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TOPOLOGICAL SENSITIVITY ANALYSIS FOR TIME-DEPENDENT PROBLEMS

Samuel Amstutz[1], Takéo Takahashi[2] and Boris Vexler[3]

Abstract. The topological sensitivity analysis consists in studying the behavior of a given shape functional when the topology of the domain is perturbed, typically by the nucleation of a small hole. This notion forms the basic ingredient of different topology optimization/reconstruction algorithms. From the theoretical viewpoint, the expression of the topological sensitivity is well-established in many situations where the governing p.d.e. system is of elliptic type. This paper focuses on the derivation of such formulas for parabolic and hyperbolic problems. Different kinds of cost functionals are considered.

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Introduction

Consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and the solution u_Ω of a system of partial differential equations defined in Ω. The topological sensitivity analysis aims at studying the asymptotic behavior of some shape functional of interest $j(\Omega) = J_\Omega(u_\Omega)$ with respect to an infinitesimal perturbation of the topology of Ω. This concept was introduced in the field of shape optimization by Schumacher *et al.* [14,15,24] and was for the first time mathematically justified in [16,25]. In these papers, the creation of holes inside the domain is considered. Given a point $x_0 \in \Omega$, a domain $\omega \subset \mathbb{R}^d$ containing the origin and a small perforation $\omega_\varepsilon = x_0 + \varepsilon\omega$, an asymptotic expansion for ε going to zero is obtained in the form:

$$j(\Omega \setminus \overline{\omega_\varepsilon}) - j(\Omega) = f(\varepsilon)g(x_0) + o(f(\varepsilon)). \tag{0.1}$$

In this expression, the function $\varepsilon \in \mathbb{R}^+ \mapsto f(\varepsilon) \in \mathbb{R}^+$ is smooth and goes to zero with ε. The number $g(x_0)$ is commonly called topological gradient, or topological derivative, at the point x_0. It gives an indication on the sensitivity of the cost functional with respect to the nucleation of a small hole around x_0. The map $x \mapsto g(x)$ forms the basis of different kinds of topology optimization algorithms. They mainly rely on the following

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[1] Laboratoire d'analyse non-linéaire et géométrie, Faculté des sciences, 33 rue Louis Pasteur, 84000 Avignon, France; samuel.amstutz@univ-avignon.fr

[2] Institut Élie Cartan de Nancy, Nancy-Université, CNRS, INRIA, BP 239, 54506 Vandœuvre-lès-Nancy cedex, France; Takeo.Takahashi@iecn.u-nancy.fr

[3] Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, 4040 Linz, Austria; boris.vexler@oeaw.ac.at

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principles. For certain problems, the interpretation in one iteration of some special features of this map, such as peaks, can provide a sufficient information (see *e.g.* [7,9,10,18]). In an iterative procedure, the topological gradient can serve as a descent direction for removing matter (see *e.g.* [16,17,22]). It can also be utilized within a level-set-based algorithm (see *e.g.* [1,6,11]).

From the theoretical point of view, most efforts for deriving the expansion (0.1) have been so far focused on problems associated with state equations of elliptic type, for which several generalizations of the above notion have been proposed (*e.g.* creation of a crack [8], exterior topological derivative [20]). To the best of our knowledge, [9] is the only publication where this issue is addressed for a time-dependent problem. But the proof presented there is merely formal. For instance, convergence theorems for integrals of multivariate functions are used without any checking of their applicability. In addition, a restricted class of cost functional is considered. In another context but still related, one should mention the paper [3], which belongs to a series of works dedicated to the reconstruction of inhomogeneities from boundary measurements (see *e.g.* [2,4] and the references therein). In these works, asymptotic expansions of the state variable u_Ω at the location of the measurements or its integrals against special test functions are derived. Then techniques borrowed from signal processing are used to recover some features of the unknown inclusions. In the frame of topology optimization, one would like to be able to deal with general cost functionals, which makes the analysis quite different. In particular, an adjoint method is generally appreciated for computational convenience.

The present paper investigates the topological sensitivity analysis of shape functionals for governing PDEs of parabolic and hyperbolic types. For simplicity, the mathematical developments are presented for model problems. The following heat and wave equations for an inclusion are considered:

$$\rho_\varepsilon \frac{\partial^p u_\varepsilon}{\partial t^p} - \operatorname{div}(\alpha_\varepsilon A \nabla u_\varepsilon) = F_\varepsilon, \qquad p = 1, 2.$$

The coefficients ρ_ε and α_ε are positive and piecewise constant, with values inside the inclusion ω_ε different from those of the background medium. The right hand side F_ε should be smooth in ω_ε and its complementary, A denotes some symmetric positive definite matrix. Dirichlet boundary conditions on the external border of Ω and null initial conditions are prescribed. For these problems, a large class of cost functionals is treated. The calculus of their sensitivity is performed by means of an adjoint state method, which, in addition to the practical interest, enables to write the expansion (0.1) in a unified form. This setting allows for some straightforward generalizations. First, the same results hold for other kinds of linear boundary conditions on $\partial\Omega$ (*e.g.* of Neumann or Robin type), since they play no role in the analysis except that of guaranteeing well-posedness and regularity properties. Second, the formulas corresponding to a vector-valued state variable can be easily inferred, provided that the expression of the first order polarization tensor (also called Pólya-Szegö polarization tensor, or virtual mass) is known. This notion is however well-documented (see *e.g.* [2,4]). Third, the case where ω_ε is a hole with Neumann boundary condition can be obtained by taking in the final formulas ρ_ε and α_ε to be zero inside ω_ε and the associated polarization tensor. This statement is proved in [5] for elliptic problems. Here, the proof, which is very similar, is omitted. We also point out that the interest of our result has already been illustrated by promising numerical experiments [7,9]. Those concern nondestructive testing in elastic media with acoustic waves and a least-square-type cost function. In [7], the expression of the topological gradient in the time domain was formally deduced from the harmonic case through the Fourier transform. This formula, identical to that found in [9], is retrieved as a particular case.

The rest of this article is organized as follows. In Section 1, we recall an abstract result which provides in a general setting the structure of the topological asymptotic expansion. In Sections 2, 3 and 4, we present our main result for the heat equation. Some examples of cost functionals are exhibited in Section 5. Sections 6 through 12 contain the proofs. Sections 13 through 17 are devoted to the wave equation, following the same outline.

1. A preliminary result

Let X and $X_0 \subset X$ be two Banach spaces. For all parameter $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we consider a function $u_\varepsilon \in X_0$ solving a variational problem of the form

$$\mathcal{A}_\varepsilon(u_\varepsilon, v) = \mathcal{L}_\varepsilon(v) \quad \forall v \in X \tag{1.1}$$

where $\mathcal{A}_\varepsilon : X \times X \to \mathbb{R}$, and $\mathcal{L}_\varepsilon : X \to \mathbb{R}$ are a bilinear form on X and a linear functional on X, respectively. We also consider a functional $\mathcal{J}_\varepsilon : X_0 \to \mathbb{R}$ and the associated reduced cost functional

$$j(\varepsilon) = \mathcal{J}_\varepsilon(u_\varepsilon) \in \mathbb{R}.$$

Suppose also that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0, \tag{1.2}$$

and such that the following holds.

(1) There exist $D\mathcal{J}_\varepsilon(u_0) \in X_0'$ and $\delta\mathcal{J} \in \mathbb{R}$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + \langle D\mathcal{J}_\varepsilon(u_0), u_\varepsilon - u_0 \rangle_{X_0', X_0} + f(\varepsilon)\delta\mathcal{J} + o(f(\varepsilon)), \tag{1.3}$$

when ε goes to zero. Here X_0' denotes the dual space of X_0 and $\langle ., . \rangle_{X_0', X_0}$ is the corresponding duality pairing. The notation $D\mathcal{J}_\varepsilon(u_0)$ has been used for the reader's convenience since in most applications, it coincides with the Fréchet derivative of $\mathcal{J}_\varepsilon$ evaluated at u_0.

(2) There exists $v_\varepsilon \in X$ solving the adjoint equation

$$\mathcal{A}_\varepsilon(\varphi, v_\varepsilon) = -\langle D\mathcal{J}_\varepsilon(u_0), \varphi \rangle_{X_0', X_0} \quad \forall \varphi \in X_0. \tag{1.4}$$

(3) There exist $\delta\mathcal{A}$, $\delta\mathcal{L} \in \mathbb{R}$ such that for ε going to zero,

$$(\mathcal{A}_\varepsilon - \mathcal{A}_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta\mathcal{A} + o(f(\varepsilon)), \tag{1.5}$$

$$(\mathcal{L}_\varepsilon - \mathcal{L}_0)(v_\varepsilon) = f(\varepsilon)\delta\mathcal{L} + o(f(\varepsilon)). \tag{1.6}$$

Proposition 1.1. Under the above assumptions, we have the following asymptotic expansion for ε tending to zero:

$$j(\varepsilon) - j(0) = f(\varepsilon)\left(\delta\mathcal{A} - \delta\mathcal{L} + \delta\mathcal{J}\right) + o(f(\varepsilon)). \tag{1.7}$$

For the proof, see [5].

Part 1. **Topological sensitivity analysis for parabolic problems**

2. Setting of the problem

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2$ or 3, with smooth (C^∞) boundary $\partial\Omega$. We consider a small subdomain $\omega_\varepsilon = x_0 + \varepsilon\omega$, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^d$ is a bounded domain containing the origin with smooth and connected boundary $\partial\omega$.

Let A be a symmetric positive definite matrix and let α_0, α_1, ρ_0, ρ_1 be some positive real numbers. For every parameter $\varepsilon \in [0, \varepsilon_0)$, ε_0 small enough, we define the piecewise constant coefficients

$$\alpha_\varepsilon = \begin{cases} \alpha_1 & \text{in } \omega_\varepsilon \\ \alpha_0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon} \end{cases}, \quad \rho_\varepsilon = \begin{cases} \rho_1 & \text{in } \omega_\varepsilon \\ \rho_0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon} \end{cases}.$$

Given $F_0, F_1 \in L^2(0,T;H^{-1}(\Omega))$, we also define the function

$$F_\varepsilon = \begin{cases} F_1 & \text{in } \omega_\varepsilon \times (0,T), \\ F_0 & \text{in } (\Omega \setminus \overline{\omega_\varepsilon}) \times (0,T). \end{cases}$$

We consider the following heat equation:

$$\begin{cases} \rho_\varepsilon \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(\alpha_\varepsilon A \nabla u_\varepsilon) = F_\varepsilon & \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

The corresponding variational formulation for

$$X = L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),$$

$$u_\varepsilon \in X_0 = \{u \in X, u(.,0) = 0\}$$

can be written as:

$$\int_0^T \left\langle \rho_\varepsilon \frac{\partial u_\varepsilon}{\partial t}, v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}t + \int_0^T a_\varepsilon(u_\varepsilon, v)\,\mathrm{d}t = \int_0^T \ell_\varepsilon(v)\,\mathrm{d}t \quad \forall v \in X. \tag{2.2}$$

Here, the bilinear form a_ε and the linear functional ℓ_ε are defined by:

$$a_\varepsilon(u,v) = \int_\Omega \alpha_\varepsilon A \nabla u \cdot \nabla v\,\mathrm{d}x, \tag{2.3}$$

$$\ell_\varepsilon(v) = \int_\Omega F_\varepsilon v\,\mathrm{d}x. \tag{2.4}$$

Equation (2.2) can be identified with the generic form (1.1) by setting

$$\mathcal{A}_\varepsilon(u,v) = \int_0^T \left(\left\langle \rho_\varepsilon \frac{\partial u}{\partial t}, v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + a_\varepsilon(u,v) \right) \mathrm{d}t,$$

$$\mathcal{L}_\varepsilon(v) = \int_0^T \ell_\varepsilon(v)\,\mathrm{d}t.$$

To apply the result of Section 1, we deal with a cost function of the form

$$j(\varepsilon) = \mathcal{J}_\varepsilon(u_\varepsilon) = \int_0^T J_\varepsilon(u_\varepsilon)\,\mathrm{d}t \tag{2.5}$$

where the functional $J_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$ satisfies the following assumptions:

$$J_\varepsilon(u) \in L^1(0,T) \quad \forall u \in X,\ \forall \varepsilon \in [0, \varepsilon_0), \tag{2.6}$$

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_\varepsilon(u_0) + \int_0^T \langle DJ_\varepsilon(u_0), u_\varepsilon - u_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\,\mathrm{d}t + \varepsilon^d \delta\mathcal{J}_1 + o(\varepsilon^d), \tag{2.7}$$

$$\mathcal{J}_\varepsilon(u_0) = \mathcal{J}_0(u_0) + \varepsilon^d \delta\mathcal{J}_2 + o(\varepsilon^d), \tag{2.8}$$

$$\|DJ_\varepsilon(u_0) - DJ_0(u_0)\|_{L^2(0,T;H^{-1}(\Omega))} = o(\varepsilon^{d/2}), \tag{2.9}$$

with $DJ_\varepsilon(u_0(t)) \in H^{-1}(\Omega)$ for almost all $t \in (0,T)$. These assumptions will be checked for some typical cost functionals in Section 5.

Remarks 2.1.

(1) Like in Section 1, we use the notation $DJ_\varepsilon(u_0(\cdot,t))$ since in most applications, it coincides with the Fréchet derivative of J_ε evaluated at $u_0(\cdot,t)$.

(2) For simplicity, we do not consider the case where the cost functional J_ε depends explicitly on time. However, all the analysis could be easily adapted to this case.

We introduce the adjoint state $v_\varepsilon \in X$ defined by (1.4), *i.e.*,

$$\int_0^T \left\langle \rho_\varepsilon \frac{\partial \varphi}{\partial t}, v_\varepsilon \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t + \int_0^T a_\varepsilon(\varphi, v_\varepsilon)\,\mathrm{d}t = -\int_0^T DJ_\varepsilon(u_0)\varphi\,\mathrm{d}t \qquad \forall \varphi \in X_0. \tag{2.10}$$

The strong formulation of the PDE associated with (2.10) reads

$$\left\{ \begin{array}{rcll} -\rho_\varepsilon \dfrac{\partial v_\varepsilon}{\partial t} - \operatorname{div}(\alpha_\varepsilon A \nabla v_\varepsilon) &=& -DJ_\varepsilon(u_0) & \text{in } \Omega \times (0,T), \\ v_\varepsilon &=& 0 & \text{on } \partial\Omega \times (0,T), \\ v_\varepsilon(\cdot,T) &=& 0 & \text{in } \Omega. \end{array} \right. \tag{2.11}$$

3. Regularity assumptions

To enable the analysis, we make additional regularity assumptions, namely: there exist two neighborhoods Ω_F and Ω_J of x_0 such that

$$F_0 \in L^2(0,T;H^4(\Omega_F)) \cap H^2(0,T;L^2(\Omega_F)), \tag{3.1}$$

$$F_1 \in L^2(0,T;W^{1,\infty}(\Omega_F)), \tag{3.2}$$

$$DJ_0(u_0) \in L^2(0,T;H^4(\Omega_J)) \cap H^2(0,T;L^2(\Omega_J)). \tag{3.3}$$

The condition (3.3) will be checked for the examples of cost functional presented in Section 5. The conditions (3.1) and (3.2) are assumed throughout all this part of the paper. Then we get the following regularity on the direct and adjoint solutions. The proof is given in Section 6.

Proposition 3.1. Assume that u_0 and v_0 solve (2.1) and (2.11), respectively, for $\varepsilon = 0$ and that the regularity assumptions (3.1), (3.3) hold. Then for all subdomains $\widetilde{\Omega}_F \subset\subset \Omega_F$, $\widetilde{\Omega}_J \subset\subset \Omega_J$, we have

$$u_0 \in L^2(0,T,H^6(\widetilde{\Omega}_F)) \cap H^3(0,T;L^2(\widetilde{\Omega}_F)), \tag{3.4}$$

$$v_0 \in L^2(0,T,H^6(\widetilde{\Omega}_J)) \cap H^3(0,T;L^2(\widetilde{\Omega}_J)). \tag{3.5}$$

For the sake of readability, we fix some subdomain $\widetilde{\Omega}$ containing x_0 and such that $\widetilde{\Omega} \subset\subset \Omega_F$, $\widetilde{\Omega} \subset\subset \Omega_J$, and we remember in the sequel that

$$F_0, F_1 \in L^2(0,T;W^{1,\infty}(\widetilde{\Omega})), \tag{3.6}$$

$$u_0, v_0 \in L^2(0,T,H^6(\widetilde{\Omega})) \cap H^3(0,T;L^2(\widetilde{\Omega})). \tag{3.7}$$

In particular, by interpolation (see [19], Chap. 4, Prop. 2.3), it follows

$$u_0, v_0 \in H^1(0,T,H^4(\widetilde{\Omega})).$$

The domains Ω_F, Ω_J, $\widetilde{\Omega}_F$ and $\widetilde{\Omega}_J$ will only be distinguished when studying special cost functionals.

4. Main result

In order to state the main result, we first introduce the polarization matrix $\mathcal{P}_{\omega,r} \in \mathbb{R}^{d\times d}$, $r \in \mathbb{R}^+$. It is defined as follows:

(1) if $r = 1$, then $\mathcal{P}_{\omega,1} = 0$;
(2) otherwise, it has the entries

$$(\mathcal{P}_{\omega,r})_{ij} = \int_{\partial\omega} p_j x_i \,\mathrm{d}s \tag{4.1}$$

where x_j is the j^{th} coordinate of the point x and the density p_i associated with the i^{th} basis vector e_i of $\mathbb{R}^d$ is the unique solution of the boundary integral equation

$$\frac{r+1}{r-1}\frac{p_i(x)}{2} + \int_{\partial\omega} p_i(y) A\nabla E(x-y).n(x)\mathrm{d}s(y) = Ae_i.n(x) \qquad \forall x \in \partial\omega. \tag{4.2}$$

Here, E denotes the fundamental solution of the operator $u \mapsto -\operatorname{div}(A\nabla u)$. We recall that the matrix $\mathcal{P}_{\omega,r}$ is symmetric (see, *e.g.*, [4]).

To apply the abstract result of Section 1, we first provide the following lemmas, which will be proved in Sections 7 through 11.

Lemma 4.1. *Assume that the bilinear form a_ε is defined by (2.3), that u_0 and v_ε solve (2.1) and (2.11), respectively, that we have the regularity assumptions (3.1)–(3.3) and that (2.9) holds true. Then*

$$\int_0^T (a_\varepsilon - a_0)(u_0, v_\varepsilon)\,\mathrm{d}t = \varepsilon^d\ \delta a + o(\varepsilon^d), \tag{4.3}$$

with

$$\delta a = \alpha_0 \int_0^T \nabla u_0(x_0,t)\cdot \mathcal{P}_{\omega,\frac{\alpha_1}{\alpha_0}} \nabla v_0(x_0,t)\,\mathrm{d}t.$$

Lemma 4.2. *Assume that u_0 and v_ε solve (2.1) and (2.11), respectively, that we have the regularity assumptions (3.1)–(3.3) and that (2.9) holds true. Then*

$$\int_0^T \left\langle (\rho_\varepsilon - \rho_0)\frac{\partial u_0}{\partial t},\ v_\varepsilon \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t = \varepsilon^d\ \delta\rho + o(\varepsilon^d), \tag{4.4}$$

with

$$\delta\rho = (\rho_1 - \rho_0)|\omega| \int_0^T \frac{\partial u_0}{\partial t}(x_0,t)\ v_0(x_0,t)\,\mathrm{d}t.$$

Lemma 4.3. *Assume that the linear functional ℓ_ε is defined by (2.4) and that u_0 and v_ε solve (2.1) and (2.11), respectively, that we have the regularity assumptions (3.1)–(3.3) and that (2.9) holds true. Then*

$$\int_0^T (\ell_\varepsilon - \ell_0)\,(v_\varepsilon)\,\mathrm{d}t = \varepsilon^d\ \delta\ell + o(\varepsilon^d), \tag{4.5}$$

with

$$\delta\ell = |\omega| \int_0^T (F_1(x_0,t) - F_0(x_0,t))\, v_0(x_0,t)\ \mathrm{d}t.$$

We are now in position to state the main result of this part.

Theorem 4.1. *Assume that the cost functional $\mathcal{J}$ satisfies (2.5)–(2.9). Suppose moreover that u_0 and v_0 solve (2.1) and (2.11), respectively, for $\varepsilon = 0$ and that the regularity assumptions (3.1)–(3.3) hold. Then we have the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = \varepsilon^d \Bigg[(\rho_1 - \rho_0)|\omega| \int_0^T \frac{\partial u_0}{\partial t}(x_0, t)\ v_0(x_0, t)\,\mathrm{d}t + \alpha_0 \int_0^T \nabla u_0(x_0, t) \cdot \mathcal{P}_{\omega, \frac{\alpha_1}{\alpha_0}} \nabla v_0(x_0, t)\,\mathrm{d}t \\ - |\omega| \int_0^T (F_1(x_0, t) - F_0(x_0, t))\, v_0(x_0, t)\ \mathrm{d}t + \delta\mathcal{J}_1 + \delta\mathcal{J}_2 \Bigg] + o(\varepsilon^d). \quad (4.6)$$

This theorem is a direct consequence of Proposition 1.1 combined with the above lemmas and the definitions $\delta\mathcal{A} = \delta\rho + \delta a$, $\delta\mathcal{L} = \delta\ell$, $\delta\mathcal{J} = \delta\mathcal{J}_1 + \delta\mathcal{J}_2$,

$$\langle D\mathcal{J}_\varepsilon(u_0), \varphi\rangle_{X_0', X_0} = \int_0^T \langle DJ_\varepsilon(u_0(\cdot, t)), \varphi(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\,\mathrm{d}t.$$

Remarks 4.1.

(1) The polarization matrix can be determined analytically in some cases. For instance, we have for the Laplace operator (A is the identity matrix) and $\omega = B(0,1)$:

$$\mathcal{P}_{\omega,r} = 2|\omega| \frac{r-1}{r+1}\ \mathrm{I}_2 \qquad \text{in 2D (disc)},$$

$$\mathcal{P}_{\omega,r} = 3|\omega| \frac{r-1}{r+2}\ \mathrm{I}_3 \qquad \text{in 3D (sphere)},$$

where I_2, I_3 denote the identity matrices in dimensions 2 and 3, respectively. For more details on polarization matrices see, *e.g.*, [2,4,5,21,23] and the references therein.

(2) Theorem 4.1 can be extended to some other situations. First, on the external boundary $\partial\Omega$, we can replace the Dirichlet condition by any kind of linear boundary condition guaranteeing well-posedness of the direct and adjoint PDEs, like the Neumann or the Robin boundary condition. Second, the proof can be easily adapted to other parabolic equations or systems like for instance the Stokes system.

(3) Theorem 4.1 remains valid in the case of a hole with Neumann condition on its boundary. The corresponding topological asymptotic expansion is given by (4.6) with $\rho_1 = 0$, $\alpha_1 = 0$, $F_1 = 0$ and the polarization matrix computed by solving (4.2) for $r = 0$ (see, *e.g.*, [4,5] for more details).

In the next section we present some examples of cost functional $\mathcal{J}$ satisfying the assumptions of the theorem.

5. Examples of cost functional

The proofs of the following results are given in Section 12.

Theorem 5.1. *Assume that $J_\varepsilon \in \mathcal{C}^2(L^2(\Omega), \mathbb{R})$ (in the sense of Fréchet) and satisfies, for all $M \geq 0$,*

$$\sup_{\|v\|_{L^2(\Omega)} \leq M} \|D^2 J_\varepsilon(v)\|_{\mathcal{B}(L^2(\Omega))} \leq C(M), \quad (5.1)$$

with a positive constant $C(M)$ which does not depend on ε and with $\mathcal{B}(L^2(\Omega))$ denoting the space of bilinear forms on $L^2(\Omega)$.

Then $\mathcal{J}_\varepsilon$ is well-defined on X and fulfills (2.7) with $\delta\mathcal{J}_1 = 0$.

Corollary 5.1. *The asymptotic expansion* (4.6) *holds true for the following cost functionals with the values of* $\delta\mathcal{J}_1$ *and* $\delta\mathcal{J}_2$ *given below.*

(1) *For the functional*

$$J_\varepsilon(u) = \int_\Omega |u - u_d|^2 \, \mathrm{d}x \tag{5.2}$$

with $u_d \in L^2(\Omega) \cap H^4(B(x_0, R))$, $R > 0$, *we have* $\delta\mathcal{J}_1 = 0$ *and* $\delta\mathcal{J}_2 = 0$. *Note that the creation of a hole with Neumann condition on its boundary cannot be considered for this functional,* cf. *Remark* 4.1 (3).

(2) *For the functional*

$$J_\varepsilon(u) = \int_\Omega \alpha_\varepsilon |u - u_d|^2 \, \mathrm{d}x \tag{5.3}$$

with $u_d \in L^2(\Omega) \cap H^4(B(x_0, R))$, $R > 0$, *we have* $\delta\mathcal{J}_1 = 0$ *and*

$$\delta\mathcal{J}_2 = (\alpha_1 - \alpha_0)|\omega| \int_0^T |u_0(x_0, t) - u_d(x_0)|^2 \, \mathrm{d}t.$$

We end this section by giving two other examples of cost functional which are not included in the setting of Theorem 5.1.

Proposition 5.1. The asymptotic expansion (4.6) holds true for the following cost functionals.

(1) For the functional

$$J_\varepsilon(u) = \int_\Omega \eta(x) A\nabla(u - u_d).\nabla(u - u_d) \, \mathrm{d}x \tag{5.4}$$

where $u_d \in L^2(0, T; H^1(\Omega))$ and η is a smooth ($\mathcal{C}^\infty$) function whose support does not contain x_0, we have $\delta\mathcal{J}_1 = 0$ and $\delta\mathcal{J}_2 = 0$.

(2) If we replace in (2.1) the Dirichlet boundary condition on $\partial\Omega$ by the Neumann boundary condition (for instance), then it makes sense to consider the functional

$$\mathcal{J}_\varepsilon(u) = \int_0^T \int_{\partial\Omega} |u - u_d|^2 \, \mathrm{d}s \, \mathrm{d}t \tag{5.5}$$

where $u_d \in L^2(0, T; L^2(\partial\Omega))$. We have $\delta\mathcal{J}_1 = 0$ and $\delta\mathcal{J}_2 = 0$.

The subsequent sections are devoted to the proofs of the results previously stated.

6. Regularity results

Proposition 3.1 is a straightforward application of the following lemma.

Lemma 6.1. *Let* $\widetilde{\Omega} \subset\subset \Omega$, k *be a positive integer,* $f \in L^2(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H^k(\widetilde{\Omega})) \cap H^{k/2}(0, T; L^2(\widetilde{\Omega}))$, $g \in L^2(0, T; H^{1/2}(\partial\Omega))$ *and* z *be the solution of the system:*

$$\left\{ \begin{aligned} \rho_0 \frac{\partial z}{\partial t} - \operatorname{div}(\alpha_0 A\nabla z) &= f && \text{in } \Omega \times (0, T), \\ z &= g && \text{on } \partial\Omega \times (0, T), \\ z(\cdot, 0) &= 0 && \text{in } \Omega. \end{aligned} \right. \tag{6.1}$$

Then, for all subdomain $\Omega_k \subset\subset \widetilde{\Omega}$, *we have*

$$z \in L^2(0, T; H^{k+2}(\Omega_k)) \cap H^{k/2+1}(0, T; L^2(\Omega_k)). \tag{6.2}$$

The same result holds if the Dirichlet boundary condition on $\partial\Omega$ *is replaced by a Neumann or Robin condition of the form* $\frac{\partial z}{\partial n} + \lambda z = g$, $\lambda \in \mathbb{R}$, $g \in L^2(0, T; H^{-1/2}(\partial\Omega))$.

Proof. The difficulty comes from the fact that the so-called compatibility relations required to apply the standard parabolic regularity theorems are not satisfied here. We will construct auxiliary functions for which those relations hold. Our proof follows a bootstrapping argument.

(1) We introduce a domain Ω_0 such that $\Omega_k \subset\subset \Omega_0 \subset\subset \widetilde{\Omega}$. Let η_0 be a smooth function with

$$\eta_0 = 0 \text{ in } \Omega \setminus \overline{\widetilde{\Omega}}$$
$$\eta_0 = 1 \text{ in } \Omega_0.$$

We consider the function

$$z_0 = \eta_0 z.$$

It solves:

$$\begin{cases} \rho_0 \dfrac{\partial z_0}{\partial t} - \operatorname{div}(\alpha_0 A \nabla z_0) = f_0 & \text{in } \Omega \times (0,T), \\ z_0 = 0 & \text{on } \partial\Omega \times (0,T), \\ z_0(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{6.3}$$

with

$$f_0 = \eta_0 f - 2\alpha_0 A \nabla \eta_0 . \nabla z - \eta_0 \operatorname{div}(\alpha_0 A \nabla \eta_0) z. \tag{6.4}$$

We are guaranteed the minimal regularity $z \in L^2(0,T;H^1(\Omega))$, from which we deduce that $f_0 \in L^2(0,T;L^2(\Omega))$. Using [19], Chapter 4, Theorem 1.1, we derive that $z_0 \in L^2(0;T,H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$, and consequently that

$$z \in L^2(0,T;H^2(\Omega_0)) \cap H^1(0,T;L^2(\Omega_0)).$$

(2) Assume that, given an integer $p \in \{0,...,k-1\}$, there exists a domain Ω_p, with $\Omega_k \subset\subset \Omega_p \subset\subset \widetilde{\Omega}$, such that

$$z \in L^2(0,T;H^{p+2}(\Omega_p)) \cap H^{p/2+1}(0,T;L^2(\Omega_p)). \tag{6.5}$$

If $p+1 < k$, we define a domain Ω_{p+1} such that $\Omega_k \subset\subset \Omega_{p+1} \subset\subset \Omega_p$. We introduce a smooth function η_{p+1} satisfying

$$\eta_{p+1} = 0 \text{ in } \Omega \setminus \overline{\Omega_p},$$
$$\eta_{p+1} = 1 \text{ in } \Omega_{p+1},$$

and we define the function

$$z_{p+1} = \eta_{p+1} z.$$

It solves

$$\begin{cases} \rho_0 \dfrac{\partial z_{p+1}}{\partial t} - \operatorname{div}(\alpha_0 A \nabla z_{p+1}) = f_{p+1} & \text{in } \Omega \times (0,T), \\ z_{p+1} = 0 & \text{on } \partial\Omega \times (0,T), \\ z_{p+1}(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{6.6}$$

with

$$f_{p+1} = \eta_{p+1} f - 2\alpha_0 A \nabla \eta_{p+1} . \nabla z - \eta_{p+1} \operatorname{div}(\alpha_0 A \nabla \eta_{p+1}) z. \tag{6.7}$$

Using [19], Chapter 4, Proposition 2.3, we obtain that $f_{p+1} \in L^2(0,T;H^{p+1}(\Omega_p)) \cap H^{(p+1)/2}(0,T;L^2(\Omega_p))$. It follows (see [19], Chap. 4, Th. 5.3) that $z_{p+1} \in L^2(0,T;H^{p+3}(\Omega_p)) \cap H^{(p+3)/2}(0,T;L^2(\Omega_p))$, and thus that

$$z \in L^2(0,T;H^{p+3}(\Omega_{p+1})) \cap H^{(p+3)/2}(0,T;L^2(\Omega_{p+1})).$$

Hence the relation (6.5) holds true at rank $p+1$. The relation (6.2) is obtained by repeating this procedure up to the rank $p+1 = k$. $\square$

7. Auxiliary results on elliptic problems

We start by introducing a vector field $H = (H_1, \dots, H_d)^\top$ where the components H_i are given as the solutions of the system:

$$\left\{\begin{array}{rcll} \operatorname{div}(A\nabla H_i) & = & 0 & \text{in } \omega, \\ \operatorname{div}(A\nabla H_i) & = & 0 & \text{in } \mathbb{R}^d \setminus \overline{\omega}, \\ H_i^+ - H_i^- & = & 0 & \text{on } \partial\omega, \\ \alpha_1 (A\nabla H_i \cdot n)^+ - \alpha_0 (A\nabla H_i \cdot n)^- & = & (\alpha_1 - \alpha_0)(An)_i & \text{on } \partial\omega, \\ H_i & \to & 0 & \text{at } \infty. \end{array}\right. \tag{7.1}$$

In the above equations, $n = (n_1, \dots, n_d)^\top$ denotes the outer unit normal of ω and the superscripts $+$ and $-$ indicate the traces of the restriction to ω and to $\mathbb{R}^d \setminus \overline{\omega}$, respectively.

The solution H_i can be expressed by means of a single layer potential (see, *e.g.*, [4,13]), namely, there exists $p_i \in H^{-1/2}(\partial\omega)$ such that

$$\int_{\partial\omega} p_i \mathrm{d}s(y) = 0, \tag{7.2}$$

$$H_i(x) = \int_{\partial\omega} p_i(y) E(x-y) \mathrm{d}s(y), \tag{7.3}$$

for all $x \in \mathbb{R}^d$. To determine the density p_i, we use the well-known formula (see, *e.g.*, [4,13]):

$$(A\nabla H_i(x) \cdot n(x))^\pm = \pm \frac{p_i(x)}{2} + \int_{\partial\omega} p_i(y)(A\nabla E(x-y) \cdot n(x)) \, \mathrm{d}s(y). \tag{7.4}$$

Substituting these expressions into the fourth equation of (7.1) leads to the integral equation

$$(\alpha_1 + \alpha_0)\frac{p_i(x)}{2} + (\alpha_1 - \alpha_0) \int_{\partial\omega} p_i(y)(A\nabla H_i(x) \cdot n(x)) \, \mathrm{d}s(y) = (\alpha_1 - \alpha_0)(An(x))_i \quad \forall x \in \partial\omega.$$

When $\alpha_1 \neq \alpha_0$, the above equation is equivalent to (4.2) with $r = \frac{\alpha_1}{\alpha_0}$. When $\alpha_1 = \alpha_0$, we get $p_i = 0$ and $H_i = 0$. In particular, the following lemma holds with the convention $\mathcal{P}_{\omega,1} = 0$.

Lemma 7.1. *Let $H = (H_1, \dots, H_d)^\top$ be the vector field defined as above and $k \in \mathbb{R}^d$. Then we have*

$$(\alpha_1 - \alpha_0) \int_{\partial\omega} (A\nabla(H \cdot k) \cdot n)^+ y \, \mathrm{d}s(y) = -\alpha_0 \mathcal{P}_{\omega, \frac{\alpha_1}{\alpha_0}} k + (\alpha_1 - \alpha_0)|\omega| Ak. \tag{7.5}$$

Proof. Let $I = (I_1, \dots, I_d)^\top$ be the vector defined by

$$I = \int_{\partial\omega} (A\nabla(H \cdot k) \cdot n)^+ y \, \mathrm{d}s(y).$$

Then for each $j \in \{1, \dots, d\}$, we have that

$$I_j = \sum_i k_i \int_{\partial\omega} (A\nabla H_i \cdot n)^+ y_j \, \mathrm{d}s(y). \tag{7.6}$$

Besides, from (7.4), we have the jump relation

$$(A\nabla H_i \cdot n)^+ - (A\nabla H_i \cdot n)^- = p_i. \tag{7.7}$$

Combining (7.7) with the third equation of (7.1) brings

$$(\alpha_0 - \alpha_1)(A\nabla H_i \cdot n)^+ = \alpha_0 p_i - (\alpha_1 - \alpha_0)(An)_i.$$

Equation (7.6) together with the above equality yield

$$(\alpha_1 - \alpha_0)I_j = \sum_i k_i \left[-\alpha_0 \int_{\partial\omega} p_i y_j \, \mathrm{d}s(y) + (\alpha_1 - \alpha_0) \int_{\partial\omega} (An)_i y_j \, \mathrm{d}s(y) \right]. \tag{7.8}$$

An integration by parts provides

$$\int_{\partial\omega} (An)_i y_j \, \mathrm{d}s(y) = |\omega| A_{ij}. \tag{7.9}$$

Gathering (7.9), (4.1) and (7.8) completes the proof. □

For all $\varepsilon \in [0, \varepsilon_0)$ and for all $x \in \mathbb{R}^d$, we define the vector field h_ε as

$$h_\varepsilon(x) = \varepsilon H\left(\frac{x - x_0}{\varepsilon}\right).$$

Then, we have the following properties. We refer to [5] for the proof.

Lemma 7.2. *Let h_ε be the vector field defined as above and R be a positive number. Then, for ε going to zero, the following relations hold:*

$$\|h_\varepsilon\|_{L^2(\Omega)} = o(\varepsilon^{d/2}), \tag{7.10}$$

$$\|\nabla h_\varepsilon\|_{L^2(\Omega)} = O(\varepsilon^{d/2}), \tag{7.11}$$

$$\|\nabla h_\varepsilon\|_{L^2(\Omega \setminus \overline{B(x_0,R)})} = O(\varepsilon^{d}). \tag{7.12}$$

8. Asymptotic behavior of the direct and adjoint states

We introduce the function

$$\widehat{h}_\varepsilon(x,t) = -h_\varepsilon(x) \cdot \nabla v_0(x_0,t) \qquad \forall (x,t) \in \mathbb{R}^d \times (0,T). \tag{8.1}$$

This function fulfills the following equations for all $t \in (0,T)$:

$$\begin{cases} \operatorname{div}\,(A\nabla\widehat{h}_\varepsilon(\cdot,t)) = 0 & \text{in } \omega_\varepsilon, \\ \operatorname{div}\,(A\nabla\widehat{h}_\varepsilon(\cdot,t)) = 0 & \text{in } (\mathbb{R}^d \setminus \overline{\omega_\varepsilon}), \\ \widehat{h}_\varepsilon^+(\cdot,t) = \widehat{h}_\varepsilon^-(\cdot,t) & \text{on } \partial\omega_\varepsilon, \\ \alpha_1 \left(A\nabla\widehat{h}_\varepsilon(\cdot,t)\cdot n\right)^+ - \alpha_0 \left(A\nabla\widehat{h}_\varepsilon(\cdot,t)\cdot n\right)^- = -(\alpha_1-\alpha_0)\left(A\nabla v_0(x_0,t)\cdot n\right) & \text{on } \partial\omega_\varepsilon, \\ \widehat{h}_\varepsilon(\cdot,t) \to 0 & \text{at } \infty. \end{cases} \tag{8.2}$$

Furthermore, let us consider the function e_ε such that

$$v_\varepsilon = v_0 + \widehat{h}_\varepsilon + e_\varepsilon. \tag{8.3}$$

With the above notations, we have the following estimate whose proof is presented at the end of this section.

Lemma 8.1. *The function e_ε defined as above satisfies*

$$\|e_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|e_\varepsilon\|_{L^2(0,T;H^1(\Omega))} = o(\varepsilon^{d/2}). \tag{8.4}$$

As a consequence of the above lemma and of Lemma 7.2 we have the following result.

Lemma 8.2. *Let v_ε and v_0 be defined by (2.10). Consider a positive number R. Then, we have the following relations*

$$\begin{aligned}
\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2(\Omega))} &= o(\varepsilon^{d/2}), &(8.5)\\
\|v_\varepsilon - v_0\|_{L^2(0,T;H^1(\Omega))} &= O(\varepsilon^{d/2}), &(8.6)\\
\|\nabla(v_\varepsilon - v_0)\|_{L^2(0,T;L^2(\Omega\setminus\overline{B(x_0,R)}))} &= o(\varepsilon^{d/2}). &(8.7)
\end{aligned}$$

We also have the corresponding result on the direct state. Indeed, it solves a similar PDE with a right hand side whose variation also satisfies $\|F_\varepsilon - F_0\|_{L^2(0,T;H^{-1}(\Omega))} = o(\varepsilon^{d/2})$. This latter statement is a straightforward consequence of (3.6).

Lemma 8.3. *Let u_ε and u_0 be defined by (2.1). Consider a positive number R. Then, we have the following relations*

$$\begin{aligned}
\|u_\varepsilon - u_0\|_{L^\infty(0,T;L^2(\Omega))} &= o(\varepsilon^{d/2}), &(8.8)\\
\|u_\varepsilon - u_0\|_{L^2(0,T;H^1(\Omega))} &= O(\varepsilon^{d/2}), &(8.9)\\
\|\nabla(u_\varepsilon - u_0)\|_{L^2(0,T;L^2(\Omega\setminus\overline{B(x_0,R)}))} &= o(\varepsilon^{d/2}). &(8.10)
\end{aligned}$$

Proof of Lemma 8.1. Using (2.11) and (8.2) and the fact that $\widehat{h}_\varepsilon(\cdot,T)=0$, we easily check that e_ε solves

$$\left\{\begin{aligned}
-\rho_1\frac{\partial e_\varepsilon}{\partial t} - \alpha_1 \operatorname{div}(A\nabla e_\varepsilon) &= Q_1+Q_2+Q_3+Q_4 && \text{in } \omega_\varepsilon\times(0,T),\\
-\rho_0\frac{\partial e_\varepsilon}{\partial t} - \alpha_0 \operatorname{div}(A\nabla e_\varepsilon) &= Q_1+Q_4 && \text{in } (\Omega\setminus\overline{\omega_\varepsilon})\times(0,T),\\
e_\varepsilon^+ &= e_\varepsilon^- && \text{on } \partial\omega_\varepsilon\times(0,T),\\
\alpha_1\,(A\nabla e_\varepsilon\cdot n)^+ - \alpha_0\,(A\nabla e_\varepsilon\cdot n)^- &= Q_5 && \text{on } \partial\omega_\varepsilon\times(0,T),\\
e_\varepsilon &= -\widehat{h}_\varepsilon && \text{on } \partial\Omega\times(0,T),\\
e_\varepsilon(\cdot,T) &= 0 && \text{in } \Omega,
\end{aligned}\right. \tag{8.11}$$

where

$$Q_1 = DJ_0(u_0) - DJ_\varepsilon(u_0), \quad Q_2 = (\rho_1-\rho_0)\frac{\partial v_0}{\partial t},$$

$$Q_3 = (\alpha_1-\alpha_0)\operatorname{div}(A\nabla v_0), \quad Q_4 = \rho_\varepsilon\frac{\partial\widehat{h}_\varepsilon}{\partial t},$$

and for all $(x,t)\in\partial\omega_\varepsilon\times(0,T)$,

$$Q_5(x,t) = -(\alpha_1-\alpha_0)\,(A\,[\nabla v_0(x,t) - \nabla v_0(x_0,t)]\cdot n)\,.$$

In order to separate difficulties, we make the splitting

$$e_\varepsilon = e_{1,\varepsilon} + e_{2,\varepsilon}$$

with

$$\begin{cases} -\rho_1 \dfrac{\partial e_{1,\varepsilon}}{\partial t} - \alpha_1 \text{ div } (A\nabla e_{1,\varepsilon}) = Q_1 + Q_2 + Q_3 + Q_4 & \text{in } \omega_\varepsilon \times (0,T), \\ -\rho_0 \dfrac{\partial e_{1,\varepsilon}}{\partial t} - \alpha_0 \text{ div } (A\nabla e_{1,\varepsilon}) = Q_1 + Q_4 & \text{in } (\Omega \setminus \overline{\omega_\varepsilon}) \times (0,T), \\ e_{1,\varepsilon}^+ = e_{1,\varepsilon}^- & \text{on } \partial\omega_\varepsilon \times (0,T), \\ \alpha_1 \left(A\nabla e_{1,\varepsilon} \cdot n\right)^+ - \alpha_0 \left(A\nabla e_{1,\varepsilon} \cdot n\right)^- = Q_5 & \text{on } \partial\omega_\varepsilon \times (0,T), \\ e_{1,\varepsilon} = 0 & \text{on } \partial\Omega \times (0,T), \\ e_{1,\varepsilon}(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \tag{8.12}$$

and

$$\begin{cases} -\rho_\varepsilon \dfrac{\partial e_{2,\varepsilon}}{\partial t} - \text{ div } (\alpha_\varepsilon A\nabla e_{2,\varepsilon}) = 0 & \text{in } \Omega \times (0,T), \\ e_{2,\varepsilon} = -\widehat{h}_\varepsilon & \text{on } \partial\Omega \times (0,T), \\ e_{2,\varepsilon}(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{8.13}$$

We estimate $e_{1,\varepsilon}$ by multiplying the first two equations of (8.12) by $e_{1,\varepsilon}$ and by integrating in space and time:

$$\begin{aligned} \frac{1}{2} \int_\Omega \rho_\varepsilon |e_{1,\varepsilon}(\cdot, t_0)|^2 \, \mathrm{d}x + \int_{t_0}^T \int_\Omega \alpha_\varepsilon A\nabla e_{1,\varepsilon} \cdot \nabla e_{1,\varepsilon} \, \mathrm{d}x \, \mathrm{d}t \leq \int_{t_0}^T \left| \int_{\partial\omega_\varepsilon} Q_5 e_{1,\varepsilon} \, \mathrm{d}s \right| \, \mathrm{d}t + \Big(\|Q_1\|_{L^2(t_0,T;H^{-1}(\Omega))} \\ + \|Q_2 \chi_{\omega_\varepsilon}\|_{L^2(t_0,T;H^{-1}(\Omega))} + \|Q_3 \chi_{\omega_\varepsilon}\|_{L^2(t_0,T;H^{-1}(\Omega))} + \|Q_4\|_{L^2(t_0,T;H^{-1}(\Omega))} \Big) \|e_{1,\varepsilon}\|_{L^2(t_0,T;H_0^1(\Omega))}, \end{aligned} \tag{8.14}$$

for almost all $t_0 \in [0,T]$. Here, $\chi_{\omega_\varepsilon}$ stands for the characteristic function of the set ω_ε.

Using the Poincaré inequality and taking the supremum for $t_0 \in [0,T]$, the above equation yields

$$\begin{aligned} \|e_{1,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|e_{1,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} \leq C \int_0^T \left| \int_{\partial\omega_\varepsilon} Q_5 e_{1,\varepsilon} \, \mathrm{d}s \right| \, \mathrm{d}t + C \Big(\|Q_1\|_{L^2(0,T;H^{-1}(\Omega))} \\ + \|Q_2 \chi_{\omega_\varepsilon}\|_{L^2(0,T;H^{-1}(\Omega))} + \|Q_3 \chi_{\omega_\varepsilon}\|_{L^2(0,T;H^{-1}(\Omega))} + \|Q_4\|_{L^2(0,T;H^{-1}(\Omega))} \Big) \|e_{1,\varepsilon}\|_{L^2(0,T;H_0^1(\Omega))}. \end{aligned} \tag{8.15}$$

Here and in the sequel, C is used to denote any constant (independent of ε), that may change from place to place. Using the regularity of ∇v_0 and the change of variables $x = x_0 + \varepsilon y$, we obtain that

$$\int_0^T \left| \int_{\partial\omega_\varepsilon} Q_5 e_{1,\varepsilon} \, \mathrm{d}s \right| \, \mathrm{d}t \leq C\varepsilon^d \|v_0\|_{L^2(0,T;W^{2,\infty}(\tilde{\Omega}))} \left(\int_0^T \int_{\partial\omega} |e_{1,\varepsilon}(\varepsilon y, t)|^2 \, \mathrm{d}s(y) \, \mathrm{d}t \right)^{1/2}. \tag{8.16}$$

By the trace theorem and the change of variables $y = \varepsilon^{-1}(x - x_0)$, it comes

$$\int_0^T \int_{\partial\omega} |e_{1,\varepsilon}(\varepsilon y, t)|^2 \, \mathrm{d}s(y) \, \mathrm{d}t \leq C \int_0^T \left(\varepsilon^{-d} \|e_{1,\varepsilon}\|^2_{L^2(\omega_\varepsilon)} + \varepsilon^{2-d} \|\nabla e_{1,\varepsilon}\|^2_{L^2(\omega_\varepsilon)} \right) \mathrm{d}t.$$

Hence, using the Sobolev inclusion $H^1(\Omega) \subset L^6(\Omega)$ (since $d = 2$ or 3) and the Hölder inequality, we obtain that

$$\int_0^T \int_{\partial\omega} |e_{1,\varepsilon}(\varepsilon y, t)|^2 \, \mathrm{d}s(y) \, \mathrm{d}t \leq C \int_0^T \left(\varepsilon^{-d/3} \|e_{1,\varepsilon}\|^2_{H^1(\Omega)} + \varepsilon^{2-d} \|\nabla e_{1,\varepsilon}\|^2_{L^2(\omega_\varepsilon)} \right) \mathrm{d}t.$$

From (8.16) and the above equation, it follows

$$\int_0^T \left| \int_{\partial\omega_\varepsilon} Q_5 e_{1,\varepsilon} \, \mathrm{d}s \right| \, \mathrm{d}t \leq C\varepsilon^{\frac{5d}{6}} \|v_0\|_{L^2(0,T;W^{2,\infty}(\tilde{\Omega}))} \|e_{1,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}. \tag{8.17}$$

Applying Lemma 7.2 leads to the following estimate on Q_4:

$$\|Q_4\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C\|\nabla v_0(x_0,\cdot)\|_{H^1(0,T)} \|h_\varepsilon\|_{H^{-1}(\Omega)} = o(\varepsilon^{d/2})\|\nabla v_0(x_0,\cdot)\|_{H^1(0,T)}. \tag{8.18}$$

The Sobolev imbedding $L^{6/5}(\Omega) \subset H^{-1}(\Omega)$ (since $d = 2$ or 3) leads to the inequalities

$$\|Q_2\chi_{\omega_\varepsilon}\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant \|Q_2\chi_{\omega_\varepsilon}\|_{L^2(0,T;L^{6/5}(\Omega))} \leqslant C \left\|\frac{\partial v_0}{\partial t}\right\|_{L^2(0,T;L^\infty(\widetilde{\Omega}))} \varepsilon^{5d/6} \tag{8.19}$$

and

$$\|Q_3\chi_{\omega_\varepsilon}\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C\|Q_3\chi_{\omega_\varepsilon}\|_{L^2(0,T;L^{6/5}(\Omega))} \leqslant C\,\|v_0\|_{L^2(0,T;W^{2,\infty}(\widetilde{\Omega}))}\,\varepsilon^{5d/6}. \tag{8.20}$$

From (2.9), we have that

$$\|Q_1\|_{L^2(0,T;H^{-1}(\Omega))} = o(\varepsilon^{d/2}). \tag{8.21}$$

Gathering (8.15), (8.17)–(8.21), we obtain that

$$\|e_{1,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|e_{1,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} \le o(\varepsilon^{d/2})\|e_{1,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}$$

which, combined with the Young inequality, provides

$$\|e_{1,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|e_{1,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} = o(\varepsilon^{d/2}). \tag{8.22}$$

In order to estimate $e_{2,\varepsilon}$, we consider a smooth function $\theta : \Omega \to \mathbb{R}$ such that $\theta = 0$ in $B(x_0, R)$ and $\theta = 1$ on $\partial\Omega$. Then we set

$$\widetilde{h}_\varepsilon(x,t) = \widehat{h}_\varepsilon(x,t)\theta(x), \tag{8.23}$$

$$\widetilde{e}_{2,\varepsilon}(x,t) = e_{2,\varepsilon}(x,t) + \widetilde{h}_\varepsilon(x,t). \tag{8.24}$$

The function $\widetilde{e}_{2,\varepsilon}$ solves

$$\left\{\begin{array}{rll} -\rho_\varepsilon \dfrac{\partial \widetilde{e}_{2,\varepsilon}}{\partial t} - \operatorname{div}(\alpha_\varepsilon A\nabla\widetilde{e}_{2,\varepsilon}) = & -\rho_\varepsilon \dfrac{\partial \widetilde{h}_\varepsilon}{\partial t} - \operatorname{div}(\alpha_\varepsilon A\nabla\widetilde{h}_\varepsilon) & \text{in } \Omega\times(0,T), \\ \widetilde{e}_{2,\varepsilon} = & 0 & \text{on } \partial\Omega\times(0,T), \\ \widetilde{e}_{2,\varepsilon}(\cdot,T) = & 0 & \text{in } \Omega. \end{array}\right. \tag{8.25}$$

By multiplying by $\widetilde{e}_{2,\varepsilon}$ and integrating by part, we obtain

$$\|\widetilde{e}_{2,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|\widetilde{e}_{2,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} \le C\left(\left\|\frac{\partial\widetilde{h}_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widetilde{h}_\varepsilon\right\|_{L^2(0,T;H^1(\Omega))}\right). \tag{8.26}$$

From (8.24), (8.26), (8.23) and (8.1), successively, it comes:

$$\begin{aligned} \|e_{2,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|e_{2,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} &\le \|\tilde{e}_{2,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|\tilde{e}_{2,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \|\tilde{h}_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\tilde{h}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le C\left(\|\tilde{h}_\varepsilon\|_{H^1(0,T;L^2(\Omega)} + \|\tilde{h}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right) \\ &\le C\,\|\nabla v_0(x_0,\cdot)\|_{H^1(0,T)}\,\|h_\varepsilon\|_{H^1(\Omega\setminus\overline{B(x_0,R)})}. \end{aligned}$$

Then using Lemma 7.2 we derive

$$\|e_{2,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|e_{2,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} = o(\varepsilon^{d/2}). \tag{8.27}$$

Combining (8.22) and (8.27) yields (8.4). □

9. Variation of the bilinear form

This section is devoted to the proof of Lemma 4.1. We study the behavior of the following quantity:

$$\int_0^T (a_\varepsilon - a_0)(u_0, v_\varepsilon)\,\mathrm{d}t = \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{9.1}$$

Adopting the decomposition (8.3), we write

$$\int_0^T (a_\varepsilon - a_0)(u_0, v_\varepsilon)\,\mathrm{d}t = \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla v_0 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla e_\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{9.2}$$

We shall prove later that:

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \varepsilon^d \alpha_0 \int_0^T \nabla u_0(x_0, t) \cdot \mathcal{P}_{\omega, \frac{\alpha_1}{\alpha_0}} \nabla v_0(x_0, t)\,\mathrm{d}t - \varepsilon^d |\omega| (\alpha_1 - \alpha_0) \int_0^T A\nabla u_0(x_0, t) \cdot \nabla v_0(x_0, t)\,\mathrm{d}t + o(\varepsilon^d). \tag{9.3}$$

Besides, we deduce from (8.4) and the Cauchy-Schwarz inequality, that

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla e_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \|\nabla u_0\|_{L^2(0,T;L^\infty(\widetilde{\Omega}))}\ o(\varepsilon^d), \tag{9.4}$$

and from the regularity of u_0 and v_0, that

$$\left| \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla v_0 \,\mathrm{d}x\,\mathrm{d}t - \varepsilon^d |\omega| \int_0^T (\alpha_1 - \alpha_0) A\nabla u_0(x_0, t) \cdot \nabla v_0(x_0, t)\,\mathrm{d}t \right| \leqslant C\varepsilon^{d+1} \|u_0\|_{L^2(0,T;W^{2,\infty}(\widetilde{\Omega}))} \|v_0\|_{L^2(0,T;W^{2,\infty}(\widetilde{\Omega}))}. \tag{9.5}$$

Gathering (9.2)–(9.5) leads to Lemma 4.1.

It remains to prove (9.3). We recall that

$$\widehat{h}_\varepsilon(x, t) = -\varepsilon H\left(\frac{x - x_0}{\varepsilon}\right) \cdot \nabla v_0(x_0, t).$$

Starting from the relation

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla (u_0(x, t) - u_0(x_0, t)) \cdot \nabla \widehat{h}_\varepsilon(x, t)\,\mathrm{d}x\,\mathrm{d}t,$$

integrating by parts and using (8.2), we obtain

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\partial\omega_\varepsilon} (\alpha_1 - \alpha_0)(u_0(x, t) - u_0(x_0, t))(A\nabla \widehat{h}_\varepsilon(x, t) \cdot n)^+ \,\mathrm{d}s(x)\,\mathrm{d}t.$$

Using the change of variables $x = x_0 + \varepsilon y$, we proceed by

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\varepsilon^{d-1}(\alpha_1 - \alpha_0) \int_0^T \int_{\partial\omega} (u_0(x_0+\varepsilon y, t) - u_0(x_0,t))(A\nabla(H(y)\cdot\nabla v_0(x_0,t))\cdot n)^+ \,\mathrm{d}s(y)\,\mathrm{d}t.$$

The regularity of u_0 leads to

$$\int_0^T \int_{\omega_\varepsilon} (\alpha_1 - \alpha_0) A\nabla u_0 \cdot \nabla \widehat{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = -\varepsilon^{d}(\alpha_1 - \alpha_0) \int_0^T \nabla u_0(x_0,t) \cdot \int_{\partial\omega} (A\nabla(H(y)\cdot\nabla v_0(x_0,t))\cdot n)^+ y \,\mathrm{d}s(y)\,\mathrm{d}t + o(\varepsilon^d).$$

Finally, applying Lemma 7.1 yields (9.3).

10. Variation of the term involving the time derivative

This section is devoted to the proof of Lemma 4.2. First we have that

$$\int_0^T \left\langle (\rho_\varepsilon - \rho_0)\frac{\partial u_0}{\partial t},\ v_\varepsilon \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t = \int_0^T \int_{\omega_\varepsilon} (\rho_1 - \rho_0)\frac{\partial u_0}{\partial t}\ v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$

and thus, we can write

$$\int_0^T \left\langle (\rho_\varepsilon - \rho_0)\frac{\partial u_0}{\partial t},\ v_\varepsilon \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t = \varepsilon^d(\rho_1 - \rho_0)|\omega| \int_0^T \frac{\partial u_0}{\partial t}(x_0,t)\ v_0(x_0,t)\,\mathrm{d}t + S_1 + S_2,$$

where

$$S_1 = \int_0^T \int_{\omega_\varepsilon} (\rho_1 - \rho_0)\frac{\partial u_0}{\partial t}(\ v_\varepsilon - v_0)\,\mathrm{d}x\,\mathrm{d}t,$$

$$S_2 = \int_0^T \int_{\omega_\varepsilon} (\rho_1 - \rho_0)\left[\frac{\partial u_0}{\partial t}(x,t)v_0(x,t) - \frac{\partial u_0}{\partial t}(x_0,t)v_0(x_0,t)\right]\mathrm{d}x\,\mathrm{d}t.$$

It stems from the regularity assumptions on u_0 and v_0 that

$$|S_2| \le C\varepsilon^{d+1}\|u_0\|_{H^1(0,T;W^{1,\infty}(\widetilde{\Omega}))}\|v_0\|_{L^2(0,T;W^{1,\infty}(\widetilde{\Omega}))}. \tag{10.1}$$

Moreover, by using the Cauchy-Schwarz inequality in time and the Hölder inequality in space together with the imbedding $H^1(\Omega) \subset L^6(\Omega)$, it comes

$$|S_1| \le C\varepsilon^{5d/6}\|u_0\|_{H^1(0,T;L^\infty(\widetilde{\Omega}))}\|v_\varepsilon - v_0\|_{L^2(0,T;H^1(\widetilde{\Omega}))}.$$

Applying (8.6), it follows

$$|S_1| = O(\varepsilon^{4d/3}), \tag{10.2}$$

which completes the proof.

11. Variation of the linear form

We turn to the variation

$$\int_0^T (\ell_\varepsilon - \ell_0)(v_\varepsilon)\ \mathrm{d}t = \int_0^T \int_{\omega_\varepsilon} (F_1 - F_0)\, v_\varepsilon\ \mathrm{d}x\ \mathrm{d}t.$$

We have that

$$\int_0^T (\ell_\varepsilon - \ell_0)(v_\varepsilon)\ \mathrm{d}t = \varepsilon^d |\omega| \int_0^T (F_1(x_0,t) - F_0(x_0,t))\, v_0(x_0,t)\ \mathrm{d}t + R_1 + R_2, \tag{11.1}$$

where

$$R_1 = \int_0^T \int_{\omega_\varepsilon} (F_1 - F_0)(v_\varepsilon - v_0)\ \mathrm{d}x\ \mathrm{d}t,$$

$$R_2 = \int_0^T \int_{\omega_\varepsilon} \left[(F_1(x,t) - F_0(x,t))\, v_0(x,t) - (F_1(x_0,t) - F_0(x_0,t))\, v_0(x_0,t) \right]\ \mathrm{d}x\ \mathrm{d}t.$$

Using the regularity assumptions on F_0 and F_1, we obtain that

$$|R_2| \leqslant C\varepsilon^{d+1} (\|F_1\|_{L^2(0,T;W^{1,\infty}(\tilde{\Omega}))} + \|F_0\|_{L^2(0,T;W^{1,\infty}(\tilde{\Omega}))}) \|v_0\|_{L^2(0,T;W^{1,\infty}(\tilde{\Omega}))}. \tag{11.2}$$

Besides, thanks to the Cauchy-Schwarz inequality, we have

$$|R_1| \leqslant C\varepsilon^{d/2} (\|F_1\|_{L^2(0,T;L^\infty(\tilde{\Omega}))} + \|F_0\|_{L^2(0,T;L^\infty(\tilde{\Omega}))}) \|v_0 - v_\varepsilon\|_{L^2(0,T;L^2(\tilde{\Omega}))}.$$

Hence, by using (8.5), we derive

$$|R_1| = o(\varepsilon^d). \tag{11.3}$$

Gathering (11.1), (11.2) and (11.3), we obtain Lemma 4.3.

12. Variation of the cost functional

Proof of Theorem 5.1. First, since $J_\varepsilon \in \mathcal{C}(L^2(\Omega); \mathbb{R})$, and

$$X \subset \mathcal{C}([0,T]; L^2(\Omega)),$$

we have that for any $v \in X$, $J_\varepsilon(v) : [0,T] \to \mathbb{R}$ is a continuous function. Therefore,

$$\mathcal{J}_\varepsilon(v) = \int_0^T J_\varepsilon(v(t))\ \mathrm{d}t$$

is well-defined.

Now, we check (2.7) with $\delta\mathcal{J}_1 = 0$. We proceed by the Taylor formula:

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_\varepsilon(u_0) - \int_0^T \langle DJ_\varepsilon(u_0(t)), u_\varepsilon(t) - u_0(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\ \mathrm{d}t$$
$$= \frac{1}{2}\int_0^T D^2 J_\varepsilon(w_\varepsilon(t))(u_\varepsilon(t) - u_0(t), u_\varepsilon(t) - u_0(t))\ \mathrm{d}t,$$

where $w_\varepsilon(t) \in [u_0(t), u_\varepsilon(t)]$ for almost all $t \in [0,T]$. From Lemma 8.3, we have that

$$\|u_\varepsilon(t) - u_0(t)\|_{L^\infty(0,T;L^2(\Omega))} = o(\varepsilon^{d/2}), \tag{12.1}$$

and thus

$$\|w_\varepsilon(t)-u_0(t)\|_{L^\infty(0,T;L^2(\Omega))}=o(\varepsilon^{d/2}).$$

Consequently, for some positive number M, we have

$$\|w_\varepsilon(t)\|_{L^2(\Omega)}\leq M \quad \forall t\in[0,T].$$

From this bound together with (5.1), we derive that

$$\|D^2J_\varepsilon(w_\varepsilon(t))\|_{\mathcal{B}(L^2(\Omega))}\leq C(M) \quad \forall t\in[0,T],$$

which implies, by using (12.1),

$$\left|\int_0^T D^2J_\varepsilon(w_\varepsilon(t))(u_\varepsilon(t)-u_0(t),u_\varepsilon(t)-u_0(t))\;\mathrm{d}t\right|\leq C(M)\int_0^T \|u_\varepsilon(t)-u_0(t)\|^2_{L^2(\Omega)}\;\mathrm{d}t=o(\varepsilon^d). \quad \square$$

Proof of Corollary 5.1.

(1) For the functional

$$J_\varepsilon(u)=\int_\Omega |u-u_d|^2\;\mathrm{d}x,$$

it is obvious that $J_\varepsilon\in\mathcal{C}^2(L^2(\Omega),\mathbb{R})$ and that (5.1) is satisfied, so that we can apply Theorem 5.1. Therefore (2.7) holds true. Since in this case J_ε does not depend on ε, relations (2.8) and (2.9) (with $\delta\mathcal{J}_2=0$) hold true. The regularity condition (3.3) is also fulfilled since u_0 satisfies (3.4) and $u_d\in H^4(B(x_0,R))$. Therefore we can apply Theorem 4.1 and we obtain the asymptotic expansion (4.6).

(2) For the functional

$$J_\varepsilon(u)=\int_\Omega \alpha_\varepsilon|u-u_d|^2\;\mathrm{d}x,$$

Theorem 5.1 can also be applied. Therefore (2.7) holds true. The condition (3.3) is fulfilled for the same reasons as before. Next, we have

$$\mathcal{J}_\varepsilon(u_0)-\mathcal{J}_0(u_0)=\int_0^T\int_{\omega_\varepsilon}(\alpha_1-\alpha_0)|u_0(x,t)-u_d(x)|^2\;\mathrm{d}x\;\mathrm{d}t.$$

From the regularity assumptions on u_0 and u_d, we have that

$$\mathcal{J}_\varepsilon(u_0)-\mathcal{J}_0(u_0)=\int_0^T|\omega_\varepsilon|(\alpha_1-\alpha_0)|u_0(x_0,t)-u_d(x_0)|^2\;\mathrm{d}t+O(\varepsilon^{d+1}),$$

which implies (2.8). Finally, for any $\varphi\in L^2(0,T;H^1_0(\Omega))$,

$$\int_0^T\langle DJ_\varepsilon(u_0(t))-DJ_0(u_0(t)),\varphi(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\;\mathrm{d}t=2\int_0^T\int_{\omega_\varepsilon}(\alpha_1-\alpha_0)(u_0(x,t)-u_d(x))\varphi(x,t)\;\mathrm{d}x\;\mathrm{d}t.$$

From the Cauchy-Schwarz inequality, it comes

$$\int_0^T\langle DJ_\varepsilon(u_0(t))-DJ_0(u_0(t)),\varphi(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\;\mathrm{d}t$$
$$\leq C\int_0^T\left(\int_{\omega_\varepsilon}|u_0(x,t)-u_d(x)|^2\;\mathrm{d}x\right)^{1/2}\left(\int_{\omega_\varepsilon}|\varphi(x,t)|^2\;\mathrm{d}x\right)^{1/2}\mathrm{d}t.$$

Using the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ and the Hölder inequality, we obtain

$$\int_0^T \langle DJ_\varepsilon(u_0(t)) - DJ_0(u_0(t)), \varphi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \mathrm{d}t \le C\varepsilon^{2d/3} \int_0^T \|u_0(t) - u_d\|_{H^1(\Omega)}\|\varphi(t)\|_{H^1(\Omega)}\ \mathrm{d}t.$$

Another application of the Cauchy-Schwarz inequality provides

$$\int_0^T \langle DJ_\varepsilon(u_0(t)) - DJ_0(u_0(t)), \varphi(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \mathrm{d}t \le C\varepsilon^{2d/3}\|u_0 - u_d\|_{L^2(0,T;H^1(\Omega))}\|\varphi\|_{L^2(0,T;H^1(\Omega))},$$

which leads to (2.9). The condition (3.3) can be checked in the same way as in the previous example. □

Proof of Proposition 5.1.

(1) For the functional

$$J_\varepsilon(u) = \int_\Omega \eta(x)A\nabla(u - u_d).\nabla(u - u_d)\ \mathrm{d}x,$$

we easily see that $\mathcal{J}_\varepsilon$ is well-defined on X and fulfills (2.8) with $\delta\mathcal{J}_2 = 0$. The condition (2.9) holds true since J_ε does not depend on ε. Next we consider the variation

$$\left|\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_\varepsilon(u_0) - \int_0^T \langle DJ_\varepsilon(u_0), (u_\varepsilon - u_0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \mathrm{d}t\right| = \int_0^T \int_\Omega \eta(x)A\nabla(u_\varepsilon - u_0).\nabla(u_\varepsilon - u_0)\ \mathrm{d}x\ \mathrm{d}t.$$

The above equation together with (8.10) yield $\delta\mathcal{J}_1 = 0$. We now check (3.3). We have

$$DJ_0(u_0) = -2\,\mathrm{div}\ (\eta A\nabla(u_0 - u_d)).$$

This function belongs to $L^2(0,T;H^4(\Omega_J)) \cap H^2(0,T;L^2(\Omega_J))$ for any $\Omega_J \subset B(x_0,R)$.

(2) For the functional

$$\mathcal{J}_\varepsilon(u) = \int_0^T \int_{\partial\Omega} |u - u_d|^2\ \mathrm{d}s\ \mathrm{d}t,$$

we easily check that $\mathcal{J}_\varepsilon$ is well-defined on X and fulfills (2.8), (2.9) with $\delta\mathcal{J}_2 = 0$. We have that

$$\left|\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_\varepsilon(u_0) - \int_0^T \langle DJ_\varepsilon(u_0), (u_\varepsilon - u_0)\rangle_{(H^1(\Omega))',H^1(\Omega)}\ \mathrm{d}t\right| = \int_0^T \int_{\partial\Omega} (u_\varepsilon - u_0)^2\ \mathrm{d}x\ \mathrm{d}t.$$

It follows from (8.8) and (8.10) that $\delta\mathcal{J}_1 = 0$. The adjoint state v_0 satisfies a non-homogeneous Neumann boundary condition with source term

$$g = 2(u_0 - u_d).$$

The regularity $v_0 \in L^2(0,T;H^6(\widetilde{\Omega}_J)) \cap H^3(0,T;L^2(\widetilde{\Omega}_J))$, for some suitable $\widetilde{\Omega}_J$, stems from Lemma 6.1. □

Part 2. Topological sensitivity analysis for hyperbolic problems

13. Setting of the problem

With the same notations as before, we consider now the wave equation:

$$\begin{cases} \rho_\varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial t^2} - \operatorname{div}(\alpha_\varepsilon A \nabla u_\varepsilon) = F_\varepsilon & \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(\cdot,0) = \frac{\partial u_\varepsilon}{\partial t}(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{13.1}$$

The corresponding variational formulation for

$$X = \mathcal{C}([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}^2([0,T]; H^{-1}(\Omega)), \tag{13.2}$$

$$u_\varepsilon \in X_0 = \left\{ u \in X, u(.,0) = \frac{\partial u_\varepsilon}{\partial t}(\cdot,0) = 0 \right\}$$

reads

$$\int_0^T \left\langle \rho_\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2}, v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}t + \int_0^T a_\varepsilon(u_\varepsilon, v)\,\mathrm{d}t = \int_0^T \ell_\varepsilon(v)\,\mathrm{d}t \quad \forall v \in X, \tag{13.3}$$

with the bilinear form a_ε and the linear functional ℓ_ε defined by (2.3) and (2.4). We write (13.3) in the general form (1.1) by setting

$$\mathcal{A}_\varepsilon(u,v) = \int_0^T \left(\left\langle \rho_\varepsilon \frac{\partial^2 u}{\partial t^2}, v \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + a_\varepsilon(u,v) \right) \mathrm{d}t,$$

$$\mathcal{L}_\varepsilon(v) = \int_0^T \ell_\varepsilon(v)\,\mathrm{d}t.$$

We consider a cost functional of the form (2.5) satisfying (2.6), (2.7), (2.8) and such that

$$\| DJ_0(u_0) - DJ_\varepsilon(u_0) \|_{W^{1,1}(0,T;H^{-1}(\Omega))} = o(\varepsilon^{d/2}). \tag{13.4}$$

The adjoint state $v_\varepsilon \in X$ defined by (1.4) solves:

$$\int_0^T \left\langle \rho_\varepsilon \frac{\partial^2 \varphi}{\partial t^2}, v_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}t + \int_0^T a_\varepsilon(\varphi, v_\varepsilon)\,\mathrm{d}t = -\int_0^T DJ_\varepsilon(u_0)\varphi\,\mathrm{d}t \qquad \forall \varphi \in X_0. \tag{13.5}$$

The associated strong formulation reads:

$$\begin{cases} \rho_\varepsilon \dfrac{\partial^2 v_\varepsilon}{\partial t^2} - \operatorname{div}(\alpha_\varepsilon A \nabla v_\varepsilon) = -DJ_\varepsilon(u_0) & \text{in } \Omega \times (0,T), \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ v_\varepsilon(\cdot,T) = \dfrac{\partial v_\varepsilon}{\partial t}(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{13.6}$$

14. Regularity assumptions

For notational simplicity, we define the differential operator

$$\Lambda : u \mapsto \operatorname{div}(\alpha_0 A \nabla u). \tag{14.1}$$

The needed regularity on the direct and adjoint solutions can be obtained from different sets of assumptions. The following one is chosen merely as an example:

$$F_0 \in \mathcal{C}^6([0,T];L^2(\Omega)) \cap \bigcap_{j=0}^{4} \mathcal{C}^j([0,T];H^{5-j}(\Omega)), \tag{14.2}$$

$$F_0, \Lambda F_0 \text{ and } \Lambda^2 F_0 \text{ vanish on } \partial\Omega, \tag{14.3}$$

$$F_1 \in L^2(0,T;W^{1,\infty}(B(x_0,R))), \qquad R>0, \tag{14.4}$$

$$DJ_0(u_0) \in \mathcal{C}^6([0,T];L^2(\Omega)) \cap \bigcap_{j=0}^{4} \mathcal{C}^j([0,T];H^{5-j}(\Omega)), \tag{14.5}$$

$$DJ_0(u_0), \Lambda DJ_0(u_0) \text{ and } \Lambda^2 DJ_0(u_0) \text{ vanish on } \partial\Omega. \tag{14.6}$$

The conditions (14.2)–(14.4) are assumed throughout all this part of the paper, whereas the conditions (14.5) and (14.6) will be checked later for some examples of cost functional. The following result is proved in Section 17.

Proposition 14.1. Assume that u_0 and v_0 solve (13.1) and (13.6), respectively, for $\varepsilon = 0$, and that the regularity assumptions (14.2)–(14.6) hold. Then

$$u_0 \in \mathcal{C}^j([0,T];H^{7-j}(\Omega)) \qquad \forall j = 0,...,7, \tag{14.7}$$

$$u_0, \Lambda u_0 \text{ and } \Lambda^2 u_0 \text{ vanish on } \partial\Omega, \tag{14.8}$$

$$v_0 \in \mathcal{C}^j([0,T];H^{7-j}(\Omega)) \qquad \forall j = 0,...,7, \tag{14.9}$$

$$v_0, \Lambda v_0 \text{ and } \Lambda^2 v_0 \text{ vanish on } \partial\Omega. \tag{14.10}$$

15. Main result

The following lemmas are proved in Section 17. The polarization matrix $\mathcal{P}_{\omega,\frac{\alpha_1}{\alpha_0}}$ involved in Lemma 15.1 is identical to that defined in the first part (see Sect. 4).

Lemma 15.1. *Assume that the bilinear form a_ε is defined by (2.3), that u_0 and v_ε solve (13.1) and (13.6), respectively, that we have the regularity assumptions (14.2)–(14.6) and that (13.4) holds true.*

Then we have

$$\int_0^T (a_\varepsilon - a_0)(u_0, v_\varepsilon)\,\mathrm{d}t = \varepsilon^d\ \delta a + o(\varepsilon^d), \tag{15.1}$$

with

$$\delta a = \alpha_0 \int_0^T \nabla u_0(x_0,t)\cdot \mathcal{P}_{\omega,\frac{\alpha_1}{\alpha_0}} \nabla v_0(x_0,t)\,\mathrm{d}t.$$

Lemma 15.2. *Assume that u_0 and v_ε solve (13.1) and (13.6), respectively, that we have the regularity assumptions (14.2)–(14.6) and that (13.4) holds true.*

Then, we have

$$\int_0^T \left\langle (\rho_\varepsilon - \rho_0)\frac{\partial^2 u_0}{\partial t^2},\ v_\varepsilon \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t = \varepsilon^d\ \delta\rho + o(\varepsilon^d), \tag{15.2}$$

with

$$\delta\rho = -(\rho_1-\rho_0)|\omega| \int_0^T \frac{\partial u_0}{\partial t}(x_0,t)\ \frac{\partial v_0}{\partial t}(x_0,t)\,\mathrm{d}t.$$

Lemma 15.3. *Assume that the linear functional ℓ_ε is defined by (2.4), that u_0 and v_ε solve (13.1) and (13.6), respectively, that we have the regularity assumptions (14.2)–(14.6) and that (13.4) holds true.*

Then, we have

$$\int_0^T (\ell_\varepsilon - \ell_0)(v_\varepsilon)\,\mathrm{d}t = \varepsilon^d\ \delta\ell + o(\varepsilon^d), \tag{15.3}$$

with

$$\delta\ell = |\omega| \int_0^T (F_1(x_0,t) - F_0(x_0,t))\, v_0(x_0,t)\ \mathrm{d}t.$$

As a consequence of Proposition 1.1 and the above lemmas, we obtain the following theorem.

Theorem 15.1. *Assume that the cost functional $\mathcal{J}_\varepsilon$ satisfies (2.5)–(2.8) and (13.4). Suppose moreover that u_0 and v_0 solve (13.1) and (13.6), respectively, for $\varepsilon = 0$, and that the regularity assumptions (14.2)–(14.6) hold. Then we have the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = \varepsilon^d \Bigg[-(\rho_1-\rho_0)|\omega| \int_0^T \frac{\partial u_0}{\partial t}(x_0,t)\ \frac{\partial v_0}{\partial t}(x_0,t)\,\mathrm{d}t + \alpha_0 \int_0^T \nabla u_0(x_0,t)\cdot \mathcal{P}_{\omega,\frac{\alpha_1}{\alpha_0}} \nabla v_0(x_0,t)\,\mathrm{d}t$$
$$- |\omega| \int_0^T (F_1(x_0,t) - F_0(x_0,t))\, v_0(x_0,t)\ \mathrm{d}t + \delta\mathcal{J}_1 + \delta\mathcal{J}_2 \Bigg] + o(\varepsilon^d). \tag{15.4}$$

16. Examples of cost functional

We consider the same examples as in the first part. The proofs, which are similar, are omitted.

Theorem 16.1. *Theorem 5.1 is valid with the current notations,* i.e. *X being defined by (13.2).*

Corollary 16.1. *The asymptotic expansion (15.4) holds true for the following cost functionals.*

(1) *For the functional*

$$J_\varepsilon(u) = \int_\Omega |u - u_d|^2\ \mathrm{d}x \tag{16.1}$$

with

$$u_d \in H^5(\Omega) \ \text{ and } \ u_d, \Lambda u_d, \Lambda^2 u_d \text{ vanishing on } \partial\Omega,$$

the operator Λ being defined by (14.1), we have $\delta\mathcal{J}_1 = 0$ and $\delta\mathcal{J}_2 = 0$. We recall that this functional cannot be considered in the case of a hole.

(2) *For the functional*

$$J_\varepsilon(u) = \int_\Omega \alpha_\varepsilon |u - u_d|^2\ \mathrm{d}x \tag{16.2}$$

with

$$u_d \in H^5(\Omega) \ \text{ and } \ u_d, \Lambda u_d, \Lambda^2 u_d \text{ vanishing on } \partial\Omega,$$

we have $\delta\mathcal{J}_1 = 0$ and

$$\delta\mathcal{J}_2 = (\alpha_1 - \alpha_0)|\omega| \int_0^T |u_0(x_0,t) - u_d(x_0)|\ \mathrm{d}t.$$

Proposition 16.1. The asymptotic expansion (15.4) holds true for the following cost functionals.

(1) For the functional

$$J_\varepsilon(u) = \int_\Omega \eta(x) A\nabla(u-u_d).\nabla(u-u_d)\ \mathrm{d}x \tag{16.3}$$

where $u_0, u_d \in \mathcal{C}^6([0,T]; H^2(\Omega)) \cap \bigcap_{j=0}^{4} \mathcal{C}^j([0,T]; H^{7-j}(\Omega))$,

$$\Lambda^j(u_0 - u_d) \text{ vanishes on } \partial\Omega \text{ for } j = 1, 2, 3, \tag{16.4}$$

and η is a smooth ($\mathcal{C}^\infty$) function whose support does not contain x_0, we have $\delta\mathcal{J}_1 = 0$ and $\delta\mathcal{J}_2 = 0$.

(2) If we replace in (13.1) the Dirichlet boundary condition on $\partial\Omega$ by the Neumann boundary condition (for instance), then we can consider the functional

$$\mathcal{J}_\varepsilon(u) = \int_0^T \eta(t) \int_{\partial\Omega} |u - u_d|^2 \, \mathrm{d}s \, \mathrm{d}t \tag{16.5}$$

where $u_{0|\partial\Omega}, u_d \in \mathcal{C}^8([0,T]; H^{-1/2}(\partial\Omega)) \cap \bigcap_{j=2}^{4} \mathcal{C}^j([0,T]; H^{11/2-j}(\partial\Omega))$ and η is a smooth function whose support is contained in $[0,T)$. Then we have $\delta\mathcal{J}_1 = 0$ and $\delta\mathcal{J}_2 = 0$.

Remark 16.1. By virtue of Lemma 17.3, a sufficient condition for (16.4) to be fulfilled is

$$F_0 \in \mathcal{C}^8([0,T]; L^2(\Omega)) \cap \bigcap_{j=0}^{6} \mathcal{C}^j([0,T]; H^{7-j}(\Omega)), \tag{16.6}$$

$$\Lambda^j F_0 \text{ vanishes on } \partial\Omega \text{ for } j = 0, ..., 3, \tag{16.7}$$

$$\Lambda^j u_d \text{ vanishes on } \partial\Omega \text{ for } j = 0, ..., 3.$$

Remark 16.2. In the second case, due to the different nature of the boundary condition, one has slightly different regularity properties. Actually, Lemma 17.2 still holds true when an homogeneous Neumann boundary condition is applied on $\partial\Omega$, which straightforwardly leads to an analogon to Lemma 17.3. Therefore, the required regularity on u_0 is guaranteed for instance if the conditions (16.6) and (16.7) are fulfilled. Concerning the regularity of the adjoint state, *i.e.* to prove that (14.9) is satisfied with the assumptions made, one has to deal with a nonhomogeneous Neumann boundary condition. This is done with the help of an adaptation of Lemma 17.3 relying on a lifting of the boundary condition and a weakening of the compatibility conditions. These latter ones can be written in a form involving values of the right hand side of the PDE together with its space and time derivatives at the initial time only (the final time T for the adjoint equation). They are satisfied by construction thanks to the cut-off function η.

17. Proofs

17.1. Preliminary lemmas

We first recall two classical results. Proofs can be found in [12].

Lemma 17.1. *For $0 \le \varepsilon < \varepsilon_0$ (ε_0 sufficiently small), let $Q_\varepsilon \in W^{1,1}(0,T; H^{-1}(\Omega))$ and z_ε be the solution of*

$$\begin{cases} \rho_\varepsilon \dfrac{\partial^2 z_\varepsilon}{\partial t^2} - \operatorname{div}(\alpha_\varepsilon A \nabla z_\varepsilon) = Q_\varepsilon & \text{in } \Omega \times (0,T), \\ z_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ z_\varepsilon(\cdot,0) = \dfrac{\partial z_\varepsilon}{\partial t}(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{17.1}$$

There exists a constant $C > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0)$,

$$\|z_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \left\| \frac{\partial z_\varepsilon}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} \le C \|Q_\varepsilon\|_{W^{1,1}(0,T;H^{-1}(\Omega))}.$$

Lemma 17.2. *Let $Q \in \mathcal{C}^1([0,T]; L^2(\Omega))$, $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $z_1 \in H_0^1(\Omega)$, and z be the solution of*

$$\left\{ \begin{array}{rll} \rho_0 \dfrac{\partial^2 z}{\partial t^2} - \Lambda z &= Q & \text{in } \Omega \times (0,T), \\ z &= 0 & \text{on } \partial\Omega \times (0,T), \\ z(\cdot,0) &= z_0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t}(\cdot,0) &= z_1 & \text{in } \Omega. \end{array} \right. \tag{17.2}$$

Then

$$z \in \mathcal{C}([0,T]; H^2(\Omega)) \cap \mathcal{C}^1([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^2([0,T]; L^2(\Omega)).$$

This latter result can be generalized as follows.

Lemma 17.3. *Let p be a nonnegative integer and $Q \in \mathcal{C}^{2p+2}([0,T]; L^2(\Omega)) \cap \bigcap_{j=0}^{2p} \mathcal{C}^j([0,T]; H^{2p+1-j}(\Omega))$ with*

$$\Lambda^j Q \text{ vanishing on } \partial\Omega \text{ for } j = 0, ..., p.$$

Let z be the solution of

$$\left\{ \begin{array}{rll} \rho_0 \dfrac{\partial^2 z}{\partial t^2} - \Lambda z &= Q & \text{in } \Omega \times (0,T), \\ z &= 0 & \text{on } \partial\Omega \times (0,T), \\ z(\cdot,0) &= \dfrac{\partial z}{\partial t}(\cdot,0) = 0 & \text{in } \Omega. \end{array} \right. \tag{17.3}$$

Then

$$z \in \mathcal{C}^j([0,T]; H^{2p+3-j}(\Omega)) \qquad \forall j = 0, ..., 2p+3,$$
$$\Lambda^j z \text{ vanishes on } \partial\Omega \text{ for } j = 0, ..., p.$$

Proof. We introduce the family of auxiliary functions

$$w_j = \frac{\partial^j z}{\partial t^j}, \qquad j = 0, ..., 2p+1. \tag{17.4}$$

Using (17.3) and (17.4), it can be checked that w_j solves:

$$\left\{ \begin{array}{rll} \rho_0 \dfrac{\partial^2 w_j}{\partial t^2} - \Lambda w_j &= \dfrac{\partial^j Q}{\partial t^j} & \text{in } \Omega \times (0,T), \\ w_j &= 0 & \text{on } \partial\Omega \times (0,T), \\ w_j(\cdot,0) &= B_j & \text{in } \Omega, \\ \dfrac{\partial w_j}{\partial t}(\cdot,0) &= B_{j+1} & \text{in } \Omega, \end{array} \right. \tag{17.5}$$

with

$$B_{2i} = \sum_{k=0}^{i-1} \rho_0^{k-i} \Lambda^{i-k-1} \frac{\partial^{2k} Q}{\partial t^{2k}}(0), \qquad i = 0, ..., p,$$
$$B_{2i+1} = \sum_{k=0}^{i-1} \rho_0^{k-i} \Lambda^{i-k-1} \frac{\partial^{2k+1} Q}{\partial t^{2k+1}}(0), \qquad i = 0, ..., p.$$

Lemma 17.2 yields

$$w_{2p+1} \in \mathcal{C}([0,T]; H^2(\Omega)) \cap \mathcal{C}^1([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^2([0,T]; L^2(\Omega)),$$

which implies by integration

$$w_{2p} \in \mathcal{C}^1([0,T]; H^2(\Omega)) \cap \mathcal{C}^2([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^3([0,T]; L^2(\Omega)).$$

Furthermore, we have

$$-\Lambda w_{2p} = \frac{\partial^{2p} Q}{\partial t^{2p}} - \rho_0 \frac{\partial^2 w_{2p}}{\partial t^2} \in \mathcal{C}([0,T]; H_0^1(\Omega)),$$

from which it follows that

$$w_{2p} \in \mathcal{C}([0,T]; H^3(\Omega)).$$

We then obtain by bootstrapping that

$$w_0 = z \in \mathcal{C}^j([0,T]; H^{2p+3-j}(\Omega)) \qquad \forall j = 0, ..., 2p+3.$$

By exploiting the first equation of (17.5), one can prove that

$$\Lambda^j z = \rho_0^j w_{2j} - \sum_{k=0}^{j-1} \rho_0^k \frac{\partial^{2k}}{\partial t^{2k}} \Lambda^{j-k-1} Q, \qquad j = 0, ..., p.$$

Due to the hypotheses, the above function vanishes on $\partial\Omega$. This completes the proof. □

Proof of Proposition 14.1. It is an application of Lemma 17.3 with $p = 2$. □

17.2. **Main estimate**

Let us consider the function e_ε such that

$$v_\varepsilon = v_0 + \widehat{h}_\varepsilon + e_\varepsilon, \tag{17.6}$$

with $\widehat{h}_\varepsilon$ defined by (8.2).

Lemma 17.4. *There holds*

$$\|e_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} = o(\varepsilon^{d/2}).$$

Proof. We easily check that e_ε solves

$$\left\{\begin{array}{rll} \rho_1 \dfrac{\partial^2 e_\varepsilon}{\partial t^2} - \alpha_1 \operatorname{div}(A\nabla e_\varepsilon) &= Q_1 + Q_2 + Q_3 + Q_4 & \text{in } \omega_\varepsilon \times (0,T), \\ \rho_0 \dfrac{\partial^2 e_\varepsilon}{\partial t^2} - \alpha_0 \operatorname{div}(A\nabla e_\varepsilon) &= Q_1 + Q_4 & \text{in } (\Omega \setminus \overline{\omega_\varepsilon}) \times (0,T), \\ e_\varepsilon^+ &= e_\varepsilon^- & \text{on } \partial\omega_\varepsilon \times (0,T), \\ \alpha_1 (A\nabla e_\varepsilon \cdot n)^+ - \alpha_0 (A\nabla e_\varepsilon \cdot n)^- &= Q_5 & \text{on } \partial\omega_\varepsilon \times (0,T), \\ e_\varepsilon &= -\widehat{h}_\varepsilon & \text{on } \partial\Omega \times (0,T), \\ e_\varepsilon(\cdot,T) &= \dfrac{\partial e_\varepsilon}{\partial t}(\cdot,T) = 0 & \text{in } \Omega, \end{array}\right. \tag{17.7}$$

where

$$Q_1 = DJ_0(u_0) - DJ_\varepsilon(u_0), \quad Q_2 = -(\rho_1 - \rho_0)\frac{\partial^2 v_0}{\partial t^2},$$

$$Q_3 = (\alpha_1 - \alpha_0) \operatorname{div}(A\nabla v_0), \quad Q_4 = -\rho_\varepsilon \frac{\partial^2 \widehat{h}_\varepsilon}{\partial t^2},$$

and for all $(x,t) \in \partial\omega_\varepsilon \times (0,T)$,

$$Q_5(x,t) = -(\alpha_1 - \alpha_0)\left(A\left[\nabla v_0(x,t) - \nabla v_0(x_0,t)\right] \cdot n\right).$$

Again we split

$$e_\varepsilon = e_{1,\varepsilon} + e_{2,\varepsilon}$$

with

$$\left\{\begin{array}{ll} \rho_1 \dfrac{\partial^2 e_{1,\varepsilon}}{\partial t^2} - \alpha_1 \text{ div } (A\nabla e_{1,\varepsilon}) = Q_1 + Q_2 + Q_3 + Q_4 & \text{in } \omega_\varepsilon \times (0,T), \\ \rho_0 \dfrac{\partial^2 e_{1,\varepsilon}}{\partial t^2} - \alpha_0 \text{ div } (A\nabla e_{1,\varepsilon}) = Q_1 + Q_4 & \text{in } (\Omega \setminus \overline{\omega_\varepsilon}) \times (0,T), \\ e_{1,\varepsilon}^+ = e_{1,\varepsilon}^- & \text{on } \partial\omega_\varepsilon \times (0,T), \\ \alpha_0 \left(A\nabla e_{1,\varepsilon} \cdot n\right)^+ - \alpha_1 \left(A\nabla e_{1,\varepsilon} \cdot n\right)^- = Q_5 & \text{on } \partial\omega_\varepsilon \times (0,T), \\ e_{1,\varepsilon} = 0 & \text{on } \partial\Omega \times (0,T), \\ e_{1,\varepsilon}(\cdot,T) = \dfrac{\partial e_{1,\varepsilon}}{\partial t}(\cdot,T) = 0 & \text{in } \Omega, \end{array}\right. \tag{17.8}$$

and

$$\left\{\begin{array}{ll} \rho_\varepsilon \dfrac{\partial^2 e_{2,\varepsilon}}{\partial t^2} - \text{ div } (\alpha_\varepsilon A\nabla e_{2,\varepsilon}) = 0 & \text{in } \Omega \times (0,T), \\ e_{2,\varepsilon} = -\widehat{h}_\varepsilon & \text{on } \partial\Omega \times (0,T), \\ e_{2,\varepsilon}(\cdot,T) = \dfrac{\partial e_{2,\varepsilon}}{\partial t}(\cdot,T) = 0 & \text{in } \Omega. \end{array}\right. \tag{17.9}$$

We first estimate $e_{1,\varepsilon}$. Lemma 17.1 provides

$$\|e_{1,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} + \left\|\frac{\partial e_{1,\varepsilon}}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} \le C\|Q_\varepsilon\|_{W^{1,1}(0,T;H^{-1}(\Omega))}, \tag{17.10}$$

with

$$Q_\varepsilon := Q_1 + Q_4 + (Q_2 + Q_3)\chi_{\omega_\varepsilon} + \hat{Q}_5, \tag{17.11}$$

and for almost all $t \in [0,T]$ and all $\varphi \in H_0^1(\Omega)$,

$$\langle \hat{Q}_5(.,t), \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} := \int_{\partial\omega_\varepsilon} Q_5(x,t)\varphi \,\mathrm{d}x.$$

Using (13.4), (14.9) and the estimate (7.10) on h_ε, we obtain that

$$\|Q_1 + Q_4\|_{W^{1,1}(0,T;H^{-1}(\Omega))} = o(\varepsilon^{d/2}). \tag{17.12}$$

Furthermore, it results from (14.9) together with the inequality

$$\|\chi_{\omega_\varepsilon}\|_{H^{-1}(\Omega)} \le C\|\chi_{\omega_\varepsilon}\|_{L^{6/5}(\Omega)} \le C\varepsilon^{5d/6}$$

that

$$\|(Q_2 + Q_3)\chi_{\omega_\varepsilon}\|_{W^{1,1}(0,T;H^{-1}(\Omega))} = O(\varepsilon^{5d/6}). \tag{17.13}$$

Finally, we find by similar arguments to the parabolic case that, for almost every $t \in [0,T]$,

$$\|\hat{Q}_5(.,t)\|_{H^{-1}(\Omega)} \le C\varepsilon^{5d/6}\|v_0(.,t)\|_{W^{2,\infty}(\Omega)},$$

$$\left\|\frac{\partial \hat{Q}_5}{\partial t}(.,t)\right\|_{H^{-1}(\Omega)} \le C\varepsilon^{5d/6}\left\|\frac{\partial v_0}{\partial t}(.,t)\right\|_{W^{2,\infty}(\Omega)}.$$

It follows that

$$\|\hat{Q}_5\|_{W^{1,1}(0,T;H^{-1}(\Omega))} \le C\varepsilon^{5d/6}\|v_0\|_{W^{1,1}(0,T;W^{2,\infty}(\Omega))}. \tag{17.14}$$

Gathering (17.10)–(17.14) yields

$$\|e_{1,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} = o(\varepsilon^{d/2}).$$

We now estimate $e_{2,\varepsilon}$. Let us consider again a smooth function $\theta : \Omega \to \mathbb{R}$ such that $\theta = 0$ in $B(x_0, R)$ and $\theta = 1$ on $\partial\Omega$, and set

$$\widetilde{h}_\varepsilon(x,t) = \widehat{h}_\varepsilon(x,t)\theta(x) \quad \text{and} \quad \widetilde{e}_{2,\varepsilon}(x,t) = e_{2,\varepsilon}(x,t) + \widetilde{h}_\varepsilon(x,t).$$

The function $\widetilde{e}_{2,\varepsilon}$ solves

$$\begin{cases} \rho_\varepsilon \dfrac{\partial^2 \widetilde{e}_{2,\varepsilon}}{\partial t^2} - \text{ div } (\alpha_\varepsilon A \nabla \widetilde{e}_{2,\varepsilon}) = \rho_\varepsilon \dfrac{\partial^2 \widetilde{h}_\varepsilon}{\partial t^2} - \text{ div } (\alpha_\varepsilon A \nabla \widetilde{h}_\varepsilon) & \text{in } \Omega \times (0,T), \\ \widetilde{e}_{2,\varepsilon} = 0 & \text{on } \partial\Omega \times (0,T), \\ \widetilde{e}_{2,\varepsilon}(\cdot, T) = \dfrac{\partial \widetilde{e}_{2,\varepsilon}}{\partial t}(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{17.15}$$

Lemma 17.1 provides

$$\|\widetilde{e}_{2,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} \le C \left(\left\| \frac{\partial^2 \widetilde{h}_\varepsilon}{\partial t^2} \right\|_{W^{1,1}(0,T;H^{-1}(\Omega))} + \left\| \widetilde{h}_\varepsilon \right\|_{W^{1,1}(0,T;H^1(\Omega))} \right).$$

Then, straightforward calculations lead to

$$\|\tilde{e}_{2,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} \le C \, \|\nabla v_0(x_0, \cdot)\|_{W^{3,1}(0,T)} \, \|h_\varepsilon\|_{H^1(\Omega \setminus \overline{B(x_0,R)})} = o(\varepsilon^{d/2}).$$

Next,

$$\begin{aligned} \|e_{2,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} &\le \|\tilde{e}_{2,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} + C \|\hat{h}_\varepsilon\|_{L^\infty(0,T;H^1(\Omega \setminus \overline{B(x_0,R)}))} \\ &\le o(\varepsilon^{d/2}) + C \, \|\nabla v_0(x_0, \cdot)\|_{L^\infty(0,T)} \, \|h_\varepsilon\|_{H^1(\Omega \setminus \overline{B(x_0,R)})} \\ &\le o(\varepsilon^{d/2}). \end{aligned}$$

This latter inequality stems from the imbedding $H^3(0,T) \subset L^\infty(0,T)$ together with Lemma 7.2. This completes the proof. □

17.3. Estimates on the direct and adjoint state

As a consequence of Lemma 17.4, the estimates provided in Lemmas 8.2 and 8.3 remain valid in this context.

17.4. Proof of Theorem 15.1

We shall prove Lemmas 15.1, 15.2 and 15.3, which lead straightforwardly to the theorem. On the basis of Lemma 17.4, Lemmas 15.1 and 15.3 can be proved following the same reasoning as in the first part. Therefore we only present the proof of Lemma 15.2. We make the splitting:

$$\int_0^T \left\langle (\rho_\varepsilon - \rho_0) \frac{\partial^2 u_0}{\partial t^2}, \; v_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}t = \varepsilon^d (\rho_1 - \rho_0) |\omega| \int_0^T \frac{\partial^2 u_0}{\partial t^2}(x_0, t) \; v_0(x_0, t) \, \mathrm{d}t + S_1 + S_2, \tag{17.16}$$

with

$$S_1 = \int_0^T \int_{\omega_\varepsilon} (\rho_1 - \rho_0) \frac{\partial^2 u_0}{\partial t^2} (\; v_\varepsilon - v_0) \, \mathrm{d}x \, \mathrm{d}t,$$

$$S_2 = \int_0^T \int_{\omega_\varepsilon} (\rho_1 - \rho_0) \left[\frac{\partial^2 u_0}{\partial t^2}(x,t) v_0(x,t) - \frac{\partial^2 u_0}{\partial t^2}(x_0,t) v_0(x_0,t) \right] \mathrm{d}x \, \mathrm{d}t.$$

From the regularity assumptions on u_0 and v_0, it comes

$$|S_2| \leq C\varepsilon^{d+1}\|u_0\|_{H^2(0,T;W^{1,\infty}(\Omega))}\|v_0\|_{L^2(0,T;W^{1,\infty}(\Omega))}. \tag{17.17}$$

Moreover, applying the Cauchy-Schwarz inequality in time and the Hölder inequality in space together with the imbedding $H^1(\Omega) \subset L^6(\Omega)$ yields

$$|S_1| \leq C\varepsilon^{5d/6}\|u_0\|_{H^2(0,T;L^\infty(\Omega))}\|v_\varepsilon - v_0\|_{L^2(0,T;H^1(\Omega))}.$$

In view of (8.6), we get that

$$|S_1| = O(\varepsilon^{4d/3}). \tag{17.18}$$

The proof of Lemma 15.2 is completed by gathering (17.16)–(17.18) as well as integrating by parts.

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