ESAIM: COCV
Vol. 13, N° 1, 2007, pp. 120–134
DOI: 10.1051/cocv:2007007

ESAIM: Control, Optimisation and Calculus of Variations
www.edpsciences.org/cocv

MODEL PROBLEMS FROM NONLINEAR ELASTICITY: PARTIAL REGULARITY RESULTS

Menita Carozza[1] and Antonia Passarelli di Napoli[2]

Abstract. In this paper we prove that every weak and strong local minimizer $u \in W^{1,2}(\Omega, \mathbb{R}^3)$ of the functional

$$I(u) = \int_\Omega |Du|^2 + f(\mathrm{Adj}Du) + g(\mathrm{det}Du),$$

where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, f grows like $|\mathrm{Adj}Du|^p$, g grows like $|\mathrm{det}Du|^q$ and $1 < q < p < 2$, is $C^{1,\alpha}$ on an open subset Ω_0 of Ω such that $meas(\Omega \setminus \Omega_0) = 0$. Such functionals naturally arise from nonlinear elasticity problems. The key point in order to obtain the partial regularity result is to establish an energy estimate of Caccioppoli type, which is based on an appropriate choice of the test functions. The limit case $p = q \leq 2$ is also treated for weak local minimizers.

Mathematics Subject Classification. 35J50, 35J60, 73C50.

Received May 11, 2005.

1. Introduction

Let us consider integral functionals of the Calculus of Variations of the type

$$I(u) = \int_\Omega F(Du)\mathrm{d}x \ ,$$

where $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^N$.

An interesting class of integral functionals which naturally arise from problems of nonlinear elasticity [3] is the one of polyconvex functionals, *i.e.* functionals in which the integrand is a convex function of the minors of the matrix $[Du]$. It is well known that polyconvex functionals are also quasiconvex, but they often satisfy anisotropic growth conditions which are not recovered by the results concerning the quasiconvex case [1, 5, 6, 8, 9, 13, 14].

For this reason [10, 11] have considered polyconvex integrals with anisotropic growth conditions, which are close to the typical examples arising from nonlinear elasticity theory. A model case included in the results of [10] is, for $n = N = 3$

$$I(u) = \int_\Omega |Du|^2 + |Du|^p + |\mathrm{Adj}Du|^p + |\mathrm{det}Du|^p$$

Keywords and phrases. Nonlinear elasticity, partial regularity, polyconvexity.

[1] Dipartimento Pe.Me.Is., Piazza Arechi II, 82100 Benevento, Italy; carozza@unisannio.it

[2] Dipartimento di Matematica e Appl. "R.Caccioppoli" Università di Napoli "Federico II" Via Cintia, 80126 Napoli, Italy; antonia.passarelli@unina.it

where $p > 2$. They proved that absolute minimizers of $I(u)$ are $C^{1,\alpha}$ except for a closed subset of Ω of zero Lebesgue measure. Recall that a function u is an absolute minimizer of $I(u)$ if $I(u) \leq I(u+\varphi)$ for every $\varphi \in C_o^\infty(\Omega)$.

Motivated by a recent paper of Ball [4], in [7, 15] partial regularity has been proved for a new class of minimizers of $I(u)$, when F is a quasiconvex integrand. In particular, they consider $W^{1,\bar{p}}$ local minimizers, defined as follows

Definition 1.1. Let $1 \leq p \leq \bar{p} \leq +\infty$. A map $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a $W^{1,\bar{p}}(\Omega; \mathbb{R}^N)$ local minimizer of $I(v)$, if there exists $\delta > 0$ such that $I(u) \leq I(v)$ whenever $v \in u + W_o^{1,p}(\Omega; \mathbb{R}^N)$ and $||Dv - Du||_{\bar{p}} \leq \delta$.

We will refer to a $W^{1,\bar{p}}(\Omega; \mathbb{R}^N)$ local minimizer of $I(v)$ with $1 \leq \bar{p} < \infty$ as a *strong local minimizer* and to a $W^{1,\infty}(\Omega; \mathbb{R}^N)$ local minimizer as a *weak local minimizer*. It has been noted in [15] that, if $p = \bar{p}$, the study of partial regularity of strong local minimizers can be reduced to the study of absolute minimizers. For this reason we confine ourselves to the case $\bar{p} > p$. Since in [4] the study of this class of minimizers is proposed for polyconvex integral functionals, as a natural continuation of the results in [15], in this paper we prove $C^{1,\alpha}$ partial regularity for weak and strong local minimizers of polyconvex functionals of the type

$$\int_\Omega |Du|^2 + f(\mathrm{Adj}Du) + g(\det Du) \ ,$$

where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, f grows like $|\mathrm{Adj}Du|^p$, g grows like $|\det Du|^q$ and $1 < q < p < 2$. We mention that, under the same assumptions on f and g of Theorem A below, the regularity result for absolute minimizers has been obtained in [16].

Theorem A. *Let us consider the functional*

$$I(v) = \int_\Omega |Dv|^2 + f(\mathrm{Adj}Dv) + g(\det Dv), \tag{1.1}$$

and suppose that $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ *are* C^2 *convex functions satisfying the following assumptions*

$$(H1) \qquad c_1(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \leq f_{z_i z_j}(z)\xi_i\xi_j \leq c_2(\mu^2 + |z|^2)^{\frac{p-2}{2}}|\xi|^2 \,,$$

$$(H2) \qquad c_3(\mu^2 + t^2)^{\frac{q-2}{2}} \leq g''(t) \leq c_4(\mu^2 + t^2)^{\frac{q-2}{2}} \,,$$

where $1 < q < p < 2$ *and* $\mu \geq 0$. *Let* $2 \leq \bar{p} \leq \infty$ *and assume that* $u \in W^{1,2}(\Omega, \mathbb{R}^3) \cap W^{1,\bar{p}}_{loc}(\Omega, \mathbb{R}^3)$ *be a* $W^{1,\bar{p}}$ *local minimizer of* $I(v)$.

If $\bar{p} = \infty$, *we assume in addition that*

$$\mathrm{limsup}_{R\to 0^+}||Du - (Du)_{x,R}||_{L^\infty(B(x,R))} < \delta \tag{1.2}$$

holds locally uniformly in $x \in \Omega$, *with* δ *as in Definition 1. Then, there exists* $\alpha \in (0,1)$ *such that* $u \in C^{1,\alpha}(\Omega_0)$ *for some open subset* Ω_0 *of* Ω *with* $\mathrm{meas}(\Omega \setminus \Omega_0) = 0$.

As far as we know, no results are available, even for absolute minimizers, if $p = q \leq 2$. However, in the special case of weak local minimizers, we are able to deal also with this assumption. Namely we have

Theorem B. *Let* f, g *satisfy the same assumptions as in Theorem A with* $1 < p = q \leq 2$. *Let* $u \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^3)$ *be a weak local minimizer for the functional (1.1) such that*

$$\mathrm{limsup}_{R\to 0^+}||Du - (Du)_{x,R}||_{L^\infty(B_R(x))} < \delta \tag{1.3}$$

holds locally uniformly in $x \in \Omega$, *with* δ *as in Definition 1. Then, there exists* $\alpha \in (0,1)$ *such that* $u \in C^{1,\alpha}(\Omega_0)$ *for some open subset* Ω_0 *of* Ω *with* $\mathrm{meas}(\Omega \setminus \Omega_0) = 0$.

We have restricted our study to the case $n = N = 3$, which is the most significant from the point of view of the applications, thus avoiding the heavy technicalities needed for the general case $n \geq 3$, $N \geq 2$, which in any case can carried on without any new idea. A fundamental tool needed to prove partial regularity is a new Caccioppoli type estimate (see Lem. 2.3).

The difficulties here are twofold. The first one is due to the anisotropic growth of the functional which requires the use of suitable test functions obtained (as in [10]) by interpolating the values of u on the boundary.

The second difficulty comes from the definition of local minimizers which imposes a bound on the $W^{1,\bar{p}}$ norm of the test functions. Kristensen and Taheri [15] discovered that this restriction is responsible of the fact that in the proof of the Caccioppoli estimate it is impossible the iteration on small radii and thus they are lead to an inequality which does not involve any two concentric balls. Here, in a different way, we also obtain a pre-iterated form of the Caccioppoli inequality involving only two balls of radii R and $\frac{R}{2}$.

This weaker form of the usual Caccioppoli estimate is however enough to establish the decay estimate by the use of an extra new iteration argument.

2. The energy estimate

Let Ω be a bounded open subset of $\mathbb{R}^3$. If $A \in Hom(\mathbb{R}^3, \mathbb{R}^3)$ we set

$$\wedge_0 A = 1, \quad \wedge_1 A = A, \quad \wedge_2 A = \mathrm{adj}A, \quad \wedge_3 A = \det A.$$

Following [11], for every $A, B \in Hom(\mathbb{R}^3, \mathbb{R}^3)$ and for $k = 2, 3$, we shall write,

$$\wedge_k (A + B) = \sum_{i=0}^{k} \wedge_{k-i} A \odot \wedge_i B, \tag{2.1}$$

where $\wedge_{k-i} A \odot \wedge_i B$ denotes a suitable linear combination of products of a component of $\wedge_{k-i} A$ times a component of $\wedge_i B$. The explicit expression of $\wedge_{k-i} A \odot \wedge_i B$ will not be needed in the sequel.

Definition 2.1. Let Ω be a smooth bounded domain of $\mathbb{R}^3$ and $p > 1$. The Sobolev class $\wedge^3 W^{1,p}(\Omega; \mathbb{R}^3)$ consists of all functions $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $\wedge_i Du \in L^p(\Omega)$ for every $1 \leq i \leq 3$.

Remark that the Sobolev class $\wedge^3 W^{1,p}$ is not a linear space. Indeed, if $u, v \in \wedge^3 W^{1,p}$ it may happen that $u + v \notin \wedge^3 W^{1,p}$.

In what follows $B_r(x)$ will be the ball centered in x of radius r. If no confusion arises, B_r will stands for a ball centered in 0 of radius r. When $r = 1$, we may use B instead of B_1. The letter C will denote a generic constant whose value may change from line to line.

Lemma 2.2. *Let* α *be a constant and let* f_h, g_h *be two sequences of functions in* $L^1(B)$ *such that* $f_h \to \alpha$ *for a.e.* $x \in B$, $g_h \to g$ *weakly in* $L^{1+\eta}(B)$, *for some* $\eta > 0$. *Assume that*

$$\int_B |f_h g_h|^{1+\delta} \mathrm{d}x < \infty,$$

where $\delta \geq \eta$. *Then* $f_h g_h \to \alpha g$ *weakly in* $L^{1+\eta}(B)$.

Proof. See [11], Lemma 5.5.

If $\Omega \subset \mathbb{R}^3$ is a bounded open set and $\lambda \in (0,1)$ is a real number, we consider, for $v \in W^{1,2}(\Omega, \mathbb{R}^3)$, the functional

$$J_\lambda(v) = \int_\Omega [|Dv|^2 + \lambda^\alpha |\mathrm{Adj}Dv|^p + \lambda^\beta |\mathrm{det}Dv|^q]\mathrm{d}x \tag{2.2}$$

where $0 < \frac{q}{p} < \frac{\beta}{\alpha}$ and $1 < q < p < 2$.

If $Q \geq 1$ is a real number we say that $u \in W^{1,2}(\Omega, \mathbb{R}^3)$ is a $W^{1,\bar{p}}(\Omega, \mathbb{R}^3)$ Q-local minimizer of J_λ if there exists a $\delta > 0$ such that $J_\lambda(u) \leq QJ_\lambda(\varphi)$ for any $\varphi \in u + W_0^{1,2}(\Omega, \mathbb{R}^3)$ with

$$||D\varphi - Du||_{L^{\bar{p}}} \leq \delta. \tag{2.3}$$

We prove the following Caccioppoli type estimate □

Lemma 2.3. *Let $2 \leq \bar{p} \leq \infty$ and let $u \in W^{1,2}(\Omega, \mathbb{R}^3) \cap W^{1,\bar{p}}_{loc}(\Omega, \mathbb{R}^3)$ be a $W^{1,\bar{p}}$ Q- local minimizer of J_λ. If $\bar{p} = \infty$, assume also that*

$$\mathrm{limsup}_{R \to 0^+} ||Du||_{L^\infty(B(x,R))} < \delta \tag{2.4}$$

where δ is the number appearing in (2.3). Then there exist a constant c depending only on Q and a radius $\bar{R}$ depending only on δ and a $\theta \in (0,1)$ such that for any $R < \bar{R}$, $B_R \subset\subset \Omega$

$$\begin{aligned} J_\lambda(u; B_{\frac{R}{2}}) &\leq \theta J_\lambda(u; B_R) + \frac{c}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} |u - u_R|^2 \mathrm{d}x \\ &+ c\lambda^\alpha R^{5-3p} \Big(\int_{B_R} |Du|^2 \mathrm{d}x \Big)^p + c\lambda^{\frac{\beta p - \alpha q}{p-q}} R^{3 - \frac{3pq}{2(p-q)}} \Big(\int_{B_R} |Du|^2 \mathrm{d}x \Big)^{\frac{pq}{2(p-q)}}. \end{aligned}$$

Proof. Fix $B_R \subset \Omega$ and define

$$E_R = \Big\{ \rho \in \Big(\frac{R}{2}, R\Big) : \int_{\partial B_\rho} |Du|^{\bar{p}} \mathrm{d}\mathcal{H}^2 \leq \frac{8}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^{\bar{p}} \mathrm{d}x$$

$$\text{and} \qquad \int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 \leq \frac{8}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \mathrm{d}x \Big\}.$$

We have that $\left(\frac{R}{2}, R\right) \setminus E_R = \mathcal{C}_1 \cup \mathcal{C}_2$ where

$$\mathcal{C}_1 = \Big\{ \rho \in \Big(\frac{R}{2}, R\Big) : \int_{\partial B_\rho} |Du|^{\bar{p}} \mathrm{d}\mathcal{H}^2 \geq \frac{8}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^{\bar{p}} \Big\}$$

$$\mathcal{C}_2 = \Big\{ \rho \in \Big(\frac{R}{2}, R\Big) : \int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 \geq \frac{8}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \Big\}.$$

One easily gets that

$$meas(\mathcal{C}_i) < \frac{R}{8}$$

and then

$$meas(E_R) \geq \frac{R}{4} \cdot \tag{2.5}$$

Now let $\omega = \frac{x}{|x|}$ and for *a.e.* $\rho \in (\frac{R}{2}, R)$ consider the function $\omega \to u(\rho\omega)$. For every $1 < m < 2$ we get

$$\left(\int_{\partial B_\rho} \left|u - u_{\partial B_\rho}\right|^{m^*} \mathrm{d}\mathcal{H}^2\right)^{\frac{1}{m^*}} \leq c\left(\int_{\partial B_\rho} |Du|^m \mathrm{d}\mathcal{H}^2\right)^{\frac{1}{m}} \tag{2.6}$$

where as usual $m^* = \frac{2m}{2-m}$ denote the Sobolev exponent of m.

For each $\rho \in E_R$, following [11], we define the function

$$\varphi(r\omega) = \begin{cases} u_{\partial B_\rho} & r \leq \frac{R}{4} \\ \frac{\rho - r}{\rho - \frac{R}{4}} u_{\partial B_\rho} + \frac{r - \frac{R}{4}}{\rho - \frac{R}{4}} u(\rho\omega) & \frac{R}{4} \leq r \leq \rho \\ u(r\omega) & \rho \leq r \leq R \end{cases} \tag{2.7}$$

and observe that, for $\frac{R}{4} < r < \rho$

$$|D\varphi(r\omega)| \leq c\left(\frac{|u(\rho\omega) - u_{\partial B_\rho}|}{\rho - \frac{R}{4}} + |Du(\rho\omega)|\right) \tag{2.8}$$

$$|\mathrm{Adj}D\varphi(r\omega)| \leq c\frac{|u(\rho\omega) - u_{\partial B_\rho}|}{\rho - \frac{R}{4}}|Du(\rho\omega)| + c|\mathrm{Adj}Du(\rho\omega)| \tag{2.9}$$

$$|\mathrm{det}D\varphi(r\omega)| \leq c\frac{|u(\rho\omega) - u_{\partial B_\rho}|}{\rho - \frac{R}{4}}|\mathrm{Adj}Du(\rho\omega)|. \tag{2.10}$$

If $\bar{p} < \infty$, by (2.8) and the assumption $u \in W^{1,\bar{p}}_{loc}$, one easily gets that there exists a $\bar{R} = \bar{R}(\delta)$ such that

$$\begin{aligned} \int_{B_R} |Du - D\varphi|^{\bar{p}} \mathrm{d}x = \int_{B_\rho} |Du - D\varphi|^{\bar{p}} \mathrm{d}x &\leq c\int_{B_\rho \setminus B_{\frac{R}{4}}} \frac{|u(\rho\omega) - u_{\partial B_\rho}|^{\bar{p}}}{(\rho - \frac{R}{4})^{\bar{p}}} \mathrm{d}x + c\int_{B_R} |Du|^{\bar{p}} \mathrm{d}x \\ &\leq \frac{c}{(\rho - \frac{R}{4})^{\bar{p}-1}} \int_{\partial B_\rho} |u(\rho\omega) - u_{\partial B_\rho}|^{\bar{p}} \mathrm{d}\mathcal{H}^2 + c\int_{B_R} |Du|^{\bar{p}} \mathrm{d}x \\ &\leq \frac{c\rho^{\bar{p}}}{(\rho - \frac{R}{4})^{\bar{p}-1}} \int_{\partial B_\rho} |Du|^{\bar{p}} \mathrm{d}\mathcal{H}^2 + c\int_{B_R} |Du|^{\bar{p}} \mathrm{d}x \\ &\leq \frac{cR^{\bar{p}-1}}{(\rho - \frac{R}{4})^{\bar{p}-1}} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^{\bar{p}} \mathrm{d}x + c\int_{B_R} |Du|^{\bar{p}} \mathrm{d}x \leq c\int_{B_R} |Du|^{\bar{p}} \end{aligned}$$

where we used also Poincaré inequality and the fact that $\frac{R}{2} < \rho < R$. Then, previous inequality ensures that φ is an admissible test function, by the absolute continuity of the integral for $\bar{p} < \infty$ and by (2.4) if $\bar{p} = \infty$.

Using (2.8), (2.9), (2.10) and the fact that u is a $W^{1,\bar{p}}$ Q- local minimizer we get

$$\begin{aligned} \int_{B_\rho} [|Du|^2 + \lambda^\alpha |\mathrm{Adj}Du|^p + \lambda^\beta |\mathrm{det}Du|^q] \mathrm{d}x &\leq Q\int_{B_\rho} [|D\varphi|^2 + \lambda^\alpha |\mathrm{Adj}D\varphi|^p + \lambda^\beta |\mathrm{det}D\varphi|^q] \mathrm{d}x \\ &\leq \frac{c}{R}\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^2 \mathrm{d}\mathcal{H}^2 + cR\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 \\ &+ \frac{c\lambda^\alpha}{R^{p-1}}\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^p |Du|^p \mathrm{d}\mathcal{H}^2 + c\lambda^\alpha R\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2 \\ &+ \frac{c\lambda^\beta}{R^{q-1}}\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^q |\mathrm{Adj}Du|^q \mathrm{d}\mathcal{H}^2 \end{aligned} \tag{2.11}$$

where we used again that $\frac{R}{2} < \rho < R$. Now, we observe that

$$\begin{aligned}\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^p |Du|^p \mathrm{d}\mathcal{H}^2 &\leq \left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^{\frac{p}{2}} \left(\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^{\frac{2p}{2-p}} \mathrm{d}\mathcal{H}^2\right)^{\frac{2-p}{2}} \\ &\leq c\left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^{\frac{p}{2}} \left(\int_{\partial B_\rho} |Du|^p \mathrm{d}\mathcal{H}^2\right) \leq cR^{2-p}\left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^p, \quad (2.12)\end{aligned}$$

where we used Hölder inequality and (2.6). Moreover using Young's inequality and (2.6) again we get

$$\begin{aligned}&\frac{c\lambda^\beta}{R^{q-1}} \int_{\partial B_\rho} |u - u_{\partial B_\rho}|^q |\mathrm{Adj}Du|^q \mathrm{d}\mathcal{H}^2 \\ &\leq \frac{c\lambda^{\frac{\beta p - \alpha q}{p-q}}}{R^{\frac{pq}{p-q}-1}} \left(\int_{\partial B_\rho} |u - u_{\partial B_\rho}|^{\frac{pq}{p-q}} \mathrm{d}\mathcal{H}^2\right) + c\lambda^\alpha R\left(\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2\right) \\ &\leq \frac{c\lambda^{\frac{\beta p - \alpha q}{p-q}}}{R^{\frac{pq}{p-q}-1}} \left(\int_{\partial B_\rho} |Du|^{\frac{2pq}{2p-2q+pq}} \mathrm{d}\mathcal{H}^2\right)^{\frac{2p-2q+pq}{2(p-q)}} + c\lambda^\alpha R\left(\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2\right) \\ &\leq c\lambda^{\frac{\beta p - \alpha q}{p-q}} R^{3-\frac{pq}{p-q}} \left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^{\frac{pq}{2(p-q)}} + c\lambda^\alpha R\left(\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2\right). \quad (2.13)\end{aligned}$$

Inserting the inequalities (2.12) and (2.13) in (2.11) we obtain

$$\begin{aligned}&\int_{B_\rho} [|Du|^2 + \lambda^\alpha |\mathrm{Adj}Du|^p + \lambda^\beta |\mathrm{det}Du|^q] \mathrm{d}x \\ &\leq \frac{c}{R} \int_{\partial B_\rho} |u - u_{\partial B_\rho}|^2 \mathrm{d}\mathcal{H}^2 + cR\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 + c\lambda^\alpha R\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2 \\ &+ c\lambda^\alpha R^{3-2p}\left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^p + c\lambda^{\frac{\beta p - \alpha q}{p-q}} R^{3-\frac{pq}{p-q}} \left(\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2\right)^{\frac{pq}{2(p-q)}}. \quad (2.14)\end{aligned}$$

Recalling that ρ is in E_R, we get

$$\begin{aligned}&\int_{B_\rho} [|Du|^2 + \lambda^\alpha |\mathrm{Adj}Du|^p + \lambda^\beta |\mathrm{det}Du|^q] \mathrm{d}x \\ &\leq \frac{c}{R} \int_{\partial B_\rho} |u - u_{\partial B_\rho}|^2 \mathrm{d}\mathcal{H}^2 + cR\int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 + c\lambda^\alpha R\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \mathrm{d}\mathcal{H}^2 \\ &+ c\lambda^\alpha R^{3-3p}\left(\int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \mathrm{d}x\right)^p + c\lambda^{\frac{\beta p - \alpha q}{p-q}} R^{3-\frac{3pq}{2(p-q)}} \left(\int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \mathrm{d}x\right)^{\frac{pq}{2(p-q)}}. \quad (2.15)\end{aligned}$$

Integrating (2.15) with respect to ρ in E_R, using (2.5) we obtain

$$R\int_{B_{\frac{R}{2}}}[|Du|^2+\lambda^\alpha|\mathrm{Adj}Du|^p+\lambda^\beta|\det Du|^q]\mathrm{d}x$$
$$\leq \frac{c}{R}\int_{B_R\setminus B_{\frac{R}{2}}}|u-u_R|^2\mathrm{d}x+cR\int_{B_R\setminus B_{\frac{R}{2}}}|Du|^2\mathrm{d}x+c\lambda^\alpha R\int_{B_R\setminus B_{\frac{R}{2}}}|\mathrm{Adj}Du|^p\mathrm{d}x$$
$$+c\lambda^\alpha R^{4-3p}\left(\int_{B_R}|Du|^2\mathrm{d}x\right)^p+c\lambda^{\frac{\beta p-\alpha q}{p-q}}R^{4-\frac{3pq}{2(p-q)}}\left(\int_{B_R}|Du|^2\mathrm{d}x\right)^{\frac{pq}{2(p-q)}}. \quad (2.16)$$

Dividing inequality (2.16) by R and using the standard trick of "hole-filling" we get

$$J_\lambda(u;B_{\frac{R}{2}})\leq\theta J_\lambda(u;B_R)+\frac{c}{R^2}\int_{B_R\setminus B_{\frac{R}{2}}}|u-u_R|^2\mathrm{d}x$$
$$+c\lambda^\alpha R^{3-3p}\Big(\int_{B_R}|Du|^2\mathrm{d}x\Big)^p+c\lambda^{\frac{\beta p-\alpha q}{p-q}}R^{3-\frac{3pq}{2(p-q)}}\Big(\int_{B_R}|Du|^2\mathrm{d}x\Big)^{\frac{pq}{2(p-q)}}$$

where $\theta=\frac{c}{c+1}\in(0,1)$ *i.e.* the conclusion follows. □

3. Proof of Theorem A

Let us consider the excess function

$$U(x,r)=\fint_{B_r(x)}|Du-(Du)_r|^2+|\mathrm{Adj}(Du-(Du)_r)|^p+|\det(Du-(Du)_r)|^q \quad (3.1)$$

we want to establish, as usual, a decay estimate for $U(x,r)$. More precisely we have the following lemma.

Lemma 3.1. *Let $u\in W^{1,2}(\Omega,\mathbb{R}^3)\cap W^{1,\bar p}_{loc}(\Omega,\mathbb{R}^3)$ be a $W^{1,\bar p}$ local minimizer of I (satisfying (1.3) if $\bar p=+\infty$). For any $M>0$ and $\tau\in(0,\frac12)$ there exist two constants $c(M)$ and $\varepsilon(\tau,M)$ such that, if*

$$|(Du)_r|\leq M \quad \text{and} \quad U(x,r)<\varepsilon, \quad (3.2)$$

then

$$U(x,\tau r)\leq c(M)\tau^\mu U(x,r), \quad (3.3)$$

for some μ independent of M and τ.

Proof. **Step 1** (*Blow up*)**.** Fix $M>0$ and $\tau\in(0,\frac12)$. Arguing by contradiction, we assume that there exists a sequence $B_{r_h}(x_h)\subset\subset\Omega$ such that

$$|(Du)_{x_h,r_h}|\leq M \qquad \text{and} \qquad \lambda_h^2=U(x_h,r_h)\to 0, \quad (3.4)$$

but

$$\frac{U(x_h,\tau r_h)}{\lambda_h^2}>c(M)\tau^\mu, \quad (3.5)$$

for some $c(M)$ to be determined later. Setting $A_h=(Du)_{x_h,r_h}$ and

$$v_h=\frac{u(x_h+r_hy)-(u)_{x_h,r_h}-r_hA_hy}{\lambda_hr_h} \quad (3.6)$$

for all $y \in B_1(0)$, we have

$$\int_{B_1(0)} |Dv_h|^2 + \lambda_h^{2p-2}|\mathrm{Adj}(Dv_h)|^p + \lambda_h^{3q-2}|\det(Dv_h)|^q = 1 \tag{3.7}$$

and $(v_h)_{0,1} = 0$. Passing to a subsequence and using the divergence structure of minors, we may assume, without loss of generality, that

$$\begin{aligned} Dv_h &\to Dv & & w - L^2(B_1) \\ v_h &\to v & & s - L^2(B_1) \\ \lambda_h^{\frac{2p-2}{p}} \mathrm{Adj} Dv_h &\to 0 & & w - L^p(B_1) \\ \lambda_h^{\frac{3q-2}{q}} \det Dv_h &\to 0 & & w - L^q(B_1) \\ A_h &\to A & & a_h \to a. \end{aligned} \tag{3.8}$$

We introduce the rescaled functionals

$$f_h(\xi) = \frac{1}{\lambda_h^2}[f(\mathrm{Adj}(A_h + \lambda_h\xi)) - f(\mathrm{Adj}(A_h)) - Df(\mathrm{Adj}(A_h))(\mathrm{Adj}(A_h + \lambda_h\xi) - \mathrm{Adj}(A_h))]$$

and

$$g_h(s) = \frac{1}{\lambda_h^2}[g(\det(A_h + \lambda_h s)) - g(\det(A_h)) - g'(\det(A_h))(\det(A_h + \lambda_h s) - \det(A_h))]$$

and for any h,we set

$$I_h(w) = \int_{B_1} |Dw|^2 + f_h(Dw) + g_h(Dw). \tag{3.9}$$

It is easy to check that $I_h(v_h) \le I_h(v_h + \varphi)$ provided $\varphi \in W_0^{1,\bar p}(B_1, R^N)$ and

$$||D\varphi||_{\bar p} \le \delta_h = \begin{cases} \frac{\delta}{\lambda_h r_h^{\frac{n}{\bar p}}} & \bar p < \infty \\ \frac{\delta}{\lambda_h} & \bar p = \infty. \end{cases}$$

Step 2 (*v solves a linear system*)**.** By formula (2.1) one easily deduces that, for every $\phi \in C_0^\infty(B_1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \wedge_i (A_h + \lambda_h(Dv_h + tD\phi))|_{t=0} = \wedge_{i-1}(A_h + \lambda_h Dv_h) \odot \lambda_h D\phi.$$

Then the minimality of v_h implies that they solve the Euler Lagrange systems:

$$\begin{aligned} &\int_{B_1(0)} Dv_h D\phi \mathrm{d}x + \int_{B_1(0)} \left(\int_0^1 D^2 f(\mathrm{Adj}(A_h) + t(\mathrm{Adj}(A_h + \lambda_h Dv_h) - \mathrm{Adj}(A_h)))\mathrm{d}t \right) \\ &\qquad \times (\lambda_h \mathrm{Adj}(Dv_h) + A_h \odot Dv_h) \cdot [(A_h + \lambda_h Dv_h) \odot D\phi]\mathrm{d}x \\ &\qquad + \int_{B_1(0)} \left(\int_0^1 g''(\det(A_h) + t(\det(A_h + \lambda_h Dv_h) - \det(A_h)))\mathrm{d}t \right) \\ &\qquad \times (\mathrm{Adj}(A_h) \odot Dv_h + \lambda_h Dv_h \odot \mathrm{adj}(Dv_h)) \cdot [\mathrm{Adj}(A_h + \lambda_h Dv_h) \odot D\phi]\mathrm{d}x = 0 \end{aligned} \tag{3.10}$$

for all $\phi \in C_0^\infty(B_1)$. Letting $h \to \infty$, using (3.8) and Lemma 2.2 we get

$$0 = \int_{B_1(0)} DvD\phi + D^2 f(\mathrm{Adj}A)(A \odot Dv)(A \odot D\phi) + D^2 g(\det A)(\mathrm{Adj}A \odot Dv)(\mathrm{Adj}A \odot D\phi)\mathrm{d}x$$

then v solves a linear elliptic system, with constant coefficients. By standard regularity result (see [12]) we have that for any $\sigma \in (0, \frac{1}{2}]$

$$\fint_{B_\sigma} |Dv - (Dv)_\sigma|^2 \le c\sigma^2 \fint_{B_1} |Dv - (Dv)_\sigma|^2 \le c\sigma^2 \tag{3.11}$$

and

$$|(Dv)_{2\sigma} - (Dv)_\sigma|^2 \le c\sigma^2 \tag{3.12}$$

where the constant c depends only on M. Setting

$$w_h(y) = v_h(y) - (Dv_h)_\sigma y - (v_h)_{2\sigma} \tag{3.13}$$

and using the fact that v_h minimizes the functional (3.9), one easily see that w_h minimizes the functional

$$w \to \int_{B_1} |Dw|^2 + f_h(Dw + (Dv_h)_\sigma) + g_h(Dw + (Dv_h)_\sigma). \tag{3.14}$$

Now, we claim that

$$\begin{aligned} |f_h(\xi)| &\le c(M)(|\xi|^2 + \lambda_h^{2p-2}|\mathrm{Adj}\xi|^p) \\ |g_h(\xi)| &\le c(M)(|\xi|^2 + \lambda_h^{3q-2}|\mathrm{det}\xi|^q). \end{aligned} \tag{3.15}$$

Namely, by definition of f_h, we have that

$$\begin{aligned} f_h(\xi) &= \frac{1}{\lambda_h^2}[f(\mathrm{Adj}(A_h + \lambda_h\xi)) - f(\mathrm{Adj}(A_h)) - Df(\mathrm{Adj}(A_h))(\mathrm{Adj}(A_h + \lambda_h\xi) - \mathrm{Adj}(A_h))] \\ &= \frac{1}{\lambda_h^2}[f(\mathrm{Adj}A_h + \lambda_h^2\mathrm{Adj}\xi + A_h \odot \lambda_h\xi) - f(\mathrm{Adj}(A_h)) - Df(\mathrm{Adj}(A_h))(\lambda_h^2\mathrm{Adj}\xi + A_h \odot \lambda_h\xi)] \end{aligned}$$

and setting

$$\zeta = \lambda_h^2\mathrm{Adj}\xi + A_h \odot \lambda_h\xi$$

we can write

$$\begin{aligned} \lambda_h^2|f_h(\xi)| &= \left|\int_0^1 (1-s)D^2f(\mathrm{Adj}A_h + s\zeta)\zeta\zeta \mathrm{d}s\right| \\ &\le |\zeta|^2 \int_0^1 \left(\mu^2 + |\mathrm{Adj}A_h + s\zeta|^2\right)^{\frac{p-2}{2}} \\ &\le c|\zeta|^2(\mu^2 + |\mathrm{Adj}A_h|^2 + |\zeta + \mathrm{Adj}A_h|^2)^{\frac{p-2}{2}}, \end{aligned} \tag{3.16}$$

where we used Lemma 2.1 in [2] and assumption (H2). On the other hand we observe that if

$$|\lambda_h^2\mathrm{Adj}(\xi)| \le |A_h \odot \lambda_h\xi|,$$

then by (3.16) we have

$$\lambda_h^2|f_h(\xi)| \le c\lambda_h^2|A_h \odot \xi|^2 \le c(M)\lambda_h^2|\xi|^2.$$

If

$$|\lambda_h^2\mathrm{Adj}(\xi)| > |A_h \odot \lambda_h\xi|,$$

then by (3.16)

$$\lambda_h^2|f_h(\xi)| \le c\lambda_h^{2p}|\mathrm{Adj}(\xi)|^p.$$

The second inequality in (3.15) is analogue. Using Lemma 2.1 in [2] again and assumptions (H1) and (H2) we obtain

$$|f_h(\xi)| \geq c(M)(|\xi|^2 + \lambda_h^{2p-2}|\mathrm{Adj}\xi|^p),$$

$$|g_h(\xi)| \geq c(M)(|\xi|^2 + \lambda_h^{3q-2}|\mathrm{det}\xi|^q).$$

Hence the functional defined at (3.14) is equivalent to the following

$$w \to \int_{B_1} |Dw|^2 + \lambda_h^{2p-2}|\mathrm{Adj}(Dw)|^p + \lambda_h^{3q-2}|\mathrm{det}(Dw)|^q$$

and then w_h is a $W^{1,\bar{p}}$ Q-local minimizer of $J_{\lambda_h}(B_1)$ for some $Q = Q(M)$, with $\alpha = 2p-2$ and $\beta = 3q-2$.

Step 3 (Conclusion)**.** Rescaling the excess function defined by (3.1), we get

$$\begin{aligned}
U(x_h, \sigma r_h) &= \fint_{B_{\sigma r_h}(x_h)} |Du - (Du)_{\sigma r_h}|^2 + |\mathrm{Adj}(Du - (Du)_{\sigma r_h})|^p + |\mathrm{det}(Du - (Du)_{\sigma r_h})|^q \\
&= \fint_{B_\sigma(0)} \lambda_h^2|Dv_h - (Dv_h)_\sigma|^2 + \lambda_h^{2p}|\mathrm{Adj}(Dv_h - (Dv_h)_\sigma)|^p + \lambda_h^{3q}|\mathrm{det}(Dv_h - (Dv_h)_\sigma)|^p \\
&= \fint_{B_\sigma(0)} \lambda_h^2|Dw_h|^2 + \lambda_h^{2p}|\mathrm{Adj}(Dw_h)|^p + \lambda_h^{3q}|\mathrm{det}(Dw_h)|^p.
\end{aligned}$$

Since w_h satisfies the assumptions of Lemma 2.3 (in particular if $\bar{p} = +\infty$ assumption (1.3) implies that w_h satisfies (2.4) with $\delta_h = \frac{\delta}{\lambda_h}$), for h sufficiently large we obtain

$$\begin{aligned}
\frac{U(x_h, \sigma r_h)}{\lambda_h^2} &= \fint_{B_\sigma(0)} |Dw_h|^2 + \lambda_h^{2p-2}|\mathrm{Adj}(Dw_h)|^p + \lambda_h^{3q-2}|\mathrm{det}(Dw_h)|^q \\
&\leq \theta \frac{U(x_h, 2\sigma r_h)}{\lambda_h^2} + \frac{c}{\sigma^2}\fint_{B_{2\sigma}} |w_h - (w_h)_{2\sigma}|^2 \\
&\quad + c\sigma^{5-3p}\lambda_h^{2p-2}\left(\fint_{B_{2\sigma}} |Dw_h|^2\right)^p + c\sigma^{3-\frac{3pq}{2(p-q)}}\lambda_h^{\frac{pq}{p-q}-2}\left(\fint_{B_{2\sigma}} |Dw_h|^2\right)^{\frac{pq}{2(p-q)}}.
\end{aligned} \tag{3.17}$$

Passing to the limit as $h \to \infty$ in (3.17) and using (3.8), (3.11), (3.12) and the assumption $1 < p < q < 2$ we get

$$\begin{aligned}
&\limsup_{h\to\infty} \frac{U(x_h, \sigma r_h)}{\lambda_h^2} \leq \theta \limsup_{h\to\infty} \frac{U(x_h, 2\sigma r_h)}{\lambda_h^2} + \frac{c}{\sigma^2}\fint_{B_{2\sigma}} |v - (v)_{2\sigma} - (Dv)_\sigma y|^2 \mathrm{d}y \\
&\leq \theta \limsup_{h\to\infty} \frac{U(x_h, 2\sigma r_h)}{\lambda_h^2} + \frac{c}{\sigma^2}\fint_{B_{2\sigma}} [|v - (v)_{2\sigma} - (Dv)_{2\sigma} y|^2 + |(Dv)_{2\sigma} y - (Dv)_\sigma y|^2] \mathrm{d}y \\
&\leq \theta \limsup_{h\to\infty} \frac{U(x_h, 2\sigma r_h)}{\lambda_h^2} + c\fint_{B_{2\sigma}} |Dv - (Dv)_{2\sigma}|^2 \mathrm{d}y + c\sigma^2 \\
&\leq \theta \limsup_{h\to\infty} \frac{U(x_h, 2\sigma r_h)}{\lambda_h^2} + c_1(M)\sigma^2,
\end{aligned} \tag{3.18}$$

where we used Poincaré inequality. Setting for $\sigma \in (0, \frac{1}{2}]$

$$\varphi(\sigma) = \limsup_{h\to\infty} \frac{U(x_h, \sigma r_h)}{\lambda_h^2}$$

inequality (3.18) can be rewritten as

$$\varphi(\sigma) \leq \theta\varphi(2\sigma) + c_1(M)\sigma^2$$

and one can easily check that the function

$$\psi(\sigma) = \varphi(\sigma) + c_1(M)\sigma^2$$

is such that, for all $\sigma \in (0, \frac{1}{2}]$,

$$\psi(\sigma) \leq \max\left\{\theta, \frac{1}{2}\right\}\psi(2\sigma) = \gamma\psi(2\sigma) \qquad 0 < \gamma < 1. \tag{3.19}$$

An iteration procedure yields that, for any $k = 0, 1, 2, \ldots$ and $\sigma \in (0, \frac{1}{2}]$

$$\psi\left(\frac{\sigma}{2^k}\right) \leq \gamma^k\psi(\sigma). \tag{3.20}$$

Now, take τ in the interval $\left(\frac{1}{2^{k+1}}; \frac{1}{2^k}\right)$ and observe, by the definition of ψ, that

$$\psi(\tau) \leq C\psi\left(\frac{1}{2^k}\right). \tag{3.21}$$

Putting together estimates (3.20) and (3.21), we obtain

$$\begin{aligned}
\varphi(\tau) \leq \psi(\tau) &\leq \gamma^{k-1}\psi\left(\frac{1}{2}\right) \\
&= 2^{(k-1)\log_2\gamma}\psi\left(\frac{1}{2}\right) = \left(\frac{2^{k+1}}{4}\right)^{\log_2\gamma}\psi\left(\frac{1}{2}\right) \\
&\leq \left(\frac{1}{4}\right)^{\log_2\gamma}\left(\frac{1}{\tau}\right)^{\log_2\gamma}\psi\left(\frac{1}{2}\right) \\
&= \frac{1}{\gamma^2}\tau^{-\log_2\gamma}\psi\left(\frac{1}{2}\right) \leq C_2(M)\tau^{\mu}.
\end{aligned} \tag{3.22}$$

Inequality (3.22) contradicts (3.5) if we choose $c(M)$ larger then $C_2(M)$. □

We now are in position to give the proof of Theorem A, that relies on a standard iteration argument involving the excess function.

Lemma 3.2. *Let $0 < \alpha < 1$ and $M > 0$. Then there exist $\tau \in (0, \frac{1}{2})$ and $\varepsilon > 0$ both depending on α and M such that if*

$$B(x, r) \subset \Omega, \quad |(Du)_r| \leq M, \quad \text{and} \quad U(x, r) < \varepsilon$$

then

$$U(x, \tau^l r) \leq (\tau^l)^{\alpha}U(x, r)$$

for every $l \in N$.

Proof. See Lemma 6.1 [10]. □

Lemma 3.3. *Let $u \in \wedge_k W^{1,p}(\Omega)$. Then*

$$\lim_{r\to 0} \fint_{B_r(x)} |\wedge_i (Du - (Du)_r|^p = 0$$

for almost every $x \in \Omega$ and $1 \le i \le k$.

Proof. See Lemma 6.2 [10]. □

The proof of Theorem A is now consequence of a iteration procedure based on Lemma 4.1. The singular set turns out to be contained in the complement of

$$\Omega_o = \left\{ x \in \Omega : \lim_{r\to 0} (Du)_r = 0,\ \lim_{r\to 0} \fint_{B_r(x)} |\mathrm{Adj}(Du - (Du)_r|^p = 0, \right.$$
$$\left. \lim_{r\to 0} \fint_{B_r(x)} |\det(Du - (Du)_r|^q = 0 \right\}. \tag{3.23}$$

4. The limit case

In this section we treat the limit case in which the growth exponents $p = q \le 2$. Also in this case the partial regularity of weak local minimizers is based on a decay estimate for the excess function

$$U(x,r) = \fint_{B_r(x)} |Du - (Du)_r|^2 + |\mathrm{Adj}(Du - (Du)_r)|^p + |\det(Du - (Du)_r)|^p. \tag{4.1}$$

The proof of Theorem B differs from the one of Theorem A only in the Caccioppoli type estimate.

Lemma 4.1. *Let $p \le 2$, and $u \in W^{1,2}(\Omega, \mathbb{R}^3) \cap W^{1,\infty}_{loc}(\Omega, \mathbb{R}^3)$ be a $W^{1,\infty}$ Q- local minimizer of*

$$J_\lambda(v) = \int_\Omega |Dv|^2 + \lambda^{2p-2} |\mathrm{Adj} Dv|^p + \lambda^{3p-2} |\det Dv|^p$$

such that

$$\mathrm{limsup}_{R\to 0^+} ||Du||_{L^\infty(B_R(x))} < \frac{\delta}{\lambda}, \tag{4.2}$$

where δ is the number appearing in (1.3). Then there exist a constant c depending only on Q, a radius $\bar{R}$ depending only on δ and a number $\theta \in (0,1)$ such that for any $R < \bar{R}$, we have

$$J_\lambda(u; B_{\frac{R}{2}}) \le \theta J_\lambda(u; B_R) + \frac{c}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} |u - u_R|^2 + c\lambda^{2p-2} R^{5-3p} \left(\int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \right)^p,$$

if $p < 2$,

$$J_\lambda(u; B_{\frac{R}{2}}) \le \theta J_\lambda(u; B_R) + \frac{c}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} |u - u_R|^2,$$

if $p = 2$.

Proof. The proof goes as the one of Lemma 2.3 with some minor changes. Assumption (4.2) implies that there exists a radius $\bar{R}$ depending only on δ such that for any $R < \bar{R}$,

$$||Du||_{L^\infty(B_R(x))} < \frac{\delta}{\lambda}.$$

Fix $R < \bar{R}$ and consider ρ in the set

$$E_R = \left\{ \rho \in \left(\frac{R}{2}, R \right) : \int_{\partial B_\rho} |Du|^2 \mathrm{d}\mathcal{H}^2 \leq \frac{4}{R} \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \mathrm{d}x \right\}.$$

Since u is Lipschitz continuous

$$|u(\rho\omega) - u_{\partial B_\rho}| \leq 2R||Du||_{L^\infty(B_\rho)} \leq 2\frac{\delta}{\lambda}R \tag{4.3}$$

holds.

By the assumption $u \in W^{1,\infty}_{loc}(\Omega)$ and (4.2), one easily gets that the function φ, defined at (2.6), is an admissible test function. Arguing as in Lemma 2.1, using that u is a $W^{1,\infty}$ Q- local minimizer, we get

$$\begin{aligned}
\int_{B_\rho} [|Du|^2 + \lambda^{2p-2}|\mathrm{Adj}Du|^p + \lambda^{3p-2}|\mathrm{det}Du|^p]\, \mathrm{d}x \\
\leq Q \int_{B_\rho \setminus B_{\frac{R}{2}}} [|D\varphi|^2 + \lambda^{2p-2}|\mathrm{Adj}D\varphi|^p + \lambda^{3p-2}|\mathrm{det}D\varphi|^p]\, \mathrm{d}x \\
\leq c\int_{\partial B_\rho} \frac{|u - u_{\partial B_\rho}|^2}{R} + cR\int_{\partial B_\rho} |Du|^2 + cR\lambda^{2p-2}\int_{\partial B_\rho} |\mathrm{Adj}Du|^p \\
+ c\lambda^{2p-2}\int_{\partial B_\rho} \frac{|u - u_{\partial B_\rho}|^p}{R^{p-1}}|Du|^p + c\lambda^{3p-2}\int_{\partial B_\rho} \frac{|u - u_{\partial B_\rho}|^p}{R^{p-1}}|\mathrm{Adj}Du|^p.
\end{aligned} \tag{4.4}$$

Let us treat separately the two cases $p < 2$ and $p = 2$. Assume that $p < 2$. Using (4.3) we get

$$\int_{\partial B_\rho} \frac{|u - u_{\partial B_\rho}|^p}{R^{p-1}}|\mathrm{Adj}Du|^p \leq 2R\frac{\delta^p}{\lambda^p}\int_{\partial B_\rho} |\mathrm{Adj}Du|^p. \tag{4.5}$$

Inserting the inequality (4.5) in (4.4) and using (2.12), we obtain

$$\begin{aligned}
\int_{B_\rho} [|Du|^2 + \lambda^{2p-2}|\mathrm{Adj}Du|^2 + \lambda^{3p-2}|\mathrm{det}Du|^p]\, \mathrm{d}x \\
\leq c\int_{\partial B_\rho} \frac{|u - u_{\partial B_\rho}|^2}{R}\mathrm{d}\mathcal{H}^2 + cR\int_{\partial B_\rho} |Du|^2\mathrm{d}\mathcal{H}^2 + c(1+\delta^p)\lambda^{2p-2}R\int_{\partial B_\rho} |\mathrm{Adj}Du|^p\mathrm{d}\mathcal{H}^2 \\
+ c\lambda^{2p-2}R^{3-2p}\left(\int_{\partial B_\rho} |Du|^2\mathrm{d}\mathcal{H}^2\right)^p.
\end{aligned} \tag{4.6}$$

Recalling that ρ is in E_R, integrating with respect to ρ in E_R and using the fact that $|E_R| \geq \frac{R}{4}$, we obtain

$$R \int_{B_{\frac{R}{2}}} |Du|^2 + \lambda^{2p-2} |\mathrm{Adj} Du|^p + \lambda^{3p-2} |\det Du|^p$$

$$\leq c \int_{B_R \setminus B_{\frac{R}{2}}} \frac{|u - u_R|^2}{R} + cR \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 + c\lambda^{2p-2} R \int_{B_R \setminus B_{\frac{R}{2}}} |\mathrm{Adj} Du|^p$$

$$+ c\lambda^{2p-2} R^{4-3p} \left(\int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \right)^p. \quad (4.7)$$

Dividing inequality (4.7) by R and using the standard trick of "hole-filling" we get

$$J_\lambda(u; B_{\frac{R}{2}}) \leq \theta J_\lambda(u; B_R) + \frac{c}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} |u - u_R|^2 + c\lambda^{2p-2} R^{3-3p} \left(\int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2 \right)^p,$$

where $\theta \in (0, 1)$. This is conclusion in the case $p < 2$.

Suppose $p = 2$. Formula (4.4) becomes

$$\int_{B_\rho} [|Du|^2 + \lambda^2 |\mathrm{Adj} Du|^2 + \lambda^4 |\det Du|^2] \, \mathrm{d}x$$

$$\leq Q \int_{B_\rho \setminus B_{\frac{R}{2}}} [|D\varphi|^2 + \lambda^2 |\mathrm{Adj} D\varphi|^2 + \lambda^4 |\det D\varphi|^2] \, \mathrm{d}x$$

$$\leq c \int_{B_\rho \setminus B_{\frac{R}{2}}} \frac{|u - u_{\partial B_\rho}|^2}{R^2} + c \int_{B_\rho \setminus B_{\frac{R}{2}}} |Du|^2 + c\lambda^2 \int_{B_\rho \setminus B_{\frac{R}{2}}} |\mathrm{Adj} Du|^2$$

$$+ c\lambda^2 \int_{B_\rho \setminus B_{\frac{R}{2}}} \frac{|u - u_{\partial B_\rho}|^2}{R^2} |Du|^2 + c\lambda^4 \int_{B_\rho \setminus B_{\frac{R}{2}}} \frac{|u - u_{\partial B_\rho}|^2}{R^2} |\mathrm{Adj} Du|^2. \quad (4.8)$$

Using (4.3) in (4.8) we find

$$\int_{B_\rho} [|Du|^2 + \lambda^2 |\mathrm{Adj} Du|^2 + \lambda^4 |\det Du|^2] \, \mathrm{d}x$$

$$\leq c \int_{B_\rho \setminus B_{\frac{R}{2}}} \frac{|u - u_{\partial B_\rho}|^2}{R^2} + c \int_{B_\rho \setminus B_{\frac{R}{2}}} |Du|^2 + c\lambda^2 \int_{B_\rho \setminus B_{\frac{R}{2}}} |\mathrm{Adj} Du|^2$$

$$+ c\lambda^2 \frac{\delta^2}{\lambda^2} \int_{B_\rho \setminus B_{\frac{R}{2}}} |Du|^2 + c\lambda^4 \frac{\delta^2}{\lambda^2} \int_{B_\rho \setminus B_{\frac{R}{2}}} |\mathrm{Adj} Du|^2$$

Arguing as before we conclude the proof. □

References

[1] E. Acerbi and N. Fusco, A regularity theorem for minimizers of quasiconvex integrals. *Arch. Rational Mech. Anal.* **99** (1987) 261–281.

[2] E. Acerbi and N. Fusco, Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$. *J. Math. Anal. Appl.* **140** (1989) 115–135.

[3] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (1977) 337–403.

[4] J.M. Ball, Some open problem in elasticity, in *Geometry, Mechanics and dynamics*, Springer, New York (2002) 3–59.

[5] M. Carozza, N. Fusco and G. Mingione, Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. *Annali Mat. Pura Appl.* **175** (1998) 141–164.

[6] M. Carozza and A. Passarelli di Napoli, A regularity theorem for minimizers of quasiconvex integrals the case $1 < p < 2$. *Proc. Roy. Soc. Edinburgh* **126A** (1996) 1181–1199.

[7] M. Carozza and A. Passarelli di Napoli, Partial regularity of local minimizers of quasiconvex integrals with sub-quadratic growth. *Proc. Roy. Soc Edinburgh* **133A** (2003) 1249–1262.

[8] B. Dacorogna, Direct methods in the calculus of variations. *Appl. Math. Sci.* **78**, Springer Verlag (1989).

[9] L.C. Evans, Quasiconvexity and partial regularity in the calculus of variations. *Arch. Rational Mech. Anal.* **95** (1986) 227–252.

[10] N. Fusco and J. Hutchinson, Partial regularity in problems motivated by nonlinear elasticity. *SIAM J. Math.* **22** (1991) 1516–1551.

[11] N. Fusco and J. Hutchinson, Partial regularity and everywhere continuity for a model problem from nonlinear elasticity. *J. Australian Math. Soc.* **57** (1994) 149–157.

[12] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems. *Ann. Math. Stud.* **105** Princeton Univ. Press (1983).

[13] M. Giaquinta and G. Modica, Partial regularity of minimizers of quasiconvex integrals. *Ann. Inst. H. Poincaré, Analyse non linéaire* **3** (1986) 185–208.

[14] E. Giusti, *Metodi diretti in calcolo delle variazioni.* U.M.I. (1994).

[15] J. Kristensen and A. Taheri, Partial regularity of strong local minimizers in the multidimensional calculus of variations. *Arch. Rational Mech. Anal.* **170** (2003) 63–89.

[16] A. Passarelli di Napoli, A regularity result for a class of polyconvex functionals. *Ricerche di Matematica* **XLVIII** (1999) 379–393.