = \Omega$ for every $n \in \mathbb{N}$, *i.e.* when we have a nonlinear homogenization problem in a fixed bounded open set $\Omega \subset \mathbb{R}^N$, where we impose Dirichlet and Neumann conditions in varying subsets of the boundary.

In this case, by Proposition 4.4 the measure μ is supported on $\partial\Omega$, and thus, if μ is sufficiently smooth, the limit problem (3.30) is equivalent to the Fourier-Robin problem

$$\begin{cases} -\operatorname{div}(a(x, Du) - G) = g & \text{in } \Omega \\ (a(x, Du) - G)\nu + F(x, u)\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 3.4. Similarly to the homogenization of nonlinear Dirichlet problems in varying domains (see e.g. [5]), some properties on a and F_n are inherited by F . Namely, we have:

(i) If $a(x, \xi)$ is linear with respect to ξ and $F_n(x, s)$ is linear with respect to s , for every $n \in \mathbb{N}$ (so, $p = 2$), then, $F(x, s)$ is linear with respect to s .

(ii) If a and F_n , $n \in \mathbb{N}$, satisfy the homogeneity assumption

$$a(x, \lambda\xi) = |\lambda|^{p-2}\lambda a(x, \xi), \quad \forall \lambda \in \mathbb{R}, \forall \xi \in \mathbb{R}^{M \times N}, \text{ a.e. } x \in \Omega,$$

$$F_n(x, \lambda s) = |\lambda|^{p-2}\lambda F_n(x, s), \quad \forall \lambda \in \mathbb{R}, \forall s \in \mathbb{R}^M, \mu_n\text{-a.e. } x \in \overline{\Omega},$$

then F also satisfies

$$F(x, \lambda s) = |\lambda|^{p-2}\lambda F(x, s), \quad \forall \lambda \in \mathbb{R}, \forall s \in \mathbb{R}^M, \mu\text{-a.e. } x \in \overline{\Omega}.$$

The proof of these results is analogous to the corresponding one of Theorems 8.1 and 8.5 in [5] and follows from the fact that the functions q_n^m of Lemma 6.1 satisfy

$$(\lambda q_1 + \tau q_2)_n^m = \lambda(q_1)_n^m + \tau(q_2)_n^m, \quad \forall q_1, q_2 \in \mathbb{Q}^M, \forall \lambda, \tau \in \mathbb{R},$$

if we assume (i), and

$$(\lambda q)_n^m = \lambda(q)_n^m, \quad \forall q \in \mathbb{Q}^M, \forall \lambda \in \mathbb{R},$$

if we assume (ii). Thus the functions T_q defined by Lemma 6.1 satisfy

$$T_{\lambda q_1 + \tau q_2} = \lambda T_{q_1} + \tau T_{q_2}, \quad \forall q_1, q_2 \in \mathbb{Q}^M, \forall \lambda, \tau \in \mathbb{R},$$

if we assume (i), and

$$T_{\lambda q} = |\lambda|^{p-2}\lambda T_q, \quad \forall q \in \mathbb{Q}^M, \forall \lambda \in \mathbb{R},$$

if we assume (ii).

4. PRELIMINARIES

In this section we recall some results related to the homogenization of the p -Laplace operator with Dirichlet boundary conditions in varying domains. From them we will obtain other results we will use later.

Throughout this section, we consider a sequence $\hat{\mu}_n \in \mathcal{M}_0^p(\hat{\Omega})$ and we denote by w_n the solution of the problem

$$\begin{cases} w_n \in W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}_n}^p(\hat{\Omega}) \\ \int_{\hat{\Omega}} |\nabla w_n|^{p-2} \nabla w_n \nabla v \, dx + \int_{\hat{\Omega}} |w_n|^{p-2} w_n v \, d\hat{\mu}_n = \int_{\hat{\Omega}} v \, dx \\ \forall v \in W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}_n}^p(\hat{\Omega}). \end{cases} \quad (4.31)$$

The following result has been proved in [8,10].

Proposition 4.1. *Let w_n be the sequence defined by (4.31). Then w_n is nonnegative q.e. in $\hat{\Omega}$ and its norm in $W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega}) \cap L_{\hat{\mu}_n}^p(\hat{\Omega})$ is bounded. Up to a subsequence, there exists a nonnegative function $w \in W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$, such that w_n converges weakly to w in $W_0^{1,p}(\hat{\Omega})$, strongly in $W_0^{1,q}(\hat{\Omega})$, $1 \leq q < p$, and weakly-* in $L^\infty(\hat{\Omega})$. Moreover, there exists a unique measure $\hat{\mu} \in \mathcal{M}_0^p(\hat{\Omega})$ such that w is the solution of*

$$\begin{cases} w \in W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}}^p(\hat{\Omega}) \\ \int_{\hat{\Omega}} |\nabla w|^{p-2} \nabla w \nabla v \, dx + \int_{\hat{\Omega}} |w|^{p-2} w v \, d\hat{\mu} = \int_{\hat{\Omega}} v \, dx \\ \forall v \in W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}}^p(\hat{\Omega}). \end{cases} \quad (4.32)$$

The interest of w_n is that for a function ψ smooth enough, the sequence $w_n \psi$ provides a corrector with limit $w \psi$ relative to the homogenization problem for the operator $v \rightarrow -\operatorname{div}(|\nabla v|^{p-2} \nabla v) + |v|^{p-2} v \, d\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions [8,10]. The following properties of w_n , w and $\hat{\mu}$ have been proved in [8,10] (see also [5]).

Proposition 4.2. *The sequence of solutions w_n of (4.31), the function w and the measure $\hat{\mu}$ given by Proposition 4.1 satisfy*

- (a) *For every Borel set $B \subset \hat{\Omega}$ with $C_p(B \cap \{w = 0\}) > 0$, it holds $\hat{\mu}(B) = +\infty$.*
- (b) *The set $\{w\varphi : \varphi \in C_c^\infty(\hat{\Omega})\}$ is dense in $W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}}^p(\hat{\Omega})$.*
- (c) *For every $\psi, \varphi \in W^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$, we have*

$$\lim_{n \rightarrow \infty} \left(\int_{\hat{\Omega}} |\nabla((w_n - w)\psi)|^p \varphi \, dx + \int_{\hat{\Omega}} |w_n \psi|^p \varphi \, d\hat{\mu}_n \right) = \int_{\hat{\Omega}} |w \psi|^p \varphi \, d\hat{\mu}. \quad (4.33)$$

- (d) *If v_n is a sequence in $W^{1,p}(\hat{\Omega})$ which converges weakly in $W^{1,p}(\hat{\Omega})$ to a function v , then it holds*

$$\liminf_{n \rightarrow \infty} \left(\int_{\hat{\Omega}} |\nabla(v_n - v)|^p \, dx + \int_{\hat{\Omega}} |v_n|^p \, d\hat{\mu}_n \right) \geq \int_{\hat{\Omega}} |v|^p \, d\hat{\mu}. \quad (4.34)$$

In particular, if $\|v_n\|_{L_{\hat{\mu}_n}^p(\hat{\Omega})}$ is bounded, the function v is in $L_{\hat{\mu}}^p(\hat{\Omega})$.

Remark 4.3. We will apply the previous results to the sequence $\hat{\mu}_n$ defined by (see Sect. 2)

$$\hat{\mu}_n(B) = \mu_n(B \cap \bar{\Omega}), \quad \forall B \subset \hat{\Omega} \text{ Borel}. \quad (4.35)$$

From Proposition 4.2, every function $v \in L_{\hat{\mu}_n}^p(\hat{\Omega})$ vanishes q.e. on $\{w_n = 0\}$. So, although in Section 2 we have considered F_n defined on $\hat{\Omega} \times \mathbb{R}^M$, only its values in $\{w_n > 0\} \times \mathbb{R}^M$ are relevant.

In the present paper, we are interested in a sequence of measures $\hat{\mu}_n$ having their supports contained in a fixed closed set (see (4.35)). We will use the following result.

Proposition 4.4. *If there exists a compact set $K \subset \hat{\Omega}$ such that $\operatorname{supp}(\hat{\mu}_n) \subset K$, for every $n \in \mathbb{N}$, then the measure $\hat{\mu}$ given by Proposition 4.1 also satisfies $\operatorname{supp}(\hat{\mu}) \subset K$.*

Proof. Since $-\operatorname{div}(|\nabla w_n|^{p-2} \nabla w_n) = 1$ in $\hat{\Omega} \setminus K$, and w_n converges weakly to w in $W_0^{1,p}(\hat{\Omega})$ and strongly in $W_0^{1,q}(\hat{\Omega})$, $1 \leq q < p$, we deduce that

$$-\operatorname{div}(|\nabla w|^{p-2} \nabla w) = 1 \quad \text{in } \hat{\Omega} \setminus K. \quad (4.36)$$

Then, taking in (4.32) $v = w\varphi$, with $\varphi \in C_c^\infty(\hat{\Omega} \setminus K)$, we get

$$\int_{\hat{\Omega}} |w|^p \varphi \, d\hat{\mu} = 0, \quad \forall \varphi \in C_c^\infty(\hat{\Omega} \setminus K). \quad (4.37)$$

On the other hand, from $w \geq 0$ q.e. in $\hat{\Omega}$, (4.36) and the strong maximum principle for the p -Laplace operator (see *e.g.* [18]), we deduce $w > 0$ in $\hat{\Omega} \setminus K$. Together with (4.37), this implies that the support of $\hat{\mu}$ is contained in K . \square

Better than Proposition 4.2 (b) and (d), we will use the following results.

Proposition 4.5. *Assume that the support of the measure $\hat{\mu}$ given by Proposition 4.1 is contained in $\overline{\Omega}$ and let ω be a Lipschitz open subset of Ω . Then, the set*

$$D_\omega = \{w\varphi : \varphi \in \mathcal{S}_\omega \cap C^\infty(\overline{\omega})\}, \quad (4.38)$$

is dense in $W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})$.

Proof. First of all, we remark that for every u in $W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})$, there exist a sequence $u_n \in W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})$ and $O_n \subset \hat{\Omega}$ open, with $\overline{\partial\omega \cap \Omega} \subset O_n$, such that $u_n = 0$ q.e. in $O_n \cap \overline{\omega}$ and u_n converges to u in $W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})$. For this purpose, we use that by definition of $W_{\partial\omega \cap \Omega}^{1,p}(\omega)$, there exists a sequence $O_n \subset \hat{\Omega}$ open, with $\overline{\partial\omega \cap \Omega} \subset O_n$ and $\psi_n \in C^\infty(\overline{\omega})$, with $\psi_n = 0$ in $O_n \cap \overline{\omega}$, such that ψ_n converges to u in $W^{1,p}(\omega)$. Then we take

$$u_n = (\psi_n \wedge u^+)^+ - ((-\psi_n) \wedge u^-)^+.$$

In order to prove Proposition 4.5, it is then enough to show that for every $u \in W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})$ such that there exists an open set $O \subset \hat{\Omega}$ with $\overline{\partial\omega \cap \Omega} \subset O$, $u = 0$ q.e. in $O \cap \overline{\omega}$, and for every $\varepsilon > 0$, there exists $\varphi \in C_c^\infty(\hat{\Omega})$ which vanishes in a neighborhood of $\overline{\partial\omega \cap \Omega}$ such that

$$\|u - w\varphi\|_{W^{1,p}(\omega) \cap L_{\hat{\mu}}^p(\overline{\omega})} < \varepsilon. \quad (4.39)$$

Using a regularization by convolution, it is enough to prove (4.39) for φ in $W_0^{1,p}(\hat{\Omega}) \cap L^\infty(\hat{\Omega})$ which vanishes on a neighborhood of $\overline{\partial\omega \cap \Omega}$.

Given u , O and ε as above, we observe that $Z_\omega(u)$ is in $W^{1,p}(\Omega) \cap L_{\hat{\mu}}^p(\overline{\Omega})$ and, since $\text{supp}(\hat{\mu}) \subset \overline{\Omega}$, we have that $Q_\omega(u) \in W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}}^p(\hat{\Omega})$ (P , Z_ω and Q_ω are defined in Sect. 2). Thus, taking an open set O' with $\overline{\partial\omega \cap \Omega} \subset O'$, $\overline{O'} \subset O$ and a function $\psi \in C^\infty(\hat{\Omega})$, with $\psi = 1$ in $\hat{\Omega} \setminus O$, $\psi = 0$ in O' , the function $u^* = Q_\omega(u)\psi$ is in $W_0^{1,p}(\hat{\Omega}) \cap L_{\hat{\mu}}^p(\hat{\Omega})$, vanishes in O' and is equal to u q.e. in $\overline{\omega}$.

We define $\mu^* \in \mathcal{M}_0^p(\hat{\Omega})$ by

$$\mu^*(B) = \begin{cases} \hat{\mu}(B) & \text{if } C_p(B \cap O') = 0 \\ +\infty & \text{if } C_p(B \cap O') > 0, \end{cases} \quad \forall B \subset \hat{\Omega} \text{ Borel,}$$

and we observe that since $u^* = 0$ in O' then $u^* \in L_{\mu^*}^p(\hat{\Omega})$. From Proposition 4.2(b) applied to μ^* we derive that taking w^* as the solution of

$$\begin{cases} w^* \in W_0^{1,p}(\hat{\Omega}) \cap L_{\mu^*}^p(\hat{\Omega}) \\ \int_{\hat{\Omega}} |\nabla w^*|^{p-2} \nabla w^* \nabla v \, dx + \int_{\hat{\Omega}} |w^*|^{p-2} w^* v \, d\mu^* = \int_{\hat{\Omega}} v \, dx \\ \forall v \in W_0^{1,p}(\hat{\Omega}) \cap L_{\mu^*}^p(\hat{\Omega}), \end{cases} \quad (4.40)$$

there exists $\varphi^* \in C_c^\infty(\hat{\Omega})$ such that

$$\|u^* - w^* \varphi^*\|_{W_0^{1,p}(\hat{\Omega}) \cap L_{\mu^*}^p(\hat{\Omega})} < \varepsilon.$$

Using then that $(w^* - \eta)^+ \varphi^*$ converges to $w^* \varphi^*$ in $W_0^{1,p}(\hat{\Omega}) \cap L_{\mu^*}^p(\hat{\Omega})$ when η tends to zero from the right, we also have, for $\eta > 0$ small enough

$$\|u^* - (w^* - \eta)^+ \varphi^*\|_{W_0^{1,p}(\hat{\Omega}) \cap L_{\mu^*}^p(\hat{\Omega})} < \varepsilon. \quad (4.41)$$

Since u^* and $(w^* - \eta)^+$ vanish in O' , this also holds replacing μ^* by $\hat{\mu}$.

From $\hat{\mu} \leq \mu^*$, the comparison principle (see Prop. 1.5 in [8]) proves that $0 \leq w^* \leq w$ and thus,

$$w \frac{(w^* - \eta)^+}{w \vee \eta} \varphi^* = (w^* - \eta)^+ \varphi^*.$$

Taking then

$$\varphi = \frac{(w^* - \eta)^+}{w \vee \eta} \varphi^* \in W_{\partial\omega \cap \Omega}^{1,p}(\omega) \cap L^\infty(\omega),$$

and using that w^* and then φ vanishes in O' , we conclude from (4.41) the proof of Proposition 4.5. \square

Proposition 4.6. *Assume that in Proposition 4.1 $\text{supp}(\hat{\mu}_n) \subset \bar{\Omega}$, for every $n \in \mathbb{N}$, and consider an open Lipschitz subset ω of Ω . Then for every sequence $v_n \in W^{1,p}(\omega)$, which converges weakly in $W^{1,p}(\omega)$ to a function v and every $\psi \in \mathcal{S}_\omega$, we have*

$$\liminf_{n \rightarrow \infty} \left(\int_\omega |\nabla(v_n - v)|^p |\psi|^p dx + \int_{\bar{\omega}} |v_n|^p |\psi|^p d\hat{\mu}_n \right) \geq \frac{1}{\|P\|} \int_{\bar{\omega}} |v|^p |\psi|^p d\hat{\mu}. \quad (4.42)$$

Proof. Similarly to the proof of Proposition 4.5, we remark that thanks to $\text{supp}(\hat{\mu}_n) \subset \bar{\Omega}$, we have that

$$\int_{\bar{\omega}} |v_n|^p |\psi|^p d\hat{\mu}_n = \int_{\hat{\Omega}} |Q_\omega(v_n \psi)|^p d\hat{\mu}_n.$$

So, using the convexity of the function $\xi \in \mathbb{R}^N \rightarrow |\xi|^p \in \mathbb{R}$, Rellich-Kondrachov's compactness theorem, $\|P\| \geq 1$, $\|Z_\omega\| = 1$ and (4.34), we obtain

$$\begin{aligned} & \int_\omega |\nabla(v_n - v)|^p |\psi|^p dx + \int_{\bar{\omega}} |v_n|^p |\psi|^p d\hat{\mu}_n \geq \\ & \|(v_n - v)\psi\|_{W^{1,p}(\omega)}^p - p \int_\omega |\nabla((v_n - v)\psi)|^{p-2} \nabla((v_n - v)\psi) \nabla \psi (v_n - v) dx \\ & - \int_\omega |(v_n - v)\psi|^p dx + \int_{\bar{\omega}} |v_n|^p |\psi|^p d\hat{\mu}_n \\ & = \|(v_n - v)\psi\|_{W^{1,p}(\omega)}^p + \int_{\bar{\omega}} |v_n|^p |\psi|^p d\hat{\mu}_n + O_n \\ & \geq \frac{1}{\|P\|} \int_{\hat{\Omega}} |\nabla Q_\omega((v_n - v)\psi)|^p dx + \int_{\hat{\Omega}} |Q_\omega(v_n \psi)|^p d\hat{\mu}_n + O_n \\ & \geq \frac{1}{\|P\|} \left(\int_{\hat{\Omega}} |\nabla Q_\omega((v_n - v)\psi)|^p dx + \int_{\hat{\Omega}} |Q_\omega(v_n \psi)|^p d\hat{\mu}_n \right) + O_n \\ & \geq \frac{1}{\|P\|} \int_{\hat{\Omega}} |Q_\omega(v \psi)|^p d\hat{\mu} + O_n = \frac{1}{\|P\|} \int_{\bar{\omega}} |v|^p |\psi|^p d\hat{\mu} + O_n. \end{aligned}$$

This proves (4.42). \square

5. ESTIMATES AND A LOCAL FIRST REPRESENTATION OF THE LIMIT PROBLEM

For Ω , a , μ_n and F_n as in Section 2 and a Lipschitz open subset ω of Ω , we will consider along this section a sequence $u_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ which satisfies (3.23) and (3.24), with g_n converging weakly in $L^{p'}(\omega)^M$ to a function g and G_n converging strongly in $L^{p'}(\omega)^{M \times N}$ to a function G . Thanks to Proposition 4.1 applied to the sequence $\hat{\mu}_n$ defined as

$$\hat{\mu}_n(B) = \mu_n(B \cap \overline{\Omega}), \quad \forall B \subset \hat{\Omega} \text{ Borel}, \quad (5.43)$$

we can also assume that there exist w and $\hat{\mu}$ in the conditions of this proposition. By Proposition 4.4, the support of this measure $\hat{\mu}$ is contained in $\overline{\Omega}$. The restriction of $\hat{\mu}$ to $\overline{\Omega}$ will be denoted by μ .

Thanks to (3.23), we can also assume that there exists $u \in W^{1,p}(\omega)^M$ such that u_n converges weakly to u in $W^{1,p}(\omega)^M$. Since by (3.23) and (4.42), the function u satisfies

$$\int_{\hat{\omega}} |u|^p |\psi|^p d\mu \leq C \|\psi\|_{L^\infty(\omega)}, \quad \forall \psi \in \mathcal{S}_\omega,$$

we also get from the monotone convergence theorem that u satisfies

$$u \in L^p_\mu(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M. \quad (5.44)$$

Our purpose in the present section is to obtain some estimates for the sequence u_n . As a consequence, we will obtain a first representation for the problem satisfied by the function u (limit problem). The fact of considering ω instead of Ω will allow us to prove the local character of the limit problem.

In order to study the asymptotic behavior of u_n , we start with the following result, which follows from Proposition 5.4 in [5].

Proposition 5.1. *The sequence u_n considered above converges to u strongly in $W^{1,q}(\omega)^M$, $1 \leq q < p$ and therefore Du_n converges in measure in ω .*

As a consequence, we have the following lemma.

Lemma 5.2. *The following convergences hold*

$$a(x, Du_n) \rightarrow a(x, Du) \text{ strongly in } L^r(\omega)^{M \times N}, \quad 1 \leq r < p', \text{ and weakly in } L^{p'}(\omega)^{M \times N}, \quad (5.45)$$

$$|a(x, Du_n) - a(x, D(u_n - u))| \rightarrow |a(x, Du)| \text{ strongly in } L^{p'}(\omega). \quad (5.46)$$

Proof. From Proposition 5.1, (2.10) and (2.12), we easily derive (5.45).

To prove (5.46) we use that Proposition 5.1 and the inequality

$$|a(x, Du_n) - a(x, D(u_n - u))| \leq \gamma(r + |Du_n| + |D(u_n - u)|)^{p-2} |Du| \text{ a.e. in } \Omega,$$

show that $|a(x, Du_n) - a(x, D(u_n - u))|$ converges in measure to $|a(x, Du)|$ and its p' -th power is equiintegrable. This implies (5.46). \square

For u_n and z_n satisfying similar conditions to u_n , the following lemma provides an estimate for $D(u_n - z_n)$. The idea is to study the difference $u_n - z_n - w_n \psi_m$, where ψ_m is such that $w \psi_m$ is close to $u - z$ (z is the limit of z_n). Recall that $w_n \psi_m$ is the corrector with limit $w \psi_m \sim u - z$ relative to the homogenization problem for the operator $v \rightarrow -\operatorname{div}(|\nabla v|^{p-2} \nabla v) + |v|^{p-2} v d\mu_n$ in $\hat{\Omega}$ with Dirichlet conditions.

Lemma 5.3. *There exists $C > 0$ which only depends on α and γ such that for every $\varphi \in \mathcal{S}_\omega$, $\varphi \geq 0$ in ω , we have*

$$\limsup_{n \rightarrow \infty} \left(\int_{\omega} |D(u_n - u)|^p \varphi dx + \int_{\overline{\omega}} |u_n|^p \varphi d\mu_n \right) \leq C \int_{\overline{\omega}} |u|^p \varphi d\mu. \quad (5.47)$$

Moreover, if besides u_n , u , g_n , g , G_n and G , we consider $z_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ with $\|z_n\|_{W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M}$ bounded, converging weakly in $W^{1,p}(\omega)^M$ to a function $z \in W^{1,p}(\omega)^M \cap L^p_{\mu}(\overline{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$, such that there exist h_n converging weakly in $L^{p'}(\omega)^M$ to a function h and H_n converging strongly in $L^{p'}(\omega)^{M \times N}$ to a function H with

$$\begin{cases} \int_{\omega} a(x, Dz_n) : Dv \, dx + \int_{\overline{\omega}} F_n(x, z_n)v \, d\mu_n = \int_{\omega} h_n v \, dx + \int_{\omega} H_n : Dv \, dx \\ \forall v \in W^{1,p}_{\partial\omega \cap \Omega}(\omega)^M \cap L^p_{\mu_n}(\overline{\omega})^M, \end{cases} \quad (5.48)$$

then, for every $\varphi \in \mathcal{S}_{\omega}$, $\varphi \geq 0$ in ω , we have

$$\limsup_{n \rightarrow \infty} \left(\int_{\omega} |D(u_n - z_n - u + z)|^p \varphi \, dx + \int_{\overline{\omega}} |u_n - z_n|^p \varphi \, d\mu_n \right) \leq C \int_{\overline{\omega}} (|u| + |z|)^{\frac{p(p-2)}{p-1}} |u - z|^{\frac{p}{p-1}} \varphi \, d\mu. \quad (5.49)$$

Proof. For $\varphi \in \mathcal{S}_{\omega}$, $\varphi \geq 0$ in ω , we consider $\phi \in \mathcal{S}_{\omega}$, such that $\phi = 1$ in $\text{supp}(\varphi)$. Then, $(u - z)\phi \in W^{1,p}_{\partial\omega \cap \Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M$ and so, from Proposition 4.5, there exists $\psi_m \in S_{\omega}^M$ such that $w\psi_m$ converges to $(u - z)\phi$ in $W^{1,p}_{\partial\omega \cap \Omega}(\omega)^M \cap L^p_{\mu}(\overline{\omega})^M$. Taking $(u_n - z_n - w_n\psi_m)\varphi$ as test function in the difference of (3.24) and (5.48), we get

$$\begin{aligned} & \int_{\omega} [a(x, Du_n) - a(x, Dz_n)] : D(u_n - z_n - w_n\psi_m)\varphi \, dx + \int_{\omega} [a(x, Du_n) - a(x, Dz_n)] : [(u_n - z_n - w_n\psi_m) \otimes \nabla\varphi] \, dx \\ & + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n - w_n\psi_m)\varphi \, d\mu_n = \int_{\omega} (g_n - h_n)(u_n - z_n - w_n\psi_m)\varphi \, dx \\ & + \int_{\omega} (G_n - H_n) : D((u_n - z_n - w_n\psi_m)\varphi) \, dx. \end{aligned}$$

Applying (5.46) to u_n and z_n and using that $(u_n - z_n - w_n\psi_m)$ converges weakly in $W^{1,p}(\omega)^M$, and then strongly in $L^p(\omega)^M$, to $(u - z)(1 - \phi)$ when n and then m tends to infinity, $g_n - h_n$ converges weakly in $L^{p'}(\omega)^M$, $G_n - H_n$ converges strongly in $L^{p'}(\omega)^{M \times N}$ and $\phi = 1$ in $\text{supp}(\varphi)$, the above equality gives

$$\begin{aligned} & \int_{\omega} [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(u_n - z_n - w_n\psi_m)\varphi \, dx \\ & + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n - w_n\psi_m)\varphi \, d\mu_n = O_{m,n}, \end{aligned}$$

or

$$\begin{aligned} & \int_{\omega} [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(u_n - z_n - u + z)\varphi \, dx + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n)\varphi \, d\mu_n = \\ & \int_{\omega} [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(w_n\psi_m - u + z)\varphi \, dx + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, z_n)]w_n\psi_m\varphi \, d\mu_n + O_{m,n}. \end{aligned}$$

By (2.9), (2.10), $|D(u_n - z_n - u + z)|\varphi$ converging weakly to zero in $L^{p'}(\omega)$ (use Lem. 5.1), (2.15) and (2.16), this gives

$$\begin{aligned} \alpha \int_{\omega} |D(u_n - u - z_n + z)|^p \varphi \, dx + \alpha \int_{\bar{\omega}} |u_n - z_n|^p \varphi \, d\mu_n \leq \\ \gamma \int_{\omega} [|D(u_n - u)| + |D(z_n - z)|]^{p-2} |D(u_n - z_n - u + z)| |D((w_n - w)\psi_m + w\psi_m - u + z)| \varphi \, dx \\ + \gamma \int_{\bar{\omega}} (|u_n| + |z_n|)^{p-2} |u_n - z_n| |w_n \psi_m| \varphi \, d\mu_n + O_{m,n}. \end{aligned}$$

Young's and Hölder's inequality allows us to write

$$\begin{aligned} \int_{\omega} |D(u_n - u - z_n + z)|^p \varphi \, dx + \int_{\bar{\omega}} |u_n - z_n|^p \varphi \, d\mu_n \leq \\ C \left(\int_{\omega} (|D(u_n - u)| + |D(z_n - z)|)^p \varphi \, dx + \int_{\bar{\omega}} (|u_n| + |z_n|)^p \varphi \, d\mu_n \right)^{\frac{p-2}{p-1}} \\ \times \left(\int_{\omega} |D((w_n - w)\psi_m)|^p \varphi \, dx + \int_{\bar{\omega}} |w_n \psi_m|^p \varphi \, d\hat{\mu}_n \right)^{\frac{1}{p-1}} + O_{m,n} \\ \leq C \left(\int_{\omega} (|D(u_n - u)| + |D(z_n - z)|)^p \varphi \, dx + \int_{\bar{\omega}} (|u_n| + |z_n|)^p \varphi \, d\mu_n \right)^{\frac{p-2}{p-1}} \\ \times \left(\int_{\Omega} |D((w_n - w)Q_{\omega}(\psi_m))|^p Q_{\omega}(\varphi) \, dx + \int_{\hat{\Omega}} |w_n Q_{\omega}(\psi_m)|^p Q_{\omega}(\varphi) \, d\hat{\mu}_n \right)^{\frac{1}{p-1}} + O_{m,n}. \quad (5.50) \end{aligned}$$

Taking $z_n = z = 0$ in (5.50), and using (4.33), $\text{supp}(\hat{\mu}) \subset \bar{\Omega}$ and μ equals to the restriction of $\hat{\mu}$ in $\bar{\Omega}$, we deduce (5.47).

Finally, to obtain (5.49), it is enough to apply (4.33) in (5.50) together with the estimation (5.47) for u_n and z_n . \square

Using Lemma 5.3, we will now obtain a first version of the limit problem satisfied by u .

Proposition 5.4. *There exists $T \in L^p_{\mu}(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ such that u satisfies*

$$\begin{cases} u \in W^{1,p}(\omega)^M \cap L^p_{\mu}(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M \\ \int_{\omega} a(x, Du) : Dv \, dx + \int_{\bar{\omega}} Tv \, d\mu = \int_{\omega} gv \, dx + \int_{\omega} G : Dv \, dx \\ \forall v \in W^{1,p}_{\partial\omega \cap \Omega}(\omega)^M \cap L^p_{\mu}(\bar{\omega})^M. \end{cases} \quad (5.51)$$

The function T is such that for every $\psi \in \mathcal{S}_{\omega}^M$, we have

$$\int_{\bar{\omega}} Tw\psi \, d\mu = \lim_{n \rightarrow \infty} \left(\int_{\omega} a(x, D(u_n - u)) : (\psi \otimes \nabla(w_n - w)) \, dx + \int_{\bar{\omega}} F_n(x, u_n) w_n \psi \, d\mu_n \right), \quad (5.52)$$

and satisfies that there exists $C > 0$, which only depends on α and γ , such that

$$|T| \leq C|u|^{p-1} \quad \mu\text{-a.e. in } \bar{\omega} \setminus \overline{\partial\omega \cap \Omega}. \quad (5.53)$$

Proof. Given $\psi \in \mathcal{S}_\omega^M$, we take $w_n\psi$ as test function in (3.24). This gives

$$\begin{aligned} \int_\omega a(x, Du_n) : D\psi w_n dx + \int_\omega a(x, Du_n) : (\psi \otimes \nabla w) dx \\ + \int_\omega a(x, Du_n) : (\psi \otimes \nabla(w_n - w)) dx + \int_{\bar{\omega}} F_n(x, u_n) w_n \psi d\mu_n = \\ \int_\omega g_n w_n \psi dx + \int_\omega G_n : D(w_n \psi) dx. \end{aligned} \quad (5.54)$$

Using that g_n converges weakly in $L^p(\omega)^M$ to g , G_n converges strongly in $L^p(\omega)^{M \times N}$ to G and u_n and w_n converge weakly in $W^{1,p}(\omega)^M$ and $W^{1,p}(\Omega)$ to u and w respectively (and then strongly in $L^p(\omega)^M$ and $L^p(\Omega)$), (5.45) and (5.46), we get

$$\begin{aligned} \int_\omega a(x, Du) : D(w\psi) dx + \int_\omega a(x, D(u_n - u)) : (\psi \otimes \nabla(w_n - w)) dx \\ + \int_{\bar{\omega}} F_n(x, u_n) w_n \psi d\mu_n = \int_\omega g w \psi dx + \int_\omega G : D(w\psi) dx + O_n. \end{aligned} \quad (5.55)$$

Since $\|w_n\|_{W_0^{1,p}(\hat{\Omega}) \cap L^p_{\mu_n}(\hat{\Omega})}$ is bounded, (3.23), (2.12) and (2.18), we have

$$\int_\omega |a(x, D(u_n - u))| |\nabla(w_n - w)| dx + \int_{\bar{\omega}} |F_n(x, u_n)| |w_n| d\mu_n \leq C.$$

Thus, there exists a vector Radon measure ρ on $\bar{\omega}$, such that for every $\varphi \in C^0(\bar{\omega})^M$, it holds

$$\lim_{n \rightarrow \infty} \left(\int_\omega a(x, D(u_n - u)) (\varphi \otimes \nabla(w_n - w)) dx + \int_{\bar{\omega}} F_n(x, u_n) w_n \varphi d\mu_n \right) = \int_{\bar{\omega}} \varphi d\rho. \quad (5.56)$$

Using the Cauchy-Schwartz inequality, (2.12), (2.18), the weak convergence of $|\nabla(w_n - w)|$ to zero in $L^p(\omega)$, (5.47) and (4.33), we deduce that ρ satisfies

$$\begin{aligned} \left| \int_{\bar{\omega}} \varphi d\rho \right| &\leq C \limsup_{n \rightarrow +\infty} \left(\int_\omega |D(u_n - u)|^p |\varphi| dx + \int_{\bar{\omega}} |u_n|^p |\varphi| d\mu_n \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_\omega |\nabla(w_n - w)|^p |\varphi| dx + \int_{\bar{\omega}} w_n^p |\varphi| d\mu_n \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{\bar{\omega}} |u|^p |\varphi| d\mu \right)^{\frac{p-1}{p}} \left(\int_{\bar{\omega}} w^p |\varphi| d\mu \right)^{\frac{1}{p}}, \quad \forall \varphi \in \mathcal{S}_\omega^M, \end{aligned} \quad (5.57)$$

where C only depends on α and γ . From the derivation measures theorem, we deduce that there exists a μ -measurable vector function $L : \bar{\omega} \setminus \overline{\partial\omega \cap \Omega} \rightarrow \mathbb{R}^M$ such that

$$\int_{\bar{\omega}} \varphi d\rho = \int_{\bar{\omega}} L \varphi d\mu, \quad \forall \varphi \in \mathcal{S}_\omega^M,$$

and

$$|L| \leq C |u|^{p-1} w \quad \mu\text{-a.e. in } \bar{\omega} \setminus \overline{\partial\omega \cap \Omega}.$$

Defining then $T = (\frac{L}{w})\chi_{\{w \neq 0\}} \in L'_\mu(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$, we have that T satisfies (5.52) and (5.53). Moreover, from (5.55) and (5.56) we obtain

$$\int_\omega a(x, Du) : D(w\psi) dx + \int_{\bar{\omega}} Tw\psi d\mu = \int_\omega gw\psi dx + \int_\omega G : D(w\psi) dx,$$

for every $\psi \in \mathcal{S}_\omega^M$, which by the density of the set D_ω given by (4.38) proves that u satisfies (5.51). \square

To finish this section, let us obtain an estimate about the dependence of the function T given by Proposition 5.4 with respect to u . For this purpose, as in Lemma 5.3, we consider $z_n \in W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ with $\|z_n\|_{W^{1,p}(\omega)^M \cap L^p_{\mu_n}(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M}$ bounded, converging weakly in $W^{1,p}(\omega)^M$ to a function $z \in W^{1,p}(\omega)^M \cap L^p_\mu(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$, such that there exist h_n converging weakly in $L^p(\omega)^M$ to a function h and H_n converging strongly in $L^p(\omega)^{M \times N}$ to a function H which satisfy (5.48). From Proposition 5.4, we also know that there exists $T' \in L'_\mu(\bar{\omega} \setminus \overline{\partial\omega \cap \Omega})^M$ such that

$$\int_{\bar{\omega}} T'w\psi d\mu = \lim_{n \rightarrow \infty} \left(\int_\omega a(x, D(z_n - z)) : (\psi \otimes \nabla(w_n - w)) dx + \int_{\bar{\omega}} F_n(x, z_n)w_n\psi d\mu_n \right), \quad (5.58)$$

for every $\psi \in \mathcal{S}_\omega^M$, and

$$\int_\omega a(x, Dz) : Dv dx + \int_{\bar{\omega}} T'v d\mu = \int_\omega hv dx + \int_\omega H : Dv dx, \quad \forall v \in W^{1,p}_{\partial\omega \cap \Omega}(\omega)^M \cap L^p_\mu(\bar{\omega})^M. \quad (5.59)$$

The following result estimates the difference between T and T' .

Lemma 5.5. *There exist $C_1, C_2 > 0$, such that T and T' satisfy*

$$|T - T'| \leq C_1(|u| + |z|)^{\frac{p(p-2)}{p-1}} |u - z|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \bar{\omega} \setminus \overline{\partial\omega \cap \Omega}, \quad (5.60)$$

and

$$(T - T')(u - z) \geq C_2|u - z|^p \quad \mu\text{-a.e. in } \bar{\omega} \setminus \overline{\partial\omega \cap \Omega}. \quad (5.61)$$

The constant C_1 only depends on α and γ . The constant C_2 only depends on α , Ω and $\hat{\Omega}$.

Proof. By (5.52), (5.58), (2.10), (2.16) and Proposition 5.1, for every $\psi \in \mathcal{S}_\omega^M$, we have

$$\begin{aligned} \left| \int_{\bar{\omega}} (T - T')w\psi d\mu \right| &\leq C \int_\omega (|D(u_n - u)| + |D(z_n - z)|)^{p-2} |D(u_n - z_n - u + z)| |\nabla(w_n - w)| |\psi| dx \\ &\quad + \int_{\bar{\omega}} (|u_n| + |z_n|)^{p-2} |u_n - z_n| |\psi| w_n d\mu_n + O_n, \end{aligned}$$

which by Hölder's inequality, (5.47) (applied to u_n and z_n) and (5.49) gives

$$\left| \int_{\bar{\omega}} (T - T')w\psi d\mu \right| \leq C \left(\int_{\bar{\omega}} (|u| + |z|)^p |\psi| d\mu \right)^{\frac{p-2}{p-1}} \left(\int_{\bar{\omega}} |u - z|^p |\psi| d\mu \right)^{\frac{1}{p(p-1)}} \left(\int_{\bar{\omega}} |w|^p |\psi| d\mu \right)^{\frac{1}{p}}.$$

From the measures derivation theorem, we then deduce (5.60).

Let us now prove (5.61). For $\varphi \in S_\omega$, $\varphi \geq 0$, we take $(u_n - z_n)\varphi^p$ as test function in the difference of (3.24) and (5.48). Using Rellich-Kondrachov's compactness theorem, the weak convergence of $g_n - h_n$ in $L^p(\omega)^M$,

the strong convergence of $G_n - H_n$ in $L^{p'}(\omega)^{M \times N}$, (5.51) and (5.59), we have

$$\begin{aligned}
\int_{\omega} [a(x, Du_n) - a(x, Dz_n)] : D(u_n - z_n) \varphi^p dx + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, v_n)](u_n - z_n) \varphi^p d\mu_n = \\
\int_{\omega} (g - h)(u - z) \varphi^p dx + \int_{\omega} (G - H) : D((u - z) \varphi^p) dx + \\
- p \int_{\omega} [a(x, Du) - a(x, Dz)] : [(u - z) \otimes \nabla \varphi] \varphi^{p-1} dx + O_n \\
= \int_{\omega} [a(x, Du) - a(x, Dz)] : D(u - z) \varphi^p dx + \int_{\overline{\omega}} (T - T')(u - z) \varphi^p d\mu + O_n, \quad (5.62)
\end{aligned}$$

which by (5.46) (applied to u_n and z_n) implies

$$\begin{aligned}
\int_{\overline{\omega}} (T - T')(u - z) \varphi^p d\mu = \lim_{n \rightarrow \infty} \left(\int_{\omega} [a(x, D(u_n - u)) - a(x, D(z_n - z))] : D(u_n - z_n - u + z) \varphi^p dx \right. \\
\left. + \int_{\overline{\omega}} [F_n(x, u_n) - F_n(x, z_n)](u_n - z_n) \varphi^p d\mu_n \right). \quad (5.63)
\end{aligned}$$

Using in the right-hand side of this inequality (2.9), (2.15) and (4.42), we deduce

$$\begin{aligned}
\int_{\overline{\omega}} (T - T')(u - z) \varphi^p d\mu &\geq \alpha \left(\int_{\omega} |D(u_n - u - z_n + z)|^p \varphi^p dx + \int_{\overline{\omega}} |u_n - z_n|^p \varphi^p d\mu_n \right) \\
&\geq \frac{\alpha}{\|P\|} \int_{\overline{\omega}} |u - z|^p \varphi^p d\mu + O_n, \quad \forall \varphi \in S_{\omega}, \varphi \geq 0. \quad (5.64)
\end{aligned}$$

From the measures derivation theorem, this proves (5.61). \square

6. PROOF OF THE MAIN RESULTS

In this section we prove that there exists a μ -Carathéodory function $F : \overline{\Omega} \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ such that the function T given by Proposition 5.4 satisfies $T(x) = F(x, u(x))$ μ -a.e. in $\overline{\omega} \setminus \overline{\partial\omega} \cap \overline{\Omega}$. We start with the following lemma. Its proof is completely similar to the one of Theorem 6.9 in [5], and thus we omit it.

Lemma 6.1. *We consider a subsequence of n such that there exists the measure μ defined in the beginning of Section 5. Then, up to another subsequence, we have that for every $q \in \mathbb{Q}^M$ and every $m \in \mathbb{N}$, the solution q_n^m of*

$$\begin{cases} q_n^m \in W^{1,p}(\Omega)^M \cap L_{\mu_n}^p(\overline{\Omega})^M \\ \int_{\Omega} a(x, Dq_n^m) : Dv dx + \int_{\overline{\Omega}} F_n(x, q_n^m) v d\mu_n = m \int_{\Omega} [|w_n q|^{p-2} w_n q - |q_n^m|^{p-2} q_n^m] v dx \\ \forall v \in W^{1,p}(\Omega)^M \cap L_{\mu_n}^p(\overline{\Omega})^M, \end{cases} \quad (6.65)$$

converges to a function q^m weakly in $W^{1,p}(\Omega)^M$. This function satisfies that there exists $T_q^m \in L^{p'}(\overline{\Omega})$ such that

$$\begin{cases} q^m \in W^{1,p}(\Omega)^M \cap L_{\mu}^p(\overline{\Omega})^M \\ \int_{\Omega} a(x, Dq^m) : Dv dx + \int_{\overline{\Omega}} T_q^m v d\mu = m \int_{\Omega} [|w q|^{p-2} w q - |q^m|^{p-2} q^m] v dx \\ \forall v \in W^{1,p}(\Omega)^M \cap L_{\mu}^p(\overline{\Omega})^M. \end{cases} \quad (6.66)$$

When m tends to infinity, the sequence q^m converges to wq strongly in $W^{1,p}(\Omega)^M \cap L_{\mu_n}^p(\overline{\Omega})^M$ and the sequence T_q^m converges strongly in $L_{\mu}^p(\overline{\Omega})^M$ to a function T_q .

Definition 6.2. We consider the subsequence of n given by Lemma 6.1. Then, we define $\mathcal{F} : \overline{\Omega} \times \mathbb{Q}^M \rightarrow \mathbb{R}^M$ by

$$\mathcal{F}(x, q) = T_q(x), \quad \forall q \in \mathbb{Q}^M, \quad \mu\text{-a.e. } x \in \overline{\Omega}.$$

By Lemma 6.1, (5.53), (5.60) and (5.61), it is easy to show that for every $q_1, q_2 \in \mathbb{Q}^M$ and μ -a.e. $x \in \overline{\Omega}$, we have

$$\mathcal{F}(x, 0) = 0, \tag{6.67}$$

$$|\mathcal{F}(x, q_2) - \mathcal{F}(x, q_1)| \leq C_1(|q_1| + |q_2|)^{\frac{p(p-2)}{p-1}} |q_2 - q_1|^{\frac{1}{p-1}} w(x)^{p-1}, \tag{6.68}$$

$$(\mathcal{F}(x, q_2) - \mathcal{F}(x, q_1))(q_2 - q_1) \geq C_2 |q_2 - q_1|^p w(x)^{p-1}. \tag{6.69}$$

Using (6.68), we can extend by continuity \mathcal{F} to $\overline{\Omega} \times \mathbb{R}^M$. Then, we define $F : \overline{\Omega} \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ by

$$F(x, s) = \begin{cases} \mathcal{F}\left(x, \frac{s}{w(x)}\right) & \text{if } w(x) > 0 \\ |s|^{p-2}s & \text{if } w(x) = 0. \end{cases}$$

Thanks to Lemma 6.1, Proposition 5.4 and estimate (5.60) we can now prove Theorem 3.1.

Proof of Theorem 3.1. We take u_n and u as in the statement of the theorem. By Proposition 5.4, there exists $T \in L_{\mu}^p(\overline{\omega})^M$ such that u is a solution of (5.51). Applying (5.60), with z replaced by q^m , we have

$$|T - T_q^m| \leq C(|u| + |q^m|)^{\frac{p(p-2)}{p-1}} |u - q^m|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}, \tag{6.70}$$

and therefore, taking the limit as m tends to infinity, we obtain

$$|T - F(x, wq)| \leq C(|u| + |wq|)^{\frac{p(p-2)}{p-1}} |u - wq|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}.$$

Thus, for every simple function $\phi(x) = \sum_{i=1}^l s_i \chi_{B_i}(x)$, with $s_i \in \mathbb{R}^M$, B_i Borel, we have

$$|T - F(x, w\phi)| \leq C(|u| + |w\phi|)^{\frac{p(p-2)}{p-1}} |u - w\phi|^{\frac{1}{p-1}} \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}.$$

Finally, taking in this inequality ϕ as a sequence ϕ_n such that $w\phi_n$ converges μ -a.e. to u in $\overline{\omega} \setminus \overline{\partial\omega \cap \Omega}$ (the existence of such sequence is an easy consequence of Prop. 4.5) and passing to the limit in n thanks to the continuity of F with respect to its second variable, we get

$$T = F(x, u) \quad \mu\text{-a.e. in } \overline{\omega} \setminus \overline{\partial\omega \cap \Omega}.$$

This proves (3.25) thanks to (5.51) and the fact that the functions of $W_{\partial\omega \cap \Omega}^{1,p}(\omega)$ are zero q.e. on $\overline{\partial\omega \cap \Omega}$. \square

Proof of Theorem 3.2. Let us just prove (ii). The proof of (i) is much simpler.

Thanks to (3.28) and the assumptions on a and F_n , problem (3.29) has a unique solution. Moreover, taking u_n as test function in (3.29) and using (3.28), (2.11) and (2.17) we get that u_n satisfies (3.23). Thus, from $W_{\partial\omega \cap \Omega}^{1,p}(\omega)$ closed, Theorem 3.1 and Proposition 5.1, we deduce that there exists a subsequence of u_n which converges weakly in $W_{\partial\omega \cap \Omega}^{1,p}(\omega)^M$ and strongly in $W_{\partial\omega \cap \Omega}^{1,q}(\omega)^M$, $1 \leq q < p$, to a solution u of (3.30). If we prove that u is unique, then the whole sequence u_n will converge to u and the proof of (ii) will be finished. For this purpose, it is enough to prove that the measure μ also satisfies (3.28) and then, from (2.9) and (3.22), we will get the uniqueness of solution of problem (3.30).

Let v be in $W_{\partial\omega\cap\Omega}^{1,p}(\omega) \cap L_{\mu}^p(\bar{\omega})$. Using Proposition 4.5, we consider $\psi_m \in \mathcal{S}_{\omega}$ such that $w\psi_m$ converges to v in $W_{\partial\omega\cap\Omega}^{1,p}(\omega) \cap L_{\mu}^p(\bar{\omega})$. From (3.28), for every $m \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\omega} |w_n \psi_m|^p dx &\leq C_P^p \left(\int_{\omega} |\nabla(w_n \psi_m)|^p dx + \int_{\bar{\omega}} |w_n \psi_m|^p d\mu_n \right) \\ &\leq C_P^p \left(\int_{\omega} (|\nabla(w_n \psi_m)|^p - |\nabla((w_n - w)\psi_m)|^p) dx \right) \\ &\quad + C_P^p \left(\int_{\hat{\Omega}} |\nabla((w_n - w)Q_{\omega}(\psi_m))|^p dx + \int_{\hat{\Omega}} |w_n Q_{\omega}(\psi_m)|^p d\hat{\mu}_n \right). \end{aligned}$$

Since w_n converges strongly to w in $W^{1,q}(\hat{\Omega})$, $1 \leq q < p$, reasoning similarly to the proof of (5.46), we have that $|\nabla(w_n \psi_m)|^p - |\nabla((w_n - w)\psi_m)|^p$ converges strongly to $|\nabla(w\psi_m)|^p$ in $L^1(\omega)$ and thus

$$\int_{\omega} (|\nabla(w_n \psi_m)|^p - |\nabla((w_n - w)\psi_m)|^p) dx \rightarrow \int_{\omega} |\nabla(w\psi_m)|^p dx,$$

whereas from (4.33) and $\mu = \hat{\mu}$ in $\text{supp}(\hat{\mu}) = \bar{\Omega}$ we have

$$\int_{\hat{\Omega}} |\nabla((w_n - w)Q_{\omega}(\psi_m))|^p dx + \int_{\hat{\Omega}} |w_n Q_{\omega}(\psi_m)|^p d\hat{\mu}_n \rightarrow \int_{\bar{\omega}} |w\psi_m|^p d\mu.$$

Thus, using also the semicontinuity of the norm in $L^p(\omega)$, we get

$$\int_{\omega} |w\psi_m|^p dx \leq C_P^p \left(\int_{\omega} |\nabla(w\psi_m)|^p dx + \int_{\bar{\omega}} |w\psi_m|^p d\mu \right).$$

Taking then m tending to infinity we derive

$$\int_{\omega} |v|^p dx \leq C_P^p \left(\int_{\omega} |\nabla v|^p dx + \int_{\bar{\omega}} |v|^p d\mu \right), \quad \forall v \in W_{\partial\omega\cap\Omega}^{1,p}(\omega) \cap L_{\mu}^p(\bar{\omega}).$$

This finishes the proof of (ii). □

Acknowledgements. This work has been partly supported by the projects MTM2005-04914 of the *Ministerio de Educación y Ciencia of Spain* and FQM-309 of the *Junta de Andalucía*.

REFERENCES

- [1] C. Calvo-Jurado and J. Casado-Díaz, The limit of Dirichlet systems for variable monotone operators in general perforated domains. *J. Math. Pures Appl.* **81** (2002) 471–493.
- [2] C. Calvo-Jurado, J. Casado-Díaz and M. Luna-Laynez, Homogenization of elliptic problems with the Dirichlet and Neumann conditions imposed on varying subsets. *Math. Meth. Appl. Sci.* **30** (2007) 1611–1625.
- [3] J. Casado-Díaz, Homogenization of general quasi-linear Dirichlet problems with quadratic growth in perforated domains. *J. Math. Pures Appl.* **76** (1997) 431–476.
- [4] J. Casado-Díaz, Homogenization of Dirichlet problems for monotone operators in varying domains. *Proc. Roy. Soc. Edinburgh A* **127** (1997) 457–478.
- [5] J. Casado-Díaz and A. Garroni, Asymptotic behavior of nonlinear elliptic systems on varying domains. *SIAM J. Math. Anal.* **31** (2000) 581–624.

- [6] D. Cionarescu and F. Murat, Un terme étrange venu d'ailleurs, in *Nonlinear partial differential equations and their applications*, Collège de France seminar, Vols. II and III, H. Brézis and J.-L. Lions Eds., *Research Notes in Math.* **60** and **70**, Pitman, London (1982) 98–138 and 154–78.
- [7] G. Dal Maso and A. Defranceschi, Limits of nonlinear Dirichlet problems in varying domains. *Manuscripta Math.* **61** (1988) 251–278.
- [8] G. Dal Maso and A. Garroni, New results on the asymptotic behaviour of Dirichlet problems in perforated domains. *Math. Mod. Meth. Appl. Sci.* **3** (1994) 373–407.
- [9] G. Dal Maso and U. Mosco, Wiener-criterion and Γ -convergence. *Appl. Math. Optim.* **15** (1987) 15–63.
- [10] G. Dal Maso and F. Murat, Asymptotic behaviour and correctors for the Dirichlet problems in perforated domains with homogeneous monotone operators. *Ann. Sc. Norm. Sup. Pisa* **7** (1997) 765–803.
- [11] G. Dal Maso and F. Murat, Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004) 445–486.
- [12] G. Dal Maso, A. Garroni and I.V. Skrypnik, A capacity method for the asymptotic analysis of Dirichlet problems for monotone operators. *J. Anal. Math.* **71** (1997) 263–313.
- [13] A. Damlamian and T. Li, Boundary homogenization for elliptic problems. *J. Math. Pures Appl.* **66** (1987) 351–361.
- [14] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton (1992).
- [15] H. Federer and W.P. Ziemer, The Lebesgue set of a function whose distribution derivatives are p -th power sumable. *Indiana Univ. Math. J.* **22** (1972) 139–158.
- [16] J. Leray and J.-L. Lions, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. *Bull. Soc. Math. France* **93** (1965) 97–107.
- [17] J. Necas, *Les méthodes directes en théorie des équations elliptiques*. Masson, Paris (1967).
- [18] J. Serrin, Local behaviour of solutions of quasilinear equations. *Acta Math.* **111** (1964) 302–347.
- [19] I.V. Skrypnik, Asymptotic behaviour of solutions of nonlinear elliptic problems in perforated domains. *Mat. Sb.* **184** (1993) 67–90.
- [20] I.V. Skrypnik, Averaging of quasilinear parabolic problems in domains with fine-grained boundary. *Diff. Equations* **31** (1995) 327–339.
- [21] W.P. Ziemer, *Weakly Differentiable Functions*. Springer-Verlag, Berlin (1989).