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EXISTENCE AND REGULARITY OF MINIMIZERS OF NONCONVEX INTEGRALS WITH $p - q$ GROWTH

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Abstract. We show that local minimizers of functionals of the form

$$\int_\Omega [f(Du(x)) + g(x\,, u(x))]\,\mathrm{d}x, \qquad u \in u_0 + W_0^{1,p}(\Omega),$$

are locally Lipschitz continuous provided f is a convex function with $p - q$ growth satisfying a condition of qualified convexity at infinity and g is Lipschitz continuous in u. As a consequence of this, we obtain an existence result for a related nonconvex functional.

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1. Introduction

We consider the integral functional

$$I(u) = \int_\Omega L(x\,, u(x)\,, Du(x))\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set ($N \geq 2$) and $L\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function satisfying the following growth hypothesis:

$$c_1|\xi|^p - a(x)(1 + |\eta|) \leq L(x\,, \eta\,, \xi) \leq c_2(1 + |\xi|)^q + a(x)(1 + |\eta|) \tag{1.1}$$

where $c_2 \geq c_1 > 0$, $a \in L^\infty_{\mathrm{loc}}(\Omega)$ is a nonnegative function and $1 < p \leq q$. As usual, we say that L has *standard growth* (or *p growth*) if $q = p$ and $p - q$ *growth* when $p < q$. For this integral I, we consider the Dirichlet problem

$$\min\left\{I(u) :\ u \in u_0 + W_0^{1,p}(\Omega)\right\} \tag{$\mathcal{P}$}$$

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where the boundary datum u_0 is in $W^{1,p}(\Omega)$ and $I(u_0) < +\infty$. If $L(x\,,u\,,Du)$ is convex with respect to the gradient variable Du, then solutions to $(\mathcal{P})$ exist by the direct method of the Calculus of Variations. Otherwise the minimum in $(\mathcal{P})$ need not be achieved and, as nonconvex integrals arise from variational models in several branches of applied sciences, the question of establishing which conditions on L, other than convexity, ensures the existence of solutions to $(\mathcal{P})$ has been receiving increasing attention in recent years.

The usual path for proving existence results for nonconvex minimum problems is based on considering the auxiliary integral

$$I^{**}(u) = \int_\Omega L^{**}(x\,,u(x)\,,Du(x))\,\mathrm{d}x \tag{1.2}$$

where $L^{**}(x\,,\eta\,,\xi)$ is the convex envelope of L with respect to ξ, *i.e.* the greatest convex function with respect to ξ satisfying $L^{**}(x\,,\eta,\xi) \leq L(x\,,\eta\,,\xi)$. Under suitable assumptions on L, I^{**} is the relaxed integral of I with respect to weak convergence of Sobolev functions and it has a minimizer v among the feasible functions $u_0 + W_0^{1,p}(\Omega)$. If it happens that the equality $L^{**}(x\,,v\,,Dv) = L(x\,,v\,,Dv)$ holds almost everywhere on Ω for the minimizer v of I^{**}, then it follows immediately from $L^{**} \leq L$ that v is a minimizer of the original integral I as well. Otherwise, one tries to modify v so as to find a new minimizer of I^{**}, say u, satisfying the required equality $L^{**}(x\,,u\,,Du) = L(x\,,u\,,Du)$ almost everywhere. The construction of u out of v is based on solving differential inclusions and the possibility of doing this obviously depends on the properties of L and L^{**}, as well as the regularity of the original minimizer v.

As far as we know, the case of functions L with p growth has been the only case studied so far. For this model, a well developed theory regarding attainment *versus* nonattainment phenomena has been set up in recent years, see for instance the results of [3–7, 16, 26, 27]. We refer to [5] and the references therein for a thorough description of all contributions to the problem. In the most important and simplest case of sum-like functions L of the form $L(x\,,u\,,Du) = f(Du) + g(u)$, this analysis shows that the minimum in $(\mathcal{P})$ is achieved when

(a) the convex envelope f^{**} of f is affine on each connected component of the detachment set $\{f^{**} < f\}$;

(b) g is piecewise monotone and moreover has no strict, local minima unless $f^{**}(0) = f(0)$;

otherwise minimizers do not likely exist, see the nonexistence results of [6, 7, 16, 23] and the examples in [5]. We refer to [5] for a discussion on how this statement for sum-like integrals translates into the general case.

The strategy for establishing existence results in the $p - q$ case goes as in the standard case, but a crucial difficulty arises. As mentioned above, the construction of a new minimizer u out of v for the relaxed integral I^{**} satisfying the equality $L^{**}(x\,,u\,,Du) = L(x\,,u\,,Du)$ almost everywhere requires continuity and differentiability almost everywhere in the classical sense for the original solution v. Both properties are shared by every local minimizer of integrals with p growth, see [3, 17] respectively. In the $p - q$ framework, continuity of minimizers cannot be granted on the ground of the growth assumptions only and classical, almost everywhere differentiability still remains an open problem.

A solution to this problem is to require stronger assumptions on L ensuring local Lipschitz continuity of minimizers of the relaxed problem, a strategy which has been successfully exploited in [2] and more recently in [15] for nonconvex, nonautonomous integrands $L(x\,,\nabla u)$. Moreover, once a minimizer u of the nonconvex problem $(\mathcal{P})$ has been obtained from a local Lipschitz continuous minimizer v of the relaxed problem, then u itself has to be locally Lipschitz continuous as well as it is a minimizer of (1.2). Thus, the crucial link in this chain of reasoning is the proof of local Lipschitz regularity of minimizers of I when L is convex in the gradient variable.

Starting from [14], the regularity of minimizers of integrals where $L = L(x\,,\eta\,,\xi)$ has standard growth but is not supposed to be differentiable with respect to ξ has been extensively studied. Because of this lack of smoothness, the usual ellipticity condition, which is required for $C^{1,\alpha}$ regularity in all classical papers such as [18, 19, 21] where L is of class C^2 with respect to ξ, is replaced at first by a qualified convexity assumption called *p-uniform convexity* which, in the simplest case $L = f(\xi)$, requires that for some $\nu > 0$ the inequality

$$f\left(\frac{\xi + \zeta}{2}\right) \leq \frac{1}{2}f(\xi) + \frac{1}{2}f(\zeta) - \nu(1 + |\xi|^2 + |\zeta|^2)^{\frac{p-2}{2}}|\xi - \zeta|^2 \tag{1.3}$$

holds for every ξ and ζ (see [15]). Later on, this condition has been weakened by assuming the so-called *p-uniform convexity condition at infinity* which means that the previous condition holds only when the segment joining ξ and ζ lies entirely outside some fixed ball $B_R(0)$. By approximating f with smooth functions, it is then possible to prove local Lipschitz regularity of local minimizers if $L = f(\xi)$ or local α-Hölder continuity for all $\alpha < 1$ in the general case, see *e.g.* [8, 12, 15].

As regards the regularity of minimizers of integrals with $p-q$ growth, this was first studied by Marcellini in [24] and many contributions have been given since then, see [1, 13] for a broad list of references. In these papers, the $C^{1,\alpha}$ regularity of minimizers is proved when L is of class C^2 and uniformly elliptic, provided p and q are not too far apart, that is $1 < q/p \leq c(N)$ where $c(N)$ goes asimptotically to 1 as N diverges, see [24]. In particular, if $L = f(\xi)$ is smooth and uniformly elliptic, this regularity is achieved if $q/p < N/(N-2)$, see [25] whereas the values of p and q must be closer in the nonhomogeneous case, namely $q/p < (N+1)/N$, otherwise counterexamples exist, see [13] for $q/p > (N+1)/N$.

In this paper we prove two results. In both of them, the $p-q$ growth of the energy density is assumed.

Our first result regards regularity (Th. 1.1): we prove local Lipschitz continuity of local minimizers of $(\mathcal{P})$ when L is the sum of two terms

$$L(x\,,u\,,Du) = f(Du) + g(x\,,u), \tag{1.4}$$

such that f satisfies the $p-q$ growth hypothesis (1.1), is convex and p-uniformly convex at infinity, see (1.3), and g is a Caratheodory function which is also Lipschitz continuous with respect to u, uniformly with respect to x ranging into a compact subset of Ω. To prove this, we approximate I by a sequence of smooth integrals with p growth and we prove that their minimizers are locally, Lipschitz continuous, uniformly with respect to the sequence (Prop. 2.2). Finally, we prove that the regularity properties of minimizers of the approximating integrals pass to minimizers of the original integral I.

As a consequence of this regularity result, we prove (Th. 5.1) the existence of locally Lipschitz continuous minimizers of $(\mathcal{P})$ when L is given by

$$L(x\,,u\,,Du) = f(Du) + a(x)h(u),$$

where now f is a possibly nonconvex, yet p-uniformly convex at infinity function with $p-q$ growth and h is Lipschitz continuous. As explained above, the corresponding relaxed functional I^{**} has locally Lipschitz continuous minimizers and, exploiting this and relying on the arguments of [3] and [5], we prove the existence of solutions to $(\mathcal{P})$ with the same regularity provided the hypotheses (a) and (b) described above hold.

As regards notation, we denote the euclidean norm and the scalar product in $\mathbb{R}^N$ by $|\cdot|$ and $\langle x, y\rangle$ respectively, the open ball in $\mathbb{R}^N$ with center at x and radius $r > 0$ by $B_r(x)$ and we just write B_r when the center is either $x = 0$ or clear by the context. Moreover, we denote the closed segment in $\mathbb{R}^N$ with endpoints x and y by $[x\,,y]$.

As usual, when $f\colon \mathbb{R}^N \to \mathbb{R}$ is a lower semicontinuous function we denote by f^{**} the convex envelope of f, *i.e.* the largest convex function below f. We use standard notation for function spaces and measures.

After these preliminaries, we now state our main result on local, Lipschitz regularity of minimizers of sum-like integrals (1.4). To this aim, let $1 < p \leq q < +\infty$ and let $f\colon \mathbb{R}^N \to [0\,,+\infty)$ be a continuous function such that

(A1) f is p-uniformly convex at infinity with constants $\nu > 0$ and $R > 0$;

(A2) f has $p-q$ growth, *i.e.* there exist $c_2 \geq c_1 > 0$ such that

$$c_1|\xi|^p \leq f(\xi) \leq c_2\,(1 + |\xi|^q)\,, \qquad \xi \in \mathbb{R}^N.$$

As we shall see in Proposition 3.1 below, the bound from below in (A2) actually follows from (A1). As regards the lower order term, we assume that $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that

(A3) $g(\cdot\,,0) \in L^1(\Omega)$ and there exists a function $a \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$|g(x\,,u) - g(x\,,v)| \leq a(x)|u - v|, \qquad u,\, v \in \mathbb{R},$$

holds for *a.e.* $x \in \Omega$.

For these functions f and g, we consider the variational integral

$$I(u) = \int_{\Omega} [f(Du(x)) + g(x\,, u(x))]\;\mathrm{d}x, \tag{1.5}$$

for those functions $u \in W^{1,1}(\Omega)$ such that the integral is well defined and we recall that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of (1.5) if $f(Du) + g(\cdot\,, u)$ is in $L^1_{\mathrm{loc}}(\Omega)$ and $I(u) \le I(u+\varphi)$ for every $\varphi \in W^{1,1}(\Omega)$ with compact support in Ω. For local minimizers of I, the following regularity result holds.

Theorem 1.1. *Let $f\colon \mathbb{R}^N \to [0\,, +\infty)$ be a convex function satisfying* (A1) *with constants ν and R and* (A2) *with*

$$1 < p \le q < \frac{N+1}{N} p \tag{1.6}$$

and let $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (A3).

Let $u \in W^{1,1}(\Omega)$ be a local minimizer of (1.5)*. Then, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and, for every ball $B_r(x_0) \subset\subset \Omega$, the following estimate holds:*

$$\sup_{x\in B_{r/2}(x_0)} |Du(x)| \le C \left\{ \int_{B_r(x_0)} [1 + f(Du(x)) + g(x, u(x))]\,\mathrm{d}x \right\}^{\alpha}, \tag{1.7}$$

where $C = C(p\,, q\,, N\,, \nu\,, R\,, r\,, \|a\|_\infty)$ and $\alpha = \alpha(p\,, q\,, N)$.

2. Regular integrals and *a priori* Lipschitz regularity

As explained in the Introduction, to prove Theorem 1.1, we first investigate the regularity properties of minimizers of regular integrals, *i.e.* integrals with smooth integrands and standard growth. Indeed, let the integral I be associated with functions f and g satisfying the following stronger properties for some $1 < p < +\infty$:

(H1) there exists a constant $c_3 > 0$ such that

$$0 \le f(\xi) \le c_3\,(1 + |\xi|^p)\,, \qquad \xi \in \mathbb{R}^N;$$

(H2) $f \in C^2(\mathbb{R}^N)$ and there exists $L_1 > 0$ such that

$$\langle D_{\xi\xi} f(\xi)\lambda, \lambda\rangle \ge L_1 \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \qquad \lambda,\, \xi \in \mathbb{R}^N;$$

(H3) $f \in C^2(\mathbb{R}^N)$ and there exists $L_2 > 0$ such that

$$\langle D_{\xi\xi} f(\xi)\lambda, \lambda\rangle \le L_2 \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \qquad \lambda,\, \xi \in \mathbb{R}^N;$$

(H4) $g \in C^2(\Omega\times\mathbb{R})$ and (A3) holds.

As we shall see in Section 3, the hypotheses (H1), (H2) and (H3) imply (A1) and (A2).

We shall prove that local minimizers of regular integrals have Hölder continuous derivatives as well as second order weak derivatives (Lem. 2.1) and that every such minimizer of I in the non regular case satisfies a Moser type inequality (Prop. 2.2). These proofs are similar, except for the lower order term g, to the corresponding proofs of [9, 12, 14] and hence we just outline the mainsteps and we refer to [9] for the details.

Lemma 2.1. *Assume that* (H1),…,(H4) *hold and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of* (1.5)*. Then, $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$.*

Proof. By classical results, u is locally bounded and hence the local $C^{1,\alpha}$ regularity follows from [21]. As to the second derivatives, let

$$\Delta_{k,\tau}v(x)=\frac{v(x+\tau e_k)-v(x)}{\tau},\qquad x\in\Omega\cap(\Omega-\tau e_k),\quad \tau\neq 0,$$

be the difference quotient of $v\colon\Omega\to\mathbb{R}$ in the direction of the unit vector e_k, $k=1,\dots,N$, and set $B_r=B_r(x_0)$ where $B_{3r}\subset\subset\Omega$. As u is a local minimizer, it is a solution of the Euler-Lagrange equation so that

$$\int_\Omega [\langle D_\xi f(Du),D\varphi\rangle+g_u(\cdot\,,u)\varphi]\,\mathrm{d}x=0,$$

holds for every test function $\varphi\in C^1_c(B_{2r})$ where $g(\cdot\,,u)$ obviously stands for the function $x\in\Omega\to g(x\,,u(x))$ and similarly for the derivative $g_u(x\,,u(x))$. Then, choosing $\varphi=\Delta_{k,-\tau}\left(\eta^2\Delta_{k,\tau}u\right)$ as a test function where $\eta\in C^2_c(B_{2r})$ is a suitable cut-off function and arguing as in Theorem 8.1 in [20], we find that

$$\int_{B_r}\left(1+|Du|^2+|Du(x+\tau e_k)|^2\right)^{\frac{p-2}{2}}|\Delta_{k,\tau}(Du)|^2\,\mathrm{d}x \\ \leq\frac{C}{r^2}\int_{B_{3r}}\left(1+|Du|^2\right)^{\frac{p}{2}}\,\mathrm{d}x+C\int_{B_r}|\Delta_{k,\tau}(g_u(\cdot\,,u))||\Delta_{k,\tau}u|\,\mathrm{d}x.$$

for some constant C. As Du is locally bounded, the coefficient of $|\Delta_{k,\tau}(Du)|^2$ in the equation above is bounded from below for every $1<p<+\infty$, so that, letting $\tau\to0$, we conclude that $u\in W^{2,2}_{\mathrm{loc}}(\Omega)$. □

Proposition 2.2. *Let $f\in C^2(\mathbb{R}^N)$ be a convex function satisfying* (A1) *with constants ν and R and* (A2) *with*

$$1<p\leq q<\frac{N+1}{N}p \tag{2.1}$$

and let $g\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy (H4).

Let $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)\cap W^{2,2}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.5). *Then, for every ball $B_r(x_0)\subset\subset\Omega$, the following estimate holds:*

$$\sup_{x\in B_{r/2}(x_0)}|Du(x)|\leq C\left\{\int_{B_r(x_0)}(1+|Du|^p)\,\mathrm{d}x\right\}^\alpha,$$

where $C=C(p\,,q\,,N\,,\nu\,,R\,,r\,,\|a\|_\infty)$ and $\alpha=\alpha(p\,,q\,,N)$.

Proof. The proof is analogous to the proof of Proposition 3.1 of [9] and it is enough we prove the following claim from which the conclusion follows by using (2.1) and Step 2 of the result mentioned above.

Claim. Set $B_r=B_r(x_0)$ and let $\eta\in C^1_c(B_r)$ be a cut-off function such that $0\leq\eta\leq1$. There exists a constant $C=C(p\,,q\,,N\,,\nu\,,R\,,r\,,\|a\|_\infty)$ such that

$$\int_{B_r}\left(1+|Du|^2\right)^{\frac{p}{2}+\delta-1}|D(|Du|^2-R^2)_+|^2\eta^2\,\mathrm{d}x\leq C\frac{\max\{1\,,1+\delta\}}{\min\{1\,,1+\delta\}}\int_{B_r}(1+|Du|^2)^{q-\frac{p}{2}+\delta+1}(\eta^2+|D\eta|^2)\,\mathrm{d}x \tag{2.2}$$

for every $\delta>-1$ where a_+ denotes the positive part of a.

For $\delta>-1$, set $\Phi(t)=(1+R^2+t)^\delta t$, $t\geq0$ and consider

$$\Psi_k(x)=\Phi\left((|Du(x)|^2-R^2)_+\right)D_ku(x),\qquad x\in\Omega,\quad k=1\,,\dots\,,N.$$

Recalling the definition of difference quotients given in Lemma 2.1, we write the Euler-Lagrange equation using

$$\Theta_{k,\tau}(x)=\eta^2(x)\triangle_{k,-\tau}\Psi_k(x),\qquad x\in\Omega,$$

as a test function for a small enough $\tau \neq 0$. Hence, writing Φ as a shorthand for $\Phi\left((|Du|^2 - R^2)_+\right)$, we find

$$\int_{B_r} \langle \triangle_{k,\tau} \left(D_\xi f(Du)\eta^2\right), D\Psi_k\rangle \,\mathrm{d}x = \int_{B_r} \left[\langle 2D_\xi f(Du), \eta D\eta\rangle + g_u(\cdot\,, u)\eta^2\right] \triangle_{k,-\tau}\Psi_k \,\mathrm{d}x.$$

Adding on k, using Einstein's summation convention and letting τ go to zero, we obtain

$$\int_{B_r} D^2_{\xi_i\xi_j} f(Du) D^2_{k,j} u\eta^2 D_i\Psi_k \,\mathrm{d}x = \int_{B_r} \left\{-2D_{\xi_i} f(Du)\eta D_k\eta D_i\Psi_k + \left[2D_{\xi_i} f(Du)\eta D_i\eta + g_u(\cdot\,, u)\eta^2\right] D_k\Psi_k\right\} \mathrm{d}x. \tag{2.3}$$

Now, we estimate from below the left hand side of the equation above. To this aim, we first recall that f, as a p-uniformly convex at infinity function of class C^2, satisfies

$$\langle D^2_{\xi\xi} f(\xi)\lambda, \lambda\rangle \geq c(\nu)(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, \qquad \lambda \in \mathbb{R}^N,$$

for every $|\xi| \geq R$ by (b) of Proposition 3.1. Hence, recalling that $\Psi_k = \Phi D_k u$, computing the derivatives and noticing that $2D^2_{k,j}uD_k u$ can be actually replaced by $D_j\left((|Du|^2 - R^2)_+\right)$ because everything is zero when $|Du| \leq R$, we find that

$$\int_{B_r} D^2_{\xi_i\xi_j} f(Du) D^2_{k,j} u\eta^2 D_i\Psi_k \,\mathrm{d}x \geq c(\nu) \int_{B_r} \left(1+|Du|^2\right)^{\frac{p-2}{2}} \left[|D^2u|^2\Phi + |D\left((|Du|^2 - R^2)_+\right)|^2\Phi'\right]\eta^2 \,\mathrm{d}x.$$

Then, we turn to the right hand side of (2.3) which we can estimate recalling that

$$|D_\xi f(\xi)| \leq c_4(1+|\xi|^2)^{\frac{q-1}{2}}$$

because f is convex, smooth and has q-growth from above by (A2). Using the fact that $|g_u(\cdot\,, u)|$ can be estimated by $\|a\|_\infty$ on B_r because of (A3) we have

$$\begin{aligned}
\int_{B_r} &\left(1+|Du|^2\right)^{\frac{p-2}{2}} \left[|D^2u|^2\Phi + |D\left((|Du|^2 - R^2)_+\right)|^2\Phi'\right]\eta^2 \,\mathrm{d}x \\
&\leq C \int_{B_r} \left(1+|Du|^2\right)^{\frac{q-1}{2}} |D^2u|\Phi(\eta + |D\eta|)\eta \,\mathrm{d}x \\
&\qquad + C \int_{B_r} \left(1+|Du|^2\right)^{\frac{q}{2}} |D\left((|Du|^2 - R^2)_+\right)||\Phi'|(\eta + |D\eta|)\eta \,\mathrm{d}x
\end{aligned}$$

with C depending only on q, c_2 of (A2), ν and the norm $\|a\|_\infty$ on B_r. Then, setting $V(x) = 1 + R^2 + (|Du(x)|^2 - R^2)_+$ for $x \in \Omega$ and recalling the definition of Φ, we exploit Young inequality to extract from the two terms at the right hand side of the previous estimate two terms similar to those at the left, *i.e.*

$$\begin{aligned}
\int_{B_r} &V^{\frac{p}{2}-1+\delta}|D^2u|^2(|Du|^2 - R^2)_+\eta^2 \,\mathrm{d}x \\
&+ \int_{B_r} V^{\frac{p}{2}-2+\delta} A_\delta(u\,, R)|D\left((|Du|^2 - R^2)_+\right)|^2\eta^2 \,\mathrm{d}x \\
&\leq C \int_{B_r} V^{q-\frac{p}{2}+\delta}\left\{A_\delta(u\,, R) + (|Du|^2 - R^2)_+\right\}\left(\eta^2 + |D\eta|^2\right) \mathrm{d}x
\end{aligned}$$

where $A_\delta(u\,, R) = 1 + R^2 + (1+\delta)(|Du|^2 - R^2)_+$. Dropping the term proportional to $|D^2u|^2$, the conclusion follows easily. □

3. Approximation of p-uniformly convex functions

In this section, we show that a p-uniformly convex function at infinity with $p-q$ growth can be approximated by a sequence of smooth, uniformly elliptic functions with p-growth. In the sequel, we agree to say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is *convex outside* B_R if

$$f\left(\frac{\xi_1+\xi_2}{2}\right) \le \frac{1}{2}f(\xi_1)+\frac{1}{2}f(\xi_2)$$

whenever $[\xi_1,\xi_2] \subset \mathbb{R}^N \setminus B_R$ and that f is convex at infinity if it is convex outside some ball B_R and similarly for p-uniformly convex functions.

We begin recalling some properties of p-uniformly convex functions at infinity for which we refer to [9].

Proposition 3.1. *Let $f\colon \mathbb{R}^N \to [0,+\infty)$ satisfy* (A1) *with constants ν and R. Then,*

(a) *there exist constants $\mu_1 = \mu_1(p,\nu) > 0$, $\mu_2 = \mu_2(p,\nu,R,M) > 0$ where $M = \max_{\partial B_R} f$ and a function $h\colon \mathbb{R}^N \to [-\mu_2,+\infty)$, convex outside B_R, such that*

$$f(\xi) = \mu_1(1+|\xi|^2)^{\frac{p}{2}} + h(\xi), \qquad \xi \in \mathbb{R}^N;$$

(b) *if $f \in C^2(\mathbb{R}^N)$, there exists $c(\nu) > 0$ such that*

$$\langle D^2_{\xi\xi} f(\xi)\lambda,\lambda\rangle \ge c(\nu)\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2, \qquad \lambda \in \mathbb{R}^N, \quad |\xi| \ge R;$$

(c) *there exist R_0, $\nu_0 > 0$ depending only on p, ν, R and M with the property that, for every $\zeta \in \mathbb{R}^N \setminus B_{R_0}$, there exists $d(\zeta) \in \mathbb{R}^N$ such that*

$$f(\xi) \ge f(\zeta) + \langle d(\zeta), \xi-\zeta\rangle + \nu_0(1+|\xi|^2+|\zeta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2, \qquad \xi \in \mathbb{R}^N;$$

(d) *if f satisfies* (A2)*, there exists $\mu_3 > 0$ such that*

$$|d(\xi)| \le \mu_3\left(1+|\xi|^2\right)^{\frac{q-1}{2}}, \qquad \xi \in \mathbb{R}^N \setminus B_{R_0};$$

(e) *$f = f^{**}$ outside B_{R_0}.*

Lemma 3.2. *Let $f\colon \mathbb{R}^N \to [0,+\infty)$ be a convex function satisfying* (A1) *with constants ν, R and* (A2)*. Then, there exist functions $f_k\colon \mathbb{R}^N \to [0,+\infty)$ with the following properties:*

(a) *f_k is convex, $f_k \le f_{k+1}$ and $f_k \to f$ uniformly on compact sets;*
(b) *each function f_k satisfies* (A1) *with constants ν' and R' independent of k;*
(c) *each function f_k satisfies* (H1) *with constant $c_3' = c_3'(k)$.*

Proof. From Proposition 3.1, there exist constants μ_1, $\mu_2 > 0$ and a function $h\colon \mathbb{R}^N \to [-\mu_2,+\infty)$, convex outside B_R, such that

$$f(\xi) = \frac{\mu_1}{2}\left(1+|\xi|^2\right)^{\frac{p}{2}} + \left[\frac{\mu_1}{2}\left(1+|\xi|^2\right)^{\frac{p}{2}} + h(\xi)\right] = \frac{\mu_1}{2}\left(1+|\xi|^2\right)^{\frac{p}{2}} + H(\xi), \quad \xi \in \mathbb{R}^N.$$

Since h is convex outside B_R and $\xi \to \left(1+|\xi|^2\right)^{\frac{p}{2}}$ is p-uniformly convex, then H is p-uniformly convex outside B_R with some constant ν' depending only on p and ν. Then, using (c) of Proposition 3.1, we find $R' = R_0$ such that

$$H(\xi) \ge H(\zeta) + \langle d(\zeta), \xi-\zeta\rangle, \qquad \xi \in \mathbb{R}^N,$$

for some vector $d(\zeta)$, $|\zeta| \geq R'$.

For $k > R'$, define $H_k\colon \mathbb{R}^N \to \mathbb{R}$ by setting

$$H_k(\xi) = \begin{cases} H(\xi) & \text{for } |\xi| \leq k, \\ \sup\{H(\zeta) + \langle d(\zeta), \xi - \zeta\rangle : \ R' \leq |\zeta| \leq k\} & \text{for } |\xi| > k. \end{cases}$$

Each function H_k is continuous, $H_k = H$ on $\mathbb{R}^N \setminus B_{R'}$, $H_k \leq H_{k+1} \leq H$ and $H_k \to H$ uniformly on compact sets. It is also easy to prove that H_k is a convex function outside $B_{R'}$. In fact, let $[\xi_1\,, \xi_2]$ be a segment lying entirely outside $B_{R'}$. For every $|\zeta| \geq R'$, we find

$$H(\zeta) + \langle d(\zeta), \frac{\xi_1 + \xi_2}{2} - \zeta\rangle = \frac{1}{2}\left[H(\zeta) + \langle d(\zeta), \xi_1 - \zeta\rangle\right] + \frac{1}{2}\left[H(\zeta) + \langle d(\zeta), \xi_2 - \zeta\rangle\right]$$

whence

$$\begin{aligned} H_k\left(\frac{\xi_1 + \xi_2}{2}\right) &= \sup_{R' \leq |\zeta| \leq k} \left\{H(\zeta) + \langle d(\zeta), \frac{\xi_1 + \xi_2}{2} - \zeta\rangle\right\} \\ &\leq \frac{1}{2} \sup_{R' \leq |\zeta| \leq k} \{H(\zeta) + \langle d(\zeta), \xi_1 - \eta\rangle\} + \frac{1}{2} \sup_{R' \leq |\zeta| \leq k} \{H(\zeta) + \langle d(\zeta), \xi_2 - \eta\rangle\} \\ &= \frac{1}{2} H_k(\xi_1) + \frac{1}{2} H_k(\xi_2). \end{aligned}$$

Finally, for $k > R'$, define $f_k\colon \mathbb{R}^N \to [0\,, +\infty)$ by setting

$$f_k(\xi) = \frac{\mu_1}{2}\left(1 + |\xi|^2\right)^{\frac{p}{2}} + H_k(\xi), \qquad \xi \in \mathbb{R}^N,$$

so that, up to an additive constant, $f_k(\xi) \geq c_1'|\xi|^p$ for some constant $c_1' > 0$ independent of k. Then, it is obvious that the sequence $\{f_k\}_k$ is increasing and converges to f uniformly on compact sets. As to the convexity of the functions f_k, it follows from the fact that f is convex, f_k is convex outside $B_{R'}$ and the equality $f_k = f$ holds in $B_k \setminus B_{R'}$. Moreover, since H_k is convex outside $B_{R'}$ and $\xi \to \left(1 + |\xi|^2\right)^{p/2}$ is p-uniformly convex, f_k is p-uniformly convex outside $B_{R'}$ and (b) follows. Finally, from (A2) and (d) of Proposition 3.1, we find that

$$|d(\zeta)| \leq \mu_3\left(1 + k^2\right)^{\frac{q-1}{2}}, \qquad R' \leq |\zeta| \leq k,$$

and this implies

$$0 \leq f_k(\xi) \leq \frac{\mu_1}{2}\left(1 + |\xi|^2\right)^{\frac{p}{2}} + |H_k(\xi)| \leq \frac{\mu_1}{2}\left(1 + |\xi|^2\right)^{\frac{p}{2}} + \mu_4\left(1 + k^2\right)^{\frac{q-1}{2}}(|\xi| + k)$$

for $\xi > R'$ whence (c) follows. □

Lemma 3.3. *For every function f_k of Lemma* 3.2*, there exists a sequence of functions $f_{k,j}\colon \mathbb{R}^N \to [0\,, +\infty)$ with the following properties:*

(a) *$f_{k,j} \to f_k$ uniformly on compact sets;*

(b) *each function $f_{k,j}$ satisfies* (A1) *and* (A2) *with constants $\nu'' > 0$, $R'' > 0$ and $c_2'' \geq c_1'' > 0$, independent of j and k;*

(c) *each function $f_{k,j}$ satisfies* (H1) *and* (H2) *with constants $c_3'' = c_3''(k)$ and $L_1'' = L_1''(j)$.*

Proof. Setting

$$f_{k,j}(\xi) = \int_{\mathbb{R}^N} \sigma(\zeta) f_k(\xi + \zeta/j)\,\mathrm{d}\zeta + \varepsilon_j \left(1 + |\xi|^2\right)^{\frac{p}{2}}, \qquad \xi \in \mathbb{R}^N,$$

where $\sigma \in C^\infty_c(B_1)$ is a nonnegative, radially symmetric mollifier and $\varepsilon_j \to 0_+$, the conclusion follows easily and we refer to either [14] or [12] for the details. □

Lemma 3.4. *For every function $f_{k,j}$ of Lemma* 3.3*, there exists a sequence of functions $f_{k,j,l} \colon \mathbb{R}^N \to [0\,,+\infty)$ with the following properties:*

(a) $f_{k,j,l} \to f_{k,j}$ *uniformly on compact sets;*

(b) *each function $f_{k,j,l}$ satisfies properties* (A1) *and* (A2) *with constants $\nu''' > 0$, $R''' > 0$ and $c_2''' \geq c_1''' > 0$, independent of j, k and l;*

(c) *each function $f_{k,j,l}$ satisfies properties* (H1) *and* (H2) *with constants $c_3''' = c_3'''(k)$ and $L_1''' = L_1'''(j)$ independent of l;*

(d) *each function $f_{k,j,l}$ satisfies* (H3) *with constant $L_2''' = L_2'''(k,j,l)$.*

Proof. Starting from $f_{k,j}$, the construction of a sequence $\{f_{k,j,l}\}_l$ satisfying (H1), (H2) with constants independent of l and such that (H3) holds can be found in [14] and [12]. The same construction implies that, if $f_{k,j}$ satisfies (A1) and (A2), the same properties pass to the $f_{k,j,l}$ with possibly new constants depending only on the corresponding constants of Lemma 3.3. □

4. Proof of the regularity result

Combining the *a priori* estimates of Section 2 and the approximation results of Section 3 we can prove our Lipschitz regularity result.

Proof of Theorem 1.1. Let $\{f_k\}_k$, $\{f_{k,j}\}_j$ and $\{f_{k,j,l}\}_l$ be the sequences of functions associated to f by Lemmas 3.2, 3.3 and 3.4. We regularize g too by considering the smooth functions $g_l \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_l(x\,,u) = \int_{B_1} \sigma(y) \left(\int_{-1}^{1} \rho(\eta) g(x + y/l\,, u + \eta/l)\,\mathrm{d}\eta \right) \mathrm{d}y, \qquad (x\,,u) \in \Omega \times \mathbb{R},$$

where g is obviously set equal to zero outside Ω and $\sigma \in C^\infty_c(B_1)$, $\rho \in C^\infty_c(-1\,,1)$ are the usual nonnegative, radially symmetric mollifiers. From (A3), for every compact set $K \subset \Omega$, there are constants C_i depending only on K and a such that

$$|g_l(x\,,u) - g_l(x\,,v)| \leq C_1 |u - v|, \qquad x \in K, \quad u,\, v \in \mathbb{R}, \tag{4.1}$$

$$\int_K |g_l(x\,,w(x))|\,\mathrm{d}x \leq C_2 \left(\int_{K+B_1} |g(x\,,0)|\,\mathrm{d}x + \int_K |w(x)|\,\mathrm{d}x \right\}, \tag{4.2}$$

for every function $w \in L^1_{\mathrm{loc}}(\Omega)$.

Then, let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.5), set $B_r = B_r(x_0) \subset\subset \Omega$ and consider the smooth functions

$$u_n(x) = \int_{B_1} \sigma(y) u(x + y/n)\,\mathrm{d}y$$

where u is zero outside Ω. We localize all integrals to the ball B_r and we write

$$I(w) = F(w) + G(w) = \int_{B_r} f(Dw(x))\,\mathrm{d}x + \int_{B_r} g(x\,,w)\,\mathrm{d}x, \qquad w \in W^{1,1}_{\mathrm{loc}}(\Omega).$$

We define in the obvious way F_k, $F_{k,j}$, $F_{k,j,l}$ and similarly G_l. Moreover, we penalize G when w is different from u_n, *i.e* we consider the integrals

$$G_n(w) = G(w) + \int_{B_r} \arctan^2 (w - u_n)\,\mathrm{d}x, \qquad w \in W^{1,1}_{\mathrm{loc}}(\Omega),$$

and similarly $G_{n,l}$. Finally, we define $I_n = F + G_n$, $I_{n,k} = F_k + G_n$, $I_{n,k,j} = F_{k,j} + G_n$ and $I_{n,k,j,l} = F_{k,j,l} + G_{n,l}$.

Then, we wish to minimize $I_{n,k,j,l}$ in the Dirichlet class $\mathcal{A} = u + W^{1,p}_0(B_r)$, *i.e.* subject to the boundary condition of u. To this aim, we first recall that Lemma 3.4 implies that there are constants c_1''', independent of k, j and l and c_3''' depending only on k such that

$$c_1'''|\xi|^p \le f_{k,j,l}(\xi) \le c_3'''(k)\,(1 + |\xi|^p)\,, \qquad \xi \in \mathbb{R}^N, \tag{4.3}$$

whence, using the first estimate from below, (4.2) and Poincaré's inequality, we obtain

$$\int_{B_r} |Dw|^p\,\mathrm{d}x \le C\,[1 + I_{n,k,j,l}(w)]\,, \qquad w \in \mathcal{A}, \tag{4.4}$$

where C depends on g, a and u but does not depend on any of the indices n, k, j and l. Thus, the existence of a minimizer of $I_{n,k,j,l}$ on $\mathcal{A}$, say $v_{n,k,j,l}$, follows immediately from this estimate and the convexity of each function $f_{k,j,l}$. Moreover, (4.4) and the minimality of $v_{n,k,j,l}$ yield that

$$\int_{B_r} |Dv_{n,k,j,l}|^p\,\mathrm{d}x \le M_k \tag{4.5}$$

for some constant M_k independent of n, j and l. Thus, up to a subsequence, there is $v_{n,k,j} \in \mathcal{A}$ such that $v_{n,k,j,l} \rightharpoonup v_{n,k,j}$ weakly in $W^{1,p}(B_r)$ and

$$\int_{B_r} |Dv_{n,k,j}|^p\,\mathrm{d}x \le M_k. \tag{4.6}$$

Now, we claim that the sequence $\{I_{n,k,j,l}\}_l$ Γ-converges to $I_{n,k,j}$ with respect to the weak $W^{1,p}$-topology induced on $\mathcal{A}$ so that $v_{n,k,j}$ is a minimizer of $I_{n,k,j}$ (Cor. 7.20 in [10]). Indeed, the convexity of the functions $f_{k,j,l}$ and (4.3) imply (Th. 5.14 in [10]) that $\{F_{k,j,l}\}_l$ Γ-converges to $F_{k,j}$ in the weak topology of $W^{1,p}$ and we only have to check that

$$\lim_l G_{n,l}(w_l) = G_n(w),$$

whenever $w_l \in \mathcal{A}$ and $w_l \rightharpoonup w$. To see this, let $w_{l'}$ be any subsequence of w_l and choose a further subsequence $w_{l''}$ such that $w_{l''} \to w$ strongly in $L^p(B_r)$ and almost everywhere by Rellich's theorem and such that the sequence $\{G_{n,l''}(w_{l''})\}_{l''}$ converges. The conclusion follows from (4.1), the chain of inequalities

$$\begin{aligned}|G_{n,l''}(w_{l''}) - G_n(w)| &\le |G_{n,l''}(w_{l''}) - G_{n,l''}(w)| + |G_{n,l''}(w) - G_n(w)| \\ &\le C_1 \int_{B_r} |w_{l''} - w|\,\mathrm{d}x + \int_{B_r} |g_{l''}(x\,,w) - g(x\,,w)|\,\mathrm{d}x\end{aligned}$$

and Lebesgue's dominated convergence theorem.

Now, we repeat the previous argument starting from (4.6). Up to a subsequence, $\{v_{n,k,j}\}_j$ converges to a function $v_{n,k} \in \mathcal{A}$ weakly in $W^{1,p}(B_r)$, the integrals $I_{n,k,j}$ Γ-converge to $I_{n,k}$ in the same topology and $v_{n,k}$ is a minimizer of $I_{n,k}$ on $\mathcal{A}$. Moreover, the same estimates used for (4.4), the minimality of $v_{n,k}$ and the inequality $F_k \le F$ which follows immediately from (a) of Lemma 3.3 yield

$$\int_{B_r} |Dv_{n,k}|^p\,\mathrm{d}x \le C\,[1 + I_{n,k}(v_{n,k})] \le C\,[1 + I(u)] \tag{4.7}$$

where C depends on g, a and u but neither on k nor n. Therefore, up to a subsequence oncemore, $v_{n,k} \rightharpoonup v_n$ weakly in $W^{1,p}(B_r)$ for some $v_n \in \mathcal{A}$ and (a) of Lemma 3.2 and Proposition 5.4 in [10] imply that $\{I_{n,k}\}_k$ Γ-converges to I_n in the weak topology of $\mathcal{A}$. Thus, v_n has to be a minimizer of I_n on $\mathcal{A}$ and, passing to the limit in (4.7), we find that

$$\int_{B_r} |Dv_n|^p \, \mathrm{d}x \leq C\,[1 + I_n(v_n)] \leq C\,[1 + I(u)]\,. \tag{4.8}$$

At last, up to a further subsequence, we have that $v_n \rightharpoonup v$ weakly in $W^{1,p}(B_r)$ for some $v \in \mathcal{A}$.

Now, we prove that v is in $W^{1,\infty}_{\mathrm{loc}}(B_r)$. In fact, we can apply Lemma 2.1 and Proposition 2.2 to the local minimizers $v_{n,k,j,l}$ of $I_{n,k,j,l}$. Therefore, (4.4) yields

$$\sup_{B_{r/2}} |Dv_{n,k,j,l}| \leq C \left\{ \int_{B_r} (1 + |Dv_{n,k,j,l}|^p) \, \mathrm{d}x \right\}^\alpha \leq C\,[1 + I_{n,k,j,l}(v_{n,k,j,l})]^\alpha$$

where C is independent of n, k, j and l. Passing to the limit with respect to l, j, k and using the Γ-convergence results proved above and (4.8), we conclude that

$$\sup_{B_{r/2}} |Dv| \leq C\,[1 + I(u)]^\alpha$$

which gives (1.7) for v.

Finally, we prove that $v = u$, thus completing the proof. Recalling the definition of I_n, the strong convergence of u_n to u and the weak convergence of v_n to v, we find by lower semicontinuity and by the minimality of v_n for I_n that

$$I(v) + \int_{B_r} \arctan^2(v-u) \, \mathrm{d}x \leq \liminf_{n\to+\infty} I_n(v_n) \leq \liminf_{n\to+\infty} I_n(u) = I(u).$$

Since u is a local minimizer of I, we conclude that $v = u$ *a.e.* on B_r. □

5. Application to nonconvex integrals

In this final section, we show how the Lipschitz regularity result of Section 1 can be applied to establish the existence of locally Lipschitz continuous minimizers for nonconvex multiple integrals with $p-q$ growth.

Indeed, let now $f\colon \mathbb{R}^N \to [0\,,+\infty)$ be a possibly nonconvex, lower semicontinuous function satisfying (A1), (A2) for some $1 < p \leq q$ and in this case let $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a Caratheodory function featuring the following special structure:

$$g(x\,,u) = a(x)h(u), \qquad x \in \Omega,\ u \in \mathbb{R},$$

where $a \in L^\infty(\Omega)$, $a \geq 0$ and $h\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function whose Lipschitz constant can be taken equal to 1 without loss of generality. It is clear that g satisfies (A3). For these functions f and g we consider the variational integral

$$I(u) = \int_\Omega [f(Du(x)) + a(x)h(u(x))] \, \mathrm{d}x, \qquad u \in W^{1,p}(\Omega),$$

and the corresponding Dirichlet problem

$$\min \left\{ I(u) :\ u \in u_0 + W^{1,p}_0(\Omega) \right\} \tag{$\mathcal{P}$}$$

where the boundary datum u_0 is any function in $W^{1,p}(\Omega)$ yielding a finite value of I, *i.e.* $I(u_0) < +\infty$.

As mentioned in the Introduction, the lack of convexity of f affects the lower semicontinuity of I along weakly converging sequences of Sobolev functions and hence the existence of solutions to $(\mathcal{P})$ cannot be established by applying Tonelli's direct method. Therefore, to prove attainment for $(\mathcal{P})$, we consider the convexified integral

$$I^{**}(u) = \int_\Omega [f^{**}(Du(x)) + a(x)h(u(x))]\,\mathrm{d}x, \qquad u \in W^{1,p}(\Omega), \tag{5.1}$$

where $f^{**}\colon \mathbb{R}^N \to [0\,,+\infty)$ is the convex envelope of f and the corresponding minimum problem

$$\min\left\{I^{**}(u):\ u \in u_0 + W_0^{1,p}(\Omega)\right\}. \tag{$\mathcal{P}^{**}$}$$

This latter problem has a solution v because I^{**} is now lower semicontinuous along weakly converging sequences of functions in $W^{1,p}(\Omega)$ as f^{**} is now convex and minimizing sequences of I^{**} are sequentially weakly compact in the same space by the growth assumptions (A2) and (A3) on f and g and by (e) of Proposition 3.1. Then, Theorem 1.1 applies yielding $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ whence, following the ideas of [3] and [5], it is possible to prove that, under appropriate hypotheses on f and g, v can be modified so as to find a new solution u to $(\mathcal{P}^{**})$ such that $f^{**}(Du) = f(Du)$ almost everywhere on Ω. As $f^{**} \le f$ by construction, it follows immediately that u is a solution also to the original Dirichlet problem $(\mathcal{P})$ and moreover that u too has to be locally Lipschitz continuous by Theorem 1.1 again as it is a solution to $(\mathcal{P}^{**})$.

Indeed, the following existence result holds. We refer to [22] for a one-dimensional version of this result.

Theorem 5.1. *Let $f\colon \mathbb{R}^N \to [0\,,+\infty)$ be a lower semicontinuous function satisfying* (A1), (A2) *for*

$$1 < p \le q < \frac{N+1}{N}p$$

and let $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that

$$g(x\,,u) = a(x)h(u), \qquad x \in \Omega, \quad u \in \mathbb{R},$$

where $a \in L^\infty(\Omega)$, $a \ge 0$ and $h\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant 1. *Assume also that*

f^{**} is affine on each connected component of $\{f^{**} < f\}$; (5.2)

for every $\eta_0 \in \mathbb{R}$, there is $\delta = \delta(\eta_0) > 0$ such that h is monotone on each interval $[\eta_0 - \delta\,,\eta_0]$ and $[\eta_0\,,\eta_0 + \delta]$; (5.3)

if $f^{**}(0) < f(0)$, h has no strict local minima; (5.4)

there exists an open set $\Omega_0 \subset \Omega$ such that $a = 0$ a.e on Ω_0 and $a > 0$ a.e. on $\Omega \setminus \Omega_0$. (5.5)

Then, the minimum problem $(\mathcal{P})$ admits a locally Lipschitz continuous solution.

In this statement, the hypotheses (A1), (A2) are related only with the existence and the regularity of solutions to $(\mathcal{P}^{**})$ whereas the hypotheses (5.2), (5.3), (5.4) and (5.5) are related with the existence of solutions to the nonconvex problem $(\mathcal{P})$. As is well known, they cannot be dropped without affecting attainment for $(\mathcal{P})$, see the examples in [23] and [5] and the results of [6] and [16]. Among them, the truly restricting ones are (5.2) and (5.4) – though the latter has to be enforced only when $f^{**}(0) < f(0)$ – whereas (5.3) and (5.5) are only mild regularity hypotheses on the coefficients a and h. Note also that (5.3) implies that h is piecewise monotone and hence has at most countably many strict, local extrema, say $\{m_i\}_i$. Moreover, (5.5) is equivalent to the requirement that a be almost everywhere null on a neighborhood of every point x where the Lebesgue value of a vanishes. If a is continuous, a sufficient condition for this to happen is that the boundary of $\{a = 0\}$ be negligible.

The proof is based on the ideas developed for Theorems 2.1 in [3] and [5] and relies on the following construction of comparison functions originally introduced by De Blasi and Pianigiani in [11]. We refer to Lemma 3.5 of [5] for the proof.

Lemma 5.2. *Let $K \subset \mathbb{R}^N$ be a compact, convex set and let $w \in W^{1,p}(\Omega)$, $1 < p < +\infty$, be a continuous, almost everywhere differentiable function such that*

(a) *w is differentiable at $x_0 \in \Omega$ with (classical) gradient $\xi_0 = \nabla w(x_0)$;*
(b) *$\xi_0 \in \operatorname{int}(K)$.*

Then, there exist $\varepsilon_0 > 0$, two families of compact sets $\{A^\pm_\varepsilon\}_\varepsilon$ such that

$$B_{r_1\varepsilon}(x_0) \subset A^\pm_\varepsilon \subset B_{r_2\varepsilon}(x_0) \subset\subset \Omega, \qquad 0 < \varepsilon \leq \varepsilon_0, \tag{5.6}$$

for some $0 < r_1 \leq r_2$ and also two corresponding families of continuous, almost everywhere differentiable functions $\{w^\pm_\varepsilon\}_\varepsilon$ in $W^{1,p}(\Omega)$ such that the following properties hold for every $0 < \varepsilon \leq \varepsilon_0$:

$$w^\pm_\varepsilon = w \text{ on } \Omega \setminus \operatorname{int}(A^\pm_\varepsilon); \tag{5.7}$$
$$w(x) < w^+_\varepsilon(x) < w(x) + 2\varepsilon \text{ for every } x \in \operatorname{int}(A^+_\varepsilon); \tag{5.8+}$$
$$w(x) - 2\varepsilon < w^-_\varepsilon(x) < w(x) \text{ for every } x \in \operatorname{int}(A^-_\varepsilon); \tag{5.8−}$$
$$\varepsilon \geq w^+_\varepsilon(x) - [w(x_0) + \langle \nabla w(x_0), x - x_0\rangle] \geq \varepsilon/2 \text{ for every } x \in B_{r_1\varepsilon}(x_0) \tag{5.9+}$$
$$-\varepsilon/2 \geq w^-_\varepsilon(x) - [w(x_0) + \langle \nabla w(x_0), x - x_0\rangle] \geq -\varepsilon \text{ for every } x \in B_{r_1\varepsilon}(x_0) \tag{5.9−}$$
$$\nabla w^\pm_\varepsilon(x) \in \partial K \text{ for a.e. } x \in A^\pm_\varepsilon. \tag{5.10}$$

Corollary 5.3. *Let K and w be as in Lemma 5.2 and let $f\colon \mathbb{R}^N \to [0,+\infty)$ be a lower semicontinuous function such that*

$$f^{**}(\xi) = \langle m, \xi\rangle + q, \qquad \xi \in K$$

for some $m \in \mathbb{R}^N$ and $q \in \mathbb{R}$. Then,

$$\int_{A^\pm_\varepsilon} f^{**}\left(Dw^\pm_\varepsilon\right) \mathrm{d}x \leq \int_{A^\pm_\varepsilon} f^{**}(Dw)\,\mathrm{d}x, \qquad 0 < \varepsilon \leq \varepsilon_0. \tag{5.11}$$

Proof. From (5.10), the fact that $w - w^\pm_\varepsilon$ is compactly supported in Ω and the convexity of f^{**}, we find that

$$\int_{A^\pm_\varepsilon} f^{**}(Dw^\pm_\varepsilon)\,\mathrm{d}x = \int_{A^\pm_\varepsilon} \left[\langle m, Dw^\pm_\varepsilon\rangle + q\right] \mathrm{d}x = \int_{A^\pm_\varepsilon} \left[\langle m, Dw\rangle + q\right] \mathrm{d}x \leq \int_{A^\pm_\varepsilon} f^{**}(Dw)\mathrm{d}x. \qquad \square$$

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we can assume that the open set $D = \{f^{**} < f\}$ has just one connected component and hence (5.2) and the growth assumption (A2) imply that $D \subset K$ for some compact, convex set $K \subset \mathbb{R}^N$. Moreover,

$$f^{**}(\xi) = \langle m, \xi\rangle + q, \qquad \xi \in K, \tag{5.12}$$
$$f^{**}(\xi) = f(\xi), \qquad \xi \in \partial K, \tag{5.13}$$

because of (5.2).

Then, let $w \in u_0 + W^{1,p}_0(\Omega)$ be a solution to the relaxed problem $(\mathcal{P}^{**})$. By Theorem 1.1, $w \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and therefore it is continuous and almost everywhere (classically) differentiable by Rademacher's theorem. Set

$$E(w) = \{x \in \Omega :\ Dw(x) \in D\}.$$

We shall prove that there exists a solution u to $(\mathcal{P}^{**})$ such that $|E(u)| = 0$, *i.e.* $f^{**}(Du) = f(Du)$ *a.e.* on Ω which shows that u is a solution to $(\mathcal{P})$ as well. To this aim, let Ω_0 be the open set of (5.5) so that, at almost every point x of $\Omega \setminus \Omega_0$, the Lebesgue value of a at x is positive, *i.e.*

$$\lim_{\rho\to 0_+} \fint_{B_\rho(x)} a \,\mathrm{d}x > 0 \quad \text{for} \quad a.e. \quad x \in \Omega \setminus \Omega_0.$$

Recall also that, by (5.3), h has at most countably many strict, local minimum or maximum point, say $\{m_i\}_i$ and that h is monotone on a neighborhood of every point $u \neq m_i$. Then, when w is a solution to $(\mathcal{P}^{**})$, write

$$E(w) = E_0(w) \cup (\cup_i E_i(w))$$

where

$$E_0(w) = \{x \in E(w) :\ h \text{ is monotone around } w(x)\},$$
$$E_i(w) = \{x \in E(w) :\ w(x) = m_i\}.$$

We shall prove the theorem by proving the following three claims.

Claim 1. There exists a solution v to $(\mathcal{P}^{**})$ such that $|E(v) \cap \Omega_0| = 0$.

Claim 2. There exists a solution u to $(\mathcal{P}^{**})$ such that $|E(u) \cap \Omega_0| = 0$ and $|E_0(u)| = 0$.

Claim 3. For the solution u of Claim 2, we have $|E_i(u)| = 0$ for every i.

In fact, they imply that $|E(u)| = 0$, *i.e.* u is a solution to $(\mathcal{P})$.

Proof of Claim 1. Let w be a solution to $(\mathcal{P}^{**})$ and assume $|E(w) \cap \Omega_0| > 0$. As Ω_0 is open and a vanishes *a.e.* on Ω_0, it follows from Corollary 5.3 that for every point $x_0 \in E(w) \cap \Omega_0$ where w is differentiable, the corresponding functions $w_\varepsilon^\pm$ are solutions to $(\mathcal{P}^{**})$ for small ε, regardless of the choice of $+$ or $-$. Moreover, $f^{**}(Dw_\varepsilon^\pm) = f(Dw_\varepsilon^\pm)$ *a.e.* on $A_\varepsilon^\pm$ because of (5.10) and (5.13). Then, exploiting (5.6), we use Vitali's covering theorem to find countably many modified functions $w_j = w_{\varepsilon_j}^\pm$ and pairwise disjoint sets $A_j = A_{\varepsilon_j}^\pm \subset \Omega_0$ such that

$$|E_0(w) \setminus (\cup_j A_j)| = 0.$$

It is then easy to check the series $v = w + \sum_j (w_j - w)$ (it is actually a finite sum at every point) is a solution to $(\mathcal{P}^{**})$ such that $|E(v) \cap \Omega_0| = 0$.

Proof of Claim 2. Let v be the solution to $(\mathcal{P}^{**})$ of Claim 1 and assume $|E_0(v)| > 0$. Pick one such point $x_0 \in E_0(v)$ where v is (classically) differentiable and recall that h is monotone, say increasing, on a neighborhood of $w(x_0)$. Then, the corresponding functions v_ε^- associated to v by Lemma 5.2 satisfy

$$\int_{A_\varepsilon^-} a(x)h(v_\varepsilon^-)\,\mathrm{d}x \le \int_{A_\varepsilon^-} a(x)h(v)\,\mathrm{d}x$$

for small ε because of (5.8$-$) and therefore (5.7) and Corollary 5.3 show that v_ε^- is a solution to $(\mathcal{P}^{**})$ such that $f^{**}(Dv_\varepsilon^-) = f(Dv_\varepsilon^-)$ *a.e.* on A_ε^- for small ε. Finally, the very same covering argument of Claim 1 yields a solution u to $(\mathcal{P}^{**})$ satisfying $|E(u) \cap \Omega_0| = 0$ and $|E_0(u)| = 0$ as expected.

Proof of Claim 3. Assume that $|E_i(u)| > 0$ for some i and, to simplify the notations, write $m = m_i$ and $E = E_i(u)$. As E is a level set of u of positive measure, $Du = 0$ *a.e.* on E and hence the very definition of E implies that $f^{**}(0) < f(0)$. Thus, m is a strict local maximum of h by (5.4) and there are left and right intervals around m where h is increasing and decreasing respectively.

Now, choose a density point $x_0 \in E$ where u is (classically) differentiable with $Du(x_0) = 0$ and, recalling that $E \subset \Omega \setminus \Omega_0$, assume also that it is a Lebesgue point of a where

$$\lim_{\rho\to 0_+} \fint_{B_\rho(x_0)} a \,\mathrm{d}x = a(x_0) > 0. \tag{5.14}$$

We shall find a contradiction by exploiting Lemma 5.2 once more. To this aim, we choose a sequence $\varepsilon_j \to 0_+$ in $(0\,,\varepsilon_0]$ where ε_0 is given by Lemma 5.2 and we set

$$\eta_j = \frac{1}{\epsilon_j} \sup\left\{|u(x)-m| : \, |x-x_0| < 2r_2\varepsilon_j\right\}$$

where r_2 comes from the same lemma. Obviously, $\eta_j \to 0_+$ since u is differentiable at x_0 with $\nabla u(x_0) = 0$ by assumption and we can assume also that $m \pm \eta_j\epsilon_j$ remains always in the intervals around m where h is monotone. Moreover, possibly extracting a subsequence still denoted by $\{\epsilon_j\}_j$, we can assume in addition that the minimum between $h(m-\eta_j\epsilon_j)$ and $h(m+\eta_j\epsilon_j)$ is actually achieved for every j by terms that always have the same sign inside, say $h(m+\eta_j\epsilon_j)$, so that

$$0 < h(m) - h(m+\eta_j\epsilon_j) = \max\left\{h(m) - h(m-\eta_j\epsilon_j)\,, h(m) - h(m+\eta_j\epsilon_j)\right\} \tag{5.15}$$

holds for every j.

According to this assumption, we choose the + functions in Lemma 5.2 and, to simplify the notations, we set $u_j = u^+_{\epsilon_j}$ and $A_j = A^+_{\epsilon_j}$ for every j. Finally, set $B_{i,j} = B_{r_i\varepsilon_j}(x_0)$ for $i = 1$, 2 and every j so that (5.6) turns into

$$B_{1,j} \subset A_j \subset B_{2,j} \tag{5.16}$$

and set also

$$J^1_j = \frac{1}{\int_{A_j} a(x)\mathrm{d}x} \int_{A_j} a(x)\left[h(m) - h(u_j(x))\right] \mathrm{d}x,$$

$$J^2_j = \frac{1}{\int_{A_j} a(x)\mathrm{d}x} \int_{A_j} a(x)\left[h(m) - h(u(x))\right] \mathrm{d}x.$$

Since $u = u_j$ on $\Omega \setminus A_j$ by (5.7) and

$$\int_{A_j} f^{**}(Du_j)\,\mathrm{d}x \le \int_{A_j} f^{**}(Du)\,\mathrm{d}x$$

by Corollary 5.3, we will get a contradiction to the minimality of u by showing that eventually $J^1_j - J^2_j > 0$.

In fact, note first that (5.9+) reduces to $\epsilon_j/2 \le u_j(x) - m \le \epsilon_j$ for every $x \in B_{1,j}$. Hence, recalling (5.14), (5.16) and that h is decreasing on an interval to the left of m, we find that

$$J^1_j \ge \frac{1}{\int_{B_{2,j}} a\,\mathrm{d}x} \int_{B_{1,j}} a\left[h(m) - h(u_j)\right] \mathrm{d}x$$

$$\ge \left(\frac{r_1}{r_2}\right)^N \frac{⨍_{B_{1,j}} a\,\mathrm{d}x}{⨍_{B_{2,j}} a\,\mathrm{d}x}\left[h(m) - h(m+\epsilon_j/2)\right] \ge \frac{1}{2}\left(\frac{r_1}{r_2}\right)^N \left[h(m) - h(m+\epsilon_j/2)\right]$$

for large enough j. As to J^2_j, we have

$$J^2_j = \frac{1}{\int_{A_j} a\,\mathrm{d}x} \int_{A_j\setminus E} a\left[h(m) - h(u)\right] \mathrm{d}x$$

for every j and $|m-u| \le \eta_j\epsilon_j$ on A_j by the definition of η_j. Hence,

$$0 \le h(m) - h(u(x)) \le \max\left\{h(m) - h(m-\eta_j\epsilon_j)\,, h(m) - h(m+\eta_j\epsilon_j)\right\} = h(m) - h(m+\eta_j\epsilon_j)$$

for every $x \in A_j$ and every j because of the behaviour of h around m and by (5.15) whence

$$0 \leq J_j^2 \leq \left(\frac{r_2}{r_1}\right)^N \frac{2\|a\|_\infty |B_{2,j} \setminus E|}{a(x_0)|B_{2,j}|} \left[h(m) - h(m + \eta_j \epsilon_j)\right]$$

for large j because of (5.14). Since $\eta_j \to 0_+$ and h is decreasing to the right of m, we conclude that eventually $h(m) - h(m + \epsilon_j/2) \geq h(m) - h(m + \eta_j \epsilon_j) > 0$. Finally, as x_0 is a density point of E by assumption, the ratio $|B_{2,j} \setminus E|/|B_{2,j}|$ goes to zero and the conclusion follows. □

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