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REGULARITY AND VARIATIONALITY OF SOLUTIONS TO HAMILTON-JACOBI EQUATIONS. PART I: REGULARITY (ERRATA)

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Abstract. This errata corrects one error in the 2004 version of this paper [Mennucci, *ESAIM: COCV* **10** (2004) 426–451].

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After the publication of [7] *in 2004, it became clear that the regularity of the form* α *in Lemma* 4.4 *had to be related to the regularity of* K *and of* u_0*; this influences the minimal regularity of* K, u_0*, as needed in hypotheses in Lemma* 4.4*, in Theorem* 4.1*, and in many following relevant discussions. This errata corrects that error; to keep the matter short, all material that is unaffected by the error is omitted; whereas care was taken so that results and discussions that are here corrected retain the original numbering as in* [7].

4.1. Regularity of conjugate points

We will prove in this section results regarding the set of *focal points*; each following result extends to the set Γ of *conjugate points* that is a subset of the focal points.

Theorem 4.1. *Assume* (CC0,H1,H2). *If* u_0, K, H *are regular enough, then, by Lemma* 4.4*, there is a (at most) countable number of* $n-1$ *dimensional submanifolds of* $\mathbb{R} \times O$ *that cover all the sets* G^i*; these submanifolds are graphs of functions* $\lambda_{i,h} : A_{i,h} \to \mathbb{R}$ *(for* $h = 1 \ldots$*) where* $A_{i,h} \subset O$ *are open sets. The least regular case is* $i = n-1$*, and the regularity of the* λ *functions is related to the regularity of* u_0, K, H*, and to the dimension* $\dim(M) = n$ *as in the following table:*

$\dim(M)$	u_0, K	H	λ
$n = 2$	$C^{(R+2,\theta)}$	$C^{(R+2,\theta)}$	$C^{(R,\theta)}$
$n \geq 3$	$C^{(R+2,\theta)}$	$C^{(R+n-1,\theta)} \cap C^n$	$C^{(R,\theta)}$

(4.1)

where $R \in \mathbb{N}, \theta \in [0,1]$.

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We now infer some explanatory results on the regularity of the *focal points* $X(\cup_i G^i)$ from the above theorem.

At the lowest regularity, when $u_0, K \in C^2$, $H \in C^n$, we know that $X \in C^1$ and that the sets G^i are graphs; we conclude that the set of focal points has measure zero. When $u_0, K \in C^{(2,\theta)}$, $H \in C^n \cap C^{(2,\theta)}$, we know that the dimension of the sets G^i does not exceed $n-\theta$; so again we conclude that the set of focal points has dimension at most $n-\theta$. In the case $\theta = 1$, we can obtain the set of all focal points is *rectifiable*; that is, if $u_0, K \in C^{(2,1)}$, $H \in C^n \cap C^{(2,1)}$, then the sets G^i are covered by Lipshitz graphs, so (by known results in [2]) the set of focal points may be covered by $(n-1)$-dimensional C^1 regular submanifolds of M, but for a set of Hausdorff $\mathcal{H}^{n-1}$ measure zero.

When we further raise the regularity, we may suppose that $u_0, K \in C^{s+3}$, $H \in C^{s+n}$ (with $s \in \mathbb{N}$)[1]; then the sets G^i are covered by graphs $(\lambda(y), y)$ inside $\mathbb{R} \times O$ of regularity C^{1+s}; while $X \in C^{2+s}$ (at least), and we restrict it to those graphs; we can then apply Theorem A.4 to state that the focal points are covered by C^{1+s} regular submanifolds of M but for a set of $\mathcal{H}^\alpha$ measure zero, where $\alpha \doteq n-2+1/(1+s)$.

[... *unchanged material deleted*...]

The main tool is this lemma; the complete proof of the lemma is in Section 6.

Lemma 4.4. *We assume that the hypotheses* (CC0,H1,H2) *hold.*

We set the regularity of the data u_0, K, H by defining parameters $R, R' \in \mathbb{N}$, $\theta, \theta' \in [0,1]$, and assuming that

$$u_0 \in C^{(R'+2,\theta')}, \quad K \in C^{(R'+2,\theta')}, \quad H \in C^{(R+2,\theta)};$$

by Proposition 3.7, *the flow $\Phi = (X, P)$ is $C^{(R+1,\theta)}$ regular; and O is a $C^{(R'+1,\theta')} \cup C^{(R+2,\theta)}$ manifold (that is, the least regular of the two).*

Lets fix $i \geq 1$, $i \leq n-1$, and fix a point $(s', y') \in \mathbb{R} \times O$, such that $(s', y') \in G^{(i)}$.

Let $\mathcal{U}$ be a neighbourhood of 0 in $\mathbb{R}^{n-1}$ and let $\phi : \mathcal{U} \to O$ be a local chart to the neighbourhood $\phi(\mathcal{U})$ of $y' = \phi(0)$. The map ϕ has regularity $C^{(R'+1,\theta')} \cup C^{(R+2,\theta)}$. In the following, y will be a point in $\phi(\mathcal{U})$.

To study $G^{(i)}$, we should study the rank of the Jacobian of the map $(t,x) \mapsto X(t, \phi(x))$; since the regularity of X is related only to the regularity of H, it will be useful to decouple this Jacobian in two parts. To this end, we define a n-form α on $\mathbb{R} \times O$, with requirement that $\alpha(t,y) = \alpha(y)$ (that is, α does not depend on t).

Writing $X^{(t,y)}$ for $X(t,y)$, let

$$X^{(t,y)^*}\alpha$$

be the push-forward of α along X; $X^{(t,y)^*}\alpha$ is then a tangent form defined on $T_{X(t,y)}M$; it will be precisely defined in equation (6.2). *We remark that $X^{(t,y)^*}\alpha = 0$ iff $(t,y) \in \bigcup_j G^j$. Note that the pushforward $X^{(t,y)^*}$ is $C^{(R,\theta)}$ regular, while the form α is as regular as TO, that is, α is $C^{(R',\theta')} \cup C^{(R+1,\theta)}$.*

Note that, since X solves an O.D.E., then X and $\frac{\partial}{\partial t}X$ have the same regularity; note moreover that

$$\frac{\partial^j}{\partial t^j}\Big(X^{(s',y')^*}\alpha\Big) = \Big(\frac{\partial^j}{\partial t^j}X\Big)^{(s',y')^*}\alpha$$

since α does not depend on t. So, by hypotheses and by the definition (6.2) *of $X^{(t,y)^*}\alpha$, the forms $X^{(t,y)^*}\alpha$ and $\frac{\partial}{\partial t}(X^{(t,y)^*}\alpha)$ have regularity $C^{(R,\theta)} \cap C^{(R',\theta')}$ (see also Eq.* (6.3)*); the derivates $\frac{\partial^j}{\partial t^j}X^{(s',y')^*}\alpha$ with $j \geq 1$ have regularity $C^{(R-j+1,\theta)} \cup C^{(R',\theta')}$.*

Then, when $R+1 \geq i$, we prove (in Sect. 6*) that*

$$X^{(s',y')^*}\alpha = 0, \quad \frac{\partial}{\partial t}X^{(s',y')^*}\alpha = 0, \quad \cdots \frac{\partial^{i-1}}{\partial t^{i-1}}X^{(s',y')^*}\alpha = 0$$

[1]A similar result may be obtained when $u_0, K \in C^{(s+3,\theta)}$, $H \in C^{(s+n,\theta)}$.

whereas

$$\frac{\partial^i}{\partial t^i} X^{(s',y')^*} \alpha \neq 0.$$

We define eventually the map $F : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ *given by*

$$F(t,x) = \frac{\partial^{i-1}}{\partial t^{i-1}} X^{(t,\phi(x))^*} \alpha;$$

since

$$\frac{\partial}{\partial t} F(t,x) \stackrel{def}{=} \frac{\partial^i}{\partial t^i} X^{(t,\phi(x))^*} \alpha \neq 0$$

the above Dini lemma implies that the set $G^{(i)}$ *is locally covered by the graph of a function* λ_i *defined on a open subset of* O*;* λ *has the same regularity of* F*, so, if* $i = 1$ *then* λ *is in* $C^{R,\theta} \cup C^{(R',\theta')}$ *while for* $i \geq 2$ *it is* $C^{(R-i+2,\theta)} \cup C^{(R',\theta')}$.

The above directly implies Theorem 4.1.
[. . . *all other results are unchanged* . . .]

5. Applications

5.1. **The Cauchy problem**

We show now how the above theorems may be used for the Cauchy problem (1.2)

$$\begin{cases} \frac{\partial}{\partial t} w(t,x') + H'(t,x', \frac{\partial}{\partial x'} w(t,x')) = 0 & \text{for } t > 0,\ x' \in M' \\ w(0,x') = w_0(x') & \forall x' \in M'. \end{cases} \tag{1.2}$$

[. . . *the preliminary discussion is unchanged* . . .]
This improves the results of 4.10, 4.12 and 4.17 in [1]; to provide for an easy comparison, we summarize these results

- if $n' = \dim(M')$, $n = n' + 1$, if $H' \in C^s$ with $s = n \vee 3$ and $w_0 \in C^2$, then the set Γ has measure zero, so the set $\overline{\Sigma}_u = \Sigma \cup \Gamma$ has measure zero;
- if $H, w_0 \in C^{(2,1)}$, then the set Γ is rectifiable, so the set $\overline{\Sigma}_u = \Sigma \cup \Gamma$ is rectifiable;
- and when $H' \in C^{R+1,\theta}$, $w_0 \in C^{R+1,\theta}$, $R \geq 2$, w is continuous, we prove that the Hausdorff dimension of $\Gamma \setminus \Sigma$ is at most β, and moreover $\mathcal{H}^\beta(\Gamma \setminus \Sigma) = 0$ if $\theta = 0$, where $\beta = n' - 1 + 2/(R + \theta)$.

In the counterexample in Section 4.4 in [1], w_0 is $C^{1,1}(M')$ and not $C^2(M')$; so our results close the gap between the counterexample, where w_0 is $C^{1,1}(M')$, and the theorem, where w_0 is $C^2(M')$; and actually, studying the counterexample, it is quite clear that, if w_0 is smoothed to become a $C^2(M')$ function, then the counterexample would not work.

5.2. **Eikonal equation and cutlocus**

As in Section 3.5, consider a smooth Riemannian manifold M, and a closed set $K \subset M$ and let $d_K(x) = d(x, K)$ be the distance to K. We set $u_0 = 0$: then O is the bundle of unit covectors that are normal to TK, and $d_K(x)$ coincides with the *min* solution $u(x)$.

We define

$$\Sigma_{d_K} \stackrel{\text{def}}{=} \{x \mid \nexists \nabla d_K(x)\}$$

If K is C^1, then Σ_{d_K} coincides with Σ as defined in (4.1).
Since d_K is semiconcave in $M \setminus K$, Σ_{d_K} is always rectifiable.

This primal problem is a good test bed to discuss the differences and synergies of the results in this paper and the results in Itoh and Tanaka [4] and Li and Nirenberg [5].

- In the example in Section 3 in [6], there is a curve $K \subset \mathbb{R}^2$, $K \in C^{1,1}$ such that $\overline{\Sigma}_{d_K}$ has positive Lebesgue measure. Note that in this example $\overline{\Sigma}_{d_K} \neq \text{Cut}(K) = \Sigma_{d_K}$, so the cutlocus $\text{Cut}(K)$ is rectifiable (but not closed).

  We do not know if there is a curve $K \in C^{1,1}$ such that $\text{Cut}(K)$ is not rectifiable. (We recall that, by Prop. 14 in [3], $\text{Cut}(K)$ has always measure zero).
- Theorem 4.1 states that if K is C^2, then Γ has measure zero, so by (1.4) and 4.11.4, we obtain that $\overline{\Sigma}_{d_K} = \text{Cut}(K)$ has measure zero; so Theorem 4.1 closes the gap between the counterexample in Section 3 [6] and the previous available results.
- In example in Remark 1.1 in [5], for all $\theta \in (0,1)$ there is a compact curve $K \in C^{2,\theta}$ such that the distance to the cutlocus is not locally Lipschitz; by Theorem 4.1, the cutlocus has dimension at most $n-\theta$.

  We do not know if there exists an example of a compact curve $K \in C^{2,\theta}$ such that $\mathcal{H}^{n-1}(\text{Cut}(K)) = \infty$
- By the results in Itoh and Tanaka [4] and Li and Nirenberg [5], when $K \in C^3$, the distance to the cutlocus is locally Lipschitz and the cutlocus is rectifiable, and moreover (by Cor 1.1 in [5]), for any B bounded $\mathcal{H}^{n-1}(\text{Cut}(K) \cap B) < \infty$. By Theorem 4.1, the set of (non optimal) focal points is rectifiable as well.

5.2.1. *Improvements*

[. . . *the discussion is unchanged* . . .]

Corollary 5.1. *Consider a 2-dimensional smooth Riemannian manifold M; suppose that K is a compact C^{3+s} embedded submanifold.*

Then, for any open bounded set $A \subset M$, the set $A \cap \Gamma$ is C^{s+1}-$M^{1/(s+1)}$-rectifiable: that is, it can be covered by at most countably many C^{s+1} curves, but for a set E such that $\mathcal{M}^{1/(s+1)}(E) = 0$.

6. Proof of 4.4

[. . . *the two lemma are unchanged*. . .]

Now we prove Lemma 4.4.

We want to define the n form α so that α does not depend on t; and so that $\alpha = e_1 \wedge \cdots \wedge e_n$ where the vectors fields $e_{n-i+1} \dots e_n$ span the kernel of $\frac{\partial}{\partial \overline{x}} X$ at the point (s', y') (kernel that we will call V) while $\frac{\partial}{\partial \overline{x}} X$ is full rank on $e_1 \dots e_{n-i}$ (that generate the space W).

One possible way to this is to fix the local chart $\phi : \mathcal{U} \subset \mathbb{R}^{n-1} \to O$, define

$$\hat{e}_1 \overset{\text{def}}{=} \phi^* \frac{\partial}{\partial x_1}, \ \dots \hat{e}_{n-1} \overset{\text{def}}{=} \phi^* \frac{\partial}{\partial x_{n-1}}, \ \hat{e}_n \overset{\text{def}}{=} \frac{\partial}{\partial t}$$

and then choose a $n \times n$ constant matrix A, so that

$$e_h \overset{\text{def}}{=} \sum_k A_{h,k} \hat{e}_k$$

satisfy the requirements.

[. . . *the rest of the proof is unchanged*. . .]

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