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GRAPH SELECTORS AND VISCOSITY SOLUTIONS ON LAGRANGIAN MANIFOLDS

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Abstract. Let Λ be a Lagrangian submanifold of T^*X for some closed manifold X. Let $S(x,\xi)$ be a generating function for Λ which is quadratic at infinity, and let $W(x)$ be the corresponding graph selector for Λ, in the sense of Chaperon-Sikorav-Viterbo, so that there exists a subset $X_0 \subset X$ of measure zero such that W is Lipschitz continuous on X, smooth on $X \backslash X_0$ and $(x, \partial W/\partial x(x)) \in \Lambda$ for $X \backslash X_0$. Let $H(x,p) = 0$ for $(x,p) \in \Lambda$. Then W is a classical solution to $H(x, \partial W/\partial x(x)) = 0$ on $X \backslash X_0$ and extends to a Lipschitz function on the whole of X. Viterbo refers to W as a variational solution. We prove that W is also a viscosity solution under some simple and natural conditions. We also prove that these conditions are satisfied in many cases, including certain commonly occuring cases where $H(x,p)$ is not convex in p.

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1. Introduction

Let Λ be a Lagrangian submanifold of T^*X for some closed manifold X. Various authors [3, 13, 19, 24] have put forward a Lusternick-Schnirelman type min-max procedure for constructing from Λ a Lipschitz continuous function $W(x)$, where $x \in X$. This construction uses the fundamental notion of a generating function for Λ which is quadratic at infinity, the details of which can be found in the above references and are summarised below. The reader is also referred to [26] for an introduction to the general notion of a generating function for a Lagrangian manifold. The function W is known as a graph selector for Λ because, if Λ is thought of as a multi-valued section of T^*X, then outside of a set X_0 of measure zero in X, the differential $dW(x)$ selects a single value of this section in a smooth way. So given some Hamiltonian $H(x,p)$ which vanishes on Λ, it follows that W is a classical solution to $H(x, \partial W/\partial x(x)) = 0$ on $X \backslash X_0$, and extends to a Lipschitz function on the whole of X. For this reason [24, 25], interprets W as a type of generalised solution to $H(x, \partial W/\partial x(x)) = 0$ on X, described in those references as a variational solution.

The question considered in this paper is under what circumstances is this variational solution W also a viscosity solution to $H(x, \partial W/\partial x(x)) = 0$. We put forward some simple and natural conditions under which this is true, and present some examples to demonstrate that these conditions are naturally satisfied in many cases, including cases where $H(x,p)$ is not convex in p. The reader is referred to standard sources such as [5,6,9] for details on viscosity solutions.

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Our results generalise and simplify two sets of existing partial results in the literature. The first set is contained in [24, 25], where it is shown that, if x is a point at which $dW(x)$ is discontinuous, then W is a viscosity solution if $\mathrm{Hess}_p H(x,p) \geq 0$ (*e.g.* if $H(x,.)$ is convex) and if there exists a surface of discontinuity in dW running through x and across which the jump in $dW(x)$ is in the direction consistent with the defining requirements of a viscosity solution. Our results clarify the precise extent to which convexity of $H(x,.)$ is required in order for W to be a viscosity solution, and this turns out to be much weaker than a general assumption of convexity at points of discontinuity of $dW(x)$. Our results also address the difficulty of achieving consistency between the orientation of the jump in dW at points of discontinuity, and the direction of the sub- and super-solution inequalities required to establish the viscosity solution property, a difficulty which also appears in other references cited below. We do this by formulating a natural condition on W in terms of the dynamics of H, which leads to the correct choice of equation $H = 0$ or $-H = 0$ to be solved in the viscosity sense.

The second set of existing results appears in [7,15,16], the latter two being earlier work of the current author. In the first reference, a construction of W from Λ is put forward for the specific case where $H(x,.)$ is convex. This construction uses an explicit formula which identifies the minimum critical value of the generating function of Λ. We show below that this formula is a special case of the general graph selector construction in the case where $H(x,.)$ is convex. Then, in [7], under a further assumption that W is Lipschitz, it is shown that W is a viscosity solution. The proofs in this reference use very natural arguments in terms of the dynamics of the problem, as well as a key result from non-smooth analysis which provides a characterisation of the sub- and super-differentials of W as subsets of convex sets defined in terms of points on Λ. These natural arguments are the basis for the proofs of our generalised results below. In [15], we showed that the *a priori* assumption of Lipschitz continuity of W in [7] was unnecessary, since this property could be proved to follow from geometrical properties of Λ, at least for $\dim \Lambda \leq 5$. The dimension restriction arose from the use of Arnold's classification of singularities of Λ up to $\dim 5$ in the proof, and did not appear to be intrinsic. This was confirmed in [16] where the approach was extended, in outline form, to arbitrary dimensions using some of the ideas of the graph selector approach applied to the explicit formula of [7], abeit still under the assumption of convexity of H. In this current paper we complete the synthesis of these two approaches by combining the natural dynamical and convexity arguments referred to above with the full power and elegance of the graph selector approach to produce general conditions under which W is a viscosity solution, without restrictive assumptions on the convexity of H or the dimension of Λ.

We briefly review some other related approaches to generalised solutions to Hamilton-Jacobi equations in the literature. First, there is the famous Hopf formula [11], which is shown in [1] to be a viscosity solution when H is convex. Also in [2] a Hopf inf-sup type construction is applied to produce a generalised solution, starting from a generating function for Λ and identifying a section of Λ. This is shown to be a viscosity solution in the special case when H is convex and depends only on p. Both of these can be interpreted as special cases of the graph selector, in that they pick out the same section of Λ. Lastly, there is the idempotent analysis approach to generalised solutions, which can also be viewed as identifying a solution in terms of a section of Λ – see [8] for details in the case of a specific physical problem. Connections between the graph selector and idempotent analysis approaches are discussed in [16], where it is shown that they give the same solution under certain circumstances.

The remainder of the paper is organised as follows. In the next two sections we briefly review the theory of generating functions quadratic at infinity and graph selectors. In Section 4, we then set out and prove the conditions under which a graph selector, or variational solution, is also a viscosity solution. In the final section, we give two sets of examples to illustrate that these conditions are naturally satisfied in many cases. The first set concerns initial and final value Cauchy problems with convex or concave Hamiltonians. The second set concerns a non-convex Hamiltonian arising from a class of differential game optimisation problems. We also give a third simple example where a variational solution exists which is not a viscosity solution.

2. Generating functions quadratic at infinity

Let Λ be a Lagrangian submanifold of T^*X for some closed manifold X. Then Λ is said to have a generating function quadratic at infinity (GFQI), denoted S, if there exists a vector bundle $P = X \times E \to X$ (with fibre space given by vector space E which we assume to be real $E = \mathbb{R}^m$ for some m) and a C^2 function $S : (x, \xi) \in P \to S(x, \xi) \in \mathbb{R}$ such that

$$\Sigma_S = \left\{(x, \xi) \in P : \frac{\partial S}{\partial \xi}(x, \xi) = 0\right\}$$

is a subvariety of P and Λ is the image of Σ_S under the embedding

$$i_S : (x, \xi) \to \left(x, \frac{\partial S}{\partial x}(x, \xi)\right)$$

with $S(x, \xi) = B_x(\xi)$ for all $|\xi| \geq R > 0$ for some family $B_x(.)$ of non-degenerate quadratic forms.

Suppose that Λ is Lagrangian isotopic to the zero section of T^*X, *i.e.* can be smoothly transformed into the zero section *via* a family of Lagragian isomorphisms, and that Λ is exact, *i.e.* the cohomology class generated by the restriction of the Liouville form to Λ vanishes. These conditions are satisfied if, for example, Λ is the image under a Hamiltonian flow of some initial Lagrangian manifold with a well-defined projection onto X. Then, by the Théorème of [19], Λ has a GFQI S, and by Proposition 1.5 of [23], S is unique up to stable equivalence.

Uniqueness up to stable equivalence means that if there are two GFQI S_1 and S_2 for Λ, then they can both be transformed into the same generating function S through application of operations of the form

$$S(x, \xi, \eta) = S_i(x, \xi) + C_i(\eta)$$

and

$$S(x, \xi) = S_i(x, \psi_i(x, \xi)),$$

where $C_i(.)$ is a non-denegerate quadratic form on $\mathbb{R}^l$ and $(x, \xi) \to (x, \psi_i(x, \xi))$ is a fibre preserving diffeomorphism of P. This still leaves an indeterminate constant in the definition of S. Note, in the example cited above of a Lagrangian manifold generated under a Hamiltonian flow from an initial Lagrangian manifold Λ_0, this constant is determined by the choice of constant in the generating function S_0 for Λ_0. This is how we will fix this constant in the examples given at the end.

The construction of a GFQI S for Λ, as set out in [13, 19], uses the so-called broken phase curves method to produce a finite-dimensional parameterisation of the Hamilton-Jacobi action functional, considered as an infinite-dimensional generating function for Λ. For the benefit of readers unfamiliar with this theory and to illustrate the essential simplicity of the idea, we give a simplified version of the proof for the case where $X = \mathbf{T}^n$, a n-dimensional torus. This version appeared in [12] and was communicated to us by Claude Viterbo.

Let Λ_0 denote the zero section of T^*X, and suppose Λ is exact and Lagrangian isotopic to Λ_0. Then there exists a time dependent Hamiltonian $H : T^*X \times [0, 1] \to T^*\mathbb{R}^n$ with compact support and with phase flow φ_t on T^*X such that $\Lambda = \varphi_1(\Lambda_0)$. Lift H to a Hamiltonian $\tilde{H}(q, p, t) : T^*\mathbb{R}^n \times [0, 1] \to T^*\mathbb{R}^n$ which is $\mathbb{Z}^n$-periodic with respect to q, and denote by $\tilde{\varphi}_t$ the corresponding phase flow on $T^*\mathbb{R}^n$. Let $0 \leq s < t \leq 1$ and consider the canonical transformation $(Q, P) = R_s^t(q, p)$ of $T^*\mathbb{R}^n$ given by $R_s^t := \tilde{\varphi}_t \circ \tilde{\varphi}_s^{-1}$. Then there exists $\delta > 0$ such that for $t - s < \delta$, R_s^t is defined by a canonical generating function $S_s^t(Q, p)$ such that

$$P = p + \delta S_s^t/\delta Q, \ q = Q + \delta S_s^t/\delta p$$

and the mapping $(Q, p) \mapsto (q, p)$ is a diffeomorphism of $T^*\mathbb{R}^n$. The function S_s^t is defined up to an additive constant by the formula

$$S_s^t(Q, p) = \int_s^t \Big((P_\tau - p)\mathrm{d}Q_\tau/\mathrm{d}\tau - \tilde{H}_\tau(Q_\tau, P_\tau)\Big)\,\mathrm{d}\tau$$

where $(Q_\tau, P_\tau) = R_s^\tau(q,p)$. So S_s^t is $\mathbb{Z}^n$-periodic with respect to Q and C^2. Choose a sufficiently large integer l such that $1/(l+1) < \delta$ and let $S_{j,t} = S_{jt/(l+1)}^{(j+1)t/(l+1)}$ for $0 \le t \le 1$ and $(Q_j, P_j) = R_{jt/(l+1)}^{(j+1)t/(l+1)}(q_j, p_j)$ for each $0 \le j \le l$. Then the function

$$S_t(Q_0, \dots, Q_l, p_0, \dots, p_l) = \sum_{j=0}^{l} \left(S_{j,t}(Q_j, p_j) + p_{j+1}(Q_{j+1} - Q_j) \right),$$

with the convention that $l + 1 = 0$, has differential

$$dS_t(Q_0, \dots, Q_l, p_0, \dots, p_l) = \sum_{j=0}^{l} \left((P_j - p_{j+1}) dQ_j + (q_{j+1} - Q_j) dp_{j+1} \right).$$

So now we take $(Q_j, p_{j+1})_{0 \le j < l}$ to be the fibre coordinates on the fibre space $E = \mathbb{R}^{2l}$ and consider the subvariety Σ_{S_t} of $P = X \times E$ on which the partial derivatives of S_t with respect to $(Q_j, p_{j+1})_{0 \le j < l}$ all vanish. Then the graph of $\tilde{\varphi}_t$ is the image of Σ_{S_t} under the embedding

$$(Q_0, \dots, Q_l, p_0, \dots, p_l) \mapsto \left((Q_l + \delta S_t / \delta p_0, p_0), (Q_l, p_0 + \delta S_t / \delta Q_l) \right).$$

To see this, note that on Σ_{S_t}, $(q_{j+1}, p_{j+1}) = R_{jt/(l+1)}^{(j+1)t/(l+1)}(q_j, p_j)$ for each $0 \le j < l$ and so $(Q_l, P_l) = \tilde{\varphi}_t(q_0, p_0)$. Now let $p_j - p_0 = z_j$ and $Q_j - Q_{j-1} = y_j$ for $1 \le j \le l$, and let $(y, z) = (y_j, z_j)_{1 \le j \le l}$ denote the fibre coordinates. Then we can re-write S_t as $\tilde{S}_t((y,z), (Q_l, p_0)) = \tilde{F}_t((y,z), (Q_l, p_0)) + \sum_{j=1}^{l} z_j y_j$ where

$$\tilde{F}_t((y,z), (Q_l, p_0)) = S_{0,t}\left(Q_l - \sum_{i=1}^{l} y_i, p_0 \right) + \sum_{j=1}^{l} S_{j,t}\left(Q_l - \sum_{j < i \le 1} y_i, p_0 + z_j \right)$$

and where $\sum_{j=1}^{l} z_j y_j$ is a non-degenerate quadratic form of index l on the fibre space $\mathbb{R}^{2l}$. Since the functions $S_{j,t}$ have compact support, $\tilde{S}_t$ reduces to this quadratic form for large (y,z). So now restricting $\tilde{S}_1((y,z), (Q_l, 0))$ to $\{p_0 = 0\}$ we get a $\mathbb{Z}^n$-periodic function with respect to Q_l, which, on taking quotients and considering $Q_l \in X$, induces the required GFQI for Λ. This argument can be generalised to an arbitrary closed manifold X by embedding it in $\mathbb{R}^m$ for sufficiently large m. This induces a symplectic embedding of T^*X in $T^*\mathbb{R}^m$. One then constructs a Hamiltonian isotopy $\tilde{\varphi}_t$ of $T^*\mathbb{R}^m$ corresponding to a compactly supported Hamiltonian such that $\Lambda = \tilde{\varphi}_1(\Lambda_0)$ and proceeds as above. The full details are given in [13, 19]. It is also shown in this latter reference that the property of having a GFQI is preserved under Hamiltonian isotopy, *i.e.* the initial exact Lagrangian manifold Λ_0 in the above argument can be generalised to be Lagrangian isotopic to the zero section and need not be the zero section itself. Also note that, under the above definition of uniqueness up to stable equivalence, we can add extra dimensions to the fibre space and add a corresponding quadratic form of arbitrary index to the GFQI. So the fact that the above proof produces a quadratic form with index equal to its co-index is not an intrinsic property of a GFQI.

3. Graph selectors

In this section we review the notion of a graph selector weak solution to Hamilton-Jacobi equations as introduced in [3] and studied in [23–25]. We continue the notation and assumptions of the previous section, and refer the reader to Chap. 8 of [14] for a good introduction to relative homology and Morse groups. In particular, we will make use of the following basic fact about relative homology groups

$$H^n(B^k, B^k \backslash \{0\}) \simeq \delta_{n,k} \mathbb{R}$$

Let $S_x(\xi) = S(x,\xi)$ be a GFQI for Λ and let

$$E^c(S_x) = E^c = \{\xi \in E : S_x(\xi) \leq c\}\,.$$

Then for c large enough

$$H^k(E^c, E^{-c}) \simeq \begin{cases} \mathbb{R} \text{ if } k \text{ is index of } B_x \\ 0 \text{ otherwise.} \end{cases}$$

Note, the family B_x has the same index for all x since it is non-degenerate. Let α be the generator of $H^*(E^c, E^{-c})$. Then define

$$\gamma(\alpha, S_x) = \inf\left\{\lambda : \alpha \text{ induces non-null class in } H^*(E^\lambda, E^{-c})\right\}.$$

Denote

$$W_S(x) = \gamma(\alpha, S_x).$$

Now, as noted in Section 2 of [23], the definition of W_S is invariant under the above defined operations of stable equivalence. Since S is unique up to stable equivalence, it follows that W is in fact independent of the particular choice of S. So we can drop the S subscript and refer to W as an (invariant) graph selector for Λ. Note that $W(x)$ is a critical value of $S_x(.)$.

Let $X_0 \subset X$ be the set of points x_0 which are either

(1) singular values for the projection $\pi : \Lambda \to X$, or
(2) non-singular values such that there exists $\xi \neq \xi'$ with

$$\frac{\partial S}{\partial \xi}(x_0,\xi) = \frac{\partial S}{\partial \xi}(x_0,\xi') = 0,\ S(x_0,\xi) = S(x_0,\xi'),\ \frac{\partial S}{\partial x}(x_0,\xi) \neq \frac{\partial S}{\partial x}(x_0,\xi')$$

(*i.e.* distinct points on Λ with the same projection onto X and s.t. $\int p\mathrm{d}x$ on a path between them is null).

Then X_0 is closed with empty interior (*i.e.* measure zero) and W is Lipschitz in X, W is C^k in $X\backslash X_0$ for some $k \geq 1$ and for $x \in X\backslash X_0$

$$\left(x, \frac{\partial W}{\partial x}(x)\right) \in \Lambda.$$

See Proposition II and Lemma V of [25] or Theorem 2.1 of [18] for details. This latter reference has a particularly clear proof of the Lipschitz continuity of W.

Hence W is called a graph selector because its differential smoothly selects a single value of the section Λ over $X\backslash X_0$, where Λ is thought of as a multi-valued section of T^*X.

Now let $H(x,p)$ be C^2 such that $H(x,p) = 0$ for all $(x,p) \in \Lambda$. Then clearly, for $x \in X\backslash X_0$

$$H\left(x, \frac{\partial W}{\partial x}(x)\right) = 0.$$

So W is a classical solution to $H(x, \partial W/\partial x(x)) = 0$ on $X\backslash X_0$, and extends to a Lipschitz function on the whole of X. For this reason, Viterbo in [24, 25] interprets the above graph selector construction as producing from Λ a type of generalised solution to $H(x, \partial W/\partial x(x)) = 0$ on X, one which he calls a variational solution. In the next section we give natural conditions under which W is also a viscosity solution on X.

4. Conditions on variational solution to be viscosity solutions

Continuing the notation of the previous section, if $x \in X\backslash X_0$, then $S_x = S(x,.)$ is a Morse function whose critical points have pairwise distinct critical values. So given a neighbourhood U of x, there exists a smooth function $\phi : U \to E$ such that $\phi(x)$ is a critical point of S_x and $W(x) = S(x, \phi(x))$.

Similarly, if $x_0 \in X_0$, then we can consider some trajectory $x_s : (-t, t) \to X$ in state space which passes through x_0 at $s = 0$ and intersects X_0 transversely. We assume here and for the remainder of the paper that Λ and S are in sufficiently general position to ensure that X_0 is embedded as a submanifold of X, so as to avoid degenerate situations. For $s < 0$, we can choose a neighbourhood U with $x_s \in U$ for all $s < 0$, $x_0 \in cl(U)$, $U \subseteq X \backslash X_0$ and such that there exists a smooth function $\phi : U \to E$ with $\phi(x_s)$ a critical point of $S(x_s, .)$, *i.e.* $\partial S/\partial\xi(x_s, \phi(x_s)) = 0$, and

$$W(x_s) = S(x_s, \phi(x_s))$$

for all $s < 0$. Now let

$$\xi_0 = \lim_{s \to 0^-} \phi(x_s)$$

which exists since ϕ is continuous, and let

$$p_0 = \lim_{s \to 0^-} \frac{\partial S}{\partial x}(x_s, \phi(x_s)) = \lim_{s \to 0^-} \frac{\partial W}{\partial x}(x_s)$$

which exists since $\partial S/\partial x$ is continuous.

Lemma 4.1. $(x_0, p_0) \in \Lambda$ *and* $W(x_0) = S(x_0, \xi_0)$.

Proof. It follows from the continuity of $\partial S/\partial x$ and $\partial S/\partial \xi$ that

$$\frac{\partial S}{\partial \xi}(x_0, \xi_0) = 0$$

and

$$p_0 = \frac{\partial S}{\partial x}(x_0, \xi_0).$$

The result now follows from the definition of S as a generating function for Λ, and from the continuity of S and W. $\square$

The same result holds true for $s > 0$.

We now set out the conditions under which a variational solution will be a viscosity solution. Let

$$C_x = \mathrm{co} \left\{ \begin{array}{c} p : (x, p) \in \Lambda \text{ where } p = \frac{\partial S}{\partial x}(x, \xi) \text{ for} \\ \text{some } \xi \in E \text{ s.t. } W(x) = S(x, \xi) \end{array} \right\} \tag{1}$$

where co denotes convex hull.

Hypothesis 4.2. *For* $x_0 \in X_0$, *there exists a convex* C^2 *function* $H^+_{x_0} : C_{x_0} \to \mathbb{R}$ *such that*

$$\begin{array}{rcl} H(x_0, q) & \leq & H^+_{x_0}(q) \; \forall q \in C_{x_0} \\ H(x_0, p_0) & = & H^+_{x_0}(p_0) \; (= 0) \end{array}$$

for some p_0 *such that* $(x_0, p_0) \in \Lambda$ *and* $H^+_{x_0}(p_0) = \max\{H^+_{x_0}(p) : p$ *is an extremal point of* $C_{x_0}\}$.

Hypothesis 4.3. *For* $x_0 \in X_0$, *let* ξ_0 *be any critical point of* S_{x_0} *such that*

$$W(x_0) = S(x_0, \xi_0).$$

Let (x_s, p_s) *be the bi-characteristic for the Hamiltonian flow corresponding to* H *such that* (x_s, p_s) *lies on* Λ *and satisfies*

$$(x_0, p_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0, \xi_0) \right).$$

Note that the parameterisation by s is in the direction of the Hamiltonian flow. We assume that (x_s) intersects X_0 transversely. Now, we can find a neighbourhood U of x_0 such that there exists a smooth function $\phi : U \to E$ with $\phi(x_s)$ a critical point of $S(x_s, .)$ and $\phi(x_0) = \xi_0$. Then the hypothesis is that either:

(1) *there exists some $t < 0$ and a convex C^2 function $H^-_{x_0} : C_{x_0} \to \mathbb{R}$ such that*

$$W(x_s) = S(x_s, \phi(x_s))$$

for all $s \in [t, 0]$ and

$$\begin{aligned} H(x_0, q) &\geq H^-_{x_0}(q)\ \forall q \in C_{x_0} \\ H(x_0, p_0) &= H^-_{x_0}(p_0)\ (= 0) \\ \frac{\partial H}{\partial p}(x_0, p_0) &= \frac{\partial H^-_{x_0}}{\partial p}(p_0) \end{aligned}$$

(2) *or there exists some $t > 0$ and a concave C^2 function $H^-_{x_0} : C_{x_0} \to \mathbb{R}$ such that*

$$W(x_s) = S(x_s, \phi(x_s))$$

for all $s \in [0, t]$ and

$$\begin{aligned} -H(x_0, q) &\leq H^-_{x_0}(q)\ \forall q \in C_{x_0} \\ H(x_0, p_0) &= H^-_{x_0}(p_0)\ (= 0) \\ \frac{\partial H}{\partial p}(x_0, p_0) &= \frac{\partial H^-_{x_0}}{\partial p}(p_0). \end{aligned}$$

Note, that in part (1) of the last hypotheses, we are assuming that $W(x_s) = S(x_s, \phi(x_s))$ for $s \leq 0$, but not necessarily for $s > 0$, *i.e.* given a Hamiltonian trajectory (x_s, p_s) on Λ on which the graph selector critical value is achieved at $s = 0$, where $x_0 \in X_0$, then we are assuming that the graph selector critical value is also achieved along the same trajectory for $s < 0$ but not necessarily for $s > 0$. In part (2), we are assuming the the graph selector critical value is achieved for $s \geq 0$ but not necessarily for $s < 0$. Note also that the last hypothesis includes a transversality or general position assumption, which is not restrictive, and which will be repeated several times below. Assuming the above hypotheses, we then have the following theorem.

Theorem 4.4. *W is a viscosity solution of $H(x, \partial W/\partial x(x)) = 0$ for all $x \in X$.*

Proof. As noted earlier, $H(x, \partial W/\partial x(x)) = 0$ for all $x \in X \backslash X_0$. So it is sufficient to prove the viscosity solution property for $x_0 \in X_0$.

Subsolution property. (The following argument is a generalisation to the GFQI setting of the proof of Th. 3 of [7].) W is Lipschitz continuous at x_0. So, by [10],

$$D^+W(x_0) \subseteq \partial W(x_0)$$

where $\partial W(x_0)$ denotes the generalised Clarke gradient. By [4], Theorem 2.5.1, for any set G of measure zero

$$\partial W(x_0) = \mathrm{co}\left\{\lim \frac{\partial W}{\partial x}(x_i) : x_i \to x_0, x_i \notin G, \frac{\partial W}{\partial x}(x_i) \text{ converges}\right\}.$$

Take $G = X_0$ which, as noted earlier, has measure zero and is such that W is smooth on $X \backslash X_0$.

Now take a sequence $x_i \to x_0$ with $x_i \notin X_0$ and $\partial W/\partial x(x_i)$ convergent. Then, as in Lemma 4.1, we can find a smooth function $\phi : U \to E$ on a neighbourhood $U \subseteq X \backslash X_0$ with $x_i \in U$ for all i and $x_0 \in cl(U)$ such that

$\phi(x_i)$ is a critical point of $S_{x_i} = S(x_i, .)$ and

$$W(x_i) = S(x_i, \phi(x_i)).$$

Now, by definition, $(x_i, p_i) \in \Lambda$ where

$$p_i = \frac{\partial S}{\partial x}(x_i, \phi(x_i)) = \frac{\partial W}{\partial x}(x_i).$$

Let $\xi \in E$ be such that $\xi = \lim_i \phi(x_i)$ and let

$$p = \frac{\partial S}{\partial x}(x_0, \xi).$$

Then by continuity of $\partial S/\partial \xi$,

$$\frac{\partial S}{\partial \xi}(x_0, \xi) = \lim_i \frac{\partial S}{\partial \xi}(x_i, \phi(x_i)) = 0,$$

from which it follows that $(x_0, p) \in \Lambda$. Further, by continuity of $\partial S/\partial x$, we have that

$$p = \lim_i \frac{\partial S}{\partial x}(x_i, \phi(x_i)) = \lim_i \frac{\partial W}{\partial x}(x_i),$$

while by continuity of W and S, we have that

$$W(x_0) = S(x_0, \xi).$$

Hence $\partial W(x_0) \subseteq C_{x_0}$.

So now let $q \in D^+W(x_0)$, then since $D^+W(x_0) \subseteq C_{x_0}$ and $H^+_{x_0}$ is convex on C_{x_0},

$$\begin{aligned} H(x_0, q) &\leq H^+_{x_0}(q) \\ &\leq \max\{H^+_{x_0}(p) : p \text{ is an extremal point of } C_{x_0}\} \\ &= H^+_{x_0}(p_0) \\ &= H(x_0, p_0) = 0 \end{aligned}$$

as required, where p_0 is as defined in Hypothesis 4.2.

Supersolution property. Let $q \in D^-W(x_0)$, so there exists a C^1 function χ such that

$$\chi(x) \leq W(x),\ \chi(x_0) = W(x_0),\ q = D\chi(x_0).$$

Now, again by [10], since W is Lipschitz at x_0, $D^-W(x_0) \subseteq \partial W(x_0)$, which as shown above is $\subseteq C_{x_0}$. So $q \in C_{x_0}$. Then with p_0 as in Hypothesis 4.3 and supposing that case (1) of that hypothesis holds,

$$\begin{aligned} H(x_0, q) &\geq H^-_{x_0}(q) \\ &\geq (q - p_0)\frac{\partial H^-_{x_0}}{\partial p}(p_0) + H^-_{x_0}(p_0) \\ &= (q - p_0)\frac{\partial H}{\partial p}(x_0, p_0) + H(x_0, p_0) \\ &= (D\chi(x_0) - p_0)\dot{x}_0 \end{aligned}$$

where (x_s, p_s) denotes the bi-characteristic defined in Hypotheses 4.3, and we are using the fact that $\dot{x}_s = \partial H/\partial p(x_s, p_s)$. Now, for $t < 0$ defined in Hypotheses 4.3(1),

$$\int_t^0 D\chi(x_s)\dot{x}_s \mathrm{d}s = \chi(x_0) - \chi(x_t) \geq W(x_0) - W(x_t)$$

and

$$\int_t^0 p_s \dot{x}_s \mathrm{d}s = S(x_0, \xi_0) - S(x_t, \phi(x_t)) = W(x_0) - W(x_t).$$

Then

$$\int_t^0 (D\chi(x_s) - p_s)\,\dot{x}_s \mathrm{d}s \geq 0.$$

Since this holds for arbitrarily small $t < 0$, it follows by continuity of $D\chi$, p_s and $\dot{x}_s$ that

$$(D\chi(x_0) - p_0)\,\dot{x}_0 \geq 0.$$

Hence $H(x_0, q) \geq 0$ as required. A similar argument with the inequalities reversed and $t > 0$ applies when case (2) of Hypothesis 4.3 holds. □

If we keep Hypothesis 4.2, but change Hypothesis 4.3 so that case (1) holds for $t > 0$ and case (2) holds for $t < 0$, then we have the following theorem.

Theorem 4.5. *W is a viscosity solution of $-H(x, \partial W/\partial x(x)) = 0$ for all $x \in X$.*

Proof.

Supersolution property. Let $q \in D^-W(x_0) \subseteq \partial W(x_0) \subseteq C_{x_0}$. Then

$$H(x_0, q) \leq H^+_{x_0}(q) \leq H^+_{x_0}(p_0) = H(x_0, p_0) = 0$$

for some p_0 as defined in Hypothesis 4.2. So W is a supersolution for $-H$ as required.

Subsolution property. Let $q \in D^+W(x_0)$, so there exists a C^1 function χ such that

$$\chi(x) \geq W(x),\ \chi(x_0) = W(x_0),\ q = D\chi(x_0).$$

Then, with p_0 as in Hypothesis 4.3 and supposing that case (1) of that hypothesis holds (for $t > 0$), we have

$$D^+W(x_0) \subseteq \partial W(x_0) \subseteq C_{x_0}$$

and so

$$\begin{aligned} H(x_0, q) &\geq H^-_{x_0}(q) \\ &\geq (q - p_0)\frac{\partial H^-_{x_0}}{\partial p}(p_0) + H^-_{x_0}(p_0) \\ &= (q - p_0)\frac{\partial H}{\partial p}(x_0, p_0) + H(x_0, p_0) \\ &= (D\chi(x_0) - p_0)\dot{x}_0. \end{aligned}$$

Now, for $t > 0$

$$\begin{aligned} \int_0^t (D\chi(x_s) - p_s)\,\dot{x}_s \mathrm{d}s &= \chi(x_t) - \chi(x_0) - S(x_t, \phi(x_t)) + S(x_0, \xi_0) \\ &\geq W(x_t) - W(x_0) - W(x_t) + W(x_0). \\ &= 0. \end{aligned}$$

Since this holds for arbitrarily small $t > 0$, it follows by continuity, that

$$(D\chi(x_0) - p_0)\,\dot{x}_0 \geq 0$$

and so $H(x_0, q) \geq 0$. W is therefore a subsolution of $-H$. □

Now consider the following hypotheses in place of Hypotheses 4.2 and 4.3 above.

Hypothesis 4.6. *For $x_0 \in X_0$, there exists a concave C^2 function $H^-_{x_0} : C_{x_0} \to \mathbb{R}$ such that*

$$\begin{aligned} H(x_0, q) &\geq H^-_{x_0}(q)\ \forall q \in C_{x_0} \\ H(x_0, p_0) &= H^-_{x_0}(p_0)\ (= 0) \end{aligned}$$

for some p_0 such that $(x_0, p_0) \in \Lambda$ and $H^-_{x_0}(p_0) = \min\{H^-_{x_0}(p) : p$ is an extremal point of $C_{x_0}\}$.

Hypothesis 4.7. *For $x_0 \in X_0$, let ξ_0 be any critical point of S_{x_0} such that*

$$W(x_0) = S(x_0, \xi_0).$$

Let (x_s, p_s) be the bi-characteristic for the Hamiltonian flow corresponding to H such that (x_s, p_s) lies on Λ and satisfies

$$(x_0, p_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0, \xi_0)\right).$$

Let $\phi : U \to E$ be a smooth function on a neighbourhood U of x_0 such that $\phi(x_s)$ is a critical point of $S(x_s, .)$ and $\phi(x_0) = \xi_0$. Then the hypothesis is that either:

(1) *there exists some $t < 0$ and a concave C^2 function $H^+_{x_0} : C_{x_0} \to \mathbb{R}$ such that*

$$W(x_s) = S(x_s, \phi(x_s))$$

for all $s \in [t, 0]$ and

$$\begin{aligned} H(x_0, q) &\leq H^+_{x_0}(q)\ \forall q \in C_{x_0} \\ H(x_0, p_0) &= H^+_{x_0}(p_0)\ (= 0) \\ \frac{\partial H}{\partial p}(x_0, p_0) &= \frac{\partial H^+_{x_0}}{\partial p}(p_0) \end{aligned}$$

(2) *or there exists some $t > 0$ and a convex C^2 function $H^+_{x_0} : C_{x_0} \to \mathbb{R}$ such that*

$$W(x_s) = S(x_s, \phi(x_s))$$

for all $s \in [0, t]$ and

$$\begin{aligned} -H(x_0, q) &\geq H^+_{x_0}(q)\ \forall q \in C_{x_0} \\ H(x_0, p_0) &= H^+_{x_0}(p_0)\ (= 0) \\ \frac{\partial H}{\partial p}(x_0, p_0) &= \frac{\partial H^+_{x_0}}{\partial p}(p_0). \end{aligned}$$

Then we have the following theorem.

Theorem 4.8. *W is a viscosity solution of $H(x, \partial W/\partial x(x)) = 0$ for all $x \in X$.*

This can be proved by re-working the proof of Theorem 4.4 and switching the roles played by the sub- and super-differentials. Alternatively, we can apply the following time-reversal argument. Suppose that case (1) of Hypothesis 4.7 holds. A similar arguments works if case (2) holds. First, reverse the direction of parameterisation of the Hamiltonian flow, *i.e.* reverse time. So we obtain a new set of Hamiltonian dynamics with respect to $\hat{H} = -H$ and with $-\hat{H}$ satisfying case (1) of 4.7 for $t > 0$. Then we can multiply the equalities and inequalities of Hypotheses 4.6 and 4.7(1) by -1 throughout, to get that $\hat{H}$ satisfies Hypotheses 4.2 and 4.3(1) for $t > 0$. Finally, we can apply Theorem 4.5 to get that W is a viscosity solution of $-\hat{H}(x, \partial W/\partial x(x)) = H(x, \partial W/\partial x(x)) = 0$.

Lastly, if we keep Hypothesis 4.6, but change Hypothesis 4.7 so that case (1) holds for $t > 0$ and case (2) holds for $t < 0$, then we have the following theorem.

Theorem 4.9. *W is a viscosity solution of $-H(x, \partial W/\partial x(x)) = 0$ for all $x \in X$.*

5. Examples

We now give two sets of examples to illustrate that the requirements of Hypotheses 4.2 and 4.3 or 4.6 and 4.7 are naturally satisfied by many cases of graph selectors, or variational solutions, defined on Lagrangian manifolds. We also give a simple one-dimensional non-convex example where a graph selector exists which does not satisfy all the hypotheses and is also not a viscosity solution.

5.1. Convex Hamiltonians

For use both in this and the following section, we start with the following set-up and prove some initial lemmas. Let $H(x, p, t)$ be convex in p for all x and t. Denote the Hamiltonian flow corresponding to H by ψ_t. Let $S_0(x)$ be some smooth function with compact support and $\Lambda_0 = \{x, \partial S_0(x)/\partial x\}$ be the corresponding Lagrangian manifold with well-defined projection onto X. For $0 \le t \le T$, let $\Lambda_t = \psi_t(\Lambda_0)$ be the exact Lagrangian isotopy of Λ_0 generated by ψ_t. The following lemma seems to be well-known amongst symplectic topologists but we could not find a published reference. It does, however, appear unpublished as Théorème 7.1 in [12]. For completeness, we include the proof from *ibid*, and thank Claude Viterbo for directing us to this.

Lemma 5.1. *For $0 \le t \le T$, there is a natural choice of GFQI $S_t(x, \xi)$ for Λ_t such that S_t has index zero at infinity, i.e. such that the quadratic form at infinity has index zero.*

Proof. For the moment, use q in place of x to denote the coordinate on X. As in the proof of existence of a GFQI given in Section 2, we start with the case where $X = \mathbf{T}^n$, a n-dimensional torus. Since H is convex in p, then by choice of a suitable partition of unity we can assume, without changing the neighbourhood of Λ_t, that for p outside of a compact set, $H(q, p, t) = 1/2\,|p|^2$. Now we apply the argument given in Section 2, noting that this is still applicable because the associated Hamiltonian field is globally Lipschitz. So there is a $\delta > 0$ such that for $0 \le s < t \le T$, if we let $(Q, P) = R_s^t(q, p)$, then the mapping $(q, p) \mapsto (q, Q)$ is a global diffeomorphism defined by a canonical generating function

$$S_s^t(q, Q) = \int_s^t \Big(P_\tau \mathrm{d}Q_\tau/\mathrm{d}\tau - \tilde{H}_\tau(Q_\tau, P_\tau)\Big)\,\mathrm{d}\tau.$$

Note in particular that for large $|Q - q|$ we have

$$S_s^t(q, Q) = \frac{1}{2}\frac{(Q-q)^2}{(t-s)}\cdot$$

Choose l sufficiently large that $T/(l+1) < \delta$, and let $S_{j,t} = S_{jt/(l+1)}^{(j+1)t/(l+1)}$ for $0 \le t \le T$. For each $1 \le j \le l$, we let $(q_{j+1}, P_j) = R_{jt/(l+1)}^{(j+1)t/(l+1)}(q_j, p_j)$ be canonically generated by $S_{j,t}(q_j, q_{j+1})$, while for $j = 0$ we let $(q_1, P_0) =$

$R_0^{t/(l+1)}(q_0, p_0)$ be canonically generated by $S_{0,t}(q_0, q_1) + S_0(q_0)$. Then the function

$$S_t(q_0, \dots, q_l, q_{l+1}) = \sum_{j=0}^{l} S_{j,t}(q_j, q_{j+1}) + S_0(q_0)$$

has differential

$$\mathrm{d}S_t(q_0, \dots, q_l, q_{l+1}) = P_l \mathrm{d}q_{l+1} + \sum_{j=0}^{l-1}(P_j - p_{j+1})\mathrm{d}q_{j+1} + \left(\frac{\partial S_0}{\partial q_0} - p_0\right)\mathrm{d}q_0.$$

Consider $(q_0, \dots, q_l)$ to be the fibre coordinates on fibre space $E = \mathbb{R}^{l+1}$ and q_{l+1} to be the coordinate on X, after taking quotients. Let Σ_{S_t} be the subvariety of $P = X \times E$ on which $\partial S_t / \partial q_j = 0$ for $0 \le j \le l$. Then $\Lambda_t = \psi_t(\Lambda_0)$ is the image of Σ_{S_t} under the embedding $(q_0, \dots, q_l, q_{l+1}) \mapsto (q_{l+1}, \partial S_t / \partial q_{l+1})$.

Now let $x = q_{l+1}$ and change the other coordinates to $\xi_j = q_{j+1} - q_j$ for $0 \le j \le l$. Then the function $S_t(x, \xi_0, \dots, \xi_l)$ is $\mathbb{Z}^n$-periodic in x and induces on $X \times \mathbb{R}^{l+1}$ a GFQI for Λ_t. Furthermore, for large values of the fibre variables ξ_j,

$$S_t(x, \xi_0, \dots, \xi_l) = \frac{(l+1)^2}{2t}\sum_{j=0}^{l}\xi_j^2 + S_0\left(x - \sum_{j=0}^{l}\xi_j\right) = \frac{(l+1)^2}{2t}\sum_{j=0}^{l}\xi_j^2.$$

So S_t is a quadratic form at infinity of index zero as required.

For an arbitrary manifold X we can repeat the above argument replacing the flat metric $1/2\,|p|^2$ by an arbitrary Riemannian metric. No matter how big we make l, it is possible for the function $S_t(., \xi_0, \dots, \xi_{l+1})$ to have more critical points than we are interested in. The extra ones correspond to conjugate points for the metric. However, for l large enough, these extra critical points correspond to very large critical values of S_t and so do not affect the deduction made in the next lemma. $\square$

Lemma 5.2. *Let S be a GFQI for a Lagrangian manifold Λ such that S has index zero at infinity. Then the graph selector takes the form*

$$W_S(x) = \min\left\{S(x,\xi) : \xi \in E \ s.t. \frac{\partial S}{\partial \xi}(x,\xi) = 0\right\}. \tag{2}$$

Proof. $S_x(.) = S(x,.)$ has a global minimum for each x, so let ξ_0 denote the globally minimising point at a given x and $\lambda_0 = S_x(\xi_0)$ be the corresponding minimum value. Then in the definition of the graph selector corresponding to S we have

$$H^0(E^{\lambda_0}, E^{-c}) = \mathbb{R}$$

and

$$H^0(E^{\lambda}, E^{-c}) = 0$$

for $\lambda < \lambda_0$. It follows that $W(x) = \lambda_0$. $\square$

It follows from the previous two lemmas that, for convex Hamiltonians, the graph selector is equivalent to the global minimum formulation put forward in [7]. Note, that [7] used inf rather than min, but then imposed various hypotheses to ensure that the infimum was always attained.

We now turn to the class of examples considered in this section. Let $K(y, q, \tau)$ be a convex Hamiltonian and consider the Cauchy problem $K(y, \partial W/\partial y(y,\tau), \tau) = -\partial W/\partial \tau(y, \tau)$ with initial data $W(y, 0) = S_0(y)$. Let $\Lambda_0 = \{y, \partial S_0(y)/\partial y\}$ be the corresponding initial Lagrangian submanifold of $\mathbb{R}^{2n}$ phase space. Then, as usual, let $\Lambda = \bigcup_{\tau > 0}\psi_\tau(\Lambda_0)$ be the Lagrangian submanifold of $\mathbb{R}^{2n+2}$ phase space traced out by Λ_0 under the Hamiltonian flow ψ_τ associated with K, and let $x = (y, \tau)$ and $p = (q, \sigma)$ be the augmented state and adjoint coordinates respectively on $\mathbb{R}^{2n+2}$. Now define an augmented Hamiltonian $H(x,p) = K(y,q,\tau) + \sigma$. Note that H is convex and Λ is contained in the hypersurface $\{H = 0\}$. By Lemma 5.1, Λ_τ has a GFQI $S_\tau(y)$ with index zero at infinity. Since Λ_τ is transverse to the flow ψ_τ, it follows that $S(x) = S(y,\tau) = S_\tau(y)$ is a GFQI for Λ with index zero at infinity. Now let W be the graph selector corresponding to S. By Lemma 5.2 this takes the form (2). We want to show that:

Theorem 5.3. *W is a viscosity solution to $H(x, \partial W/\partial x(x)) = 0$.*

As outlined in the Introduction, for H convex as here, this result appears in [24, 25], under an additional assumption on the orientation of jumps in $dW(x)$ across surfaces of discontinuity. For W explicitly represented in the form (2), it also appears in [7], under an additional assumption of Lispchtiz continuity of W. We show here that the result in fact holds without these extra assumptions by demonstrating that the hypotheses of the previous section are satisfied. Now clearly, Hypothesis 4.2 is satisfied trivially by H and Λ. The result will therefore follow by Theorem 4.4 once we have shown that:

Lemma 5.4. *For the above choice of S and W, Hypothesis 4.3(1) is satisfied with $H = H^-_{x_0}$ and $t < 0$ for all $x_0 \in X_0$.*

Proof. Let $x_0 \in X_0$ and let (x_s, p_s) denote the Hamiltonian bi-characteristic of Hypothesis 4.3 which lies on Λ and satisfies

$$(x_0, p_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0, \xi_0) \right),$$

where ξ_0 is an arbitrary critical point of S_{x_0} at which $S(x_0, \xi_0) = W(x_0)$, *i.e.* at which the min of $S_{x_0}(.)$ over E is achieved. For some neighbourhood U of x_0, let $\phi : U \to E$ be a smooth function with $\phi(x_s)$ a critical point of S_{x_s} and $\phi(x_0) = \xi_0$. We need to prove that, for some $t < 0$, $S(x_s, \phi(x_s)) = W(x_s)$ for $s \in [t, 0]$, which we will do by showing that the min of $S_{x_s}(.)$ over E is achieved at $\phi(x_s)$.

Let $\hat{\xi}_0 \neq \xi_0$ be some other critical point of S_{x_0}, *i.e.* with

$$\frac{\partial S}{\partial \xi}(x_0, \xi_0) = \frac{\partial S}{\partial \xi}\left(x_0, \hat{\xi}_0\right) = 0$$

but

$$p_0 = \frac{\partial S}{\partial x}(x_0, \xi_0) \neq \frac{\partial S}{\partial x}(x_0, \hat{\xi}_0) = \hat{p}_0.$$

Then there exists a second smooth function $\hat{\phi} : U \to E$ such that $\hat{\phi}(x_s)$ is a critical point of S_{x_s} with $\hat{\phi}(x_s) \neq \phi(x_s)$ and $\hat{\phi}(x_0) = \hat{\xi}_0$. If we denote

$$\hat{p}_s = \frac{\partial S}{\partial x}(x_s, \hat{\phi}(x_s))$$

then there is a trajectory $(x_s, \hat{p}_s)$ lying over (x_s) on the branch of Λ corresponding to $\hat{\xi}_0$, with

$$(x_0, \hat{p}_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0, \hat{\xi}_0) \right).$$

If $S(x_0, \hat{\xi}_0) \neq S(x_0, \xi_0) = W(x_0)$, then by definition $S(x_0, \hat{\xi}_0) > S(x_0, \xi_0)$, and so for some $t < 0$, we have $S(x_s, \hat{\phi}(x_s)) > S(x_s, \phi(x_s))$ for $s \in [t, 0]$.

Suppose, on the other hand, that $S(x_0, \hat{\xi}_0) = S(x_0, \xi_0) = W(x_0)$, *i.e.* the min of $S_{x_0}(.)$ over E is also achieved at $\hat{\xi}_0$. Then

$$H(x_s, \hat{p}_s) \geq (\hat{p}_s - p_s) \frac{\partial H}{\partial p}(x_s, p_s) + H(x_s, p_s).$$

Since both (x_s, p_s) and $(x_s, \hat{p}_s) \in \Lambda$, $H(x_s, p_s) = H(x_s, \hat{p}_s) = 0$, and so, for some $t < 0$ and $t_1 \in [t, 0)$

$$\begin{aligned}
0 &\geq \int_{t_1}^{0} (\hat{p}_s - p_s) \frac{\partial H}{\partial p}(x_s, p_s) \mathrm{d}s \\
&= \int_{t_1}^{0} (\hat{p}_s - p_s) \dot{x}_s \mathrm{d}s \\
&= S(x_0, \hat{\xi}_0) - S(x_{t_1}, \hat{\phi}(x_{t_1})) - S(x_0, \xi_0) + S(x_{t_1}, \phi(x_{t_1})) \\
&= -S(x_{t_1}, \hat{\phi}(x_{t_1})) + S(x_{t_1}, \phi(x_{t_1})).
\end{aligned}$$

Since the choice of t_1 was arbitrary, it follows that for some $t < 0$ and all $s \in [t, 0]$,

$$S(x_s, \phi(x_s)) = \min \left\{ S(x_s, \xi) : \xi \in E \text{ s.t.} \frac{\partial S}{\partial \xi}(x_s, \xi) = 0 \right\}$$

and so, by (2), $S(x_s, \phi(x_s)) = W(x_s)$. □

Note, if $H(x, .)$ is concave for all x, then there is a natural choice of GFQI $S(x, \xi)$ for Λ such that S is negative definite at infinity. It follows in this case that $S_x(.)$ has a global maximum for all x. The graph selector therefore reduces to a global maximum formulation. Hypothesis 4.7(1) is then satisfied for $t < 0$ by $H^+_{x_0} = H$, and the corresponding graph selector W is a viscosity solution of $H(x, \partial W/\partial x(x)) = 0$. Similarly, for final value problems, either Hypothesis 4.3(1) or 4.7(1) is satisfied for $t > 0$ depending on whether H is respectively convex or concave, and W is then a viscosity solution of $-H(x, \partial W/\partial x(x)) = 0$.

5.2. A class of non-convex Hamiltonians

Let $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^q$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $k : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $r : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ be C^2 functions with $f(0) = 0$, $h(0) = 0$, $h(x) \neq 0$ for $x \neq 0$ and $r(x)$ positive definite for all x. Let $\gamma > 0$ be a scalar. We consider the following Hamiltonian

$$\begin{aligned}
H(x,p) &= \min_{w} \max_{u} \left\{ p^T(f(x) + g(x)u + k(x)w) - \frac{1}{2}|h(x)|^2 - \frac{1}{2}u^T r(x) u + \frac{1}{2}\gamma^2 |w|^2 \right\} \\
&= \frac{1}{2} p^T g(x) r(x)^{-1} g(x)^T p - \frac{1}{2\gamma^2} p^T k(x) k(x)^T p + p^T f(x) - \frac{1}{2}|h(x)|^2. \qquad (3)
\end{aligned}$$

Clearly this Hamiltonian is in general neither convex in p for all x, nor concave in p for all x. For fixed x, it is a quadratic form in p, but the rank and index of this form will, in general, vary with x.

This Hamiltonian arises from the application of the maximum principle to the differential game formulation of a particular class of, so-called, control-affine nonlinear H_∞-optimal control problems – see [21, 22] for details of this class of problems, and [20] for details of its formulation as a differential game. It is sufficient to note here that there is an underlying dynamical system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u + k(x)w, \quad x(0) = \zeta \\
z &= h(x)
\end{aligned}$$

which can be thought of as being disturbed by one player via inputs w, and controlled by the other player *via* inputs u. The aim of the second player is to achieve an attenuation level γ between the L_2 norms of the disturbance input and controlled system output.

The Hamiltonian dynamics

$$\dot{x} = \partial H/\partial p, \ \dot{p} = -\partial H/\partial x$$

corresponding to the above H have an equilibrium point at the origin in phase space. For γ above some minimum level, and given certain controllability and observability assumptions on the linearisation of these dynamics around this equilibrium point, it can be shown that the equilibrium is hyperbolic. So, by the local stable manifold theorem, there exist n-dimensional stable and unstable planes T^+ and T^- for the linearised dynamics around the equilibrium point. Further, by the global stable manifold theorem, these can be extended to global n-dimensional stable and unstable manifolds Λ^+ and Λ^- for the Hamiltonian dynamics. From now on, we concentrate just on Λ^+, which for brevity we denote by Λ. This can be shown to be Lagrangian with $H = 0$ on Λ. Furthermore, Λ is tangent to T^+ at the origin, and so in a neighbourhood Ω_0 of the origin, Λ is a graph $\{x, \partial S_0(x)/\partial x : x \in \Omega_0\}$ over state space for some classical generating function $S_0(x)$. The details of the arguments of this paragraph can be found in [21, 22].

Now let Ω be the largest open region in state space containing 0 such that Ω is covered by Λ and is forward invariant with respect to the projection of the Hamiltonian dynamics on Λ. Note that Ω will in general be strictly larger than Ω_0, but may not equal the whole of $\mathbb{R}^n$, *e.g.* in cases where Λ does not cover $\mathbb{R}^n$ or is not simply connected. To deal with these cases, we restrict consideration to the submanifold of Λ consisting of those $(x, p) \in \Lambda$ with $x \in \Omega$. This submanifold, which we continue to denote as Λ, is simply connected, and we can then lift Ω to a closed manifold X and Λ to a Lagrangian submanifold of T^*X such that Λ is exact and Lagrangian isotopic to the zero section of T^*X.

Now consider the final value Cauchy problem

$$-H(x, \partial V(x,t)/\partial x) = \partial V(x,t)/\partial t, \quad V(x,T) = 0$$

where $x \in \Omega_0$ and H is given by (3). This has a smooth solution $V_T(x,t)$, and it can be shown that as $T \to \infty$, the sequence $V_T(x,t)$ tends to a steady state limit $V(x) = S_0(x)$ which solves the steady state equation

$$-H(x, \partial V(x)/\partial x) = 0$$

for $x \in \Omega_0$. The Hamiltonian bi-characteristics corresponding to this steady state solution lie on the stable manifold Λ and tend asymptotically to the origin as $t \to \infty$.

Now let $S(x,\xi)$ be a GFQI for Λ which satisfies $S(x,\xi) = S_0(x)$ for $x \in \Omega_0$. Let W be the graph selector corresponding to S. Then we have the following, where the set X_0 is as defined in Section 3:

Theorem 5.5. *Suppose that for $x_0 \in X_0$, $H(x_0, p)$ is a non-degenerate quadratic form. Then W is a viscosity solution of $-H(x, \partial W/\partial x(x)) = 0$ for $x \in X$.*

Note that X_0 is not necessarily connected, and so the hypothesis of this theorem still allows variation in the index of $H(x_0, p)$ on X_0, as well as variation in the rank of $H(x, p)$ for $x \notin X_0$. We prove this theorem by showing that, for $x_0 \in X_0$, the conditions of either Hypotheses 4.2 and 4.3(1) or of 4.6 and 4.7(1) are satisfied for $t > 0$. We start with the following lemma and proposition which will show that the first part of both Hypotheses 4.3(1) and 4.7(1) holds true for $t > 0$.

Lemma 5.6. *For $x_0 \in X_0$, let $\hat{C}_{x_0} = \mathrm{co}\{p_1, p_2 : (x_0, p_i) \in \Lambda\}$. Then $H(x_0, p)$ is either convex in p for all $p \in \hat{C}_{x_0}$ or concave in p for all $p \in \hat{C}_{x_0}$.*

Proof. $H(x_0, p)$ is a quadratic form in p of rank n and some index i. So the level set $H(x_0, p) = 0$ defines a non-degenerate conic in adjoint (*i.e.* p) space. Since Λ is contained the hypersurface $H = 0$, it follows that both p_1 and p_2 lie on this conic. So $\hat{C}_{x_0}$ is a chord between two points on a conic, and therefore lies either wholly in an affine space on which $H(x_0, .)$ is convex, or wholly in an affine space on which $H(x_0, .)$ is concave. □

As an example to illustrate the above, suppose that Λ is two-dimensional and that, for some $x_0, H(x_0, p) = p_1^2 - p_2^2 - 1$. Then, for instance, on the lines $p_2 = 1$ or $p_1 + 2p_2 = 2, H(x_0, .)$ is convex while on the lines $p_1 = 2$ or $2p_1 + p_2 = 4, H(x_0, .)$ is concave.

Proposition 5.7. *For $x_0 \in X_0$, let ξ_0 be an arbitrary critical point of S_{x_0} at which $S(x_0,\xi_0) = W(x_0)$. Let (x_s,p_s) denote the Hamiltonian bi-characteristic which lies on Λ and satisfies*

$$(x_0,p_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0,\xi_0)\right).$$

We assume that (x_s) intersects X_0 transversely. For some neighbourhood U of x_0, let $\phi : U \to E$ be a smooth function with $\phi(x_s)$ a critical point of S_{x_s} and $\phi(x_0) = \xi_0$. Then there exists $t > 0$ such that $S(x_s,\phi(x_s)) = W(x_s)$ for $s \in [0,t]$.

Proof. Consider the set C_{x_0} defined above in (1). When this is a singleton, *i.e.* contains only p_0, then the graph selector critical value $W(x_0)$ is achieved uniquely at the critical point ξ_0 of S_{x_0}. Now since the trajectory (x_s) intersects X_0 transversely, since the critical value $W(x_s)$ must be achieved at some critical point of S_{x_s} and since $W(x_s)$ must be continuous with respect to s at x_0, it follows that $S(x_s,\phi(x_s)) = W(x_s)$ for $s \in [-t,t]$ and for some $t > 0$.

Now suppose that C_{x_0} is not a singleton. Then there exists at least one other extremal point $\hat{p}_0 \neq p_0$ in C_{x_0}, in addition to p_0. Let $\hat{\xi}_0 \neq \xi_0$ be the critical point of S_{x_0} corresponding to $\hat{p}_0$, *i.e.* with

$$\frac{\partial S}{\partial \xi}(x_0,\xi_0) = \frac{\partial S}{\partial \xi}\left(x_0,\hat{\xi}_0\right) = 0,\ S(x_0,\xi_0) = S\left(x_0,\hat{\xi}_0\right) = W(x_0)$$

and

$$p_0 = \frac{\partial S}{\partial x}(x_0,\xi_0) \neq \frac{\partial S}{\partial x}\left(x_0,\hat{\xi}_0\right) = \hat{p}_0.$$

Then there exists a second smooth function $\hat{\phi} : U \to E$ such that $\hat{\phi}(x_s)$ is a critical point of S_{x_s} with $\hat{\phi}(x_s) \neq \phi(x_s)$ and $\hat{\phi}(x_0) = \hat{\xi}_0$. If we denote

$$\hat{p}_s = \frac{\partial S}{\partial x}\left(x_s,\hat{\phi}(x_s)\right)$$

then there is a (non-Hamiltonian) trajectory $(x_s,\hat{p}_s)$ lying over (x_s) on the branch of Λ corresponding to $\hat{\xi}_0$, with

$$(x_0,\hat{p}_0) = \left(x_0, \frac{\partial S}{\partial x}(x_0,\hat{\xi}_0)\right).$$

Suppose, with a view to a contradiction, that $S(x_s,\hat{\phi}(x_s)) = W(x_s)$ for $s \in [0,t]$ and for some small $t > 0$, *i.e.* the graph selector critical value over the trajectory (x_s) is obtained on the branch of Λ corresponding to $\hat{\xi}_0$. So now, consider the section Λ' of Λ lying over a neighbourhood U' around the trajectory (x_s) for $s \in [0,t]$, and with Λ' restricted to just the two branches determined by the functions ϕ and $\hat{\phi} : U' \to E$. Since for all $x \in U'$, $W(x)$ is achieved at either $S(x,\phi(x))$ or $S(x,\hat{\phi}(x))$ on Λ', then W is a graph selector for Λ'.

Now, along the trajectory (x_s) for $s \in [0,t]$, consider the line segment in adjoint space defined by

$$\hat{C}_{x_s} = \mathrm{co}\left\{p_s,\hat{p}_s : p_s = \frac{\partial S}{\partial x}(x_s,\phi(x_s)),\ \hat{p}_s = \frac{\partial S}{\partial x}\left(x_s,\hat{\phi}(x_s)\right)\right\}.$$

At $s = 0$, $\hat{C}_{x_0}$ is the subset of C_{x_0} defined by the two extremal points p_0 and $\hat{p}_0$. Since both (x_s,p_s) and $(x_s,\hat{p}_s) \in \Lambda$, Lemma 5.6 shows that $H(x_s,.)$ is either convex or concave on $\hat{C}_{x_s}$ for $s \in [0,t]$. Furthermore, by smoothness of H and the hypothesis that $H(x_0,.)$ is a non-degenerate quadratic form for $x_0 \in X_0$, it follows that if $H(x_0,.)$ is convex on $\hat{C}_{x_0}$, then $H(x_s,.)$ is convex on $\hat{C}_{x_s}$ for $s \in (0,t]$, and *vice versa.* Note, if $H(x_0,.)$ was degenerate then we could have the situation where $\hat{C}_{x_0}$ lies wholly in the degenerate conic defined by $H(x_0,p) = 0$ and then it is not clear whether $H(x_s,.)$ is convex or concave on C_{x_s} for $s > 0$. This situation has to be excluded, hence the non-degeneracy condition.

So suppose that $H(x_s,.)$ is convex on $\hat{C}_{x_s}$ for $s \in [0,t]$. Let ψ_τ denote the flow corresponding to H and consider the Lagrangian section $\Lambda'_t = \psi_t(\Lambda')$ generated as the image of Λ' under ψ_t. Let U'_t be the state space neighbourhood over which Λ'_t lies. Then since the trajectory (x_s) is transverse to X_0, it follows that for t small enough, we can assume that the function

$$\max\{S(x,\phi(x)), S(x,\hat{\phi}(x))\}$$

is smooth on U'_t and Λ'_t is the graph of the differential of this function over U'_t. Now consider $\Lambda' = \psi_{-t}(\Lambda'_t)$ to be the image of Λ'_t under the reverse flow. Since this flow is associated with a convex Hamiltonian, we can apply the argument of Lemma 5.1 to obtain a GFQI S' for Λ' which is negative definite at infinity. We then apply the argument of Lemma 5.2 to obtain a graph selector W' for Λ' with the property that $W'(x)$ is the global maximum of the critical values of S'_x over Λ', for $x \in U'$.

Note that, since the bi-characteristic (x_s,p_s) lies on the stable manifold Λ, the function

$$\max\left\{S(x_s,\phi(x_s)), S\left(x_s,\hat{\phi}(x_s)\right)\right\} \to S_0(x_s)$$

as $s \to \infty$. So

$$S'(x,\phi(x)) = S(x,\phi(x)) \tag{4}$$

for $x \in U'$, since they can both be obtained by integrating backwards along the same Hamiltonian flow from the same final data S_0. Also,

$$W'(x) = W(x) \tag{5}$$

for $x \in U'$ by the uniqueness of the graph selector for Λ', referenced earlier in Section 3.

Then, working with S' and W' on Λ', we have that for $s \in [0,t]$, by the convexity of $H(x_s,.)$ on $\hat{C}_{x_s}$

$$H(x_s,\hat{p}_s) \geq (\hat{p}_s - p_s)\frac{\partial H}{\partial p}(x_s,p_s) + H(x_s,p_s).$$

So, for $t_1 \in (0,t]$

$$\begin{aligned} 0 &\geq \int_0^{t_1} (\hat{p}_s - p_s)\frac{\partial H}{\partial p}(x_s,p_s)\mathrm{d}s \\ &= \int_0^{t_1} (\hat{p}_s - p_s)\dot{x}_s\mathrm{d}s \\ &= S'(x_{t_1},\hat{\phi}(x_{t_1})) - S'(x_0,\hat{\xi}_0) - S'(x_{t_1},\phi(x_{t_1})) + S'(x_0,\xi_0) \\ &= S'(x_{t_1},\hat{\phi}(x_{t_1})) - S'(x_{t_1},\phi(x_{t_1})). \end{aligned}$$

Since the choice of t_1 was arbitrary and Λ' consists of only the two branches determined by ϕ and $\hat{\phi}$ over U', it follows that $S'(x_s,\phi(x_s))$ is the global maximum of the critical values of S'_x for $s \in [0,t]$, and hence

$$S'(x_s,\phi(x_s)) = W'(x_s)$$

for $s \in [0,t]$.

Then, by (4) and (5), we have

$$S(x_s,\phi(x_s)) = W(x_s)$$

for $s \in [0,t]$, which contradicts the assumption that $S(x_s,\hat{\phi}(x_s)) = W(x_s)$ and (x_s) intersects X_0 transversely at x_0.

In the remaining case, where $H(x_s,.)$ is concave on $\hat{C}_{x_s}$ for $s \in [0,t]$, we consider $\Lambda' = \psi_{-t}(\Lambda'_t)$ to be the image under the reverse flow of a Lagrangian section of Λ'_t defined as the graph of the differential of the smooth function

$$\min\{S(x,\phi(x)), S(x,\hat{\phi}(x))\}$$

on U'_t. We then apply the arguments of Lemmas 5.1 and 5.2 to obtain a GFQI S' with index 0 at infinity and a graph selector W' such that $W'(x)$ is the global minimum of the critical values of S'_x over Λ'. The above calculation can then be repeated using the concavity of $H(x_s,.)$ on C_{x_s} to obtain a contradiction.

Finally, since the choice of the other extremal point $\hat{p}_0$ leading to a contradiction was arbitrary, and since the critical value $W(x_s)$ must be achieved at some critical point of S_{x_s} over the trajectory (x_s), it follows that it must be achieved on the branch of Λ corresponding to ξ_0, *i.e.* for some $t > 0$, $S(x_s,\phi(x_s)) = W(x_s)$ for $s \in [0,t]$ as required. □

Proof. (of Theorem 5.5). Let $x_0 \in X_0$ and recall the definition of the convex set C_{x_0} from (1). Let $\{p_i : i \in I\}$ denote the set of extremal points of C_{x_0}. So for each i, $(x_0,p_i) \in \Lambda$ where $p_i = \frac{\partial S}{\partial x}(x_0,\xi_i)$ for some $\xi_i \in E$ s.t. $W(x_0) = S(x_0,\xi_i)$. Then for each pairwise selection $i \neq j \in I$, let $C^{ij}_{x_0} = \mathrm{co}\{p_i,p_j\}$ be the edge of C_{x_0} lying between p_i and p_j. By Lemma 5.6, $H_{x_0}(.) = H(x_0,.)$ is either convex or concave on $C^{ij}_{x_0}$.

Now let $q \in C_{x_0}$. As in the proof of Theorem 4.4, $D^{\pm}W(x_0) \subseteq C_{x_0}$ and so either $q \notin D^{\pm}W(x_0)$, in which case we are not interested in it, or $q \in D^{-}W(x_0)$ and we have to show $H(x_0,q) \leq 0$, or $q \in D^{+}W(x_0)$ and we have to show $H(x_0,q) \geq 0$. Since this covers all points in $D^{\pm}W(x_0)$, it will then follow that W is a viscosity solution of $-H(x_0,\partial W(x_0)/\partial x) = 0$ as required.

The remainder of the proof proceeds by induction on the dimension of C_{x_0}, and is notationally complicated but geometrically simple. We outline the arguments up to dimension two and leave the reader to formalise the induction step. For dim $C_{x_0} = 0$, $q = p_0$ for some $(x_0,p_0) \in \Lambda$, so $H(x_0,q) = 0$ and we are trivially done.

For dim$C_{x_0} = 1$, $C_{x_0} = \mathrm{co}\{p_0,p_1\}$ on which H_{x_0} is either convex or concave. Suppose it is convex. Then if $q \in D^{-}W(x_0)$, we apply the proof of the supersolution property from Theorem 4.5 with $H^{+}_{x_0}(.) = H(x_0,.)$ to get that $H(x_0,q) \leq 0$, while if $q \in D^{+}W(x_0)$ then we take $H^{-}_{x_0}(.) = H(x_0,.)$ together with the property established in the Proposition 5.7, and apply the proof of the subsolution property from Theorem 4.5 to get that $H(x_0,q) \geq 0$. If H_{x_0} is concave on C_{x_0}, then we switch the roles of sub- and super-differentials above and apply the proof of Theorem 4.9.

Now suppose dim$C_{x_0} = 2$. Then it is sufficient to prove the result for the 2-simplex $C_{x_0} = \mathrm{co}\{p_1,p_2,p_3\}$, where $\{p_i : i = 1,2,3\}$ are non-colinear extremal points. For $s \in [0,1]$, let $C^{is}_{x_0} = \mathrm{co}\{p_i,p(s)\}$ denote the interior line seqment from one extremal point p_i to the point $p(s) = sp_j + (1-s)p_k$ lying on the opposite edge $C^{jk}_{x_0}$. Note that $(x_0,p(s)) \notin \Lambda$ for $s \in (0,1)$. Let $C^{is_q}_{x_0} = \mathrm{co}\{p_i,p(s_q)\}$ denote the particular interior line segment from p_i which passes through q. Recall that, by Lemma 5.6, H_{x_0} is either convex or concave on each edge $C^{ij}_{x_0}$. The same argument shows that H_{x_0} is either convex or concave (or both) on each $C^{is}_{x_0}$. A further analysis of the properties of the non-degenerate quadratic form H_{x_0} shows that if, for some $s_0 \in (0,1)$, H_{x_0} is identically zero (*i.e.* both convex and concave) on $C^{is_0}_{x_0}$, then either H_{x_0} is convex on $C^{is}_{x_0}$ for all $0 \leq s < s_0$ and concave on $C^{is}_{x_0}$ for all $s_0 < s \leq 1$, or *vice versa.* We now distinguish three cases.

In the first case, suppose that H_{x_0} is convex on each edge $C^{ij}_{x_0}$. (A similar argument works if H_{x_0} is concave on each edge.) Then for each i, H_{x_0} is convex on both edges $C^{ij}_{x_0}$ and $C^{ik}_{x_0}$ emanating from p_i, and so by the previous paragraph, H_{x_0} is also convex on all interior line segments $C^{is}_{x_0}$, $s \in (0,1)$, emanating from p_i. In particular, for each i, H_{x_0} is convex on the interior line segment $C^{is_q}_{x_0}$ through q. It then follows that, for some $i, H(x_0,q) \leq H(x_0,p_i) = 0$.

To see this last statement, suppose that it is not true, *i.e.* $H(x_0,q) > H(x_0,p_i) = 0$ for each i. Since H_{x_0} is convex on the convex set $C^{is_q}_{x_0}$, then $H(x_0,q)$ must be less than or equal to the value of H_{x_0} at the other extreme point of $C^{is_q}_{x_0}$, *i.e.* $0 < H(x_0,q) \leq H(x_0,p(s_q))$. But $p(s_q)$ is an interior point on the edge $C^{jk}_{x_0}$ opposite p_i. Since H_{x_0} is convex on this edge, it follows that $H(x_0,p(s_q)) \leq H(x_0,p_j) = H(x_0,p_k) = 0$, which is a contradiction.

So for some i, H_{x_0} is convex on the interior line segment $C_{x_0}^{is_q}$ through q with $H(x_0, q) \leq H(x_0, p_i) = 0$. Now, we repeat the argument used above for the dimension 1 case. If $q \in D^-W(x_0)$, apply the proof of the supersolution property from Theorem 4.5 with $H_{x_0}^+(.) = H(x_0, .)$ on $C_{x_0}^{is_q}$ to get that $H(x_0, q) \leq 0$, while if $q \in D^+W(x_0)$ then take $H_{x_0}^-(.) = H(x_0, .)$ on $C_{x_0}^{is_q}$ together with Proposition 5.7, and apply the proof of the subsolution property from Theorem 4.5 to get that $H(x_0, q) \geq 0$.

In the second case, suppose that H_{x_0} is convex on at least one edge $C_{x_0}^{ij}$ and concave on at least one edge $C_{x_0}^{ik}$. Suppose also in this case that, for each i, H_{x_0} is convex on the interior line segment $C_{x_0}^{is_q}$ through q. It then follows that, for some $i, H(x_0, q) \leq H(x_0, p_i) = 0$. Again, to see why, suppose otherwise and reason as above to get that each edge $C_{x_0}^{jk}$ contains an interior point $p(s_q)$ at which $H(x_0, p(s_q)) > 0$. This is inconsistent with H_{x_0} being convex on at least one edge. So again, for some i, H_{x_0} is convex on $C_{x_0}^{is_q}$ with $H(x_0, q) \leq H(x_0, p_i) = 0$, and we can repeat the argument of the previous paragraph to get the required result. A similar argument, using Theorem 4.9, holds if we suppose that, for each i, H_{x_0} is concave on $C_{x_0}^{is_q}$.

In the third and final case, suppose that there exists at least one i and one j such that H_{x_0} is convex on the interior line segment $C_{x_0}^{is_q}$ through q, and concave on $C_{x_0}^{js_q}$. Suppose $q \in D^+W(x_0)$. Then by Proposition 5.7, the conditions of Hypothesis 4.3(1) hold for $t > 0$ and for $H_{x_0}^-(.)$ equal to the convex function $H(x_0, .)$ on $C_{x_0}^{is_q}$. We then apply the proof of the subsolution property from Theorem 4.5 to get that $H(x_0, q) \geq 0$. Suppose on the other hand that $q \in D^-W(x_0)$. Then by Proposition 5.7, the conditions of Hypothesis 4.7(1) hold for $t > 0$ and for $H_{x_0}^+(.)$ equal to the concave function $H(x_0, .)$ on $C_{x_0}^{js_q}$. We then apply the proof of the supersolution property from Theorem 4.9 to get that $H(x_0, q) \leq 0$.

The proof for $\dim C_{x_0} > 2$ then follows by generalising the above argument to an n-simplex. □

Note that the result of Theorem 5.5, namely the graph selector W being a viscosity solution to $-H(x, \partial W/\partial x) = 0$, is equivalent to $V = -W$ being a viscosity solution to $H(x, -\partial V/\partial x) = 0$. If, in addition, $V \geq 0$, then by the results of [20], this is sufficient for existence of a solution to the H_∞-control problem described above. In fact, we will show directly in a later paper [17] that, under this additional non-negativity assumption, the variational solution $-W$ is the value function (*i.e.* the optimal solution) for the H_∞-control problem in a weak sense, and that a weak set valued feedback control can be constructed from W and Λ.

5.3. A non-convex one-dimensional example

We give in this subsection a simple example of a Lipschitz graph selector on a one-dimensional Lagrangian manifold Λ, and a non-convex H vanishing on Λ. This example satisfies Hypothesis 4.3 and is a trivial supersolution, since the sub-differential at the point of non-differentiability is empty. However, it fails to satisfy Hypothesis 4.2 and also fails to be a subsolution. This latter fact can be seen directly.

In 2-dimensional phase space, take as generating function

$$S(x, p) = xp - \frac{1}{2}p^2 + \frac{1}{4}p^4.$$

So the 1-dimensional Lagrangian manifold is given by

$$\Lambda = \left\{(x, p) : x = p - p^3\right\}.$$

The graph selector W on Λ picks out, at each x, the minimum value of S over the branches of Λ. It has the following analytic form.

For $x < 0$, let $p^+(x)$ denote the unique positive root of the cubic polynomial $x - p + p^3 = 0$ (for $x < -\frac{2}{3\sqrt{3}}$ there is only one real root, for $-\frac{2}{3\sqrt{3}} \leq x < 0$ there is one positive real root and two negative). Then

$$W(x) = xp^+(x) - \frac{1}{2}p^+(x)^2 + \frac{1}{4}p^+(x)^4.$$

For $x > 0$, let $p^-(x)$ denote the unique negative root of the cubic polynomial $x - p + p^3 = 0$. Then

$$W(x) = xp^-(x) - \frac{1}{2}p^-(x)^2 + \frac{1}{4}p^-(x)^4.$$

At $x = 0$, $W(0) = -\frac{1}{4}$, this being the maximum value of W over x. So W has a single point of non-differentiability at $x = 0$, at which the superdifferential $D^+W(0) = [-1, +1]$ and the subdifferential $D^-W(0) = \emptyset$.

The Hamiltonian

$$H(x, p) = p - p^3 - x$$

vanishes on Λ. This corresponds to dynamics on Λ which, when projected onto the x-axis go monotonically from $+\infty$ to the left, come to a stop at $x = -\frac{2}{3\sqrt{3}}$, reverse back to $x = +\frac{2}{3\sqrt{3}}$, stop again, reverse and then proceed monotonically to $-\infty$. The p-axis projection goes at constant velocity from $-\infty$ to $+\infty$. There are "cut" dynamics corresponding to W which involve jumping from the lower to the upper branch of Λ at $x = 0$. These "cut" dynamics are smooth when projected onto the x-axis, so in that sense the variational solution W does provide a generalised action on the projected "cut" dynamics.

However, W is not a viscosity solution to $H(x, DW(x)) = 0$ at $x = 0$. The subsolution property

$$H(0, D^+W(0)) \le 0$$

fails. For example, take $q = \frac{1}{\sqrt{3}}$. Then $q \in D^+W(0) = [-1, +1]$, but $H(0, q) = \frac{2}{3\sqrt{3}} > 0$. The supersolution property is satisfied trivially since $D^-W(0) = \emptyset$.

It can be seen also that Hypothesis 4.2 is not satisfied by H and Λ, since in this case $C_0 = [-1, +1]$ and we need to find a convex function $H_0^+(q)$ on C_0 such that $H_0^+(q) \ge H(0, q)$ for all $q \in C_0$ and which must satisfy $H_0^+(+1) = H(0, +1) = 0$ if $H_0^+(+1) \ge H_0^+(-1)$, or must satisfy $H_0^+(-1) = H(0, -1) = 0$ if $H_0^+(-1) \ge H_0^+(+1)$, *i.e.* H_0^+ must take maximum value 0 on C_0. Since the maximum value of $H(0, q)$ on C_0 is $\frac{2}{3\sqrt{3}} > 0$, it is not possible to find such a function.

A trivial calculation shows that Hypothesis 4.3 is satisfied at $x = 0$, corresponding to the fact that W is a supersolution. In fact both case (1) and (2) of this hypothesis are satisfied at $x = 0$; case (1) with $p_0 = -1$ for $t < 0$ and case (2) with $p_0 = +1$ for $t > 0$.

Note that since H vanishes on Λ, W is a generalised solution to $H(x, \partial W/\partial x) = 0$ in the variational sense put forward by Viterbo [24, 25]. The work of Viterbo and others cited in the Introduction has established existence and uniqueness properties for variational solutions. Also, as this example shows, variational solutions can exist which are not viscosity solutions, and can be given a meaningful dynamical interpretation. It would be interesting to see if some of the other nice properties of viscosity solutions, such as dynamic programming principles, can be extended to variational solutions.

6. Conclusion

We have proved conditions under which a graph selector W defined on a Lagrangian manifold Λ is a viscosity solution to $H(x, \partial W/\partial x(x)) = 0$, for a Hamiltonian H vanishing on Λ. We have then presented examples where these conditions are satisfied, including where $H(x, p)$ is not convex in p.

References

[1] M. Bardi and L.C. Evans, On Hopf's formula for solutions of Hamilton-Jacobi equations. *Nonlinear Anal. Th. Meth. Appl.* **8** (1984) 1373–1381.

[2] F. Cardin, On viscosity solutions and geometrical solutions of Hamilton-Jacobi equations. *Nonlinear Anal. Th. Meth. Appl.* **20** (1993) 713–719.

[3] M. Chaperon, Lois de conservation et geometrie symplectique. *C.R. Acad. Sci. Paris Ser. I Math.*, **312** (1991) 345–348.

[4] F.H. Clarke, *Optimization and Nonsmooth Analysis.* J. Wiley, New York (1983).

[5] M.G. Crandall and P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS* **277** (1983) 1–42.

[6] M.G. Crandall, L.C. Evans and P.L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS* **282** (1984) 487–502.

[7] M.V. Day, On Lagrange manifolds and viscosity solutions. *J. Math. Syst. Estim. Contr.* **8** (1998) `http://www.math.vt.edu/people/day/research/LMVS.pdf`

[8] S.Yu. Dobrokhotov, V.N. Kolokoltsov and V.P. Maslov, Quantization of the Bellman equation, exponential asymptotics and tunneling, in *Advances in Soviet Mathematics*, V.P. Maslov and S.N. Samborskii, Eds., American Mathematical Society, Providence, Rhode Island **13** (1992) 1–46 .

[9] W.H Fleming and H.M. Soner, *Controlled markov processes and viscosity solutions*. Springer-Verlag, New York (1993).

[10] H. Frankowska, Hamilton-Jacobi equations: viscosity solutions and generalised gradients. *J. Math. Anal. Appl.* **141** (1989) 21–26.

[11] E. Hopf, Generalized solutions of non-linear equations of first order. *J. Math. Mech.* **14** (1965) 951–973.

[12] T. Joukovskaia, Thèse de Doctorat, Université de Paris VII, Denis Diderot (1993).

[13] F. Laudenbach and J.C. Sikorav, Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibre cotangent. *Invent. Math.* **82** (1985) 349–357.

[14] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems. Applied Mathematical Sciences Series* **74**, Springer-Verlag, Berlin (1989).

[15] D. McCaffrey and S.P. Banks, Lagrangian Manifolds, Viscosity Solutions and Maslov Index. *J. Convex Anal.* **9** (2002) 185–224.

[16] D. McCaffrey, Viscosity Solutions on Lagrangian Manifolds and Connections with Tunnelling Operators, in *Idempotent Mathematics and Mathematical Physics*, V.P. Maslov and G.L. Litvinov Eds., *Contemp. Math.* **377**, American Mathematical Society, Providence, Rhode Island (2005).

[17] D. McCaffrey, Geometric existence theory for the control-affine H_∞ problem, to appear in *J. Math. Anal & Appl.* (August 2005).

[18] G.P. Paternain, L. Polterovich and K.F. Siburg, Boundary rigidity for Lagrangian submanifolds, non-removable intersections and Aubry-Mather theory. *Moscow Math. J.* **3** (2003) 593–619.

[19] J.C. Sikorav, Sur les immersions lagrangiennes dans un fibre cotangent admettant une phase generatrice globale. *C. R. Acad. Sci. Paris, Ser. I Math.* **302** (1986) 119–122.

[20] P. Soravia, H_∞ control of nonlinear systems: differential games and viscosity solutions. *SIAM J. Contr. Opt.* **34** (1996) 1071–1097.

[21] A.J. van der Schaft, On a state space approach to nonlinear H_∞ control. *Syst. Contr. Lett.* **16** (1991) 1–8.

[22] A.J. van der Schaft, L_2 gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Trans. Automatic Control* **AC-37** (1992) 770–784.

[23] C. Viterbo, Symplectic topology as the geometry of generating functions. *Math. Ann.* **292** (1992) 685–710.

[24] C. Viterbo, Solutions d'equations d'Hamilton-Jacobi et geometrie symplectique, Addendum to: *Séminaire sur les équations aux Dérivés Partielles 1994–1995*, École Polytech., Palaiseau (1996).

[25] A. Ottolengi and C. Viterbo, Solutions généralisées pour l'équation de Hamilton-Jacobi dans le cas d'évolution, *unpublished*.

[26] A. Weinstein, *Lectures on symplectic manifolds*, Regional Conference Series in Mathematics **29**, Conference Board of the Mathematical Sciences, AMS, Providence, Rhode Island (1977).