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TOWARDS A TWO-SCALE CALCULUS

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Abstract. We define and characterize weak and strong *two-scale convergence* in L^p, C^0 and other spaces *via* a transformation of variable, extending Nguetseng's definition. We derive several properties, including weak and strong two-scale compactness; in particular we prove two-scale versions of theorems of Ascoli-Arzelà, Chacon, Riesz, and Vitali. We then approximate two-scale derivatives, and define two-scale convergence in spaces of either weakly or strongly differentiable functions. We also derive two-scale versions of the classic theorems of Rellich, Sobolev, and Morrey.

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Introduction

Let Ω be a domain of $\mathbf{R}^N$ $(N \geq 1)$, and set $Y := [0,1[^N$. In the seminal work [25], Nguetseng introduced the following concept: a bounded sequence $\{u_\varepsilon\}$ of $L^2(\Omega)$ is said (weakly) *two-scale convergent* to $u \in L^2(\Omega \times Y)$ if and only if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\, \psi\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \iint_{\Omega\times Y} u(x,y)\, \psi(x,y)\, \mathrm{d}x\mathrm{d}y, \tag{1}$$

for any smooth function $\psi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ that is Y-periodic w.r.t. the second argument. It should be noticed that $u_\varepsilon : \Omega \to \mathbf{R}$ for any ε, whereas $u : \Omega \times Y \to \mathbf{R}$.

This notion was then analyzed in detail and applied to a number of problems by Allaire [1] and others. It can account for occurrence of a fine-scale periodic structure, and indeed has been and is still extensively applied to *homogenization,* see *e.g.* [2, 5, 8, 13, 17, 20, 21, 26, 35, 36], just to mention some papers of a growing literature. In the framework of periodic homogenization, two-scale convergence can represent an alternative to the classic *energy method* of Tartar, see *e.g.* [3, 7, 16, 19, 24, 28–31]. Extensions to the nonperiodic setting have been proposed by Casado-Diaz and Gayte [11, 12] and by Nguetseng [27]. Multi-scale convergence has been studied by Allaire and Briane [4] and by others.

In this paper we investigate some properties of two-scale convergence, and extend it in several ways. In Section 1 we set $u_\varepsilon = u = 0$ outside Ω and define a family of scale transformations $S_\varepsilon : \mathbf{R}^N \times Y \to \mathbf{R}^N$. Denoting weak one-scale (two-scale, resp.) convergence by $\rightharpoonup$ ($\underset{2}{\rightharpoonup}$, resp.), along the lines of [5, 8, 13, 15, 20, 21] we set

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(\Omega\times Y) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u \quad \text{in } L^2(\mathbf{R}^N \times Y); \tag{2}$$

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we then prove the equivalence to the standard definition (1). (This procedure has been named *periodic unfolding* in [15].) We thus represent two-scale convergence by means of a single function space; we also define strong two-scale convergence (that we denote by $\underset{2}{\to}$) *via* an analogous characterization. The extension to either weak and strong two-scale convergence in $L^p(\mathbf{R}^N \times Y)$ for any $p \in [1, +\infty]$ is obvious.

In this theory the periodicity w.r.t. the fine-scale variable y plays an important role. We then denote by $\mathcal{Y}$ the set Y equipped with the topology of the N-dimensional torus. After a simple modification of the discontinuous transformation S_ε, we also define weak and strong two-scale convergence in the Fréchet space $C^0(\mathbf{R}^N \times \mathcal{Y})$.

In Section 2 we derive some properties of two-scale convergence. Some of these results are already known, *cf. e.g.* [1, 15, 20, 21, 23, 25]; in particular this is the case for several either equivalent or sufficient conditions for two-scale convergence in L^p. Here we organize their derivation by using the tool of two-scale decomposition, and also deal with two-scale convergence in C^0 and in $\mathcal{D}'$, with the Fourier transform, and with two-scale convolution.

In Section 3 we study weak and strong two-scale compactness. We prove a two-scale version of a result of Chacon, known as *the biting lemma, cf.* [10]; we characterize strong two-scale compactness in L^p and in C^0, generalizing classic criteria of Riesz and Ascoli-Arzelà. Along the same lines, we also extend Vitali's convergence theorem.

Differential properties of two-scale convergence are the main concern of this paper. Even by simple examples it appears that the gradient of the two-scale limit of u_ε need not coincide with the two-scale limit of the gradient of u_ε. The two-scale limit of sequences bounded in $H^1(\Omega)$ has already been studied in [1, 25]; the present analysis moves towards a different direction. In Section 4 we show that it is possible to express the gradient of the two-scale limit without the need of evaluating the limit itself, *via* what we name *approximate two-scale derivatives.* More specifically, we define an approximate gradient Λ_ε such that, denoting the weak two-scale limit by $\lim_{\varepsilon \to 0}{}^{(2)}$,

$$\lim_{\varepsilon \to 0}{}^{(2)} \Lambda_\varepsilon u_\varepsilon = (\nabla_x, \nabla_y) \lim_{\varepsilon \to 0}{}^{(2)} u_\varepsilon \qquad \text{in } L^p(\mathbf{R}^N \times Y)^{2N}. \tag{3}$$

(The fact that $\varepsilon \nabla u_\varepsilon \underset{2}{\rightharpoonup} \nabla_y u$ was already known, *cf.* [1].)

By means of two-scale approximate derivatives, in Section 5 we define two-scale convergence in spaces of differentiable functions: $W^{m,p}$, C^m, $C^{m,\lambda}$, $\mathcal{D}$. For instance, for any Caratheodory function $w \in W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$, $w(x, x/\varepsilon)$ two-scale converges to $w(x,y)$ in that space. We then derive two-scale versions of the Rellich compactness theorem and of the Sobolev and Morrey imbedding theorems. Indeed several classic results have a two-scale counterpart, which does not concern single functions but sequences of functions (loosely speaking, these properties are dynamic rather than static...).

This paper reports on a research on multi-scale analysis and modelling; some of the present results were announced in [33]. This point of view induced this author to amend the vector Preisach model of ferromagnetic hysteresis in [32]. A work apart, [34], deals with the identification of the two-scale limit of first-order differential operators.

1. Two-scale convergence *via* two-scale decomposition

In this section we introduce a family of variable transformations, and use it to define two-scale convergence, along the lines of [5, 8, 13, 15, 20, 21].

Throughout this paper we denote by $\mathcal{Y}$ the set $Y = [0,1[^N$ equipped with the topology of the N-dimensional torus, and identify any function on $\mathcal{Y}$ with its Y-periodic extension to $\mathbf{R}^N$. In passing we notice that $\mathcal{D}(\mathcal{Y}) \neq \mathcal{D}(Y)$, whereas $L^p(\mathcal{Y}) = L^p(Y)$ for any $p \in [1, +\infty]$.

Two-scale decomposition. Let B be a complex separable Banach space, denote its norm by $\|\cdot\|_B$ and the duality pairing between B and B' by $\langle \cdot, \cdot \rangle$. We set $p' := p/(p-1)$ for any $p \in]1, +\infty[$, $1' := \infty$ and $\infty' := 1$; we assume that either B is reflexive or B' is separable, so that $\big(L^p(\mathbf{R}^N; B)\big)' = L^{p'}(\mathbf{R}^N; B')$ for any $p \in [1, +\infty[$, *cf. e.g.* [18]. For any $\varepsilon > 0$, we decompose real numbers and real vectors as follows:

$$\begin{aligned} \hat{n}(x) &:= \max\{n \in \mathbf{Z} : n \le x\}, \quad \hat{r}(x) := x - \hat{n}(x) \ (\in [0,1[) \qquad \forall x \in \mathbf{R}, \\ \mathcal{N}(x) &:= (\hat{n}(x_1), ..., \hat{n}(x_N)) \in \mathbf{Z}^N, \quad \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \quad \forall x \in \mathbf{R}^N. \end{aligned} \tag{1.1}$$

Thus $x = \varepsilon[\mathcal{N}(x/\varepsilon) + \mathcal{R}(x/\varepsilon)]$ for any $x \in \mathbf{R}^N$. In applications the variable x often expresses the ratio between some dimensional quantity and a given scale. If ε represents the ratio between a finer scale and the given one, $\mathcal{N}(x/\varepsilon)$ and $\mathcal{R}(x/\varepsilon)$ may then be regarded as coarse-scale and fine-scale variables, resp. Besides the above two-scale decomposition, we define a *two-scale composition* function:

$$S_\varepsilon(x, y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \qquad \forall (x, y) \in \mathbf{R}^N \times \mathcal{Y}, \qquad \forall \varepsilon > 0. \tag{1.2}$$

As $S_\varepsilon(x, y) = x + \varepsilon[y - \mathcal{R}(x/\varepsilon)]$,

$$S_\varepsilon(x, y) \to x \quad \text{ uniformly in } \mathbf{R}^N \times \mathcal{Y}, \text{ as } \varepsilon \to 0. \tag{1.3}$$

The next result is at the basis of our approach to two-scale convergence. First let us denote by $\mathcal{L}(\mathbf{R}^N)$ ($\mathcal{B}(\mathbf{R}^N)$, resp.) the σ-algebra of Lebesgue- (Borel-, resp.) measurable subsets of $\mathbf{R}^N$, define $\mathcal{L}(\mathcal{Y})$ and $\mathcal{B}(\mathcal{Y})$ similarly, and set

$$\begin{aligned} \mathcal{F} := \big\{ f : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} \ &\text{ measurable either w.r.t. the } \sigma\text{-algebra generated} \\ &\text{by } \mathcal{B}(\mathbf{R}^N) \times \mathcal{L}(\mathcal{Y}), \text{ or w.r.t. that generated by } \mathcal{L}(\mathbf{R}^N) \times \mathcal{B}(\mathcal{Y}) \big\}. \end{aligned} \tag{1.4}$$

This class includes all Caratheodory functions, *cf. e.g.* [9], p. 30. Henceforth by writing any sum over $\mathbf{Z}^N$ we shall implicitly assume that it is absolutely convergent.

Lemma 1.1. *Let $f \in \mathcal{F}$, and assume that either $f \in L^1\big(\mathcal{Y}; L^\infty(\mathbf{R}^N)\big)$ and has compact support, or $f \in L^1\big(\mathbf{R}^N; L^\infty(\mathcal{Y})\big)$. Let us extend $f(x,\cdot)$ by Y-periodicity to $\mathbf{R}^N$ for a.a. $x \in \mathbf{R}^N$.*

Then, for any $\varepsilon > 0$, the functions $\mathbf{R}^N \to \mathbf{R} : x \mapsto f(x, x/\varepsilon)$ and $\mathbf{R}^N \times \mathcal{Y} \to \mathbf{R} : (x, y) \mapsto f(S_\varepsilon(x, y), y)$ are integrable, and

$$\int_{\mathbf{R}^N} f(x, x/\varepsilon)\,\mathrm{d}x = \varepsilon^N \sum_{m \in \mathbf{Z}^N} \int_Y f(\varepsilon[m + y], y)\,\mathrm{d}y = \iint_{\mathbf{R}^N \times Y} f(S_\varepsilon(x, y), y)\,\mathrm{d}x\mathrm{d}y \qquad \forall \varepsilon > 0. \tag{1.5}$$

For any $p \in [1, +\infty]$ and $\varepsilon > 0$, the operator $A_\varepsilon : g \mapsto g \circ S_\varepsilon$ is then a (nonsurjective) linear isometry $L^p(\mathbf{R}^N; B) \to L^p(\mathbf{R}^N \times \mathcal{Y}; B)$.

Proof. The function $x \mapsto f(x, x/\varepsilon)$ is obviously measurable. The function $(x, y) \mapsto f(S_\varepsilon(x, y), y)$ is also measurable, for the mapping $(x, y) \mapsto (S_\varepsilon(x, y), y)$ is piecewise constant w.r.t. x and affine w.r.t. y. As $\mathbf{R}^N = \bigcup_{m \in \mathbf{Z}^N} (\varepsilon m + \varepsilon\mathcal{Y})$ and $\mathcal{N}(x/\varepsilon) = m$ for any $x \in \varepsilon m + \varepsilon\mathcal{Y}$, we have

$$\begin{aligned} \int_{\mathbf{R}^N} f(x, x/\varepsilon)\,\mathrm{d}x &= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} f(x, x/\varepsilon)\,\mathrm{d}x = \varepsilon^N \sum_{m \in \mathbf{Z}^N} \int_Y f(\varepsilon[m + y], y)\,\mathrm{d}y \\ &= \sum_{m \in \mathbf{Z}^N} \int_{\varepsilon m + \varepsilon Y} \mathrm{d}x \int_Y f(\varepsilon[\mathcal{N}(x/\varepsilon) + y], y)\,\mathrm{d}y = \int_{\mathbf{R}^N} \mathrm{d}x \int_Y f(S_\varepsilon(x, y), y)\,\mathrm{d}y. \end{aligned}$$

Writing (1.5) for $f(x, y) = \|g(x)\|_B$ for a.a. (x, y), we get the final statement for any $p \in [1, +\infty[$, and then by passage to the limit also for $p = \infty$. The operator A_ε is not onto, for $g \circ S_\varepsilon$ is piecewise constant w.r.t. x for any $g \in L^p(\mathbf{R}^N; B)$. □

Corollary 1.2. *Denoting by $|\cdot|_M$ the M-dimensional Lebesgue measure, for any measurable set $A \subset \mathbf{R}^N$ of finite measure and any measurable set $C \subset \mathcal{Y}$,*

$$|\{x \in A : \mathcal{R}(x/\varepsilon) \in C\}|_N = |\{(x, y) \in \mathbf{R}^N \times \mathcal{Y} : S_\varepsilon(x, y) \in A,\ y \in C\}|_{2N} \qquad \forall \varepsilon > 0. \tag{1.6}$$

In particular,

$$|A|_N = |\{(x,y) \in \mathbf{R}^N \times \mathcal{Y} : S_\varepsilon(x,y) \in A\}|_{2N} \qquad \forall \varepsilon > 0. \tag{1.7}$$

Proof. Let us define the characteristic function

$$\chi_A(v) := 1 \quad \forall v \in A, \qquad \chi_A(v) := 0 \quad \forall v \in \mathbf{R}^N \setminus A,$$

and define χ_C similarly. By Lemma 1.1, we have

$$\begin{aligned} |\{x \in A : \mathcal{R}(x/\varepsilon) \in C\}|_N &= \int_{\mathbf{R}^N} \chi_A(x)\, \chi_C(\mathcal{R}(x/\varepsilon))\,\mathrm{d}x \\ &= \iint_{\mathbf{R}^N \times Y} \chi_A(S_\varepsilon(x,y))\, \chi_C(y)\,\mathrm{d}x\mathrm{d}y = |\{(x,y) : S_\varepsilon(x,y) \in A, y \in C\}|_{2N}. \end{aligned}$$

□

Two-scale convergence in L^p. In this paper we deal with sequences of functions, which we label by the index ε, as it is customary in the literature about two-scale convergence. More precisely, ε will represent an arbitrary but prescribed, positive and vanishing sequence of real numbers; for instance, $\varepsilon = \{1, 1/2, ..., 1/n, ...\}$. The results of this paper do not depend on the specific choice of this sequence, that we regard as fixed.

Let B, $\mathcal{Y}$, and S_ε be defined as above. Along the lines of [15], for any sequence of measurable functions, $u_\varepsilon : \mathbf{R}^N \to B$, and any measurable function, $u : \mathbf{R}^N \times \mathcal{Y} \to B$, we say that u_ε *two-scale converges* to u (w.r.t. the prescribed positive vanishing sequence $\{\varepsilon_n\}$) in some specific sense, whenever $u_\varepsilon \circ S_\varepsilon \to u$ in the corresponding standard sense. In this way we define strong and weak (weak star for $p = \infty$) two-scale convergence ($1 \le p \le +\infty$), that we denote by $u_\varepsilon \underset{2}{\to} u$, $u_\varepsilon \underset{2}{\rightharpoonup} u$, $u_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} u$ (resp.):

$$u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \to u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \quad \forall p \in [1, +\infty]; \tag{1.8}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \quad \forall p \in [1, +\infty[; \tag{1.9}$$

$$u_\varepsilon \overset{*}{\underset{2}{\rightharpoonup}} u \text{ in } L^\infty(\mathbf{R}^N \times \mathcal{Y}; B') \quad \Leftrightarrow \quad u_\varepsilon \circ S_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\mathbf{R}^N \times \mathcal{Y}; B')\ (= L^1(\mathbf{R}^N \times \mathcal{Y}; B)'). \tag{1.10}$$

For any domain $\Omega \subset \mathbf{R}^N$, two-scale convergence in $L^p(\Omega \times \mathcal{Y}; B)$ is then defined by extending functions to $\mathbf{R}^N \setminus \Omega$ with vanishing value. We similarly define *a.e.* (*i.e.*, almost everywhere) two-scale convergence, quasi-uniform two-scale convergence, two-scale convergence in measure, and so on. In all of these cases the limit is obviously unique. We refer to the usual convergence over $\mathbf{R}^N$ as *one-scale* convergence.

For instance, for any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, $u_\varepsilon(x) := \psi(x, x/\varepsilon) \underset{2}{\to} \psi(x,y)$. Examples of this sort play an important role, for they often represent the best behaviour one may expect for this type of convergence. By the next result, weak and strong two-scale convergence can be regarded as intermediate properties between the usual (one-scale) weak and strong convergence.

Theorem 1.3. *Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N; B)$ and $u \in L^p(\mathbf{R}^N \times \mathcal{Y}; B)$. Then*

$$\text{whenever } u \text{ is independent of } y, \qquad u_\varepsilon \to u \text{ in } L^p(\mathbf{R}^N; B) \ \Leftrightarrow\ u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \tag{1.11}$$

$$u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \ \Rightarrow\ u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \tag{1.12}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \ \Rightarrow\ u_\varepsilon \rightharpoonup \int_Y u(\cdot, y)\,\mathrm{d}y \text{ in } L^p(\mathbf{R}^N; B). \tag{1.13}$$

For any Lipschitz-continuous function $f : B \to B$,

$$u_\varepsilon \underset{2}{\to} u \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \ \Rightarrow\ f(u_\varepsilon) \underset{2}{\to} f(u) \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B). \tag{1.14}$$

Proof. For any $u \in L^p(\mathbf{R}^N; B)$, by Lemma 1.1 we have

$$\begin{aligned}\|u_\varepsilon - u\|^p_{L^p(\mathbf{R}^N;B)} &= \int_{\mathbf{R}^N} \|u_\varepsilon(x) - u(x)\|^p_B \,\mathrm{d}x \\ &= \iint_{\mathbf{R}^N\times\mathcal{Y}} \|u_\varepsilon(S_\varepsilon(x,y)) - u(S_\varepsilon(x,y))\|^p_B \,\mathrm{d}x\mathrm{d}y = \|u_\varepsilon \circ S_\varepsilon - u \circ S_\varepsilon\|^p_{L^p(\mathbf{R}^N\times\mathcal{Y};B)}.\end{aligned}$$

Hence

$$\begin{aligned}\big|\|u_\varepsilon \circ S_\varepsilon - u\|_{L^p(\mathbf{R}^N\times\mathcal{Y};B)} - \|u_\varepsilon - u\|_{L^p(\mathbf{R}^N;B)}\big| &= \big|\|u_\varepsilon \circ S_\varepsilon - u\|_{L^p(\mathbf{R}^N\times\mathcal{Y};B)} - \|u_\varepsilon \circ S_\varepsilon - u \circ S_\varepsilon\|_{L^p(\mathbf{R}^N\times\mathcal{Y};B)}\big| \\ &\le \|u - u\circ S_\varepsilon\|_{L^p(\mathbf{R}^N\times\mathcal{Y};B)} \to 0 \quad \text{(by (1.3))}.\end{aligned}$$

(1.11) is thus established. (1.12) and (1.14) are straightforward.

Let us now come to (1.13), assume that $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, and fix any (bounded if $p > 1$) Lebesgue measurable set $A \subset \mathbf{R}^N$. Applying Lemma 1.1 to $f = u_\varepsilon \chi_A$ $(\in L^1(\mathbf{R}^N))$, we have

$$\int_A u_\varepsilon(x)\,\mathrm{d}x = \iint_{A\times Y} u_\varepsilon(S_\varepsilon(x,y))\,\mathrm{d}x\mathrm{d}y \rightharpoonup \iint_{A\times Y} u(x,y)\,\mathrm{d}x\mathrm{d}y = \int_A \mathrm{d}x \int_Y u(x,y)\,\mathrm{d}y \qquad \text{in } B.$$

As the finite linear combinations of indicator functions χ_A are dense in $L^{p'}(\mathbf{R}^N)$, we conclude that $u_\varepsilon \rightharpoonup \int_Y u(\cdot,y)\,\mathrm{d}y$ in $L^p(\mathbf{R}^N; B)$. □

Remark. For $p = \infty$ the implication (1.11) may fail. As a counterexample it suffices to select any real a that is no integral multiple of ε_n for any n, and set $u_{\varepsilon_n} = \chi_{[a,+\infty[}$ for any $n \in \mathbf{N}$. This constant sequence does not two-scale converge in $L^\infty(\mathbf{R}^N \times \mathcal{Y})$, as $u_{\varepsilon_n} \circ S_{\varepsilon_n}$ is constant w.r.t. x in a small neighbourhood of a for any n. This shows that strong two-scale convergence in $L^\infty(\mathbf{R}^N\times\mathcal{Y}; B)$ to discontinuous functions is a rather restrictive property. See however Proposition 1.5 below.

On the other hand it is easy to see that for $p = \infty$ (1.13) holds with $\overset{*}{\rightharpoonup}$ ($\underset{2}{\overset{*}{\rightharpoonup}}$, resp.) in place of $\rightharpoonup$ ($\underset{2}{\rightharpoonup}$, resp.), provided that B is the dual of a separable Banach space.

Limit decomposition and orthogonality. Let $p \in [1, +\infty[$, $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, and set

$$\left.\begin{aligned} u_{0\varepsilon} &:= \int_Y u_\varepsilon(\varepsilon\mathcal{N}(\cdot/\varepsilon) + \varepsilon\xi)\,\mathrm{d}\xi, \quad u_{1\varepsilon} := u_\varepsilon - u_{0\varepsilon} \\ u_0 &:= \int_Y u(\cdot,y)\,\mathrm{d}y, \quad u_1 := u - u_0 \end{aligned}\right\} \quad a.e. \text{ in } \mathbf{R}^N. \tag{1.15}$$

Via Lemma 1.1 it is easy to see that

$$u_{0\varepsilon} \underset{2}{\rightharpoonup} u_0, \quad u_{1\varepsilon} \underset{2}{\rightharpoonup} u_1 \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B).$$

Notice that $u_\varepsilon \rightharpoonup u_0$ in $L^p(\mathbf{R}^N; B)$, for u_0 is independent of y, *cf.* (1.11); hence $u_{1\varepsilon} \rightharpoonup 0$ in $L^p(\mathbf{R}^N; B)$. This yields the *limit two-scale decomposition*

$$\begin{aligned} &u(x,y) = u_0(x) + u_1(x,y) \quad \text{for a.a. } (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \\ &\text{with } \int_Y u_1(x,y)\,\mathrm{d}y = 0 \quad \text{for a.a. } x \in \mathbf{R}^N. \end{aligned} \tag{1.16}$$

Proposition 1.4. *Let* $p \in [1,+\infty[$*, the sequence* $\{u_\varepsilon\}$ *and* u_0, u_1 *be as above. Assume that* $\varphi_\varepsilon \underset{2}{\to} \varphi$ *in* $L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B')$*, and decompose* φ *in the form* $\varphi = \varphi_0 + \varphi_1$*, with* $\varphi_\varepsilon \rightharpoonup \varphi_0$ *in* $L^{p'}(\mathbf{R}^N; B')$*. Then*

$$\begin{aligned}\int_{\mathbf{R}^N} \langle u_\varepsilon(x), \varphi_\varepsilon(x)\rangle \,\mathrm{d}x &\to \iint_{\mathbf{R}^N\times Y} \langle u(x,y), \varphi(x,y)\rangle \,\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbf{R}^N} \langle u_0(x), \varphi_0(x)\rangle \,\mathrm{d}x + \iint_{\mathbf{R}^N\times Y} \langle u_1(x,y), \varphi_1(x,y)\rangle \,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.17}$$

Proof. By Lemma 1.1, by the decomposition formula (1.16) and the analogous formula for φ, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\mathbf{R}^N} \langle u_\varepsilon(x), \varphi_\varepsilon(x)\rangle \,\mathrm{d}x &= \lim_{\varepsilon\to 0} \iint_{\mathbf{R}^N\times Y} \langle u_\varepsilon(S_\varepsilon(x,y)), \varphi_\varepsilon(S_\varepsilon(x,y))\rangle \,\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathbf{R}^N\times Y} \langle u(x,y), \varphi(x,y)\rangle \,\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbf{R}^N} \langle u_0(x), \varphi_0(x)\rangle \,\mathrm{d}x + \iint_{\mathbf{R}^N\times Y} \langle u_1(x,y), \varphi_1(x,y)\rangle \,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

□

Let us denote the duality mapping $B \to 2^{B'}$ by F. If u_0 and u_1 are as above, we have the following orthogonality-type property:

$$\int_{\mathbf{R}^N} \mathrm{d}x \int_Y \langle F(u_0(x)), u_1(x,y)\rangle \,\mathrm{d}y = \int_{\mathbf{R}^N} \mathrm{d}x \left\langle F(u_0(x)), \int_Y u_1(x,y)\,\mathrm{d}y \right\rangle = 0, \tag{1.18}$$

as $\int_Y u_1(\cdot,y)\,\mathrm{d}y = 0$. If $p = 2$ and B is a Hilbert space, the decomposition (1.16) is actually orthogonal: u_1 is the projection of u onto the subspace $\{v \in L^2(\mathbf{R}^N \times \mathcal{Y}; B) : \int_Y u_1(x,y)\,\mathrm{d}y = 0$ for a.a. $x \in \mathbf{R}^N\}$, and

$$\|u\|^2_{L^2(\mathbf{R}^N\times\mathcal{Y};B)} = \|u_0\|^2_{L^2(\mathbf{R}^N;B)} + \|u_1\|^2_{L^2(\mathbf{R}^N\times\mathcal{Y};B)}. \tag{1.19}$$

Interpolation. Modifications are needed to extend the previous definitions to C^0, for in general the function $u_\varepsilon \circ S_\varepsilon$ is discontinuous w.r.t. x and w.r.t. $y \in \mathcal{Y}$, even if u_ε is continuous. We then replace $u_\varepsilon \circ S_\varepsilon$ by a continuous function, $L_\varepsilon u_\varepsilon$, that we construct *via* linear interpolation w.r.t. each coordinate axis as follows. For $i = 1, ..., N$, let us denote by e_i the unit vector of the x_i-axis, set $x_{[i]} := x - x_i e_i$ for any $x \in \mathbf{R}$, $y_{[i]} := y - y_i e_i$ for any $y \in \mathbf{R}$ (thus $0 \le y_i < 1$), and (*cf.* (1.1))

$$\begin{aligned}&(I_{\varepsilon,i}w)(x,y) := w(x_{[i]} + \varepsilon\hat{n}(x_i/\varepsilon)e_i, y) \\ &\qquad\qquad + r(x_i/\varepsilon)\big[w(x_{[i]} + \varepsilon\hat{n}(x_i/\varepsilon)e_i + \varepsilon e_i, y) - w(x_{[i]} + \varepsilon\hat{n}(x_i/\varepsilon)e_i, y)\big], \\ &(J_i w)(x,y) := w(x,y) - y_i\Big(\lim\nolimits_{t\to 1^-} w(x, y_{[i]} + te_i) - w(x, y_{[i]})\Big) \\ &\forall (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \forall w : \mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}, \text{ for } i = 1, ..., N, \\ &L_\varepsilon v := (J_1 \circ ... \circ J_N \circ I_{\varepsilon,1} \circ ... \circ I_{\varepsilon,N})(v \circ S_\varepsilon) \qquad \forall v : \mathbf{R}^N \to \mathbf{R}.\end{aligned} \tag{1.20}$$

Thus $v \circ S_\varepsilon$ is piecewise constant w.r.t. x, whereas $L_\varepsilon v$ is piecewise linear and continuous w.r.t. x_i for any i. Moreover

$$\lim_{t\to 1^-} (L_\varepsilon v)(x, y_{[i]} + te_i) = (L_\varepsilon v)(x, y_{[i]}) \qquad \forall (x,y) \in \mathbf{R}^N \times \mathcal{Y}, \forall i.$$

If $v \in C^0(\mathbf{R}^N; B)$ then $L_\varepsilon v \in C^0(\mathbf{R}^N \times \mathcal{Y}; B)$.

The interpolation procedure that here has been applied w.r.t. x is labelled Q_1, and is widely used in the finite-element theory. A Q_1-interpolation was also applied to two-scale convergence in [15].

Two-scale convergence in C^0. It is well known that $C^0(\mathbf{R}^N;B)$ and $C^0(\mathbf{R}^N\times\mathcal{Y};B)$ are Fréchet spaces: *e.g.*, $C^0(\mathbf{R}^N;B)$ is equipped with the family of seminorms $\{v\mapsto \sup_K \|v\|_B : K\subset\subset \mathbf{R}^N\}$.

For any sequence $\{u_\varepsilon\}$ in the Fréchet space $C^0(\mathbf{R}^N;B)$ and any $u\in C^0(\mathbf{R}^N\times\mathcal{Y};B)$, we say that u_ε *strongly (weakly,* resp.) *two-scale converges* to u in $C^0(\mathbf{R}^N\times\mathcal{Y};B)$ iff (*i.e.*, if and only if)

$$L_\varepsilon u_\varepsilon \to u \quad (L_\varepsilon u_\varepsilon \rightharpoonup u \text{ resp.}) \qquad \text{in } C^0(\mathbf{R}^N\times\mathcal{Y};B) \tag{1.21}$$

w.r.t. the Fréchet topology. A result (here omitted) analogous to Theorem 1.3 holds in $C^0(\mathbf{R}^N\times\mathcal{Y};B)$.

Proposition 1.5. *Let $\{u_\varepsilon\}$ be a bounded sequence in $C^0(\mathbf{R}^N;B)$ and $u\in C^0(\mathbf{R}^N\times\mathcal{Y};B)$. Then*
(i) $u_\varepsilon \underset{2}{\to} u$ in $C^0(\mathbf{R}^N\times\mathcal{Y};B)$ iff $u_\varepsilon \underset{2}{\to} u$ in $L^\infty(K\times\mathcal{Y};B)$ for any compact subset K of $\mathbf{R}^N$.
(ii) $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $C^0(\mathbf{R}^N\times\mathcal{Y};B)$ iff $u_\varepsilon \underset{2}{\rightharpoonup} u$ in B pointwise in $\mathbf{R}^N\times\mathcal{Y}$.

Proof. Part (i) follows from the definition of convergence in the Fréchet space $C^0(\mathbf{R}^N\times\mathcal{Y};B)$. By the continuity of u, it is easy to see that $u_\varepsilon \underset{2}{\rightharpoonup} u$ (*i.e.*, $u_\varepsilon\circ S_\varepsilon\to u$) in B pointwise in $\mathbf{R}^N\times\mathcal{Y}$ iff $L_\varepsilon u_\varepsilon \rightharpoonup u$ in B pointwise in the same set. This yields part (ii) as, for any compact set K, weak convergence in $C^0(K\times\mathcal{Y})$ is equivalent to boundedness and pointwise convergence (under the assumption that the limit is also continuous), *cf. e.g.* [18], p. 269. □

Parameters and scales. So far we dealt with sequences indexed by a parameter ε, that we assumed to coincide with the ratio between two scales. But this coincidence is not really needed: we illustrate this issue dealing with two sequences of parameters. First let us define the set of all *scale sequences,* $\mathcal{E}$, namely the set of all positive vanishing sequences. Let us fix any $\hat\varepsilon := \{\varepsilon_1,...,\varepsilon_n,...\}$, $\hat\varepsilon' := \{\varepsilon'_1,...,\varepsilon'_n,...\}\in\mathcal{E}$. We say that $u_\varepsilon \underset{2}{\to} u$ in $L^p(\mathbf{R}^N\times\mathcal{Y})$ w.r.t. ε', and write $u_\varepsilon \overset{\varepsilon'}{\underset{2}{\to}} u$, iff $u_{\varepsilon_n}\circ S_{\varepsilon'_n}\to u$ in $L^p(\mathbf{R}^N\times\mathcal{Y})$ as $n\to\infty$. For instance, if $\varepsilon'_n := \varepsilon_n^2$ for any n, as $n\to\infty$ we have

$$\begin{gathered}
\cos(2\pi S_{\varepsilon_n}(x,y)/\varepsilon_n) = \cos(2\pi[\mathcal{N}(x/\varepsilon_n)+y]) = \cos(2\pi y)\\
\cos(2\pi S_{\varepsilon_n^2}(x,y)/\varepsilon_n) = \cos(2\pi\varepsilon_n[\mathcal{N}(x/\varepsilon_n^2)+y]) = \cos(2\pi[x/\varepsilon_n + O(\varepsilon_n)]) \rightharpoonup 0\\
\cos(2\pi S_{\varepsilon_n^2}(x,y)/\varepsilon_n^2) = \cos(2\pi[\mathcal{N}(x/\varepsilon_n^2)+y]) = \cos(2\pi y)\\
\cos(2\pi S_{\varepsilon_n}(x,y)/\varepsilon_n^2) = \cos(2\pi[\mathcal{N}(x/\varepsilon_n)+y]/\varepsilon_n) \rightharpoonup 0
\end{gathered}$$

in the Fréchet space $L^p_{\rm loc}(\mathbf{R}^N\times\mathcal{Y})$ for any $p<+\infty$. Hence

$$\begin{matrix}
\cos(2\pi x/\varepsilon) \overset{\varepsilon}{\underset{2}{\to}} \cos(2\pi y), & \cos(2\pi x/\varepsilon) \overset{\varepsilon^2}{\underset{2}{\rightharpoonup}} 0\\
\cos(2\pi x/\varepsilon^2) \overset{\varepsilon^2}{\underset{2}{\longrightarrow}} \cos(2\pi y), & \cos(2\pi x/\varepsilon^2) \overset{\varepsilon}{\underset{2}{\rightharpoonup}} 0
\end{matrix} \qquad \text{in } L^p_{\rm loc}(\mathbf{R}^N\times\mathcal{Y}), \forall p<+\infty. \tag{1.22}$$

Two-scale convergence is indeed invariant upon rescaling: for any $\hat\varepsilon\in\mathcal{E}$ and any strictly increasing function $\alpha:\mathbf{R}^+\to\mathbf{R}^+$ such that $\alpha(v)\to 0$ as $v\to 0$, setting $\varepsilon'_n = \alpha(\varepsilon_n)$ for any n, we have $\hat\varepsilon'\in\mathcal{E}$; moreover, $u_\varepsilon \overset{\varepsilon}{\underset{2}{\to}} u$ iff $u_{\varepsilon'} \overset{\varepsilon'}{\underset{2}{\to}} u$. Henceforth we deal with a single sequence $\varepsilon\in\mathcal{E}$, and omit the *hat,* ˆ.

2. Some properties of two-scale convergence

In this section we study several necessary and/or sufficient properties for two-scale convergence in the spaces L^p and C^0, partially revisiting known results. We then define *two-scale convolution,* and generalize two-scale convergence to distributions. Several other notions have a natural two-scale extension: for instance, a *two-scale Fourier transform* will be studied apart.

2.1. Characterization of two-scale convergence in L^p and in C^0

We still assume that the Banach space B is separable, and that either it is reflexive or B' is separable.

Is two-scale convergence invariant upon traslations? For any $u \in \mathcal{F}$ (*cf.* (1.4)) let us set $u_{(\varepsilon)}(x) = u(x, x/\varepsilon)$ for any $x \in \mathbf{R}^N$. We wonder whether a relation may be established between $u_\varepsilon \underset{2}{\to} u$ and $u_\varepsilon - u_{(\varepsilon)} \underset{2}{\to} 0$ either in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ ($p \in [1, +\infty[$) or in $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$, and similarly for weak two-scale convergence. We address this question in Propositions 2.3 and 2.4.

First we notice that if u is just an element of $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, $u_{(\varepsilon)}$ need not be measurable. After [15], we then define the *coarse-scale averaging* operator M_ε:

$$(M_\varepsilon u)(x, y) := \int_Y u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon\xi, y)\, \mathrm{d}\xi \qquad \text{for a.a. } (x, y) \in \mathbf{R}^N \times \mathcal{Y}. \tag{2.1}$$

This function is piecewise constant w.r.t. x; it is also measurable w.r.t. y, for it is the average of a family of measurable functions. More precisely, $M_\varepsilon u$ is measurable w.r.t. (x, y), and $(M_\varepsilon u)(x, x/\varepsilon)$ is measurable as well. For any $p \in [1, +\infty[$, M_ε is a (linear and) continuous operator in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$. Indeed for any $u \in L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, by Jensen's inequality,

$$\begin{aligned}\|M_\varepsilon u\|^p_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)} &= \iint_{\mathbf{R}^N \times Y} \Big\| \int_Y u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon\xi, y)\, \mathrm{d}\xi \Big\|^p_B \mathrm{d}x\mathrm{d}y \\ &\le \iint_{\mathbf{R}^N \times Y} \left(\int_Y \|u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon\xi, y)\|^p_B\, \mathrm{d}\xi \right) \mathrm{d}x\mathrm{d}y = \|u\|^p_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)}.\end{aligned}$$

Lemma 2.1. *Let $p \in [1, +\infty[$. For any $u \in L^p(\mathbf{R}^N \times \mathcal{Y}; B)$,*

$$(M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\to} u(x, y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B). \tag{2.2}$$

If $u \in \mathcal{F}$ (cf. (1.4)) the operator M_ε may be dropped.

Proof. By the definitions of M_ε and S_ε (*cf.* (1.2)), by the Y-periodicity of the function $u(x, \cdot)$, and by a classic theorem of Lebesgue on the pointwise convergence of averages, we have

$$(M_\varepsilon u)(S_\varepsilon(x, y), S_\varepsilon(x, y)/\varepsilon) = (M_\varepsilon u)(x, y) \to u(x, y) \qquad \text{in } B, \text{ a.e. in } \mathbf{R}^N \times \mathcal{Y}.$$

As $\{\|M_\varepsilon u\|^p_B\}$ is equi-integrable, Vitali's theorem yields the convergence in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, *i.e.* (2.2).

If $u \in \mathcal{F}$ then $u(S_\varepsilon(x, y), S_\varepsilon(x, y)/\varepsilon) = u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon y, y)$ is a measurable function of (x, y). Moreover, by (1.3) and by L^p-continuity w.r.t. shift of the argument,

$$u(\varepsilon \mathcal{N}(x/\varepsilon) + \varepsilon y, y) \to u(x, y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B). \qquad \square$$

Lemma 2.2. *For any $u \in C^0(\mathbf{R}^N \times \mathcal{Y}; B)$,*

$$u(x, x/\varepsilon) \underset{2}{\to} u(x, y) \qquad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B). \tag{2.3}$$

Proof. Denoting the modulus of continuity of u by m_u and setting $u_\varepsilon(x, y) := u(S_\varepsilon(x, y), y)$, by (1.20) we have

$$\begin{aligned}\|(I_{\varepsilon,i} u_\varepsilon)(x, y) - u(x, y)\|_B &\le \|(I_{\varepsilon,i} u_\varepsilon)(x, y) - u_\varepsilon(x, y)\|_B + \|u_\varepsilon(x, y) - u(x, y)\|_B \\ &\le 2m_u(\varepsilon) \qquad \forall (x, y) \in \mathbf{R}^N \times \mathcal{Y}, \text{ for } i = 1, ..., N.\end{aligned}$$

Hence $\|(L_\varepsilon u_\varepsilon)(x, y) - u(x, y)\|_B \le 2N m_u(\varepsilon) \to 0$, and (2.3) follows. $\square$

Proposition 2.3. *Let* $p \in [1, +\infty[$. *For any sequence* $\{u_\varepsilon\}$ *in* $L^p(\mathbf{R}^N; B)$ *and any* $u \in L^p(\mathbf{R}^N \times \mathcal{Y}; B)$,

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\to} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \tag{2.4}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B), \tag{2.5}$$

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \to 0 \quad \text{in } L^p(\mathbf{R}^N; B), \tag{2.6}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \underset{\not\Leftarrow}{\Rightarrow} u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) \rightharpoonup 0 \quad \text{in } L^p(\mathbf{R}^N; B). \tag{2.7}$$

If $u \in \mathcal{F}$ *(cf. (1.4)), then the operator* M_ε *may be dropped.*

The equivalence (2.6) was already stated in the second part of Theorem 3 of [15].

Proof. (2.4) and (2.5) directly follow from Lemma 2.1. In view of proving (2.6), let us set $w_\varepsilon := u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon)$. By Lemma 1.1, $\|w_\varepsilon \circ S_\varepsilon\|_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)} = \|w_\varepsilon\|_{L^p(\mathbf{R}^N; B)}$; thus

$$w_\varepsilon \underset{2}{\to} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Leftrightarrow \quad w_\varepsilon(x) \to 0 \quad \text{in } L^p(\mathbf{R}^N; B),$$

and (2.6) is established. Let us now come to (2.7). For any $g \in L^{p'}(\mathbf{R}^N; B')$, by Lemma 1.1

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^N} \langle w_\varepsilon(x), g(x) \rangle \, \mathrm{d}x = \lim_{\varepsilon \to 0} \iint_{\mathbf{R}^N \times Y} \langle w_\varepsilon(S_\varepsilon(x, y)), g(S_\varepsilon(x, y)) \rangle \, \mathrm{d}x\mathrm{d}y$$

provided that one of these limits exists. As $g \circ S_\varepsilon \to g$ in $\in L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B')$, we conclude that

$$w_\varepsilon \underset{2}{\rightharpoonup} 0 \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Rightarrow \quad w_\varepsilon(x) \rightharpoonup 0 \quad \text{in } L^p(\mathbf{R}^N; B),$$

that is, the implication "⇒" of (2.7). We show that the converse may fail by means of a counterexample. Let us set $u_\varepsilon(x) = \mathrm{e}^{-x^2} \sin(2\pi x/\varepsilon)$ and $u(x, y) = 0$ for any $x \in \mathbf{R}$ and any $y \in [0, 1[$. Then $u_\varepsilon(x) - (M_\varepsilon u)(x, x/\varepsilon) = u_\varepsilon(x) \rightharpoonup 0$ in $L^p(\mathbf{R})$ for any $p \in [1, +\infty[$, but $u_\varepsilon \underset{2}{\rightharpoonup} \mathrm{e}^{-x^2} \sin(2\pi y)$ in $L^p(\mathbf{R} \times \mathcal{Y})$. □

Proposition 2.4. *For any sequence* $\{u_\varepsilon\}$ *in* $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$ *and any* $u \in C^0(\mathbf{R}^N \times \mathcal{Y}; B)$,

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow \quad u_\varepsilon(x) - u(x, x/\varepsilon) \underset{2}{\to} 0 \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B), \tag{2.8}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow u_\varepsilon(x) - u(x, x/\varepsilon) \underset{2}{\rightharpoonup} 0 \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B), \tag{2.9}$$

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B) \Leftrightarrow u_\varepsilon(x) - u(x, x/\varepsilon) \to 0 \quad \text{in } C^0(\mathbf{R}^N; B). \tag{2.10}$$

Proof. (2.8) and (2.9) follow from Lemma 2.2. Let us now set $w_\varepsilon := u_\varepsilon(x) - u(x, x/\varepsilon)$, and notice that $\|L_\varepsilon w_\varepsilon\|_{C^0(\mathbf{R}^N \times \mathcal{Y}; B)} = \|w_\varepsilon\|_{C^0(\mathbf{R}^N; B)}$. Thus

$$w_\varepsilon \underset{2}{\to} 0 \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Leftrightarrow \quad w_\varepsilon(x) \to 0 \quad \text{in } C^0(\mathbf{R}^N; B),$$

and (2.10) holds. □

Remark.

(i) Here we do not address the possible relation between $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$ and $u_\varepsilon(x) - u(x, x/\varepsilon) \rightharpoonup 0$ in $C^0(\mathbf{R}^N; B)$.

(ii) There exist *pathologic* functions $u \in L^p(\mathbf{R}^N \times \mathcal{Y})$ such that the mapping $x \mapsto u(x, x/\varepsilon)$ is not measurable. This issue has been investigated in some detail in [1]; see also references therein.

(iii) By Lemmata 2.1 and 2.2, any function of $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ ($p \in [1, +\infty[$) or of $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$ is the two-scale limit of some sequence.

We now retrieve the original definition of (weak) two-scale convergence of Nguetseng [25], for any $p \neq \infty$.

Proposition 2.5. *Let $p \in [1, +\infty[$. For any bounded sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N; B)$ and any $u \in L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ iff*

$$\int_{\mathbf{R}^N} \langle u_\varepsilon(x), \psi(x, x/\varepsilon)\rangle \, \mathrm{d}x \to \iint_{\mathbf{R}^N \times Y} \langle u(x,y), \psi(x,y)\rangle \, \mathrm{d}x\mathrm{d}y \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}; B'). \tag{2.11}$$

Proof. For any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}; B')$, $\langle u_\varepsilon(x), \psi(x,y)\rangle \in L^1\big(\mathbf{R}^N; C^0(\mathcal{Y})\big)$, so that we can apply Lemma 1.1. As $\psi(S_\varepsilon(x,y), y) \to \psi(x,y)$ in $L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B')$, we have

$$\int_{\mathbf{R}^N} \langle u_\varepsilon(x), \psi(x, x/\varepsilon)\rangle \, \mathrm{d}x - \iint_{\mathbf{R}^N \times Y} \langle u_\varepsilon(S_\varepsilon(x,y)), \psi(x,y)\rangle \, \mathrm{d}x\mathrm{d}y$$
$$= \iint_{\mathbf{R}^N \times Y} \langle u_\varepsilon(S_\varepsilon(x,y)), \psi(S_\varepsilon(x,y), y) - \psi(x,y)\rangle \, \mathrm{d}x\mathrm{d}y \to 0.$$

Hence

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^N} \langle u_\varepsilon(x), \psi(x, x/\varepsilon)\rangle \, \mathrm{d}x = \lim_{\varepsilon \to 0} \iint_{\mathbf{R}^N \times Y} \langle u_\varepsilon(S_\varepsilon(x,y)), \psi(x,y)\rangle \, \mathrm{d}x\mathrm{d}y. \qquad \square$$

Remark. As the tensor product $\mathcal{D}(\mathbf{R}^N; B') \otimes \mathcal{D}(\mathcal{Y})$ is dense in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y}; B')$, (2.11) is equivalent to

$$\int_{\mathbf{R}^N} \langle u_\varepsilon(x), \psi(x)\rangle \varphi(x/\varepsilon) \, \mathrm{d}x \to \iint_{\mathbf{R}^N \times Y} \langle u(x,y), \psi(x)\rangle \varphi(y) \, \mathrm{d}x\mathrm{d}y$$
$$\forall \psi \in \mathcal{D}(\mathbf{R}^N; B'), \forall \varphi \in \mathcal{D}(\mathcal{Y}).$$

Here φ might equivalently be confined to (the real and immaginary parts of) the Fourier basis $\{\phi_n\}_{n \in \mathbf{Z}^N}$, where $\phi_n(y) := \exp(2\pi i \, n \cdot y)$ for any $y \in \mathcal{Y}$ and any $n \in \mathbf{Z}^N$.

In the next statement we assume that B is a complex Hilbert space equipped with a Hilbert basis $\{\phi_n\}_{n \in \mathbf{N}}$; we denote this space by H and its scalar product by $(\cdot,\cdot)_H$. We also denote by ℓ^2_H the complex Hilbert space of square-summable sequences $\mathbf{N} \to H$.

Theorem 2.6 (generalized Fourier expansion w.r.t. y)**.** *Let $\{u_\varepsilon\}$ be a sequence in $L^2(\mathbf{R}^N; H)$, $u \in L^2(\mathbf{R}^N \times \mathcal{Y}; H)$, define S_ε as in (1.2), and set*

$$a_{n,\varepsilon}(x) := \int_Y (u_\varepsilon(S_\varepsilon(x,y)), \phi_n(y))_H \, \mathrm{d}y, \qquad a_n(x) := \int_Y (u(x,y), \phi_n(y))_H \, \mathrm{d}y$$
$$\text{for a.a. } x \in \mathbf{R}^N, \forall n \in \mathbf{N}, \forall \varepsilon. \tag{2.12}$$

Then

$$u_\varepsilon(x) \underset{2}{\rightharpoonup} u(x,y) \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; H) \quad \Leftrightarrow \quad \{a_{n,\varepsilon}(x)\} \rightharpoonup \{a_n(x)\} \quad \text{in } L^2(\mathbf{R}^N; \ell^2_H), \tag{2.13}$$
$$u_\varepsilon(x) \underset{2}{\to} u(x,y) \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; H) \quad \Leftrightarrow \quad \{a_{n,\varepsilon}(x)\} \to \{a_n(x)\} \quad \text{in } L^2(\mathbf{R}^N; \ell^2_H). \tag{2.14}$$

The examples of (1.22) might be interpreted within this framework.

The statements (2.13) and (2.14) might be reformulated in terms of the (generalized) Fourier expansion of $a_{n,\varepsilon}$ and a_n as functions of x, thus achieving the global Fourier expansion of $u_\varepsilon \circ S_\varepsilon$ and u w.r.t. (x,y).

Proof. The $a_{n,\varepsilon}$'s and the a_n's are the coefficients of the partial Fourier expansion of $u_\varepsilon \circ S_\varepsilon$ and u, resp., in the sense that

$$\begin{aligned} u_\varepsilon(S_\varepsilon(x,y)) &= \sum_{n=0}^{\infty} a_{n,\varepsilon}(x)\phi_n(y) \qquad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; H), \forall \varepsilon, \\ u(x,y) &= \sum_{n=0}^{\infty} a_n(x)\phi_n(y) \qquad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}; H). \end{aligned} \tag{2.15}$$

By definition $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\mathbf{R}^N \times \mathcal{Y}; H)$ iff $u_\varepsilon \circ S_\varepsilon \rightharpoonup u$ in $L^2(\mathbf{R}^N \times \mathcal{Y}; H)$, or equivalently

$$\int_{\mathbf{R}^N} (u_\varepsilon(S_\varepsilon(x,y)), g(x))_H \,\mathrm{d}x \rightharpoonup \int_{\mathbf{R}^N} (u(x,y), g(x))_H \,\mathrm{d}x \qquad \text{in } L^2(\mathcal{Y}), \forall g \in L^2(\mathbf{R}^N; H).$$

Setting $b_{g,n,\varepsilon} := \int_{\mathbf{R}^N} (a_{n,\varepsilon}(x), g(x))_H \,\mathrm{d}x$ and $b_{g,n} := \int_{\mathbf{R}^N} (a_n(x), g(x))_H \,\mathrm{d}x$ for any n, ε, this reads

$$\sum_{n=0}^{\infty} b_{g,n,\varepsilon}\phi_n(y) \rightharpoonup \sum_{n=0}^{\infty} b_{g,n}\phi_n(y) \qquad \text{in } L^2(\mathcal{Y}), \forall g \in L^2(\mathbf{R}^N; H),$$

that is, $\{b_{g,n,\varepsilon}\} \rightharpoonup \{b_{g,n}\}$ in ℓ^2_H for any $g \in L^2(\mathbf{R}^N; H)$. This means that $\{a_{n,\varepsilon}(x)\} \rightharpoonup \{a_n(x)\}$ in $L^2(\mathbf{R}^N; \ell^2_H)$, and (2.13) is thus established.

Let us now come to strong convergence. By (2.15)

$$\|u_\varepsilon \circ S_\varepsilon\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; H)} = \|\{a_{n,\varepsilon}\}\|_{L^2(\mathbf{R}^N; \ell^2_H)} \quad \forall \varepsilon, \quad \|u\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; H)} = \|\{a_n\}\|_{L^2(\mathbf{R}^N; \ell^2_H)}; \tag{2.16}$$

thus

$$\|u_\varepsilon \circ S_\varepsilon\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; H)} \to \|u\|_{L^2(\mathbf{R}^N \times \mathcal{Y}; H)} \quad \Leftrightarrow \quad \|\{a_{n,\varepsilon}\}\|_{L^2(\mathbf{R}^N; \ell^2_H)} \to \|\{a_n\}\|_{L^2(\mathbf{R}^N; \ell^2_H)}. \tag{2.17}$$

This statement and (2.13) entail (2.14). □

Proposition 2.7 (norm semicontinuity and continuity)**.** *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N; B)$. Then*

$$\begin{aligned} &u_\varepsilon \underset{2}{\rightharpoonup} u \ \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Rightarrow \\ &\liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\mathbf{R}^N; B)} \geq \|u\|_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)} \Big(\geq \Big\| \int_Y u(\cdot, y)\,\mathrm{d}y \Big\|_{L^p(\mathbf{R}^N; B)}\Big), \\ &u_\varepsilon \underset{2}{\to} u \ \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Rightarrow \quad \begin{cases} u_\varepsilon \underset{2}{\rightharpoonup} u & \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \\ \|u_\varepsilon\|_{L^p(\mathbf{R}^N; B)} \to \|u\|_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)}. \end{cases} \end{aligned} \tag{2.19}$$

If $p \in]1, +\infty[$ and the space B is uniformly convex, then the latter implication can be inverted.

Proof. By Lemma 1.1, $\|u_\varepsilon\|_{L^p(\mathbf{R}^N; B)} = \|u_\varepsilon \circ S_\varepsilon\|_{L^p(\mathbf{R}^N \times \mathcal{Y}; B)}$ for any ε. It then suffices to recall the definitions of weak and strong two-scale convergence and to apply standard properties. □

Proposition 2.8. *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\mathbf{R}^N; B)$.*

(i) If $u_\varepsilon \underset{2}{\to} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$, then

$$\begin{aligned} &\forall \{v_\varepsilon\} \subset L^{p'}(\mathbf{R}^N; B'), \text{if } v_\varepsilon \underset{2}{\rightharpoonup} v \ \text{ in } \ L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B') \ \ (v_\varepsilon \underset{2}{\overset{*}{\rightharpoonup}} v \ \text{ if } p' = \infty) \\ &\text{then } \int_{\mathbf{R}^N} \langle u_\varepsilon(x), v_\varepsilon(x) \rangle \,\mathrm{d}x \to \iint_{\mathbf{R}^N \times Y} \langle u(x,y), v(x,y) \rangle \,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.20}$$

(ii) If $p = 2$ and B is uniformly convex, conversely (2.20) entails $u_\varepsilon \underset{2}{\to} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$.

For $p = 2$ and $B = \mathbf{R}$, we thus retrieve the definition of strong two-scale convergence of [1].

Proof. Part (i) directly follows from the definitions of weak and strong two-scale convergence.

Let us come to part (ii). For any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}; B')$, by Lemma 2.2 we can take $v_\varepsilon(x) := \psi(x, x/\varepsilon)$ in (2.20). By Proposition 2.5 we then get $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^2(\mathbf{R}^N \times \mathcal{Y}; B)$. Denoting the duality mapping $B \to 2^{B'}$ by F and taking $v_\varepsilon \in F(u_\varepsilon)$ in (2.20), we get $\|u_\varepsilon\|_{L^2(\mathbf{R}^N;B)} \to \|u\|_{L^2(\mathbf{R}^N \times \mathcal{Y};B)}$. By the final statement of Proposition 2.7, we conclude that $u_\varepsilon \underset{2}{\to} u$ in $L^2(\mathbf{R}^N \times \mathcal{Y}; B)$. □

Remark. Part (ii) of Proposition 2.8 holds for any $p \in]1, +\infty[$; this will be proved in a work apart, in the framework of the analysis of the two-scale behaviour of convex functionals.

An analogous characterization holds for *weak* two-scale convergence, and generalizes Proposition 2.5.

Proposition 2.9. *Let $p \in]1, +\infty[$ and $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\mathbf{R}^N; B)$. Then $u_\varepsilon \underset{2}{\rightharpoonup} u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ iff*

$$\begin{gathered} \forall \{v_\varepsilon\} \subset L^{p'}(\mathbf{R}^N; B'), \text{ if } v_\varepsilon \underset{2}{\to} v \ \text{ in } \ L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B'), \text{ then} \\ \int_{\mathbf{R}^N} \langle u_\varepsilon(x), v_\varepsilon(x) \rangle \, \mathrm{d}x \to \iint_{\mathbf{R}^N \times Y} \langle u(x,y), v(x,y) \rangle \, \mathrm{d}x\mathrm{d}y. \end{gathered} \tag{2.21}$$

Proof. The "only if" part is straightforward. To prove the converse, it suffices to choose $v_\varepsilon(x) := \psi(x, x/\varepsilon)$ for any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}; B')$, and then apply Proposition 2.5, since $\psi(x, x/\varepsilon) \underset{2}{\to} \psi(x, y)$ in $L^{p'}(\mathbf{R}^N \times \mathcal{Y}; B')$ by Lemma 2.1. □

2.2. Some sufficient conditions for two-scale convergence in L^p and in C^0

The next statement extends Lemmata 2.1 and 2.2.

Proposition 2.10.

(i) For any sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ ($p \in [1, +\infty[$), defining M_ε as in (2.1),

$$u_\varepsilon \to u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Rightarrow \quad (M_\varepsilon u_\varepsilon)(x, x/\varepsilon) \underset{2}{\to} u(x,y) \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}; B). \tag{2.22}$$

(ii) For any sequence $\{u_\varepsilon\}$ in $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$,

$$u_\varepsilon \to u \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B) \quad \Rightarrow \quad u_\varepsilon(x, x/\varepsilon) \underset{2}{\to} u(x,y) \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}; B). \tag{2.23}$$

Proof.

(i) If $u_\varepsilon \to u$ in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ then $M_\varepsilon u_\varepsilon \to u$ in the same space. Hence

$$\iint_{\mathbf{R}^N \times Y} \|(M_\varepsilon u_\varepsilon)(S_\varepsilon(x,y), y) - u(x,y)\|_B^p \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbf{R}^N \times Y} \|(M_\varepsilon u_\varepsilon)(x,y) - u(x,y)\|_B^p \, \mathrm{d}x\mathrm{d}y \to 0.$$

Thus (2.22) holds.

(ii) By the hypothesis, $u_\varepsilon(\varepsilon \mathcal{N}(x/\varepsilon), y) \to u(x,y)$ locally uniformly in $\mathbf{R}^N \times \mathcal{Y}$. Let us set $v_\varepsilon(x) := u_\varepsilon(x, x/\varepsilon)$ for any $x \in \mathbf{R}^N$. As the function $L_\varepsilon v_\varepsilon$ linearly interpolates the nodal values $\{v_\varepsilon(\varepsilon m + \varepsilon y) = u_\varepsilon(\varepsilon m, y) : m \in \mathbf{Z}^N\}$ w.r.t. the first argument and along the coordinate axes (*cf.* (1.20)), we infer that $L_\varepsilon v_\varepsilon \to u$ locally uniformly in $\mathbf{R}^N \times \mathcal{Y}$. Thus $v_\varepsilon \underset{2}{\to} u$ in $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$. □

Analogous statements for *weak* convergence either in $L^p(\mathbf{R}^N \times \mathcal{Y}; B)$ or in $C^0(\mathbf{R}^N \times \mathcal{Y}; B)$ fail. As a counterexample for both, it suffices to take $u_\varepsilon(x,y) := \cos(2\pi x/\varepsilon)$ for any $(x,y) \in]0,1[\times \mathcal{Y}$. However, the two next proposition easily follows from Lemma 1.1.

Proposition 2.11. *Let $p \in [1, +\infty[$. For any sequences $\{u_\varepsilon\}$ in $L^p(\mathcal{Y}; B)$,*

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^p(\mathcal{Y}; B) \quad \Leftrightarrow \quad u_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} u(y) \quad \text{in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathcal{Y}; B), \tag{2.24}$$

$$u_\varepsilon \to u \quad \text{in } L^p(\mathcal{Y}; B) \quad \Leftrightarrow \quad u_\varepsilon(x/\varepsilon) \underset{2}{\to} u(y) \quad \text{in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathcal{Y}; B). \tag{2.25}$$

This result and (1.11) yield the next statement.

Proposition 2.12. *Let $p, q, r \in [1, +\infty[$ be such that $1/p + 1/q = 1/r$. Let $\{v_\varepsilon\}$ and $\{w_\varepsilon\}$ be sequences in $L^p(\mathbf{R}^N)$ and $L^q(\mathcal{Y}; B)$, resp. Then*

$$\begin{aligned} &v_\varepsilon \to v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \rightharpoonup w \ \text{ in } L^q(\mathcal{Y}; B) \\ &\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} v(x) w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}; B); \end{aligned} \tag{2.26}$$

$$\begin{aligned} &v_\varepsilon \to v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \to w \ \text{ in } L^q(\mathcal{Y}; B) \\ &\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\to} v(x) w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}; B). \end{aligned} \tag{2.27}$$

Remark. An analogous result holds if $\{v_\varepsilon\} \subset L^p(\mathbf{R}^N; B)$ and $\{w_\varepsilon\} \subset L^q(\mathcal{Y})$. On the other hand, still for $p, q, r \in [1, +\infty[$ such that $1/p + 1/q = 1/r$,

$$\begin{aligned} &v_\varepsilon \rightharpoonup v \ \text{ in } L^p(\mathbf{R}^N) \ \text{ and } \ w_\varepsilon \to w \ \text{ in } L^q(\mathcal{Y}) \\ &\not\Rightarrow \ v_\varepsilon(x) w_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} v(x) w(y) \ \text{ in } L^r(\mathbf{R}^N \times \mathcal{Y}). \end{aligned} \tag{2.28}$$

As a counterexample it suffices to take $v_\varepsilon(x) = \cos(2\pi x/\varepsilon)$ for any $x \in \mathbf{R}$, $w_\varepsilon \equiv 1$.

2.3. **Two-scale convolution**

Proposition 2.13. *Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence of $L^p(\Omega)$ and $\{w_\varepsilon\}$ be a sequence of $L^1(\mathbf{R}^N)$ such that*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathcal{Y}), \qquad w_\varepsilon \underset{2}{\to} w \quad \text{in } L^1(\mathbf{R}^N \times \mathcal{Y}). \tag{2.29}$$

Then

$$\begin{aligned} &(u_\varepsilon * w_\varepsilon)(x) := \int_{\mathbf{R}^N} u_\varepsilon(\xi) w_\varepsilon(x - \xi)\, \mathrm{d}\xi \ \underset{2}{\rightharpoonup} \\ &(u ** w)(x, y) := \iint_{\mathbf{R}^N \times Y} u(\xi, \eta) w(x - \xi, y - \eta)\, \mathrm{d}\xi \mathrm{d}\eta \qquad \text{in } L^p(\Omega \times \mathcal{Y}). \end{aligned} \tag{2.30}$$

If moreover $u_\varepsilon \underset{2}{\to} u$ in $L^2(\Omega \times \mathcal{Y})$ then $u_\varepsilon * w_\varepsilon \underset{2}{\to} u ** w$ in $L^2(\Omega \times \mathcal{Y})$.

We shall refer to $u ** w$ as a *two-scale convolution.* This tool may be used for *two-scale regularization.*

Proof. First notice that by (1.1)

$$\begin{aligned} S_\varepsilon(x - \xi, y - \eta) - S_\varepsilon(x, y) + S_\varepsilon(\xi, \eta) \ &= \ \varepsilon[\mathcal{N}((x - \xi)/\varepsilon) - \mathcal{N}(x/\varepsilon) + \mathcal{N}(\xi/\varepsilon)] \\ &=: N_{x,y,\xi,\eta}\, \varepsilon \qquad \forall (x, y), (\xi, \eta) \in \mathbf{R}^N \times Y, \end{aligned} \tag{2.31}$$

and each component of $N_{x,y,\xi,\eta}$ is an element of the set $\{-1,0,1\}$. By Lemma 1.1 and by Fubini's theorem, (2.29) entails that, for any $g \in L^{p'}(\Omega \times \mathcal{Y})$,

$$\begin{aligned}
&\iint_{\mathbf{R}^N \times Y} (u_\varepsilon * w_\varepsilon)(S_\varepsilon(x,y)) g(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad = \iint_{\mathbf{R}^N \times Y} \Big(g(x,y) \int_{\mathbf{R}^N} u_\varepsilon(\xi)\; w_\varepsilon(S_\varepsilon(x,y)-\xi)\,\mathrm{d}\xi \Big) \mathrm{d}x\mathrm{d}y \\
&\quad = \iint_{\mathbf{R}^N \times Y} \Big(g(x,y) \iint_{\mathbf{R}^N \times Y} u_\varepsilon(S_\varepsilon(\xi,\eta))\; w_\varepsilon(S_\varepsilon(x,y) - S_\varepsilon(\xi,\eta))\,\mathrm{d}\xi\mathrm{d}\eta \Big) \mathrm{d}x\mathrm{d}y \\
&\quad = \iint_{\mathbf{R}^N \times Y} \Big(u_\varepsilon(S_\varepsilon(\xi,\eta)) \iint_{\mathbf{R}^N \times Y} w_\varepsilon(S_\varepsilon(x,y) - S_\varepsilon(\xi,\eta)) g(x,y)\,\mathrm{d}x\mathrm{d}y \Big) \mathrm{d}\xi\mathrm{d}\eta \\
&\quad = \iint_{\mathbf{R}^N \times Y} \Big(u_\varepsilon(S_\varepsilon(\xi,\eta)) \iint_{\mathbf{R}^N \times Y} w_\varepsilon(S_\varepsilon(x-\xi,y-\eta) - N_{x,y,\xi,\eta}\varepsilon) g(x,y)\,\mathrm{d}x\mathrm{d}y \Big) \mathrm{d}\xi\mathrm{d}\eta \\
&\quad \rightharpoonup \iint_{\mathbf{R}^N \times Y} \Big(u(\xi,\eta) \iint_{\mathbf{R}^N \times Y} w(x-\xi, y-\eta) g(x,y)\,\mathrm{d}x\mathrm{d}y \Big) \mathrm{d}\xi\mathrm{d}\eta \\
&\quad = \iint_{\mathbf{R}^N \times Y} \Big(g(x,y) \iint_{\mathbf{R}^N \times Y} u(\xi,\eta)\; w(x-\xi,y-\eta)\,\mathrm{d}\xi\mathrm{d}\eta \Big) \mathrm{d}x\mathrm{d}y \\
&\quad = \iint_{\mathbf{R}^N \times Y} (u ** w)(x,y) g(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{in } L^p(\Omega \times \mathcal{Y}).
\end{aligned} \tag{2.32}$$

(2.30) thus holds. To prove the final statement, we replace $g(x,y)$ by $g_\varepsilon(S_\varepsilon(x,y))$ for any weakly two-scale convergent sequence $\{g_\varepsilon\}$ of $L^2(\Omega \times \mathcal{Y})$, and apply part (ii) of Proposition 2.8. □

Remark. The final property of Proposition 2.13 can be extended to any $p \in\,]1,+\infty[$, after the remark that follows Proposition 2.8.

2.4. Two-scale convergence of distributions

Let us assume that $B = \mathbf{R}$, for the sake of simplicity, and denote by $\langle \cdot,\cdot \rangle$ ($\langle\langle \cdot,\cdot \rangle\rangle$, resp.) the duality pairing between $\mathcal{D}(\mathbf{R}^N)$ ($\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, resp.) and its dual space. For any sequence $\{u_\varepsilon\}$ in $\mathcal{D}'(\mathbf{R}^N)$ and any $u \in \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$, we say that u_ε two-scale converges to u in $\mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$ iff

$$\langle u_\varepsilon(x), \psi(x, x/\varepsilon) \rangle \to \langle\langle u(x,y), \psi(x,y) \rangle\rangle \qquad \forall \psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}). \tag{2.33}$$

We similarly define two-scale convergence in the sense of Radon measures, for $\psi \in C^0_c(\mathbf{R}^N \times \mathcal{Y})$. By Proposition 2.5, (2.33) extends the weak two-scale convergence of $L^p(\mathbf{R}^N \times \mathcal{Y})$. In Section 5 we shall define two-scale convergence in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$ in such a way that $\psi(x, x/\varepsilon) \underset{2}{\to} \psi(x,y)$ in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$ for any $\psi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$; in (2.23) we already saw the analogous property for $C^0(\mathbf{R}^N \times \mathcal{Y})$.

For instance, for $N = 1$, fix any $y_0 \in \mathcal{Y}$ and let $\{\varphi_\varepsilon\}$ be a sequence in $L^1(\mathcal{Y})$ such that $\varphi_\varepsilon(y) \rightharpoonup \delta_{y_0}(y)$ (the Dirac measure concentrated at y_0) in $\mathcal{D}'(\mathcal{Y})$. After extending φ_ε to $\mathbf{R}$ by Y-periodicity, it is easy to see that *e.g.*

$$\begin{aligned}
&u_\varepsilon(x) := x\varphi_\varepsilon(x/\varepsilon) \rightharpoonup x \qquad \text{in } \mathcal{D}'(\mathbf{R}), \\
&x\varphi_\varepsilon(x/\varepsilon) \underset{2}{\rightharpoonup} x\delta_{y_0}(y) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y}).
\end{aligned} \tag{2.34}$$

Let us now denote by $\langle \cdot,\cdot \rangle_{\mathcal{Y}}$ the duality pairing between $\mathcal{D}(\mathcal{Y})$ and its dual, and by $\hat{1}$ ($\in \mathcal{D}(\mathcal{Y})$) the function identically equal to 1. As $\langle\langle u(x,y), v(x) \rangle\rangle = \langle \big(\langle u(x,y), \hat{1}(y) \rangle_{\mathcal{Y}}\big), v(x) \rangle$ for any $v \in \mathcal{D}(\mathbf{R}^N)$, we get the following statement, which may be compared with (1.13).

Proposition 2.14. *For any sequence* $\{u_\varepsilon\}$ *in* $\mathcal{D}'(\mathbf{R}^N)$,

$$u_\varepsilon(x) \underset{2}{\rightharpoonup} u(x,y) \quad \text{in } \mathcal{D}'(\mathbf{R}^N \times \mathcal{Y}) \;\; \Rightarrow \;\; u_\varepsilon(x) \rightharpoonup \langle u(x,y), \hat{1}(y) \rangle_{\mathcal{Y}} \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{2.35}$$

It would be unnatural to define two-scale convergence in $\mathcal{D}'(\mathbf{R}^N \times Y)$ (with Y in place of $\mathcal{Y}$) by letting ψ range in $\mathcal{D}(\mathbf{R}^N \times Y)$ in (2.33). For instance, taking $y_0 = 0$ and defining u_ε as in (2.34), this would yield $u_\varepsilon(x) \underset{2}{\rightharpoonup} 0$ in $\mathcal{D}'(\mathbf{R}^N \times Y)$.

Other notions have a natural extension to two-scale convergence, and will be dealt apart.

3. Two-scale compactness

In this section we extend some classic compactness theorems to two-scale convergence in the spaces L^p and C^0. Henceforth we confine ourselves to scalar-valued functions, although most of our results take over to vector-valued functions. We shall say that a sequence $\{u_\varepsilon\}$ is relatively compact iff it is possible to extract a convergent subsequence from any of its subsequences. Theorem 1.3 yields the following result.

Proposition 3.1. *Let $p \in [1, +\infty[$. For any sequence $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N)$,*

if $\{u_\varepsilon\}$ is strongly one-scale relatively compact in $L^p(\mathbf{R}^N)$,
then it is strongly two-scale relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$; (3.1)

if $\{u_\varepsilon\}$ is strongly two-scale relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$,
then it is weakly two-scale relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$; (3.2)

if $\{u_\varepsilon\}$ is weakly two-scale relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$,
then it is weakly one-scale relatively compact in $L^p(\mathbf{R}^N)$. (3.3)

The same holds for $C^0(\mathbf{R}^N)$, and (replacing weak compactness by weak star compactness) for $L^\infty(\mathbf{R}^N)$.

Weak two-scale compactness in L^p

Proposition 3.2. *(i) Let $p \in]1, +\infty]$. Any sequence $\{u_\varepsilon\}$ of $L^p(\mathbf{R}^N)$ is weakly star two-scale relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$ iff it is bounded, hence iff it is weakly (weakly star if $p = \infty$) one-scale relatively compact in $L^p(\mathbf{R}^N)$.*

(ii) Similarly, any sequence of $L^1(\mathbf{R}^N)$ is weakly star two-scale relatively compact in $C^0_c(\mathbf{R}^N \times \mathcal{Y})'$ iff it is bounded, hence iff it is weakly star one-scale relatively compact in $C^0_c(\mathbf{R}^N)'$.

(iii) Any sequence of $L^1(\mathbf{R}^N)$ is weakly two-scale relatively compact in $L^1(\mathbf{R}^N \times \mathcal{Y})$ iff it is weakly one-scale relatively compact in $L^1(\mathbf{R}^N)$.

Proof. For any $p \in [1, +\infty]$, by Lemma 1.1, $\{u_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N)$ iff $\{u_\varepsilon \circ S_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N \times \mathcal{Y})$. Parts (i) and (ii) then follow from the classic Banach-Alaoglu theorem.

If $p = 1$, by the classic de la Vallée Poussin criterion, $\{u_\varepsilon\}$ is weakly relatively compact in $L^1(\mathbf{R}^N)$ iff there exists a Borel function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty, \qquad \sup_\varepsilon \int_{\mathbf{R}^N} \psi(|u_\varepsilon(x)|)\, \mathrm{d}x < +\infty.$$

By (1.5), $\int_{\mathbf{R}^N} \psi(|u_\varepsilon(x)|)\, \mathrm{d}x = \iint_{\mathbf{R}^N \times Y} \psi(|u_\varepsilon(S_\varepsilon(x,y))|)\, \mathrm{d}x\mathrm{d}y$. The property of *$\psi$-boundness* then holds for $\{u_\varepsilon\}$ in $\mathbf{R}^N$ iff it holds for $\{u_\varepsilon \circ S_\varepsilon\}$ in $\mathbf{R}^N \times \mathcal{Y}$, and part (iii) follows. □

By the latter result now we derive a two-scale version of the Chacon *biting lemma, cf. e.g.* [6,10].

Theorem 3.3. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^1(\mathbf{R}^N)$. There exist $u \in L^1(\mathbf{R}^N \times \mathcal{Y})$, a subsequence $\{u_{\tilde\varepsilon}\}$, and a nondecreasing sequence $\{\Omega_k\}$ of measurable subsets of $\mathbf{R}^N$ such that, denoting by $|\cdot|_N$ the N-dimensional Lebesgue measure,*

$$\begin{cases} |\mathbf{R}^N \setminus \Omega_k|_N \to 0 & \text{as } k \to \infty, \\ u_{\tilde\varepsilon}|_{\Omega_k} \underset{2}{\rightharpoonup} u|_{\Omega_k \times Y} & \text{in } L^1(\Omega_k \times \mathcal{Y}), \text{ as } \tilde\varepsilon \to 0, \forall k \in \mathbf{N}. \end{cases} \tag{3.4}$$

Proof. The standard Chacon's biting lemma states that there exist $\hat u \in L^1(\mathbf{R}^N)$, a subsequence that we still denote by u_ε, and a sequence $\{\Omega_k\}$ as above, such that

$$u_\varepsilon|_{\Omega_k} \rightharpoonup \hat u|_{\Omega_k} \qquad \text{in } L^1(\Omega_k), \text{ as } \varepsilon \to 0, \forall k \in \mathbf{N}.$$

Let us denote by $\varepsilon(0)$ the sequence ε, and successively extract subsequences $\varepsilon(1) \supset \varepsilon(2) \supset ...$ as follows. By part (iii) of Proposition 3.2, for any $k \ge 1$ there exist $\tilde u_k \in L^1(\Omega_k \times \mathcal{Y})$ and a subsequence $\varepsilon(k) := \{\varepsilon(k)_n\}_{n\in\mathbf{N}}$ of $\varepsilon(k-1)$ such that $u_{\varepsilon(k)_n}|_{\Omega_k} \underset{2}{\rightharpoonup} \tilde u_k$ in $L^1(\Omega_k \times \mathcal{Y})$ as $n \to \infty$. (Any function defined on a subset of $\mathbf{R}^N$ is here extended to $\mathbf{R}^N$ with vanishing value.)

As a.a. $x \in \mathbf{R}^N$ is element of Ω_k for some k and because of the monotonicity of $\{\Omega_k\}$, by setting $u(x) := \tilde u_k(x)$, u is defined consistently *a.e.* in $\mathbf{R}^N$. Moreover, $u \in L^1(\mathbf{R}^N \times \mathcal{Y})$, as the sequence $\{\|\tilde u_k\|_{L^1(\Omega_k \times \mathcal{Y})}\}$ is uniformly bounded as k ranges in $\mathbf{N}$. Finally, a subsequence $\{u_{\tilde\varepsilon}\}$ as in (3.4) is constructed by applying a diagonalization procedure to the family of sequences $\{\{u_{\varepsilon(k)_n}\} : k \in \mathbf{N}\}$. □

Strong two-scale compactness in L^p. (1.22) and other simple examples show that in L^p the relative strong two-scale compactness is strictly weaker than the relative strong one-scale compactness. Here we provide a sequential version of the classic Riesz compactness theorem. (Analogous sequential versions can also be given for other classic theorems: *e.g.*, the Ascoli-Arzelà theorem, see Lemma 3.6 below, the Dunford-Pettis theorem of weak compactness in L^1, and so on.) By means of this result, we then characterize the relative strong two-scale compactness in L^p, for any $p \in [1, +\infty[$.

Lemma 3.4. *Let $p \in [1, +\infty[$. A sequence $\{f_n\}$ in $L^p(\mathbf{R}^N)$ is strongly relatively compact iff it is bounded and*

$$\int_{\mathbf{R}^N} |f_n(x+h) - f_n(x)|^p \, \mathrm{d}x \to 0 \qquad \text{as } (h, 1/n) \to (0,0), \tag{3.5}$$

$$\sup_{n\in\mathbf{N}} \int_{\mathbf{R}^N \setminus B(0,R)} |f_n(x)|^p \, \mathrm{d}x \to 0 \qquad \text{as } R \to +\infty. \tag{3.6}$$

Proof. $(h, 1/n) \to (0,0)$ means that $h \to 0$ and $n \to \infty$ independently. (3.5) thus reads

$$\forall\delta > 0, \exists\tilde n \in \mathbf{N}, \exists\tilde h > 0 : \forall n > \tilde n, \forall h \in]0, \tilde h[, \quad \int_{\mathbf{R}^N} |f_n(x+h) - f_n(x)|^p \, \mathrm{d}x \le \delta.$$

A priori this inequality might fail for $n \le \tilde n$. However, for any $n \le \tilde n$, $\int_{\mathbf{R}^N} |f_n(x+h) - f_n(x)|^p \, \mathrm{d}x \to 0$ as $h \to 0$; hence there exists $\tilde h_n > 0$ such that the above inequality holds for any $h \in]0, \tilde h_n[$. Settings $\hat h := \min\{\tilde h, \tilde h_1, ..., \tilde h_{\tilde n}\}$, we then get

$$\forall\delta > 0, \exists\hat h > 0 : \forall h \in]0, \hat h[, \forall n \in \mathbf{N}, \quad \int_{\mathbf{R}^N} |f_n(x+h) - f_n(x)|^p \, \mathrm{d}x \le \delta,$$

i.e., $\lim_{h\to 0} \sup_{n\in\mathbf{N}} \int_{\mathbf{R}^N} |f_n(x+h) - f_n(x)|^p \mathrm{d}x = 0$. It then suffices to apply the classic Riesz theorem. □

Theorem 3.5. *Let $p \in [1, +\infty[$. A sequence $\{u_\varepsilon\}$ of $L^p(\mathbf{R}^N)$ is strongly relatively two-scale compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$ iff it is bounded and (defining S_ε as in (1.2))*

$$\int_{\mathbf{R}^N} |u_\varepsilon(x + S_\varepsilon(h,k)) - u_\varepsilon(x)|^p \, \mathrm{d}x \to 0 \qquad \text{as } (h,k,\varepsilon) \to (0,0,0), \tag{3.7}$$

$$\sup_\varepsilon \int_{\mathbf{R}^N \setminus B(0,R)} |u_\varepsilon(x)|^p \, \mathrm{d}x \to 0 \qquad \text{as } R \to +\infty. \tag{3.8}$$

(If we drop the hypothesis (3.8), then $\{u_\varepsilon\}$ is just strongly two-scale relatively compact in the Fréchet space $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathcal{Y})$.)

Proof. By Lemma 3.4, the sequence $\{u_\varepsilon \circ S_\varepsilon\}$ is strongly relatively compact in $L^p(\mathbf{R}^N \times \mathcal{Y})$ iff (3.8) holds and

$$\iint_{\mathbf{R}^N \times Y} |u_\varepsilon(S_\varepsilon(x+h, y+k)) - u_\varepsilon(S_\varepsilon(x,y))|^p \, \mathrm{d}x\mathrm{d}y \to 0 \qquad \text{as } (h,k,\varepsilon) \to (0,0,0). \tag{3.9}$$

Notice that by (1.1)

$$\begin{aligned} S_\varepsilon(x+h, y+k) - S_\varepsilon(x,y) + S_\varepsilon(h,k) &= \varepsilon[\mathcal{N}((x+h)/\varepsilon) - \mathcal{N}(x/\varepsilon) + \mathcal{N}(h/\varepsilon)] \\ &=: \varepsilon N_{x,y,h,k} \quad \forall (x,y), (h,k) \in \mathbf{R}^N \times \mathcal{Y}, \end{aligned} \tag{3.10}$$

and each component of $N_{x,y,\xi,\eta}$ is an element of the set $\{-1, 0, 1\}$. As $(h,k,\varepsilon) \to (0,0,0)$ iff $(h + \varepsilon J, k, \varepsilon) \to (0,0,0)$, (3.9) is then equivalent to

$$\iint_{\mathbf{R}^N \times Y} |u_\varepsilon(S_\varepsilon(x,y) + S_\varepsilon(h,k)) - u_\varepsilon(S_\varepsilon(x,y))|^p \, \mathrm{d}x\mathrm{d}y \to 0 \qquad \text{as } (h,k,\varepsilon) \to (0,0,0),$$

which is in turn equivalent to (3.7), by Lemma 1.1. □

Strong two-scale compactness in C^0. Although this property is strictly weaker than strong one-scale compactness, we can prove a two-scale version of the Ascoli-Arzelà compactness theorem. First we need a sequential version of this classic result, which can be proved mimicking the argument of Lemma 3.4.

Lemma 3.6. *Let K be a compact topological space. A sequence $\{f_n\}$ in $C^0(K)$ is relatively compact iff it is bounded and*

$$\sup_{x \in K} |f_n(x+h) - f_n(x)| \to 0 \qquad \text{as } (h, 1/n) \to (0,0). \tag{3.11}$$

An argument analogous to that of Theorem 3.5, that we omit here, then yields the following result.

Theorem 3.7. *A sequence $\{u_\varepsilon\}$ of $C^0(\mathbf{R}^N)$ is strongly two-scale relatively compact in the Fréchet space $C^0(\mathbf{R}^N \times \mathcal{Y})$ iff it is bounded and*

$$\sup_{x \in K} |u_\varepsilon(x + S_\varepsilon(h,k)) - u_\varepsilon(x)| \to 0 \qquad \text{as } (h,k,\varepsilon) \to (0,0,0), \forall \text{compact } K \subset \mathbf{R}^N. \tag{3.12}$$

Remark. In (3.7) ((3.12), resp.) $S_\varepsilon(h,k) := \varepsilon\mathcal{N}(h/\varepsilon) + \varepsilon k$ cannot be replaced by $h + \varepsilon k$. This would make the hypothesis more restrictive, and would entail the relative strong one-scale compactness of $\{u_\varepsilon\}$ in $L^p(\mathbf{R}^N)$ ($C^0(\mathbf{R}^N)$, resp.).

Two-scale Vitali's theorem

Theorem 3.8. *Let $p \in [1,+\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbf{R}^N)$, such that $\sup_\varepsilon \int_{\mathbf{R}^N \setminus B(0,R)} |u_\varepsilon(x)|^p \,\mathrm{d}x \to 0$ as $R \to +\infty$ and $u_\varepsilon \underset{2}{\to} u$ a.e. in $\mathbf{R}^N \times Y$. Then*

$$u \in L^p(\mathbf{R}^N \times \mathcal{Y}), \qquad u_\varepsilon \underset{2}{\to} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}) \tag{3.13}$$

iff $\{|u_\varepsilon|^p\}$ is equi-integrable, in the sense that, for any sequence $\{A_n\}$ of measurable subsets of $\mathbf{R}^N$,

$$\sup_\varepsilon \int_{A_n} |u_\varepsilon(x)|^p \,\mathrm{d}x \to 0 \qquad \text{as } |A_n|_N \to 0. \tag{3.14}$$

(By ε we still denote the running parameter of a vanishing sequence.)

Proof. By the classic Vitali theorem, (3.13) is equivalent to the equi-integrability of the sequence $\{|u_\varepsilon \circ S_\varepsilon|^p\}$. By the argument of Lemma 3.4, one can see that this is tantamount to

$$\iint_B |u_\varepsilon(S_\varepsilon(x,y))|^p \,\mathrm{d}x\mathrm{d}y \to 0 \qquad \text{as } (|B|_{2N}, \varepsilon) \to (0,0); \tag{3.15}$$

by the same token, (3.14) is equivalent to

$$\int_A |u_\varepsilon(x)|^p \,\mathrm{d}x \to 0 \qquad \text{as } (|A|_N, \varepsilon) \to (0,0). \tag{3.16}$$

By Lemma 1.1, (3.15) is equivalent to

$$\int_{S_\varepsilon(B)} |u_\varepsilon(x)|^p \,\mathrm{d}x \to 0 \qquad \text{as } (|S_\varepsilon(B)|_N, \varepsilon) \to (0,0). \tag{3.17}$$

Now for any measurable set $A \subset \mathbf{R}^N$, $B := S_\varepsilon^{-1}(A)$ is measurable and $|B|_{2N} = |A|_N$, *cf.* Corollary 1.2; hence (3.15) entails (3.16), which is equivalent to (3.14). On the other hand (3.16) entails (3.17), which is equivalent to (3.15), and thus to (3.13). In conclusion (3.13) is equivalent to (3.14). □

4. Two-scale differentiation I

Let $p \in [1,+\infty[$, $w \in W^{1,p}(\mathbf{R}^N \times \mathcal{Y}) \cap \mathcal{F}$ (*cf.* (1.4)), and set $u_\varepsilon(x) := w(x, x/\varepsilon)$ for a.a. x. Although $u_\varepsilon(x) \underset{2}{\to} w(x,y)$ in $L^p(\mathbf{R}^N \times \mathcal{Y})$, in general $\nabla w(x,y)$ is not the (weak) two-scale limit of $\nabla u_\varepsilon(x)$; actually this sequence is bounded in $L^p(\mathbf{R}^N)^N$ only if $w(x,y)$ does not depend from y. In this section we show that nevertheless it is possible to express the derivatives of the two-scale limit without evaluating the limit itself, *via* what we name *approximate two-scale derivatives.*

Preliminarly, for $i = 1, ..., N$, let us denote by $\nabla_i \varphi$ the partial derivative w.r.t. x_i of any function $\varphi(x)$, by $\nabla_{x_i} \psi$ ($\nabla_{y_i} \psi$, resp.) the partial derivative w.r.t. x_i (y_i, resp.) of any function $\psi(x,y)$, by e_i the unit vector of the x_i-axis. Let us also define the shift operator $(\tau_\xi v)(x) := v(x+\xi)$ for any $x, \xi \in \mathbf{R}^N$, set

$$\nabla_{\varepsilon,i} := \frac{\tau_{\varepsilon e_i} - I}{\varepsilon}, \quad \nabla_\varepsilon^\alpha := \prod_{i=1}^N \nabla_{\varepsilon,i}^{\alpha_i}, \quad \nabla^\alpha = \prod_{i=1}^N \nabla_i^{\alpha_i} \qquad \forall \alpha \in \mathbf{N}^N, \forall \varepsilon > 0, \tag{4.1}$$

and define $\nabla_{-\varepsilon}$, ∇_x^α, ∇_y^α similarly. Notice that $\nabla_{-\varepsilon}$ is the adjoint of $-\nabla_\varepsilon$ in $\mathbf{R}^N$, for

$$\textstyle\int_{\mathbf{R}^N} (\nabla_\varepsilon u)\, v \,\mathrm{d}x = -\int_{\mathbf{R}^N} u \nabla_{-\varepsilon} v \,\mathrm{d}x \qquad \forall u, v \in H^1(\mathbf{R}^N). \tag{4.2}$$

After [1] it is known that $\varepsilon\nabla$ approximates ∇_y in the sense of two-scale convergence. We intend to show that ∇_ε approximates ∇_x.

Lemma 4.1. *Let $m \in \mathbf{N}$, $p \in [1,+\infty[$, $w \in W^{m,p}(\mathbf{R}^N \times \mathcal{Y}) \cap \mathcal{F}$ (cf. (1.4)), and set $u_\varepsilon(x) := w(x, x/\varepsilon)$ for any $x \in \mathbf{R}^N$. Then*

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\to} \nabla_x^\alpha \nabla_y^\beta w \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \forall \alpha, \beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m. \tag{4.3}$$

Proof. If $m \ge 1$, for any $\varepsilon > 0$ by the Y-periodicity of w, for $i = 1, ..., N$ and for a.a. $x \in \mathbf{R}^N$ we have

$$\nabla_{\varepsilon,i}\, u_\varepsilon(x) = \frac{w(x + \varepsilon e_i, x/\varepsilon) - w(x, x/\varepsilon)}{\varepsilon} = \frac{1}{\varepsilon}\int_0^\varepsilon (\nabla_{x_i} w)(x + t e_i, x/\varepsilon)\, \mathrm{d}t.$$

Defining S_ε as in (1.2), we then get

$$(\nabla_{\varepsilon,i} u_\varepsilon)(S_\varepsilon(x,y)) = \frac{1}{\varepsilon}\int_0^\varepsilon (\nabla_{x_i} w)(\varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y + t e_i, y)\, \mathrm{d}t \to \nabla_{x_i} w(x,y) \text{ in } L^p(\mathbf{R}^N \times \mathcal{Y}), \forall i,$$

that is, $\nabla_\varepsilon u_\varepsilon \underset{2}{\to} \nabla_x w$ in $L^p(\mathbf{R}^N \times \mathcal{Y})^N$. Moreover,

$$\varepsilon\nabla u_\varepsilon(x) = \varepsilon\nabla_x w(x, x/\varepsilon) + \nabla_y w(x, x/\varepsilon) \underset{2}{\to} \nabla_y w(x,y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y})^N.$$

If $m \ge 2$, this can easily be extended to second-order derivatives; for instance,

$$\begin{aligned} \varepsilon\nabla_j \nabla_{\varepsilon,i}\, u_\varepsilon(x) &= (\varepsilon\nabla_{x_j} + \nabla_{y_j})\, \frac{1}{\varepsilon}\int_0^\varepsilon (\nabla_{x_i} w)(x + \varepsilon y + t e_i, x/\varepsilon)\, \mathrm{d}t \\ &\underset{2}{\to} \nabla_{y_j}\nabla_{x_i} w(x,y) \qquad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \forall i, j. \end{aligned}$$

Similarly, one can check that

$$\nabla_{\varepsilon,i}\nabla_{\varepsilon,j} u_\varepsilon(x) \underset{2}{\to} \nabla_{x_i}\nabla_{x_j} w(x,y), \qquad \varepsilon^2 \nabla_i \nabla_j u_\varepsilon(x) \underset{2}{\to} \nabla_{y_i}\nabla_{y_j} w(x,y).$$

This can easily be extended to higher-order derivatives, too. □

Now we deal with the general case, in which u_ε need not be of the form $u_\varepsilon(x) = w(x, x/\varepsilon)$.

Proposition 4.2. *Let $p \in]1,+\infty[$, and $\alpha, \beta \in \mathbf{N}^N$. If $\{u_\varepsilon\}$ is a sequence in $W^{|\beta|,p}(\mathbf{R}^N)$ and*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}), \qquad \sup_\varepsilon \|\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon\|_{L^p(\mathbf{R}^N)} < +\infty, \tag{4.4}$$

then, denoting by $W^{\beta,p}(\mathcal{Y})$ the Banach space of functions $v : \mathcal{Y} \to \mathbf{R}$ such that $v, \nabla^\beta v \in L^p(\mathcal{Y})$,

$$\nabla_x^\alpha u \in L^p\big(\mathbf{R}^N; W^{\beta,p}(\mathcal{Y})\big), \qquad \nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } L^p(\mathbf{R}^N \times \mathcal{Y}). \tag{4.5}$$

This also holds for $p = \infty$, provided that $\underset{2}{\rightharpoonup}$ is replaced by $\underset{2}{\overset{*}{\rightharpoonup}}$ in (4.4) and (4.5).

Proof. By Proposition 3.1(i), there exists $z \in L^p(\mathbf{R}^N \times \mathcal{Y})$ such that

$$\int_{\mathbf{R}^N} \left[\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon(x)\right] \varphi(x, x/\varepsilon)\, \mathrm{d}x \to \iint_{\mathbf{R}^N \times Y} z(x,y)\varphi(x,y)\, \mathrm{d}x\mathrm{d}y \qquad \forall \varphi \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y}).$$

On the other hand, as the operator $(-1)^{|\alpha|+|\beta|}\nabla^{\alpha}_{-\varepsilon}(\varepsilon\nabla)^{\beta}$ is the adjoint of $\nabla^{\alpha}_{\varepsilon}(\varepsilon\nabla)^{\beta}$, *cf.* (4.2), a formula analogous to (4.3) yields

$$\int_{\mathbf{R}^N}\left[\nabla^{\alpha}_{\varepsilon}(\varepsilon\nabla)^{\beta}u_\varepsilon(x)\right]\varphi(x,x/\varepsilon)\,\mathrm{d}x=(-1)^{|\alpha|+|\beta|}\int_{\mathbf{R}^N}u_\varepsilon(x)\,\nabla^{\alpha}_{-\varepsilon}(\varepsilon\nabla)^{\beta}\varphi(x,x/\varepsilon)\,\mathrm{d}x$$
$$\to(-1)^{|\alpha|+|\beta|}\iint_{\mathbf{R}^N\times Y}u(x,y)\,\nabla^{\alpha}_{x}\nabla^{\beta}_{y}\varphi(x,y)\,\mathrm{d}x\mathrm{d}y=\iint_{\mathbf{R}^N\times Y}\nabla^{\alpha}_{x}\nabla^{\beta}_{y}u(x,y)\,\varphi(x,y)\,\mathrm{d}x\mathrm{d}y.$$

By comparing the two latter formulas we infer that $z=\nabla^{\alpha}_{x}\nabla^{\beta}_{y}u$ *a.e.* in $\mathbf{R}^N\times\mathcal{Y}$. □

Similar results hold for linear differential *vector* operators with constant coefficients. Now we see an example; analogous statements apply to the approximation of $\nabla_x\cdot$ and to the curl operator. First let us set $L^p_{\mathrm{div}}(\mathbf{R}^N)^N:=\left\{v\in L^p(\mathbf{R}^N)^N:\nabla\cdot v\in L^p(\mathbf{R}^N)\right\}$ $(\nabla\cdot:=\mathrm{div})$.

Proposition 4.3. *Let* $p\in\,]1,+\infty[$. *If* $\{u_\varepsilon\}$ *is a sequence in* $L^p_{\mathrm{div}}(\mathbf{R}^N)^N$ *and*

$$u_\varepsilon\underset{2}{\rightharpoonup}u\quad in\ L^p(\mathbf{R}^N\times\mathcal{Y})^N,\qquad\sup_{\varepsilon}\varepsilon\|\nabla\cdot u_\varepsilon\|_{L^p(\mathbf{R}^N)}<+\infty\quad(\nabla\cdot:=\mathrm{div}),\tag{4.6}$$

then

$$\nabla_y\cdot u\in L^p(\mathbf{R}^N\times\mathcal{Y}),\qquad\varepsilon\nabla\cdot u_\varepsilon\underset{2}{\rightharpoonup}\nabla_y\cdot u\quad in\ L^p(\mathbf{R}^N\times\mathcal{Y}).\tag{4.7}$$

This also holds for $p=\infty$, *provided that* $\underset{2}{\rightharpoonup}$ *is replaced by* $\underset{2}{\overset{*}{\rightharpoonup}}$ *in (4.6) and (4.7).*

If the forward incremental ratio, $\nabla_{\varepsilon,i}$, is replaced either by the backward incremental ratio, $\frac{1}{\varepsilon}(I-\tau_{-\varepsilon e_i})$, or by the centered ratio, $\frac{1}{\varepsilon}(\tau_{\varepsilon e_i/2}-\tau_{-\varepsilon e_i/2})$ $(i=1,...,N)$, formula (4.3) and the other results of this section can easily be extended. One might also approximate ∇_x by $\nabla_{(\varepsilon)}v(x):=\int_{\mathcal{Y}}\nabla v(x+\varepsilon\lambda)\,\mathrm{d}\lambda$.

We also define an *approximate two-scale Fréchet differential:* for any $v\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$, at a.a. $x_0\in\mathbf{R}^N$,

$$d_\varepsilon v(x_0):(\mathbf{R}^N)^2\to\mathbf{R}:(h,k)\mapsto v(x_0)+\nabla_\varepsilon v(x_0)\cdot h+\varepsilon\nabla u_\varepsilon(x_0)\cdot k.\tag{4.8}$$

Two-scale boundedness in Sobolev spaces. Let us now define the *approximate two-scale gradient* operator $\Lambda_\varepsilon:=(\nabla_\varepsilon,\varepsilon\nabla)$. For any $p\in[1,+\infty]$ we also say that a sequence $\{u_\varepsilon\}$ is *two-scale bounded in* $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$ whenever $\{u_\varepsilon\}$ and the sequence $\{\Lambda_\varepsilon u_\varepsilon\}$ are bounded in $L^p(\mathbf{R}^N)$ and in $L^p(\mathbf{R}^N)^{2N}$, resp. More generally, one might say that $\{u_\varepsilon\}$ is *two-scale bounded in* $W^{1,p}\big(\mathbf{R}^N;L^p(\mathcal{Y})\big)$ (in $L^p\big(\mathbf{R}^N;W^{1,p}(\mathcal{Y})\big)$, resp.) whenever $\{u_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N)$ and $\{\nabla_\varepsilon u_\varepsilon\}$ ($\{\varepsilon\nabla u_\varepsilon\}$, resp.) is bounded in $L^p(\mathbf{R}^N)^N$.

In $W^{1,p}$ *two-scale* boundedness is strictly weaker than *one-scale* boundedness, at variance with what we saw for L^p. For instance, for any $w\in W^{1,p}(\mathbf{R}^N\times\mathcal{Y})\cap\mathcal{F}$ (*cf.* (1.4)):

(i) the sequence $\{u_\varepsilon(x):=w(x,x/\varepsilon)\}$ of $W^{1,p}(\mathbf{R}^N)$ is *two-scale* bounded in $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$;

(ii) the same sequence is *one-scale* bounded in $W^{1,p}(\mathbf{R}^N)$ only if $w(x,y)$ is independent of y.

Now we see that two-scale boundedness in $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$ entails the relative strong *two-scale* compactness in the Fréchet space $L^p_{\mathrm{loc}}(\mathbf{R}^N\times\mathcal{Y})$; however, the latter example shows that this does not entail the relative strong *one-scale* compactness in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$.

Theorem 4.4 (two-scale Rellich-type theorem)**.** *Let* $p\in[1,+\infty]$. *Any sequence* $\{u_\varepsilon\}$ *of* $W^{1,p}(\mathbf{R}^N)$ *that is two-scale bounded in* $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$ *is strongly two-scale relatively compact in* $L^p_{\mathrm{loc}}(\mathbf{R}^N\times\mathcal{Y})$.

Proof. Let $p \in [1,+\infty[$ (the argument for $p=\infty$ is analogous). Let us fix any $i \in \{1,...,N\}$, any $h \in \mathbf{R}$, and any $k \in [0,1[$. Recalling (1.1) and (1.2), we have

$$\begin{aligned}\|u_\varepsilon(x+S_\varepsilon(he_i,ke_i))-u_\varepsilon(x)\|_{L^p(\mathbf{R}^N)} &= \|u_\varepsilon(x+\varepsilon\hat{n}(h/\varepsilon)e_i+\varepsilon ke_i)-u_\varepsilon(x)\|_{L^p(\mathbf{R}^N)}\\ &\le \|u_\varepsilon(x+\varepsilon\hat{n}(h/\varepsilon)e_i+\varepsilon ke_i)-u_\varepsilon(x+\varepsilon\hat{n}(h/\varepsilon)e_i)\|_{L^p(\mathbf{R}^N)}\\ &\quad +\|u_\varepsilon(x+\varepsilon\hat{n}(h/\varepsilon)e_i)-u_\varepsilon(x)\|_{L^p(\mathbf{R}^N)} =: A_1+A_2,\end{aligned}$$

$$A_1 \le \frac{1}{\varepsilon}\int_0^{\varepsilon k}\|\varepsilon\nabla_i u_\varepsilon(x+\varepsilon\hat{n}(h/\varepsilon)e_i+te_i)\|_{L^p(\mathbf{R}^N)}\,\mathrm{d}t,$$

$$A_2 = \|(\tau_{\varepsilon e_i}-I)^{\hat{n}(h/\varepsilon)}u_\varepsilon\|_{L^p(\mathbf{R}^N)} \le \varepsilon\hat{n}(h/\varepsilon)\|\nabla_{\varepsilon e_i}u_\varepsilon\|_{L^p(\mathbf{R}^N)} \le (h+\varepsilon)\|\nabla_{\varepsilon e_i}u_\varepsilon\|_{L^p(\mathbf{R}^N)}.$$

If $\{u_\varepsilon\}$ is two-scale bounded in $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$ then $A_1+A_2\to 0$ as $(h,k,\varepsilon)\to(0,0,0)$, and (3.7) holds. By Theorem 3.5 the sequence $\{u_\varepsilon\}$ is then relatively compact in the Fréchet space $L^p_{\mathrm{loc}}(\mathbf{R}^N\times\mathcal{Y})$. □

The next formulas easily follow from the definitions (1.20).

Lemma 4.5. *For any $p\in[1,+\infty]$ and any $v\in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N)$,*

$$\begin{aligned}\nabla_{x_i}I_{\varepsilon,i}(v\circ S_\varepsilon) &= (\nabla_{\varepsilon,i}v)\circ S_\varepsilon \;\big(=\nabla_{\varepsilon,i}(v\circ S_\varepsilon)\big)\\ \nabla_{y_i}I_{\varepsilon,i}(v\circ S_\varepsilon) &= I_{\varepsilon,i}[\varepsilon(\nabla_i v)\circ S_\varepsilon]\;\big(=I_{\varepsilon,i}[\nabla_{y_i}(v\circ S_\varepsilon)]\big),\end{aligned}\tag{4.9}$$

$$\begin{aligned}\nabla_{x_i}(I_{\varepsilon,j}\circ I_{\varepsilon,i})(v\circ S_\varepsilon) &= I_{\varepsilon,j}\nabla_{x_i}I_{\varepsilon,i}(v\circ S_\varepsilon) = I_{\varepsilon,j}\nabla_{\varepsilon,i}(v\circ S_\varepsilon)\\ \nabla_{y_i}(I_{\varepsilon,j}\circ I_{\varepsilon,i})(v\circ S_\varepsilon) &= I_{\varepsilon,j}\nabla_{y_i}I_{\varepsilon,i}(v\circ S_\varepsilon) = I_{\varepsilon,j}I_{\varepsilon,i}[\nabla_{y_i}(v\circ S_\varepsilon)].\end{aligned}\tag{4.10}$$

These equalities hold a.e. in $\mathbf{R}^N\times\mathcal{Y}$ and for any $i,j\in\{1,...,N\}$ with $i\neq j$.

The next statement is an easy consequence of these formulas.

Proposition 4.6. *Let $p\in[1,+\infty]$, $\{u_\varepsilon\}$ be a sequence in $W^{1,p}(\mathbf{R}^N)$, and $u\in W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$. Then (using the notation (1.20))*

$$\begin{aligned}&u_\varepsilon \text{ is two-scale bounded in } W^{1,p}(\mathbf{R}^N\times\mathcal{Y})\\ &\Leftrightarrow\; L_\varepsilon u_\varepsilon \text{ is one-scale bounded in this space.}\end{aligned}\tag{4.11}$$

Proof. For the sake of simplicity, let us assume that $N=2$. By (4.9) and (4.10),

$$\begin{aligned}&\nabla_{x_1}L_\varepsilon u_\varepsilon = (J_1\circ J_2\circ I_{\varepsilon,2}\nabla_{\varepsilon,1}u_\varepsilon)\circ S_\varepsilon, \qquad \nabla_{x_2}L_\varepsilon u_\varepsilon = (J_1\circ J_2\circ I_{\varepsilon,1}\nabla_{\varepsilon,2}u_\varepsilon)\circ S_\varepsilon,\\ &\nabla_{y_1}L_\varepsilon u_\varepsilon = \varepsilon(J_1\circ J_2\circ I_{\varepsilon,1}\circ I_{\varepsilon,2}\nabla_{x_1}u_\varepsilon)\circ S_\varepsilon, \qquad \nabla_{y_2}L_\varepsilon u_\varepsilon = \varepsilon(J_1\circ J_2\circ I_{\varepsilon,1}\circ I_{\varepsilon,2}\nabla_{x_2}u_\varepsilon)\circ S_\varepsilon.\end{aligned}\tag{4.12}$$

As the interpolation operators $I_{\varepsilon,i}$'s are bounded, and the composition with S_ε is an isometry (*cf.* Lem. 1.1), (4.11) follows. The extension to $N\neq 2$ is straightforward. □

Two-scale convergence in Euclidean domains. Let Ω be a domain of $\mathbf{R}^N$, and $B(\Omega)$ ($B(\Omega\times\mathcal{Y})$, resp.) be a space (either L^p, or C^0, or $W^{m,p}$, etc.) of functions over Ω (over $\Omega\times\mathcal{Y}$, resp.). Generally speaking, we say that a sequence $\{u_\varepsilon\}$ of functions of $B(\Omega)$ two-scale converges to a function $u\in B(\Omega\times\mathcal{Y})$ iff there exist extensions $\tilde{u}_\varepsilon:\mathbf{R}^N\to\mathbf{R}$ of u_ε and $\tilde{u}:\mathbf{R}^N\times\mathcal{Y}\to\mathbf{R}$ of u, such that $\tilde{u}_\varepsilon$ two-scale converges to $\tilde{u}$ in $B(\mathbf{R}^N\times\mathcal{Y})$. This applies to either weak and strong two-scale convergence. Obviously, the regularity that is natural to assume for the domain Ω depends on the function space under consideration.

Let now Ω be a Lipschitz domain of $\mathbf{R}^N$, denote by ν the outward-oriented, unit, normal vector to $\Gamma:=\partial\Omega$, and by $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. For the sake of simplicity here we assume that $p=2$; however these developments might easily be extended to any $p\in[1,+\infty[$.

We recall the reader that $L^2_{\rm div}(\Omega)^N$ is a Hilbert space equipped with the graph norm; moreover there exists a unique linear and continuous trace operator $\gamma_\nu : L^2_{\rm div}(\Omega)^N \to H^{-1/2}(\Gamma)$ such that $\gamma_\nu u = u|_\Gamma \cdot \nu$ for any $u \in C^1(\bar\Omega)^N$, *cf. e.g.* [22]. A generalized Gauss theorem holds, $\int_\Omega \nabla\cdot u \,{\rm d}x = \langle \gamma_\nu u, 1\rangle_\Gamma$ for any $u \in L^2_{\rm div}(\Omega)^N$, as well as the following formula of *integration by parts:*

$$\int_\Omega (\nabla\cdot u) v \,{\rm d}x + \int_\Omega u\cdot\nabla v \,{\rm d}x = \langle \gamma_\nu u, v\rangle_\Gamma \qquad \forall u \in L^2_{\rm div}(\Omega)^N, \forall v \in H^1(\Omega).$$

Now we extend this formula to two-scale convergence (see also Prop. 4.3).

Proposition 4.7. *Let a sequence* $\{u_\varepsilon\}$ *of* $L^2(\mathbf{R}^N)^N$ *and* $u \in L^2(\mathbf{R}^N \times \mathcal{Y})^N$ *be such that*

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y})^N, \qquad \|\nabla_\varepsilon \cdot u_\varepsilon\|_{L^2(\mathbf{R}^N)} \le \text{ Constant.} \tag{4.13}$$

Then

$$u \in L^2\big(\mathcal{Y}; L^2_{\rm div}(\mathbf{R}^N)^N\big), \qquad \nabla_\varepsilon \cdot u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x \cdot u \quad \text{in } L^2(\mathbf{R}^N \times \mathcal{Y}). \tag{4.14}$$

Moreover, for any bounded domain Ω *of* $\mathbf{R}^N$ *of Lipschitz class and any sequence* $\{v_\varepsilon\}$ *of* $L^2(\mathbf{R}^N)$ *such that* $v_\varepsilon \underset{2}{\to} v$ *in* $L^2(\mathbf{R}^N \times \mathcal{Y})$ *and* $\nabla_\varepsilon v_\varepsilon \underset{2}{\to} \nabla_x v$ *in* $L^2(\mathbf{R}^N \times \mathcal{Y})^N$*, (omitting restrictions)*

$$\begin{aligned} \int_\Omega \big(\nabla_\varepsilon \cdot u_\varepsilon(x)\big)\, v_\varepsilon(x) \,{\rm d}x + \int_\Omega u_\varepsilon(x) \cdot \nabla_\varepsilon v_\varepsilon(x) \,{\rm d}x \to \iint_{\Omega\times Y} \nabla_x \cdot [u(x,y) v(x,y)] \,{\rm d}x{\rm d}y \\ = \int_Y \langle \gamma_\nu u(\cdot,y), v(\cdot,y)\rangle_\Gamma \,{\rm d}y. \end{aligned} \tag{4.15}$$

The latter formula can be applied also if $u_\varepsilon, v_\varepsilon$ are defined just in Ω, after they have been suitably extended to $\mathbf{R}^N$. (An extension is needed, for $(\nabla_\varepsilon \cdot u_\varepsilon)|_\Omega$ and $(\nabla_\varepsilon v_\varepsilon)|_\Omega$ also depend on the values of u_ε and v_ε outside Ω.) An analogous result holds for the curl operator, with a corresponding formula of integration by parts.

Proof. (4.14) can be proved *via* the procedure of Proposition 4.2. Notice that

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) \cdot \nabla_\varepsilon v_\varepsilon(x) \,{\rm d}x = \lim_{\varepsilon\to 0} \sum_i \int_\Omega u_{\varepsilon,i}(x + \varepsilon e_i) \nabla_{\varepsilon,i} v_\varepsilon(x) \,{\rm d}x.$$

By the above-mentioned extention of the Gauss theorem, we then have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \Big(\int_\Omega \big(\nabla_\varepsilon \cdot u_\varepsilon(x)\big)\, v_\varepsilon(x) \,{\rm d}x + \int_\Omega u_\varepsilon(x) \cdot \nabla_\varepsilon v_\varepsilon(x) \,{\rm d}x \Big) \\ &\quad = \lim_{\varepsilon\to 0} \sum_i \int_\Omega \big[\big(\nabla_{\varepsilon,i} u_{\varepsilon,i}(x)\big) v_\varepsilon(x) + u_{\varepsilon,i}(x+\varepsilon e_i) \nabla_{\varepsilon,i} v_\varepsilon(x) \big] \,{\rm d}x \\ &\quad = \lim_{\varepsilon\to 0} \sum_i \int_\Omega \nabla_{\varepsilon,i} (u_{\varepsilon,i} v_\varepsilon)(x) \,{\rm d}x = \iint_{\Omega\times Y} \nabla_x \cdot [u(x,y) v(x,y)] \,{\rm d}x{\rm d}y \\ &\quad = \int_Y \langle \gamma_\nu (uv)(\cdot,y), 1\rangle_\Gamma \,{\rm d}y = \int_Y \langle \gamma_\nu u(\cdot,y), v(\cdot,y)\rangle_\Gamma \,{\rm d}y. \end{aligned}$$

□

5. Two-scale convergence in spaces of differentiable functions

In this section we define two-scale convergence in spaces of either weakly or strongly differentiable functions, by means of the approximate two-scale derivatives ∇_ε and $\varepsilon\nabla$, that we defined in (4.1).

Two-scale convergence in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$**.** Let $m \in \mathbf{N}$ and $p \in [1,+\infty]$. For any sequence $\{u_\varepsilon\}$ in $W^{m,p}(\mathbf{R}^N)$ and any $u \in W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$, we say that u_ε *strongly two-scale converges* to u in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$ iff

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\to} \nabla_x^\alpha \nabla_y^\beta u \qquad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y}), \forall\alpha,\beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m, \tag{5.1}$$

and similarly for weak (weak star if $p = \infty$) two-scale convergence. Strong two-scale convergence in $W^{m,\infty}(\mathbf{R}^N\times\mathcal{Y})$ is rather restrictive, consistently with what we remarked for $L^\infty(\mathbf{R}^N\times\mathcal{Y})$ in Section 1.

Generalizing the definition we gave for $m = 1$ in Section 4, we say that a sequence $\{u_\varepsilon\}$ in $W^{m,p}(\mathbf{R}^N)$ is two-scale bounded in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$ whenever the sequence $\{\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon\}$ is bounded in $L^p(\mathbf{R}^N)$ for any $\alpha, \beta \in \mathbf{N}^N$ such that $|\alpha| + |\beta| \le m$. Propositions 3.2 and 4.2 entail the following result.

Proposition 5.1. *For any $m \in \mathbf{N}$ and any $p \in]1, +\infty]$, any sequence of $W^{m,p}(\mathbf{R}^N)$ that is two-scale bounded in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$ has a weakly (weakly star if $p = \infty$) two-scale convergent subsequence in the latter space.*

One might also define two-scale convergence in fractional Sobolev spaces, but here we omit that issue.

The property (5.1) does not entail that $u_\varepsilon \circ S_\varepsilon \rightharpoonup u$ in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$; actually in general $u_\varepsilon \circ S_\varepsilon \notin W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$. For $m = 1$ however we have the next result.

Proposition 5.2. *Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence in $W^{1,p}(\mathbf{R}^N)$, and $u \in W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$. Then (using the notation (1.20))*

$$u_\varepsilon \underset{2}{\to} u \quad \text{in } W^{1,p}(\mathbf{R}^N\times\mathcal{Y}) \quad \Leftrightarrow \quad L_\varepsilon u_\varepsilon \to u \quad \text{in } W^{1,p}(\mathbf{R}^N\times\mathcal{Y}), \tag{5.2}$$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } W^{1,p}(\mathbf{R}^N\times\mathcal{Y}) \quad \Leftrightarrow \quad L_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}(\mathbf{R}^N\times\mathcal{Y}). \tag{5.3}$$

The latter equivalence also holds for $p = \infty$, with $\underset{2}{\overset{*}{\rightharpoonup}}$ ($\overset{*}{\rightharpoonup}$, resp.) in place of $\underset{2}{\rightharpoonup}$ ($\rightharpoonup$, resp.).

Proof. By Proposition 4.6 any of these convergences entails that u_ε is two-scale bounded in $W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$ and that $L_\varepsilon u_\varepsilon$ is one-scale bounded in the same space. By (4.9) and (4.10), this boundedness entails that

$$\nabla_{x_i} L_\varepsilon v - (\nabla_{\varepsilon,i} v) \circ S_\varepsilon \to 0, \qquad \nabla_{y_i} L_\varepsilon v - (\nabla_{y_i} v) \circ S_\varepsilon \to 0 \qquad \text{in } L^p(\mathbf{R}^N\times\mathcal{Y}), \forall i,$$

for any $v \in W^{1,p}(\mathbf{R}^N\times\mathcal{Y})$. The equivalences (5.2) and (5.3) then follow. □

Weak two-scale convergence in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})'$ (the dual space of $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$). Let us fix any integer $m > 0$, any $p \in [1, +\infty[$, and denote by $\langle\cdot,\cdot\rangle$ ($\langle\langle\cdot,\cdot\rangle\rangle$, resp.) the duality pairing between $W^{m,p}(\mathbf{R}^N)$ ($W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$, resp.) and the respective dual space. For any sequence $\{u_\varepsilon\}$ in $W^{m,p}(\mathbf{R}^N)'$ and any $u \in W^{m,p}(\mathbf{R}^N\times\mathcal{Y})'$, we say that u_ε *weakly two-scale converges* to u in $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})'$ iff

$$\begin{aligned} &\langle u_\varepsilon(x), \psi_\varepsilon(x)\rangle \to \langle\langle u(x,y), \psi(x,y)\rangle\rangle \\ &\forall\{\psi_\varepsilon\} \subset W^{m,p}(\mathbf{R}^N) \text{ such that } \psi_\varepsilon \underset{2}{\to} \psi \ \text{ in } W^{m,p}(\mathbf{R}^N\times\mathcal{Y}). \end{aligned} \tag{5.4}$$

We define the *strong* two-scale convergence in the same space simply by replacing $\psi_\varepsilon \underset{2}{\to} \psi$ with $\psi_\varepsilon \underset{2}{\rightharpoonup} \psi$ in (5.4). The next statement can easily be proved by transposing approximate derivatives, and applying the above definitions of two-scale convergence in the spaces $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})$ and $W^{m,p}(\mathbf{R}^N\times\mathcal{Y})'$.

Proposition 5.3. *For any $p \in]1,+\infty[$, any sequence $\{u_\varepsilon\}$ in $L^{p'}(\mathbf{R}^N)$ and any $u \in L^{p'}(\mathbf{R}^N \times \mathcal{Y})$,*

$$\begin{aligned} &u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^{p'}(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \\ &\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta u_\varepsilon \underset{2}{\rightharpoonup} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } W^{|\alpha|+|\beta|,p}(\mathbf{R}^N \times \mathcal{Y})', \forall \alpha,\beta \in \mathbf{N}^N. \end{aligned} \tag{5.5}$$

Moreover, any bounded sequence in $L^{p'}(\mathbf{R}^N)$ is weakly two-scale relatively compact in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})'$ for any integer $m > 0$.

Two-scale convergence in $C^m(\mathbf{R}^N \times \mathcal{Y})$. For any integer $m > 0$, any sequence $\{u_\varepsilon\}$ in $C^m(\mathbf{R}^N)$ and any $u \in C^m(\mathbf{R}^N \times \mathcal{Y})$, we say that u_ε *strongly two-scale converges* to u in $C^m(\mathbf{R}^N \times \mathcal{Y})$ iff, defining the interpolation operator L_ε as in (1.20),

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon \underset{2}{\to} \nabla_x^\alpha \nabla_y^\beta u \qquad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}), \forall \alpha,\beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m, \tag{5.6}$$

and analogously for weak two-scale convergence. The extension of two-scale convergence to the space of infinitely differentiable functions, $C^\infty(\mathbf{R}^N \times \mathcal{Y})$, is obvious.

Two-scale convergence in $C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$. For any $\lambda \in]0,1]$, any sequence $\{u_\varepsilon\}$ in $C^{0,\lambda}(\mathbf{R}^N)$ and any $u \in C^{0,\lambda}(\mathbf{R}^N \times \mathcal{Y})$, we say that u_ε *strongly two-scale converges* to u in $C^{0,\lambda}(\mathbf{R}^N \times \mathcal{Y})$ iff

$$L_\varepsilon u_\varepsilon \to u \qquad \text{in } C^{0,\lambda}(\mathbf{R}^N \times \mathcal{Y}), \tag{5.7}$$

and analogously for weak and weak star two-scale convergence.

For any integer $m > 0$, any $\lambda \in]0,1]$, any sequence $\{u_\varepsilon\}$ in $C^{m,\lambda}(\mathbf{R}^N)$ and any $u \in C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$, we then say that u_ε *strongly two-scale converges* to u in $C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$ iff

$$\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon \underset{2}{\to} \nabla_x^\alpha \nabla_y^\beta u \qquad \text{in } C^{0,\lambda}(\mathbf{R}^N \times \mathcal{Y}), \forall \alpha,\beta \in \mathbf{N}^N, |\alpha| + |\beta| \le m, \tag{5.8}$$

and analogously for weak and weak star two-scale convergence.

Two-scale convergence can be similarly defined in $C^{m,\lambda}(\overline{\mathbf{R}^N} \times \mathcal{Y})$, namely the Banach space of functions $\mathbf{R}^N \times \mathcal{Y} \to \mathbf{R}$ that are uniformly Hölder-continuous of exponent λ $(\in]0,1])$ with all derivatives up to the order m.

We also say that a sequence $\{u_\varepsilon\}$ of $C^{m,\lambda}(\mathbf{R}^N)$ is two-scale bounded in $C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$ whenever the sequence $\{\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon\}$ is bounded in $C^{0,\lambda}(\mathbf{R}^N)$ for any $\alpha,\beta \in \mathbf{N}^N$ such that $|\alpha| + |\beta| \le m$. The next statement follows from the relative weak star (one-scale) compactness of bounded subsets of Hölder spaces.

Proposition 5.4. *For any $m \in \mathbf{N}$ and any $\lambda \in]0,1]$, any sequence of $C^{m,\lambda}(\mathbf{R}^N)$ that is two-scale bounded in $C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$ has a weakly star two-scale convergent subsequence in the latter space.*

Two-scale convergence in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$. If $\{u_\varepsilon\}$ is a sequence in $\mathcal{D}(\mathbf{R}^N)$ and $u \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, we say that u_ε two-scale converges to u in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$ iff

$$\begin{aligned} &\exists \text{ compact } K \subset \mathbf{R}^N \text{ such that } u_\varepsilon \equiv 0 \text{ in } \mathbf{R}^N \setminus K \text{ for any } \varepsilon, \text{ and} \\ &\nabla_\varepsilon^\alpha (\varepsilon\nabla)^\beta L_\varepsilon u_\varepsilon \underset{2}{\to} \nabla_x^\alpha \nabla_y^\beta u \quad \text{in } C^0(\mathbf{R}^N \times \mathcal{Y}), \forall \alpha,\beta \in \mathbf{N}^N. \end{aligned} \tag{5.9}$$

E.g., for any $w \in \mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$, $w(x, x/\varepsilon) \underset{2}{\to} w(x,y)$ in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$. Similarly, for any $w \in C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$ and any $w \in W^{m,p}(\mathbf{R}^N \times \mathcal{Y}) \cap \mathcal{F}$ (*cf.* (1.4)), $w(x, x/\varepsilon) \underset{2}{\to} w(x,y)$ in the respective space. This justifies the definition (2.33) of two-scale convergence in $\mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$, and allows us to extended to two-scale convergence classic density results, as stated in the next proposition.

Proposition 5.5. *For any $m \in \mathbf{N}$, any $p \in [1,+\infty[$ and any $u \in W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$, there exists a sequence $\{u_\varepsilon\}$ in $\mathcal{D}(\mathbf{R}^N)$ such that $u_\varepsilon \underset{2}{\to} u$ in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$.*

An analogous statement applies to $C^{m,\lambda}(\mathbf{R}^N \times \mathcal{Y})$, as well as to weak two-scale convergence in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})'$ and in $\mathcal{D}'(\mathbf{R}^N \times \mathcal{Y})$.

Proof. For any $u \in W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$ there exists a sequence $\{u_n\}$ in $\mathcal{D}(\mathbf{R}^N \times \mathcal{Y})$ such that $u_n \to u$ in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$. As we saw, $u_n(x, x/\varepsilon) \underset{2}{\to} u_n(x,y)$ in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$ as $\varepsilon \to 0$, for any n. One can then extract a diagonalized sequence $\{\tilde{u}_\varepsilon := u_{n_\varepsilon}\}$ such that $\tilde{u}_\varepsilon(x, x/\varepsilon) \underset{2}{\to} u(x,y)$ in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$ as $\varepsilon \to 0$. The remainder can be proved similarly. □

The relation between two-scale and one-scale convergence in $W^{m,p}(\mathbf{R}^N \times \mathcal{Y})$ is easily established by applying Theorem 1.3 to the approximate two-scale derivatives. Two-scale convergence in the asymmetric spaces

$$C^0\big(\mathcal{Y}; C^{m,\lambda}(\mathbf{R}^N)\big), \quad C^0\big(\mathbf{R}^N; C^{m,\lambda}(\mathcal{Y})\big), \quad L^p\big(\mathcal{Y}; W^{m,p}(\mathbf{R}^N)\big), \quad L^p\big(\mathbf{R}^N; W^{m,p}(\mathcal{Y})\big)$$

might also be defined *via* approximate two-scale derivatives, Λ_ε. Anyway we refrain from adding further generalizations.

Two-scale Sobolev and Morrey imbeddings. We now extend these two classic imbedding theorems to two-scale convergence.

Theorem 5.6.

(i) For any $p \in [1, 2N]$ there exists a positive constant $C_{N,p}$ such that the following occurs for any q such that

$$p \le q \le \frac{2Np}{2N-p} \quad \text{if } p < 2N, \qquad p \le q < +\infty \quad \text{if } p = 2N. \tag{5.10}$$

If a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbf{R}^N)$ is two-scale bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$ then it is bounded in $L^q(\mathbf{R}^N)$, and, defining L_ε as in (1.20),

$$\|u_\varepsilon\|_{L^q(\mathbf{R}^N)} \le C_{N,p} \|L_\varepsilon u_\varepsilon\|_{W^{1,p}(\mathbf{R}^N \times \mathcal{Y})} \qquad \forall \varepsilon. \tag{5.11}$$

(ii) For any $p \in \,]2N, +\infty[$ there exists a positive constant $C_{N,p}$ such that the following occurs for any $\lambda \in \,]0, 1-2N/p]$. If a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbf{R}^N)$ is two-scale bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$ then it is bounded in $C^{0,\lambda}(\overline{\mathbf{R}^N})$, and, defining L_ε as in (1.20),

$$\|u_\varepsilon\|_{C^{0,\lambda}(\overline{\mathbf{R}^N})} \le C_{N,p} \|L_\varepsilon u_\varepsilon\|_{W^{1,p}(\mathbf{R}^N \times \mathcal{Y})} \qquad \forall \varepsilon. \tag{5.12}$$

Proof. By Proposition 4.6, $L_\varepsilon u_\varepsilon$ is one-scale bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$. Under the hypotheses of part (i), the classic Sobolev inequality then yields $\|L_\varepsilon u_\varepsilon\|_{L^q(\mathbf{R}^N \times \mathcal{Y})} \le C \|L_\varepsilon u_\varepsilon\|_{W^{1,p}(\mathbf{R}^N \times \mathcal{Y})}$, for a suitable constant C that only depends on $2N$ (the dimension of $\mathbf{R}^N \times \mathcal{Y}$) and p. By Lemma 1.1, it is easy to see that $\|u_\varepsilon\|_{L^q(\mathbf{R}^N)} \le 2\|L_\varepsilon u_\varepsilon\|_{L^q(\mathbf{R}^N \times \mathcal{Y})}$. Part (i) then follows. The argument of part (ii) is similarly based on the classic Morrrey theorem. □

Notice that the thresholds $\tilde{p} := 2Np/(2N-p)$ (for $p < 2N$) and $\tilde{\lambda} := 1 - 2N/p$ (for $p > 2N$) are smaller than the corresponding *one-scale exponents* prescribed by the classic Sobolev and Morrey theorems in $\mathbf{R}^N$. This is consistent with the fact that two-scale boundedness in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$ is a weaker condition than one-scale boundedness in $W^{1,p}(\mathbf{R}^N)$.

Corollary 5.7.

(i) Let $1 \leq p \leq 2N$, and (5.10) be fulfilled. Then for any sequence $\{u_\varepsilon\}$ in $W^{1,p}(\mathbf{R}^N)$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } W^{1,p}(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \quad u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } L^q(\mathbf{R}^N \times \mathcal{Y}). \tag{5.13}$$

(ii) Let $2N < p < +\infty$ and $0 < \lambda \leq 1 - 2N/p$. Then for any sequence $\{u_\varepsilon\}$ in $W^{1,p}(\mathbf{R}^N)$

$$u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } W^{1,p}(\mathbf{R}^N \times \mathcal{Y}) \quad \Rightarrow \quad u_\varepsilon \underset{2}{\rightharpoonup} u \quad \text{in } C^{0,\lambda}(\overline{\mathbf{R}^N} \times \mathcal{Y}). \tag{5.14}$$

The same applies if weak two-scale convergence is everywhere replaced by strong two-scale convergence.

Proof. Under the hypotheses of part (i), by Proposition 4.6 the sequence $\{L_\varepsilon u_\varepsilon\}$ is (one-scale) bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$. By (5.11) the sequence $\{u_\varepsilon\}$ is then bounded in $L^q(\mathbf{R}^N \times \mathcal{Y})$, and (5.13) follows. (5.14) can be proved similarly.

Let us now assume that $u_\varepsilon \underset{2}{\to} u$ in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$, whence $L_\varepsilon u_\varepsilon \to u$ in the same space, *cf.* (5.2). The classic Sobolev inequality then yields $L_\varepsilon u_\varepsilon \to u$ in $L^q(\mathbf{R}^N \times \mathcal{Y})$, namely $u_\varepsilon \underset{2}{\to} u$ in the latter space. Strong two-scale convergence in (5.14) can be derived similarly. □

Corollary 5.8.

(i) Let $1 \leq p \leq 2N$ and $p \leq q < 2Np/(2N - p)$ (setting $1/0 := +\infty$). For any sequence $\{u_\varepsilon\}$ in $W^{1,p}(\mathbf{R}^N)$ that is two-scale bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$,

$$\{u_\varepsilon\} \text{ is strongly two-scale relatively compact in } L^q_{\mathrm{loc}}(\mathbf{R}^N \times \mathcal{Y}). \tag{5.15}$$

(ii) Let $2N < p < +\infty$ and $0 < \lambda < 1 - 2N/p$. For any sequence $\{u_\varepsilon\}$ in $W^{1,p}(\mathbf{R}^N)$ that is two-scale bounded in $W^{1,p}(\mathbf{R}^N \times \mathcal{Y})$,

$$\{u_\varepsilon\} \text{ is strongly two-scale relatively compact in } C^{0,\lambda}_{\mathrm{loc}}(\mathbf{R}^N \times \mathcal{Y}). \tag{5.16}$$

Proof. We mimic a standard argument of one-scale convergence. Under the hypotheses of part (i), by Theorem 4.4 the sequence $\{u_\varepsilon \circ S_\varepsilon\}$ is strongly relatively compact in the Fréchet space $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathcal{Y})$. By (5.11) $\{u_\varepsilon \circ S_\varepsilon\}$ is weakly relatively compact in $L^r(\mathbf{R}^N \times \mathcal{Y})$ for any $r \in]q, 2Np/(2N-p)[$. By a well-known argument based on the Egorov theorem, $\{u_\varepsilon \circ S_\varepsilon\}$ is then strongly relatively compact in $L^q_{\mathrm{loc}}(\mathbf{R}^N \times \mathcal{Y})$, that is, $\{u_\varepsilon\}$ is strongly two-scale relatively compact in this space. (5.15) is thus established. (5.16) can be proved similarly. □

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