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CONICAL DIFFERENTIABILITY FOR BONE REMODELING CONTACT ROD MODELS *

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Abstract. We prove the conical differentiability of the solution to a bone remodeling contact rod model, for given data (applied loads and rigid obstacle), with respect to small perturbations of the cross section of the rod. The proof is based on the special structure of the model, composed of a variational inequality coupled with an ordinary differential equation with respect to time. This structure enables the verification of the two following fundamental results: the polyhedricity of a modified displacement constraint set defined by the obstacle and the differentiability of the two forms associated to the variational inequality.

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Introduction

We consider a bone remodeling model, for a rod that may come into contact without friction with a rigid obstacle, due to the action of external loads, and we characterize the conical differentiability of the solution to this model, with respect to small variations of the geometry of the cross section of the rod. The knowledge of this conical differentiability is important, for example, in shape optimization bone remodeling problems, where the purpose is to control the geometry of the rod. In this introduction we describe the model and summarize the essential results of this paper.

Let $s \in [0, \delta]$, $\delta > 0$, be a small parameter and $\Omega_s = \omega_s \times]0, L[$ a domain, representing the reference configuration of a rod with cross section $\omega_s \subset \mathbb{R}^2$ and axis length $L > 0$. For each $s \in [0, \delta]$, $\omega_s = \omega + s\theta(\omega)$ is a perturbation of $\omega \subset \mathbb{R}^2$ in the direction of the vector field $\theta = (\theta_1, \theta_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, that is regular enough. Consequently, the set Ω_s is a perturbation of the rod $\Omega = \Omega_0 = \omega \times]0, L[$. Let V be a Hilbert space, representing the admissible displacements of the rod and $K \subset V$ a convex and closed subset of V, defining the constraints imposed on the admissible displacements of the rod. This set K represents the possible contact, without friction, of the rod with the rigid obstacle. Let $< ., >$ denote the duality between V' and V, where V' is the dual of V, let $x = (x_1, x_2, x_3)$ be a generic element of $\overline{\Omega}$, and let t be the time variable in the interval $[0, T]$, with $T > 0$ a

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real positive parameter. Given a function $g^s(x,t)$, depending on $s \in [0,\delta]$ and defined in $\overline{\Omega} \times [0,T]$, we denote by $\dot{g}^s$ and $\partial_i g^s$ its partial derivatives, with respect to time t and to x_i for $i = 1, 2, 3$, respectively.

For each perturbed rod Ω_s, with $s \in [0,\delta]$, the bone remodeling rod model that we consider is the elastic adaptive reduced rod model derived by Figueiredo and Trabucho [5], but with different boundary conditions and additional constraints (we recall that the theory of adaptive elasticity was introduced by Cowin and Hegedus [2, 7] and describes the physiological process of bone remodeling, that is, the continual process of growth, reinforcement, deposition and absorption of material, which occurs in living bone). Moreover, the bone remodeling model that we adopt in this paper, can be mathematically justified by the asymptotic expansion method as in Figueiredo and Trabucho [5] (*cf.* also Trabucho and Viãno [13], for an explanation of the asymptotic expansion method applied to elastic rod contact models), and is defined by the following system, formulated in the set $\overline{\Omega} \times [0,T]$ independent of s (*cf.* (1.9))

$$\left[\begin{array}{l} \text{Find} \quad (u^s,d^s) \quad \text{such that:} \\ u^s = (u_1^s,u_2^s,u_3^s) : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d^s : \overline{\Omega} \times [0,T] \to \mathbb{R}, \\ u^s(.,t) \in K \subset V, \\ a_{d^s}(u^s, v-u^s) \geq \langle L_{d^s}, v-u^s \rangle, \qquad \forall v \in K \subset V, \\ \dot{d}^s = h(s,\theta,d^s,u^s), \quad \text{in} \quad \Omega \times (0,T), \\ d^s(x,0) = \overline{d}^s(x), \quad \text{in} \quad \overline{\Omega}. \end{array}\right. \tag{0.1}$$

The pair (u^s,d^s) is the unknown of the model: the vector field $u^s(.,t)$ represents the displacement of the rod Ω at time t and the scalar field $d^s(.,t)$ is the measure of change in volume fraction of the elastic material of the rod Ω at time t (from a reference volume fraction of elastic material present in the porous bone, identified with the set Ω). The variational inequality, where $a_{d^s}(.,.)$ is a bilinear form and L_{d^s} a linear form that depend on d^s, expresses the equilibrium of the rod Ω under the action of external forces, and subjected to the displacement constraints defined by the set K, that represents the possible contact of the rod with the rigid obstacle. The ordinary differential equation with respect to time, where h is a function that depends on u^s, d^s, θ and s (*cf.* (1.9) and (1.17)), is the so-called remodeling rate equation and models the physiological process of bone remodeling – if $\dot{d}^s$ is positive (respectively negative) it means that the volume fraction of elastic material is increasing (respectively decreasing). The unknowns u^s and d^s are interdependent: the displacement u^s is the solution of the variational inequality and depends on d^s and the unknown d^s depends on u^s and is the solution of the ordinary differential equation with respect to time.

The aim is to analyze the right-derivative of the solution to problem (0.1), with respect to s, at $s = 0$. To compute this derivative we mainly use the regularity hypotheses for the solution to problem (0.1), convenient *a priori* norm bound estimates for the families $\{(u^s,d^s)\}_{s>0}$ and $\{(\frac{u^s-u^0}{s}, \frac{d^s-d^0}{s})\}_{s>0}$, where (u^0,d^0) is the solution to problem (0.1) with $s = 0$, Theorem 4.14 of Sokolowski and Zolesio [12], p. 178 (or equivalentely, Th. 4.30 of Sokolowski and Zolesio [12] p. 210), the Schauder's fixed point theorem and uniqueness results. We remark that, in order to be able to apply the above mentioned Theorem 4.14 of Sokolowski and Zolesio [12], p. 178, we prove the polyhedricity of a modified constraint displacement set, using a technique described in Sokolowski and Zolesio [12], p. 209, and assuming an appropriate additional condition imposed to a non-negative Radon measure, as indicated in Proposition 3.4.

The main theorem of the paper can be formulated as follows.

Theorem 0.1. *For each $t \in [0,T]$, let $A_s(.,t) \in \mathcal{L}(V;V')$ be the linear operator defined by $\langle A_s v, u \rangle = a_{d^s}(v,u)$ for all v, u in V. Then the following three statements i), ii) and iii) are verified.*

i) For each $t \in [0,T]$, there exists $A'(.,t) \in \mathcal{L}(V;V')$ such that

$$\lim_{s\to 0^+} \left\| \left(\frac{A_s - A_0}{s} - A' \right)(.,t) \right\|_{\mathcal{L}(V;V')} = 0. \tag{0.2}$$

ii) For each $t \in [0,T]$, there exists $L'(.,t) \in V'$ such that

$$\lim_{s \to 0^+} \left\| \left(\frac{L_{d^s} - L_{d^0}}{s} - L' \right)(.,t) \right\|_{V'} = 0. \tag{0.3}$$

iii) With the hypothesis of Proposition 3.4, for each time $t \in [0,T]$ the solution $\Pi(L_{d^0})(.,t)$ of the variational inequality

$$\left[\begin{array}{l} u^0(.,t) = \Pi(L_{d^0})(.,t), \\ a_{d^0}(u^0, v - u^0) \geq \langle L_{d^0}, v - u^0 \rangle, \qquad \forall v \in K \subset V, \end{array} \right. \tag{0.4}$$

is conical differentiable at $L_{d^0}(.,t)$, that is

$$\forall l \in V', \qquad \Pi(L_{d^0} + sl)(.,t) = \Pi(L_{d^0})(.,t) + sQ(l)(.,t) + o(s) \tag{0.5}$$

for all $s > 0$, small enough, where for each t, the mapping $Q(.,t) : V' \to V$ is continuous and positively homogeneous and $\frac{\|o(s)\|_V}{s} \to 0$, as $s \to 0^+$.

Consequently, the properties i), ii) and iii) imply that, for each $t \in [0,T]$, the solution $(u^s, d^s)(.,t)$ to the problem (0.1) is right-differentiable with respect to s, at $s = 0$

$$u^s(.,t) = u^0(.,t) + su'(.,t) + o(s), \quad \text{in} \quad V, \quad \text{and} \quad u' = Q(L' - A'u^0), \tag{0.6}$$

$$d^s(.,t) = d^0(.,t) + sd'(.,t) + r(s), \quad \text{in} \quad L^2(\Omega), \tag{0.7}$$

for all $s > 0$, small enough, where (u^0, d^0) is the solution of (0.1) for $s = 0$ and as $s \to 0^+$, $\frac{\|o(s)\|_V}{s} \to 0$ and $\frac{\int_\Omega r(s) v \mathrm{d}\Omega}{s} \to 0$, for any $v \in L^2(\Omega)$.

In particular A' and L' are defined by (2.25) and (2.26), Q is defined by (3.28) and the pair (u', d') is the unique solution of problem (5.1).

Finally let us briefly explain the contents of this paper. In Section 1 we introduce the family of bone remodeling rod models. In Sections 2 and 4 we give partial proofs of the conditions (0.2)–(0.3) and (0.6)–(0.7), respectively. In Section 3 we prove the property (0.5). Finally in Section 5 we completely prove Theorem 0.1.

1. The family of rod models

In this section we introduce some notations, definitions and hypotheses, we define the family of rod models depending on the parameter s, we redefine this family on a set independent of s and finally we give some results concerning the existence and uniqueness of solution.

1.1. Notations, definitions and hypotheses

Let $\delta > 0$ be a small parameter and for each $s \in [0, \delta]$ we consider the perturbation I_s of the identity operator I in $\mathbb{R}^2$, defined by $I_s(x_1, x_2) = (I + s\theta)(x_1, x_2) = (x_{s1}, x_{s2})$, for all $(x_1, x_2) \in \mathbb{R}^2$, where $\theta = (\theta_1, \theta_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field regular enough (at least $\theta \in [W^{2,\infty}(\mathbb{R}^2)]^2$). Let ω be an open, bounded and connected subset of $\mathbb{R}^2$, with a boundary $\partial\omega$ regular enough. For each $s \in [0, \delta]$ we define $\omega_s = I_s(\omega)$, which is the perturbation of ω in the direction of the vector field θ. We also denote by $\overline{\Omega}_s$ the set occupied by a cylindrical rod, in its reference configuration, with length $L > 0$ and cross section ω_s, that is $\overline{\Omega}_s = \overline{\omega}_s \times [0, L] = I_s(\overline{\omega}) \times [0, L] \subset \mathbb{R}^3$. Moreover we denote by $x_s = (x_{s1}, x_{s2}, x_3)$ a generic element of $\overline{\Omega}_s$ and define the sets $\Gamma_s = \partial\omega_s \times]0, L[$, $\Gamma_{s0} = \overline{\omega}_s \times \{0\}$, $\Gamma_{sL} = \overline{\omega}_s \times \{L\}$, where $\partial\omega_s$ is the boundary of ω_s. These three sets represent, respectively, the lateral boundary of the rod $\overline{\Omega}_s$ and its extremities. We assume that the boundary $\partial\omega_s$ is divided into two nonempty disjoint parts denoted by $\partial\omega_{sc}$ and $\partial\omega_{sg}$ and consequently we denote $\Gamma_{sc} = \partial\omega_{sc} \times]0, L[$ and $\Gamma_{sg} = \partial\omega_{sg} \times]0, L[$.

We assume that, for each $s \in [0,\delta]$ the coordinate system (O, x_{s1}, x_{s2}, x_3) is a principal system of inertia associated with the rod Ω_s. Consequently, axis Ox_3 passes through the centroid of each section $\omega_s \times \{x_3\}$ and we have $\int_{\omega_s} x_{s1}\,\mathrm{d}\omega_s = \int_{\omega_s} x_{s2}\,\mathrm{d}\omega_s = \int_{\omega_s} x_{s1}x_{s2}\,\mathrm{d}\omega_s = 0$ (we observe that the choice of the vector field θ, that realizes the shape variation of the cross section ω, must be admissible with this last condition).

The set $C^m(\overline{\Omega}_s)$ stands for the space of real functions m times continuously differentiable in $\overline{\Omega}_s$. The spaces $H^m(\Omega_s) = W^{m,2}(\Omega_s)$ and $W^{0,2}(\Omega_s) = L^2(\Omega_s)$ are the usual Sobolev spaces, where $m \geq 0$ is a positive integer. The norms in these Sobolev spaces are denoted by $\|.\|_{W^{m,2}(\Omega_s)}$.

Throughout the paper, the latin indices i, j, k, l... belong to the set $\{1,2,3\}$, the greek indices α, β, μ... vary in the set $\{1,2\}$ and the summation convention with respect to repeated indices is employed, that is, for example, $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

Let $T > 0$ be a real parameter and we denote by t the time variable in the interval $[0,T]$. If V is a topological vectorial space, the set $C^m([0,T];V)$ is the space of functions $g : t \in [0,T] \to g(t) \in V$, such that g is m times continuously differentiable with respect to t. If V is a Banach space we denote $\|.\|_{C^m([0,T];V)}$ the usual norm in $C^m([0,T];V)$. Moreover, given a function $g_s(x_s,t)$ defined in $\overline{\Omega}_s \times [0,T]$ we denote by $\dot{g}_s$ its partial derivative with respect to time, by $\partial_{s\alpha} g_s$ and $\partial_3 g_s$ its partial derivatives with respect to $x_{s\alpha}$ and x_3, that is, $\dot{g}_s = \frac{\partial g_s}{\partial t}$, $\partial_{s\alpha} g_s = \frac{\partial g_s}{\partial x_{s\alpha}}$ and $\partial_3 g_s = \frac{\partial g_s}{\partial x_3}$.

For each $s \in [0,\delta]$ we consider the following model for the rod Ω_s, that can be mathematically justified by the asymptotic expansion method as in Figueiredo and Trabucho [5].

$$\left[\begin{array}{l} \text{Find} \quad (u_s, d_s) \quad \text{such that:} \\ u_s = (u_{s1}, u_{s2}, u_{s3}) : \overline{\Omega}_s \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R}, \\ u_s(.,t) \in K_s \subset V_s, \\ a_{d_s}(u_s, v_s - u_s) \geq \langle L_{d_s}, v_s - u_s \rangle, \qquad \forall v_s \in K_s \subset V_s, \\ \dot{d}_s = b(d_s) + c(d_s) e_{33}(u_s), \quad \text{in} \quad \Omega_s \times (0,T), \\ d_s(x,0) = \overline{d}_s(x), \quad \text{in} \quad \overline{\Omega}_s. \end{array}\right. \tag{1.1}$$

The unknowns of the model (1.1) are the displacement vector field $u_s(x_s,t)$, corresponding to the displacement of the point x_s of the rod $\overline{\Omega}_s$ at time t and the measure of change in volume fraction of the elastic material (from a reference volume fraction denoted in the sequel by ξ_{s0}) $d_s(x_s,t)$ at (x_s,t). In particular $e_{33}(u_s)$ is an element of the linear strain tensor $\big(e_{ij}(u_s)\big) = \big(\frac{1}{2}(\partial_{si} u_{sj} + \partial_{sj} u_{si})\big)$, and it is a function of u_s.

On the other hand, the data of the model (1.1) are the following: the space V_s of admissible displacements, the set $K_s \subset V_s$ of displacement constraints, the bilinear form $a_{d_s}(.,.) : V_s \times V_s \to \mathbb{R}$ and the element $L_{d_s}(.) \in V'$, that depend on the unknown d_s and represent, respectively, the elastic equilibrium equations and the external forces acting on the rod, the initial value of the change in volume fraction $\overline{d}_s(.) = d_s(.,0)$, and the coefficients $b(d_s)$ and $c(d_s)$ which are material coefficients depending upon the change in volume fraction d_s.

Assuming that the rod is clamped at its extremities $\Gamma_{s0} = \overline{\omega}_s \times \{0\}$ and $\Gamma_{sL} = \overline{\omega}_s \times \{L\}$, the space V_s of admissible displacements is defined by

$$V_s = \left\{ v_s \in \left[W_0^{2,2}(]0,L[) \right]^2 \times W^{1,2}(\Omega_s) : \quad e_{\alpha\beta}(v_s) = e_{3\beta}(v_s) = 0 \right\} \tag{1.2}$$

which is identified with the set

$$\Big\{ v_s = (v_{s1}, v_{s2}, v_{s3}) \in \left[W_0^{2,2}(]0,L[) \right]^2 \times W^{1,2}(\Omega_s) : \quad v_{s\alpha}(x_s) = v_{s\alpha}(x_3), \\ v_{s3}(x_s) = \underline{v}_{s3}(x_3) - x_{s\alpha} \partial_3 v_\alpha(x_3), \quad \underline{v}_{s3} \in W_0^{1,2}(]0,L[) \Big\}, \tag{1.3}$$

that is, $V_s \subset [W^{1,2}(\Omega_s)]^3$ is the space of Bernoulli-Navier displacements. We remark that $W_0^{1,2}(]0,L[) = \{\xi \in W^{1,2}(]0,L[) : \ \xi(0) = \xi(L) = 0\}$, and $W_0^{2,2}(]0,L[) = \{\xi \in W^{2,2}(]0,L[) : \ \xi(0) = \xi(L) = 0, \quad \xi'(0) = \xi'(L) = 0\}$, where ξ' is the first derivative of ξ.

The bilinear form $a_{d_s}(.,.)$ is defined

$$a_{d_s}(u_s, v_s) = \int_{\Omega_s} \frac{1}{b_{3333}(d_s)}\, e_{33}(u_s)\, e_{33}(v_s) \mathrm{d}\Omega_s, \quad \forall u_s,\ v_s \in V_s, \tag{1.4}$$

where $e_{33}(v_s) = \partial_3 v_{s3} = \partial_3 \underline{v}_{s3} - x_{s\alpha}\partial_{33} v_{s\alpha}$ and $b_{3333}(d_s)$ is a material coefficient that depends on d_s (in fact it is an element of the matrix $\big(b_{ijkl}(d_s)\big)$ which is the inverse of the matrix composed of the three-dimensional elastic coefficients of the rod $\overline{\Omega}_s$, as explained in Figueiredo and Trabucho [5]).

The element L_{d_s} is defined by

$$\langle L_{d_s}, v_s\rangle = \int_{\Omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{si}\, v_{si}\, \mathrm{d}\Omega_s + \int_{\Gamma_{sg}} g_{si}\, v_{si}\, \mathrm{d}\Gamma_s, \quad \forall v_s \in V_s, \tag{1.5}$$

where γ is the density of the full elastic material, which is supposed to be a constant independent of s, ξ_{s0} is the reference volume fraction of the elastic material (already mentioned immediately after the definition of the problem (1.1)) that belongs to $C^1(\overline{\Omega}_s)$, $f_s = (f_{si})$ and $g = (g_{si})$ are, respectively, the density of body loads and normal tractions on the lateral boundary Γ_{sg} of the rod $\overline{\Omega}_s$, and $P_\eta(.)$ is a truncation operator. We suppose that $0 < \xi_{s0}^{\min} \le \xi_{s0}(x_s) \le \xi_{s0}^{\max} < 1$, for all $x_s \in \overline{\Omega}_s$, and the truncation operator P_η is of class C^1 and satisfies $0 < \frac{\eta}{2} \le (\xi_{s0} + P_\eta(d_s))(x_s) \le 1$ for all $x_s \in \overline{\Omega}_s$, where $\eta > 0$ is a small parameter. We also assume that $f_{si} \in C^1([0,T])$ and $g_{si} \in C^1([0,T]; W^{1-1/p,p}(\Gamma_{sg}))$, with $p > 3$. These hypotheses of regularity on the forces are necessary to obtain existence results.

The set $K_s \subset V_s$ is a nonempty, closed and convex subset of V_s, representing the additional constraints imposed on the admissible displacements. Due to the action of the applied loads we assume that the lateral surface Γ_{sc} of the rod may come into contact, without friction, with a rigid obstacle. Moreover, we suppose that the candidate contact surface Γ_{sc} is plane and perpendicular to the inertia axis Ox_{s1} of the rod. Therefore, from these assumptions we deduce that the set K_s of the reduced elastic adaptive rod model (1.1) is of the form (*cf.* also Trabucho and Viãno [13], Chap. VI, p. 770 (28.46))

$$K_s = \{v_s \in V_s : \qquad v_{s1} \ge \psi \quad \text{in }]0,L[\} \tag{1.6}$$

where $\psi : [0,L] \to \mathbb{R}$ is a smooth enough scalar function, such that $\psi(x_3) < 0$, for all $x_3 \in [0,L]$. Then the set K_s physically imposes that the bending component v_{s1}, of the admissible displacement v_s, can touch but not penetrate the obstacle represented by the function ψ.

Finally, we suppose that the initial value $\overline{d}_s(.) = d_s(.,0)$ of the change in volume fraction verifies $\overline{d}_s \in C^0(\overline{\Omega}_s)$ and the material coefficients $b(d_s)$, $c(d_s)$ and $b_{3333}(d_s)$ appearing in the right hand side of the remodeling rate equation are continuously differentiable with respect to d_s. In addition we also assume that there exist strictly positive constants C_1, C_2, C_3, C_4, C_5 and C_6 independent of s and t such that for any $(x_s,t) \in \Omega_s \times [0,T]$

$$\begin{aligned} & 0 \le C_1 \le \frac{1}{b_{3333}(d_s)} \le C_2, \qquad \forall s \in [0,\delta], \\ & |b(d_s)| \le C_3, \ \ |b'(d_s)| \le C_4, \ \ |c(d_s)| \le C_5, \ \ |c'(d_s)| \le C_6, \qquad \forall s \in [0,\delta], \end{aligned} \tag{1.7}$$

where $b'(.)$ and $c'(.)$ are the derivatives of the scalar functions $b(.)$ and $c(.)$, respectively.

We observe that we could have considered in (1.1) a remodeling rate equation depending nonlinearly on $e_{33}(u_s)$, that is (*cf.* Figueiredo and Trabucho [5])

$$\dot{d}_s = b(d_s) + c(d_s)e_{33}(u_s) + \frac{1}{b_{3333}(d_s)} e_{33}(u_s)e_{33}(u_s) \tag{1.8}$$

which is an equation that seems to be more suitable to represent the remodeling rate process, from the mechanical view-point, even in the case of small strains (*cf.* Hegedus and Cowin [7]). In fact, all the results of Theorem 0.1 can also be derived for this type of nonlinear remodeling rate equation; the nonlinear term $\frac{1}{b_{3333}(d_s)} e_{33}(u_s) e_{33}(u_s)$ in (1.8) only originates more complicated calculus.

1.2. The family of rod models formulated in Ω

In order to derive the results stated in Theorem 0.1 we reformulate now, for each $s \in [0,\delta]$, the problem (1.1) in the fixed rod $\overline{\Omega}$ independent of s.

We consider the perturbation map I_s defined in Section 1.1 that maps Ω onto Ω_s. For a function v_s defined in Ω_s we associate the corresponding function v^s (with upper index s) defined in Ω by $v^s = v_s o I_s$. Performing this change of variables and observing that $e_{33}(v_s) = e_{33}(v^s) - s\theta_\alpha \partial_{33} v_\alpha^s$, for any $v_s \in V_s$, the problem (1.1) is equivalent to the following problem defined in the rod Ω independent of s

$$\left[\begin{array}{l} \text{Find} \quad (u^s, d^s) \quad \text{such that:} \\ u^s = (u_1^s, u_2^s, u_3^s) : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d^s : \overline{\Omega} \times [0,T] \to \mathbb{R}, \\ u^s(.,t) \in K \subset V, \\ a_{d^s}(u^s, v - u^s) \geq \langle L_{d^s}, v - u^s \rangle, \qquad \forall v \in K, \\ \dot{d^s} = b(d^s) + c(d^s) e_{33}(u^s) - s c(d^s) \theta_\alpha \partial_{33} u_\alpha^s, \qquad \text{in} \quad \Omega \times (0,T), \\ d^s(x,0) = \overline{d}(x), \quad \text{in} \quad \overline{\Omega}, \end{array}\right. \tag{1.9}$$

where we suppose that $\overline{d}$ is independent of $s \in [0,\delta]$, and, for all u and v in V

$$a_{d^s}(u,v) = a_0^s(u,v) + s a_1^s(u,v) + s^2 a_2^s(u,v) + s^3 a_3^s(u,v) + s^4 a_4^s(u,v), \tag{1.10}$$

and

$$\left\{\begin{array}{l} \langle L_{d^s}, v \rangle = F_0^s(v) + G_0^s(v) + s\left(F_1^s(v) + G_1^s(v)\right) \\ \qquad\qquad + s^2\left(F_2^s(v) + G_2^s(v)\right) + s^3\left(F_3^s(v) + G_3^s(v)\right). \end{array}\right. \tag{1.11}$$

The bilinear forms $a_i^s(.,.)$, for $i = 0,1,2,3,4$, are defined by

$$\begin{aligned} a_0^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(u) e_{33}(v) \mathrm{d}\Omega, \\ a_1^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[-\theta_\alpha \big(e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha \big) + e_{33}(u) e_{33}(v)\, div\, \theta \Big] \mathrm{d}\Omega, \\ a_2^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[e_{33}(u) e_{33}(v) \det \nabla\theta + \theta_\alpha\, \theta_\beta\, \partial_{33} u_\alpha\, \partial_{33} v_\beta - (div\,\theta)\, \theta_\alpha \big(e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha \big) \Big] \mathrm{d}\Omega, \\ a_3^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[\theta_\alpha \theta_\beta \partial_{33} u_\alpha \partial_{33} v_\beta\, div\,\theta - (\det \nabla\theta)\, \theta_\alpha \big(e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha \big) \Big] \mathrm{d}\Omega, \\ a_4^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[\theta_\alpha\, \theta_\beta\, \partial_{33} u_\alpha\, \partial_{33} v_\beta\, \det \nabla\theta \Big] \mathrm{d}\Omega. \end{aligned} \tag{1.12}$$

The forms $F_j^s(v)$ and $G_j^s(v)$, for $j = 0, 1, 2, 3$, are defined by

$$\begin{aligned}
F_0^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,(f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\,\mathrm{d}\Omega,\\
F_1^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,\Big[(f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\, div\,\theta - f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\Big]\,\mathrm{d}\Omega,\\
F_2^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,\Big[(f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\,\det\nabla\theta - f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, div\,\theta\Big]\,\mathrm{d}\Omega,\\
F_3^s(v) &= -\int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\,\det\nabla\theta\,\mathrm{d}\Omega,
\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}
G_0^s(v) &= \int_{\Gamma_g} \big(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\big)\,\mathrm{d}\Gamma,\\
G_1^s(v) &= \int_{\Gamma_g} \Big[(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3)\, G_1(\theta, n) - g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\Big]\,\mathrm{d}\Gamma,\\
G_2^s(v) &= \int_{\Gamma_g} \Big[(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3)\, G_2(\theta, n) - g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, G_1(\theta, n)\Big]\,\mathrm{d}\Gamma,\\
G_3^s(v) &= -\int_{\Gamma_g} g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, G_3(\theta, n)\,\mathrm{d}\Gamma,
\end{aligned} \tag{1.14}$$

where $\Gamma = \Gamma_0$, $\Gamma_g = \Gamma_{0g}$, and $G_1(\theta, n)$, $G_2(\theta, n)$, $G_3(\theta, n)$ are bounded scalar functions of θ and n (the unit outer normal vector to the lateral boundary Γ_s for $s = 0$). The space V is a subspace of $[H_0^2(]0, L[)]^2 \times H^1(\Omega) = [W_0^{2,2}(]0, L[)]^2 \times W^{1,2}(\Omega)$ defined by

$$\begin{aligned}
V = \Big\{u \in [H_0^2(]0, L[)]^2 \times H^1(\Omega) :\quad & v = \big(v_1(x_3), v_2(x_3), v_3(x_1, x_2, x_3)\big),\\
& v_3(x_1, x_2, x_3) = \underline{v}_3(x_3) - x_\alpha \partial_3\, v_\alpha(x_3),\ \text{with } \underline{v}_3 \in H_0^1(]0, L[)\Big\}.
\end{aligned} \tag{1.15}$$

We consider that V is equipped with the usual norm of $[H^1(\Omega)]^3$. Finally, the closed convex K is defined by

$$K = \{v \in V :\quad v_1(x_3) \geq \psi(x_3), \quad \text{in} \quad]0, L[\}. \tag{1.16}$$

We remark that if we have considered the remodeling rate equation (1.8), then in (1.9) the ordinary differential equation would be the following

$$\left[\begin{aligned}
\dot{d}^s = {} & c(d^s) e_{33}(u^s) + b(d^s) + \frac{1}{b_{3333}(d^s)} e_{33}(u^s) e_{33}(u^s)\\
& + s\left(\frac{-2}{b_{3333}(d^s)}\theta_\alpha \partial_{33} u_\alpha^s e_{33}(u^s) - c(d^s)\theta_\alpha \partial_{33} u_\alpha^s\right)\\
& + s^2 \frac{1}{b_{3333}(d^s)} (\theta_\alpha \partial_{33} u_\alpha^s)\,(\theta_\beta \partial_{33} u_\beta^s), \qquad \text{in} \quad \Omega \times (0, T).
\end{aligned}\right. \tag{1.17}$$

In the sequel we represent by (u^0, d^0) the solution of problem (1.9) for $s = 0$, that is:

$$\left[\begin{array}{l} \text{Find} \quad (u^0, d^0) \quad \text{such that:} \\ u^0 = (u_1^0, u_2^0, u_3^0) : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d^0 : \overline{\Omega} \times [0,T] \to \mathbb{R}, \\ u^0(.,t) \in K \subset V, \\ a_{d^0}(u^0, v - u^0) \geq \left\langle L_{d^0}, v - u^0 \right\rangle, \qquad \forall v \in K, \\ \dot{d}^0 = b(d^0) + c(d^0) e_{33}(u^0), \quad \text{in} \quad \Omega \times (0,T), \\ d^0(x,0) = \overline{d}(x), \quad \text{in} \quad \overline{\Omega}, \end{array}\right. \tag{1.18}$$

where $a_{d^0}(.,.)$ and $L_{d^0}(.)$ are independent of s and defined by

$$\begin{aligned} a_{d^0}(z,v) &= \int_\Omega \frac{1}{b_{3333}(d^0)}\, e_{33}(z)\, e_{33}(v)\, \mathrm{d}\Omega, \\ L_{d^0}(v) &= F_0(v) + G_0(v), \end{aligned} \tag{1.19}$$

where

$$\begin{aligned} F_0(v) &= \int_\Omega \gamma(\xi_0 + P_\eta(d^0))\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, \mathrm{d}\Omega, \\ G_0(v) &= \int_{\Gamma_g} \left(g_\alpha\, v_\alpha + g_3\, \underline{v}_3\right) \mathrm{d}\Gamma, \end{aligned} \tag{1.20}$$

for all z and v in V, with $f = (f_i)$ and $g = (g_i)$ independent of s. For the case where the remodeling rate equation is defined by (1.17) then for $s = 0$

$$\dot{d}^0 = b(d^0) + c(d^0) e_{33}(u^0) + \frac{1}{b_{3333}(d^0)} e_{33}(u^0) e_{33}(u^0), \quad \text{in} \quad \Omega \times (0,T). \tag{1.21}$$

We also observe that because of the following Korn's type inequality in the space V (*cf.* Ciarlet [1] or Valent [14])

$$\exists c > 0 : \quad \|v\|^2_{[H^1(\Omega)]^3} \leq c\, \|e_{33}(v)\|^2_{L^2(\Omega)}, \quad \forall v \in V, \tag{1.22}$$

where

$$\|e_{33}(v)\|^2_{L^2(\Omega)} = \|\partial_3 \underline{v}_3\|^2_{L^2(0,L)} + \left(\int_\omega x_\alpha^2 \mathrm{d}\omega\right) \|\partial_{33} v_\alpha\|^2_{L^2(0,L)}. \tag{1.23}$$

Then $\|e_{33}(.)\|_{L^2(\Omega)}$ is a norm in the space V, equivalent to the usual norm induced in V by $\|.\|_{[H^1(\Omega)]^3}$. So in the sequel and for all $v \in V$, we denote by $\|v\|_V$ the norm $\|e_{33}(v)\|_{L^2(\Omega)}$ or the norm $\|v\|_{[H^1(\Omega)]^3}$. Moreover, V is a Hilbert space with the norm $\|e_{33}(.)\|_{L^2(\Omega)}$ and for each s, the bilinear form $a_0^s(.,.)$ is continuous and elliptic in V (this statement is also a consequence of the condition imposed on the coefficient $b_{3333}(d^s)$ in (1.7)), that is, there exist positive constants C_1 and C_2 independent of s, for all z and v in V and for all $s \in [0, \delta]$, such that

$$\begin{aligned} &a_0^s(z,v) \leq C_2 \|e_{33}(z)\|_{L^2(\Omega)} \|e_{33}(v)\|_{L^2(\Omega)} = C_2 \|z\|_V \|v\|_V \text{ (continuity)}, \\ &a_0^s(v,v) \geq C_1 \|e_{33}(v)\|^2_{L^2(\Omega)} = C_1 \|v\|^2_V \text{ (ellipticity)}. \end{aligned} \tag{1.24}$$

The existence and uniqueness of solution to the family of bone remodeling rod models defined by (1.9) or (1.18) can be proved using the same arguments of Figueiredo and Trabucho [5] and also Monnier and Trabucho [9]. The proof of existence relies on Schauder's fixed point theorem together with the Cauchy-Lipschitz-Picard theorem (used to solve the remodeling rate equation, for a fixed dispacement), the Stampacchia theorem (that is necessary to guarantee the existence of solution to the variational inequality, for a fixed change of volume fraction) and regularity results. The proof of uniqueness is based on arguments similar to those of Cowin and Nachlinger [3]. The next theorem summarizes this statement of existence and uniqueness.

Theorem 1.1 (Solution of (1.9)). *Let $s \in [0,\delta]$ and we assume that, for each fixed $\hat{d}^s$, the unique solution $\hat{u}^s$ of the equilibrium problem*

$$\begin{cases} \text{Find} \quad \hat{u}^s(.,t) \in K \subset V, \quad \text{such that:} \\ a_{\hat{d}^s}(\hat{u}^s, v - \hat{u}^s) \geq \langle L_{\hat{d}^s}, v - \hat{u}^s \rangle, \quad \forall v \in K, \end{cases} \tag{1.25}$$

has components with the regularity $\hat{u}^s_\alpha(.,t) \in W_0^{2,2}(]0,L[) \cap W^{3,2}(]0,L[)$ and $\underline{\hat{u}}^s_3(.,t) \in W_0^{1,2}(]0,L[) \cap W^{2,2}(]0,L[)$, for any $t \in [0,T]$ (which implies that $\hat{u}^s(.,t) \in W^{2,2}(\Omega)$). Then, there exists a unique pair (u^s, d^s) solution of problem (1.9), verifying

$$u^s \in C^1([0,T];V) \qquad \text{and} \qquad d^s \in C^1\left([0,T]; C^0\left(\overline{\Omega}\right)\right). \tag{1.26}$$

2. Partial Proof of Conditions (0.2) and (0.3)

We prove in this section that the conditions (0.2) and (0.3) are satisfied for sub-families $\{A_{s_j}\}_{j=1}^\infty$ and $\{L_{s_j}\}_{j=1}^\infty$, where $s_j \in [0,\delta]$, of $\{A_s\}_{s>0}$ and $\{L_s\}_{s>0}$. Then, in Section 5 we conclude that (0.2) and (0.3) are true for the entire families $\{A_s\}_{s>0}$ and $\{L_s\}_{s>0}$.

Theorem 2.1. *Let (u^s, d^s) and (u^0, d^0) be the solutions of problem (1.9) and (1.18), respectively. We assume that the conditions (1.7) are verified, and, for each s, $\xi_0^s = \xi_0$, $f_i^s = f_i$, $g_i^s = g_i$, where ξ_0, f_i and g_i are independent of s. Moreover we suppose that there exists a constant $c > 0$, such that $\|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} \leq c$, for all $s \in [0,\delta]$. Then, for each t, there exists a subsequence of $\{(u^s, d^s)(.,t)\}$, denoted by $\{(u^{s_j}, d^{s_j})(.,t)\}$, and elements $\bar{u}(.,t) \in V$ and $\bar{d}(.,t) \in L^2(\Omega)$, such that, when $s_j \to 0^+$*

$$\frac{u^{s_j} - u^0}{s_j}(.,t) \rightharpoonup \bar{u}(.,t) \quad \text{weakly in} \quad V, \tag{2.1}$$

$$e_{33}\left(\frac{u^{s_j} - u^0}{s_j}\right)(.,t) \rightharpoonup e_{33}(\bar{u})(.,t) \quad \text{weakly in} \quad L^2(\Omega), \tag{2.2}$$

$$\frac{d^{s_j} - d^0}{s_j}(.,t) \rightharpoonup \bar{d}(.,t) \quad \text{weakly in} \quad L^2(\Omega), \tag{2.3}$$

$$(u^{s_j} - u^0)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad V, \tag{2.4}$$

$$e_{33}(u^{s_j} - u^0)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad L^2(\Omega), \tag{2.5}$$

$$(d^{s_j} - d^0)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad L^2(\Omega), \tag{2.6}$$

$$e_{33}(u^{s_j} - u^0) \longrightarrow 0 \quad \text{strongly in} \quad C^0([0,T]; C^0(\bar{\Omega})), \tag{2.7}$$

$$d^{s_j} - d^0 \longrightarrow 0 \quad \text{strongly in} \quad C^0\left([0,T]; C^0(\bar{\Omega})\right). \tag{2.8}$$

In addition the limit $\bar{d}$ depends implicity on $\bar{u}$ and is the solution of the following ordinary differential equation with respect to time

$$\left[\begin{array}{l} \dot{\bar{d}} = c(d^0)\, e_{33}(\bar{u}) + \bar{d}\left[c'(d^0) e_{33}(u^0) + b'(d^0)\right] - c(d^0)\, \theta_\alpha\, \partial_{33} u^0_\alpha, \\ \bar{d}(x,0) = 0, \quad \text{in} \quad \overline{\Omega}. \end{array} \right. \tag{2.9}$$

Proof. The proof consists of four steps. The first two steps are preliminary results that prepare the proof of (2.1)–(2.8) in steps 3 and 4.

Step 1. There exist positive constants c_1 and c_2 independent of s, such that

$$\|u^s\|_{C^0([0,T];V)} \leq \|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} \leq c_1, \quad \forall s \in [0,\delta], \tag{2.10}$$

$$\|d^s\|_{C^0([0,T];L^2(\Omega))} \leq c_2, \quad \forall s \in [0,\delta]. \tag{2.11}$$

The estimate (2.10) is a consequence of the hypotheses. Then, taking the integral, with respect to time in the remodeling rate equation we get

$$d^s(x,t) = \int_0^t \Big[c(d^s)e_{33}(u^s) + b(d^s) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha\Big](x,r)\,\mathrm{d}r + \overline{d}(x). \tag{2.12}$$

Then we immediately deduce (2.11) taking the $L^2(\Omega)$ norm in the last equation and using (2.10) and (1.7).
Step 2. There exist positive constants c_3 and c_4 independents of s, such that

$$\left\|\frac{u^s - u^0}{s}\right\|_{C^0([0,T];V)} \leq c_3, \quad \forall s \in [0,\delta], \tag{2.13}$$

$$\left\|\frac{d^s - d^0}{s}\right\|_{C^0([0,T];L^2(\Omega))} \leq c_4, \quad \forall s \in [0,\delta]. \tag{2.14}$$

Choosing $v = u^0$ in problem (1.9) and $v = u^s$ in problem (1.18) and subtracting the two corresponding variational inequalities we obtain

$$a_{d^s}(u^s, u^s - u^0) - a_{d^0}(u^0, u^s - u^0) \leq L_{d^s}(u^s - u^0) - L_{d^0}(u^s - u^0). \tag{2.15}$$

Dividing by s^2 and using the definitions of $a_{d^s}(.,.)$ and L_{d^s} we have

$$\left[\begin{array}{l} \dfrac{a_0^s\left(u^s, \dfrac{u^s-u^0}{s}\right) - a_0^s\left(u^0, \dfrac{u^s-u^0}{s}\right)}{s} + \dfrac{a_0^s\left(u^0, \dfrac{u^s-u^0}{s}\right) - a_{d^0}\left(u^0, \dfrac{u^s-u^0}{s}\right)}{s} \leq \\ \dfrac{\big[(F_0^s + G_0^s) - (F_0 + G_0)\big]}{s}\left(\dfrac{u^s-u^0}{s}\right) - a_1^s\left(u^s, \dfrac{u^s-u^0}{s}\right) + (F_1^s + G_1^s)\left(\dfrac{u^s-u^0}{s}\right) + o(s). \end{array}\right. \tag{2.16}$$

Now using the estimates (2.10)–(2.11), the last inequality yields, for each $t \in [0,T]$

$$\left\{\begin{array}{l} a_0^s\left(\dfrac{u^s-u^0}{s}(.,t), \dfrac{u^s-u^0}{s}(.,t)\right) \leq \\ \qquad c\left\|\dfrac{u^s-u^0}{s}(.,t)\right\|_V + \overline{c}\left\|\dfrac{d^s-d^0}{s}(.,t)\right\|_{L^2(\Omega)}\left\|\dfrac{u^s-u^0}{s}(.,t)\right\|_V + o(s), \end{array}\right. \tag{2.17}$$

where c and $\overline{c}$ are positive constants independent of s and t, and $|o(s)| \to 0$, as $s \to 0^+$. Consequently, because of the ellipticity of $a_0^s(.,.)$, *cf.* (1.24), we have

$$\left\|\frac{u^s-u^0}{s}(.,t)\right\|_V \leq c\left\|\frac{d^s-d^0}{s}(.,t)\right\|_{L^2(\Omega))} + \overline{c}, \tag{2.18}$$

where c and $\overline{c}$ are other positive constants independent of s and t. But subtracting the two remodeling rate equations in problems (1.9) and (1.18), and taking the integral with respect to time we obtain

$$\left\{\begin{array}{l} (d^s - d^0)(x,t) = \displaystyle\int_0^t \Big[c(d^s)e_{33}(u^s - u^0) + [c(d^s) - c(d^0)]e_{33}(u^0) \\ \qquad +[b(d^s) - b(d^0)] - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha\Big]\,(x,r)\,\mathrm{d}r, \end{array}\right. \tag{2.19}$$

and therefore, using (1.7), the mean value theorem for the terms $c(d^s) - c(d^0)$ and $b(d^s) - b(d^0)$, and dividing by s, we obtain

$$\left\|\frac{d^s - d^0}{s}(.,t)\right\|_{L^2(\Omega)} \le \int_0^t \left[c_1 \left\|\frac{u^s - u^0}{s}(.,r)\right\|_V + c_2 \left\|\frac{d^s - d^0}{s}(.,r)\right\|_{L^2(\Omega)} + c_3\right] \mathrm{d}r, \tag{2.20}$$

where c_1, c_2 and c_3 are other positive constants independent of s and t. Using now (2.18) and the integral Gronwall's inequality (*cf.* Evans [4], p. 625) we have (2.14). Then, the property (2.13) is a consequence of (2.14) and (2.18).

Step 3. Because of the norm estimates (2.13)–(2.14) we directly obtain the weak convergences (2.1)–(2.3). The strong convergences (2.4)–(2.6) are a consequence of these weak convergences.

The strong convergence (2.7) is a consequence of (2.5) and the fact that $\partial_3(\underline{u}_3^{s_j} - \underline{u}_3^0)$ and $\partial_{33}(u_\alpha^{s_j} - u_\alpha^0)$ are bounded in the space $C^0([0,T]; W^{1,2}(]0,L[))$ and $W^{1,2}(]0,L[)$ is compactly imbedded in $C^0([0,L])$.

Taking into account the definition of $d^{s_j} - d^0$ given by (2.19), the strong convergence (2.8) is a consequence of (2.7) and the integral's Gronwall inequality.

Step 4. To prove (2.9) we consider in (2.19) $s = s_j$ and we divide by s_j. Then for each t, when $s_j \to 0^+$

$$\left\{\begin{array}{l} c(d^{s_j})(.,t) \longrightarrow c(d^0)(.,t) \quad \text{strongly in} \quad C^0(\overline{\Omega}), \\ e_{33}\left(\dfrac{u^{s_j} - u^0}{s_j}\right)(.,t) \rightharpoonup e_{33}(\bar{u})(.,t) \quad \text{weakly in} \quad L^2(\Omega), \\ \dfrac{c(d^{s_j}) - c(d^0)}{s_j}\, e_{33}(u^0)(.,t) \rightharpoonup \bar{d}\, c'(d^0)\, e_{33}(u^0)(.,t) \quad \text{weakly in} \quad L^2(\Omega), \\ \dfrac{b(d^{s_j}) - b(d^0)}{s_j}(.,t) \rightharpoonup \bar{d}\, b'(d^0)(.,t) \quad \text{weakly in} \quad L^2(\Omega), \\ \partial_{33} u_\alpha^{s_j}(.,t) \longrightarrow \partial_{33} u_\alpha^0(.,t) \quad \text{strongly in} \quad L^2(\Omega). \end{array}\right. \tag{2.21}$$

Hence we conclude that, for each t, and for all $v \in L^2(\Omega)$

$$\left\{\begin{array}{l} \displaystyle\lim_{s_j \to 0^+} \int_\Omega \frac{d^{s_j} - d^0}{s}(.,t)\, v\, \mathrm{d}\Omega = \int_\Omega \left(\int_0^t \Big[c(d^0)e_{33}(\bar{u}) \right. \\ \qquad\qquad \left. +\bar{d}\, c'(d^0)\, e_{33}(u^0) + \bar{d}\, b'(d^0) - c(d^0)\,\theta_\alpha \partial_{33} u_\alpha^0\Big](x,r)\,\mathrm{d}r \right) v\,\mathrm{d}\Omega. \end{array}\right. \tag{2.22}$$

Therefore $\bar{d}(.,t)$ must verify (2.9), since the weak limit is unique. □

Theorem 2.2. *With the hypotheses of the previous Theorem 2.1, there exist $A' = A'_{\bar{d}}$ and $L' = L'_{\bar{d}}$ depending explicitly on $\bar{d}$ and verifying, respectively, the conditions (0.2) and (0.3) for $s = s_j$, that is,*

$$\lim_{s_j \to 0^+} \left\| \left(\frac{A_{s_j} - A_0}{s_j} - A' \right)(.,t) \right\|_{\mathcal{L}(V;V')} = 0, \tag{2.23}$$

$$\lim_{s_j \to 0^+} \left\| \left(\frac{L_{s_j} - L_0}{s_j} - L' \right)(.,t) \right\|_{V'} = 0. \tag{2.24}$$

For any u and v in V, $A'(.,t) \in \mathcal{L}(V', V)$ is defined by

$$\left[\begin{aligned} \langle A'u, v \rangle = & - \int_\Omega b'_{3333}(d^0) \frac{1}{b^2_{3333}(d^0)} \, \bar{d} \, e_{33}(u) e_{33}(v) \mathrm{d}\Omega \\ & + \int_\Omega \frac{-\theta_\alpha}{b_{3333}(d^0)} \, (e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha) \mathrm{d}\Omega \\ & + \int_\Omega \frac{1}{b_{3333}(d^0)} e_{33}(u) e_{33}(v) div\theta \mathrm{d}\Omega, \end{aligned} \right. \tag{2.25}$$

where b'_{3333} is the first derivative of the scalar function b_{3333}. The element $L'(.,t) \in V'$ is defined by

$$\left[\begin{aligned} L'(v) = & \int_\Omega \gamma \, \bar{d} \, P'_\eta(d^0) \, (f_\alpha \, v_\alpha + f_3 \, \underline{v}_3) \, \mathrm{d}\Omega \\ & + \int_\Omega \gamma(\xi_0 + P_\eta(d^0)) \Big[(f_\alpha \, v_\alpha + f_3 \, \underline{v}_3) \, div \, \theta - f_3 \, \theta_\alpha \, \partial_3 v_\alpha \Big] \, \mathrm{d}\Omega \\ & + \int_{\Gamma_g} \Big[(g_\alpha \, v_\alpha + g_3 \, \underline{v}_3) \, G_1(\theta, n) - g_3 \, \theta_\alpha \, \partial_3 v_\alpha \Big] \, \mathrm{d}\Gamma, \end{aligned} \right. \tag{2.26}$$

for any v in V, where P'_η is the first derivative of the scalar function P_η.

Proof. We consider in the sequel $s = s_j$. Using the definitions of A_s and A_0 we obtain

$$\left[\begin{aligned} & \frac{\langle A_s u, v \rangle - \langle A_0 u, v \rangle}{s} = \\ & \frac{a_{d^s}(u,v) - a_{d^0}(u,v)}{s} = \frac{a^s_0(u,v) - a_{d^0}(u,v)}{s} + a^s_1(u,v) + o(s), \end{aligned} \right. \tag{2.27}$$

where $|o(s)|$ tends to zero when $s \to 0^+$.

The calculus of the limit $a^s_1(u,v)$, when $s \to 0^+$ is immediate. To compute the limit $\frac{a^s_0(u,v) - a_{d^0}(u,v)}{s}$, when $s \to 0^+$, we remark that the space $C^\infty_0([0,L])$ is dense in $H^2_0(]0,L[)$ and $H^1_0(]0,L[)$, for the norms $\|.\|_{H^2(]0,L[)}$ and $\|.\|_{H^1(]0,L[)}$, respectively. So by density, we only prove (2.25) for $u \in V$ and $v = (v_1, v_2, \underline{v}_3 - x_\alpha \partial_3 v_\alpha) \in V$, such that, $v_\alpha \in C_0^\infty([0,L])$ and $\underline{v}_3 \in C_0^\infty([0,L])$. Thus, for each $t \in [0,T]$, when $s \to 0^+$, we obtain

$$\left[\begin{array}{l} \dfrac{a_0^s(u,v) - a_{d^0}(u,v)}{s} = \\ \displaystyle\int_\Omega \left(\frac{1}{b_{3333}(d^s)} - \frac{1}{b_{3333}(d^0)}\right) e_{33}(u) e_{33}(v)\,\mathrm{d}\Omega = \\ \displaystyle\int_\Omega \frac{b_{3333}(d^0) - b_{3333}(d^s)}{d^s - d^0} \big(b_{3333}(d^s)\, b_{3333}(d^0)\big)^{-1} \frac{d^s - d^0}{s}\, e_{33}(u)\, e_{33}(v)\,\mathrm{d}\Omega \end{array}\right. \tag{2.28}$$

$$\downarrow$$

$$-\int_\Omega b'_{3333}(d^0)\, b_{3333}(d^0)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\,\mathrm{d}\Omega,$$

because $e_{33}(u)\, e_{33}(v) \in L^2(\Omega)$,

$$\frac{b_{3333}(d^0) - b_{3333}(d^s)}{d^s - d^0}(.,t) \longrightarrow b'_{3333}(d^0)(.,t), \quad \text{in} \quad C^0(\overline{\Omega}), \tag{2.29}$$

and $\frac{d^s - d^0}{s}(.,t)$ converges weakly to $\bar{d}(.,t)$ in $L^2(\Omega)$. Therefore (2.25) is proved.

Applying the definitions of L_{d^s} and L_{d^0} we get

$$\left[\begin{array}{l} \dfrac{L_{d^s}(v) - L_{d^0}(v)}{s} = \\ \dfrac{F_0^s(v) + G_0^s(v) - F_0(v) - G_0(v)}{s} + F_1^s(v) + G_1^s(v) + o(s), \end{array}\right. \tag{2.30}$$

where $|o(s)|$ tends to zero when $s \to 0^+$. So we obtain (2.26) by taking the limit in the latter expressions when $s \to 0^+$, using the definitions of F_0^s, G_0^s, F_1^s, G_1^s, and remarking that

$$\frac{F_0^s + G_0^s - F_0 - G_0}{s}(v) \longrightarrow \int_\Omega \gamma \bar{d} P'_\eta(d^0)(f_\alpha v_\alpha + f_3 \underline{v}_3)\mathrm{d}\Omega. \tag{2.31}$$

□

So we conclude that the conditions (0.2) and (0.3) are proved for $s = s_j$.

3. Proof of Condition (0.5)

We show that condition (0.5) is verified, using a technique described in Sokolowski and Zolesio [12] p. 209, that consists in proving the polyhedricity of a modified constraint displacement set, and assuming an appropriate additional condition imposed to a non-negative Radon measure, as indicated in Proposition 3.4.

We consider the closed and convex subset S of $H_0^2(]0,L[)$ defined by

$$S = \{\varphi \in H_0^2(]0,L[) : \quad \varphi(x_3) \geq \psi(x_3) \quad \text{in} \quad [0,L]\} \tag{3.1}$$

and the operator

$$\begin{array}{rccl} R: & V & \longrightarrow & H_0^2(]0,L[) \\ & v = (v_1, v_2, v_3) & \longrightarrow & R(v) = v_1. \end{array} \tag{3.2}$$

It is clear that the constraint set K verifies

$$K = \{v \in V : \quad R(v) \in S\}. \tag{3.3}$$

Moreover since R maps V onto $H_0^2(]0,L[)$ and $0 \in S \subset H_0^2(]0,L[)$, we have $KerR = KerR \cap K$, where $KerR = \{v \in V : \ Rv = 0\}$. In addition $V = KerR \oplus (KerR)^{\perp}$, where $(KerR)^{\perp} = \{v \in V : \ a_{d^0}(v,u) = 0, \ \forall u \in KerR\}$. The next proposition defines the operator $R^{-1} \in \mathcal{L}(H_0^2(]0,L[), (KerR)^{\perp})$, which is the right inverse of R, that is, $R \circ R^{-1} = id_{H_0^2(]0,L[)}$.

Proposition 3.1. *The operator R^{-1} is defined by*

$$R^{-1}(\varphi) = (\varphi, v_2, \underline{v}_3 - x_1\partial_3\varphi - x_2\partial_3 v_2) = v + u, \qquad \forall \varphi \in H_0^2(]0,L[), \tag{3.4}$$

where $v = (0, v_2, \underline{v}_3 - x_2\partial_3 v_2)$ is the element of $KerR$ solution of the equation

$$a_{d^0}(v,z) = -a_{d^0}(u,z), \qquad \forall z \in KerR, \tag{3.5}$$

and $u = (\varphi, 0, -x_1\partial_3\varphi)$.

Proof. We define $R^{-1}(\varphi)$ by (3.4), because $R \circ R^{-1}(\varphi) = \varphi$ and $R^{-1}(\varphi)$ must be in V. Moreover, as $R^{-1}(\varphi)$ must be in $(KerR)^{\perp}$ we impose

$$a_{d^0}(R^{-1}(\varphi), z) = 0, \qquad \forall z \in KerR. \tag{3.6}$$

This is equivalent to find a $v = (0, v_2, \underline{v}_3 - x_2\partial_3 v_2) \in KerR$, such that $a_{d^0}(v+u, z) = 0$, for all $z \in KerR$, where $u = (\varphi, 0, -x_1\partial_3\varphi)$. Hence (3.5) is an immediate consequence of the linearity of $a_{d^0}(.,.)$ with respect to the first component. □

Obviously we can define a scalar product $((.,.))$ in $H_0^2(]0,L[)$ in the following way

$$((\zeta,\xi)) = a_{d^0}(R^{-1}\zeta, R^{-1}\xi), \qquad \forall \zeta, \ \xi \in H_0^2(]0,L[), \tag{3.7}$$

and the orthogonal projection P_S associated to this new scalar product is defined by

$$\begin{aligned} P_S : H_0^2(]0,L[) &\to S \subset H_0^2(]0,L[) \\ \xi &\to P_S(\xi) \end{aligned} \tag{3.8}$$

where $\varphi = P_S(\xi)$ is the unique solution of the following variational inequality

$$\left[\begin{array}{l} \varphi = P_S(\xi) \in S: \\ ((\varphi - \xi, \zeta - \varphi)) \geq 0, \qquad \forall \zeta \in S. \end{array} \right. \tag{3.9}$$

Then, accordingly to Sokolowski and Zolesio [12], p. 209, for each $t \in [0,T]$, the unique solution $\Pi(L_{d^0})(.,t) = u^0(.,t)$ of the variational inequality

$$\left[\begin{array}{l} u^0(.,t) \in K \subset V, \\ a_{d^0}(u^0, v - u^0) \geq \langle L_{d^0}, v - u^0 \rangle, \qquad \forall v \in K, \end{array} \right. \tag{3.10}$$

of problem (1.18), satisfies

$$\left[\begin{array}{l} \Pi(L_{d^0})(.,t) = \Upsilon(L_{d^0})(.,t) + R^{-1}P_S(\Phi(L_{d^0}))(.,t), \\ \text{with} \quad \Upsilon(L_{d^0})(.,t) \in KerR, \quad \text{and} \quad R^{-1}P_S(\Phi(L_{d^0}))(.,t) \in (KerR)^{\perp}. \end{array} \right. \tag{3.11}$$

For any $l \in V'$, the operator $\Upsilon : V' \to KerR$ is defined by

$$\left[\begin{array}{l} \Upsilon(l) \in KerR \subset V : \\ a_{d^0}(\Upsilon(l), z) = \langle l, z \rangle, \qquad \forall z \in KerR \subset V, \end{array} \right. \tag{3.12}$$

and the operator $\Phi : V' \to H_0^2(]0, L[)$ is defined as follows

$$\begin{bmatrix} \Phi(l) \in H_0^2(]0,L[) : \\ ((\Phi(l), \varphi)) = \langle l, R^{-1}\varphi\rangle, \qquad \forall \varphi \in H_0^2(]0,L[). \end{bmatrix} \tag{3.13}$$

Due to the decomposition (3.11) and also because the mappings Υ, R^{-1} and Φ are linear and continuous we immediately conclude that, for each $t \in [0,T]$, Π is conically differentiable at $L_{d^0}(.,t)$, *cf.* (0.5), if and only if, P_S is conically differentiable at $\Phi(L_{d^0})(.,t)$.

We prove now that the orthogonal projection P_S, with respect to the scalar product $((.,.))$ defined in (3.7), is conical differentiable.

It is well known that the polyhedricity of the set S at a given point $\varphi \in S$ implies the conical differentiability of P_S at φ. For convenience of the reader we include in the paper the next statement, that recalls the definition of polyhedric set and the relation between polyhedricity and conical differentiability, applied to the set S and the projection P_S (*cf.* Haraux [6], or Mignot [8], or Rao and Sokolowski [11]).

Proposition 3.2. *The set $S \subset H_0^2(]0,L[)$ is polyhedric at $\varphi \in S$, if for any $\xi \in H_0^2(]0,L[)$, such that $\varphi = P_S(\xi)$ it follows*

$$T_S(\varphi) \cap [\varphi - \xi]^\perp = \overline{C_S(\varphi) \cap [\varphi - \xi]^\perp}, \tag{3.14}$$

where $^\perp$ denotes the orthogonal with respect to the inner product $((.,.))$, the closure is in the space $H_0^2(]0,L[)$, $C_S(\varphi)$ is the convex cone defined by

$$C_S(\varphi) = \{\zeta \in H_0^2(]0,L[) : \quad \exists_{r>0}, \quad \varphi(x_3) + r\zeta(x_3) \geq \psi(x_3) \quad in \quad]0,L[\}, \tag{3.15}$$

and $T_S(\varphi) = \overline{C_S(\varphi)}$ is the tangent cone to S at $\varphi \in S$, that is, the closure in the space $H_0^2(]0,L[)$ of the convex cone $C_S(\varphi)$.

If condition (3.14) is satisfied, for a pair (φ, ξ) in the space $H_0^2(]0,L[) \times H_0^2(]0,L[)$, with $\varphi = P_S(\xi)$, then for all $\zeta \in H_0^2(]0,L[)$ and for $s > 0$ small enough

$$P_S(\xi + s\zeta) = P_S(\xi) + sP_M(\zeta) + o(s) \quad and \quad M = T_S(\varphi) \cap [\varphi - \xi]^\perp, \tag{3.16}$$

where P_M is the orthogonal projection on M, and $\|o(s)\|_{H^2(]0,L[)}/s \to 0$ as $s \to 0$. The condition (3.16) means that P_S is conical differentiable at $\varphi \in S$.

Thus to conclude that P_S is conical differentiable at a point $\varphi \in S$ it is enough to provide sufficient conditions under which the set S is polyhedric at a point $\varphi \in S$. These sufficient conditions are summarized in the next proposition.

Proposition 3.3. *The set S is polyhedric at a point $\varphi \in S$, if the Radon measure μ defined by*

$$((\varphi - P_S(\varphi), \zeta)) = -\int_0^L \zeta \mathrm{d}\mu, \qquad \forall \zeta \in C_0^\infty(]0,L[) \tag{3.17}$$

is non-negative and its support denoted by suppμ, that is a compact subset of $[0,L]$ and verifies supp$\mu \subset \Xi_\psi = \{x_3 \in]0,L[: \ \varphi(x_3) = \psi(x_3)\}$, is admissible in the following sense

$$\begin{cases} \forall \zeta \in H_0^2(]0,L[), \quad such\ that \quad \zeta = 0 \quad C_2 - q.e\ on\ supp\mu, \\ implies\ that \quad \zeta \in H_0^2(]0,L[\backslash supp\mu). \end{cases} \tag{3.18}$$

In consequence the set M defined in (3.16) is the following convex cone

$$M = \Big\{\zeta \in H_0^2(]0,L[\backslash supp\mu) : \quad \zeta(x_3) \geq 0, \quad C_2 - q.e.\ on \quad \Xi_\psi\Big\}. \tag{3.19}$$

(Note – we recall that a statement holds $C_2 - q.e.$ if it holds except for a set of C_2-capacity zero, where the C_2-capacity of a compact set N, $C_2(N)$, is defined by $C_2(N) = \inf\left\{\int_0^L |\partial_{33}\zeta(x_3)|^2\, \mathrm{d}x_3 : \quad \zeta \geq 1 \text{ on } N, \quad 0 \leq \zeta \in C_0^\infty(]0,L[)\right\}$.)

Proof. We first prove the two following statements i) and ii):
i) the scalar product $((.,.))$ is equivalent to the usual scalar product $(.,.)$ defined in $H_0^2(]0,L[)$ by

$$(\varphi, \xi) = \int_0^L \partial_{33}\varphi(x_3)\, \partial_{33}\xi(x_3)\, \mathrm{d}x_3, \qquad \forall \varphi,\ \xi \in H_0^2(]0,L[); \tag{3.20}$$

ii) the Radon measure μ defined in (3.17) is non-negative.

To prove i) we show that the norms $\|.\|_{a_{d^0}}$ and $\|.\|_{H_0^2(]0,L[)}$ associated to the scalar products $((.,.))$ and $(.,.)$ defined by (3.7) and (3.20), respectively, are equivalent. For any $\varphi \in H_0^2(]0,L[)$ we have (see Prop. 3.1)

$$\|\varphi\|_{a_{d^0}}^2 = a_{d^0}(R^{-1}\varphi, R^{-1}\varphi) = \int_\Omega \frac{1}{b_{3333(d^0)}}(\partial_3 \underline{v}_3 - x_1\partial_{33}\varphi - x_2\partial_{33}v_2)^2 \mathrm{d}\Omega \tag{3.21}$$

where $v = (0, v_2, \underline{v}_3 - x_2\partial_3 v_2) \in KerR$ is such that

$$a_{d^0}(v,z) = -a_{d^0}(u,z), \qquad \forall z \in KerR, \tag{3.22}$$

with $u = (\varphi, 0, -x_1\partial_3\varphi)$, and thus

$$\|\varphi\|_{a_{d^0}}^2 = a_{d^0}(u+v, u+v) = a_{d^0}(u,u) + 2a_{d^0}(u,v) + a_{d^0}(v,v). \tag{3.23}$$

Choosing $z = v$ in (3.22) and using condition (1.7) we obtain

$$\begin{aligned} \|\varphi\|_{a_{d^0}}^2 &= -a_{d^0}(v,v) + a_{d^0}(u,u) \leq a_{d^0}(u,u) \\ &= \int_\Omega \frac{1}{b_{3333}(d^0)} x_1^2 |\partial_{33}\varphi|^2 \mathrm{d}\Omega \leq c\|\varphi\|_{H_0^2(]0,L[)}^2, \end{aligned} \tag{3.24}$$

where c is a positive constant. On the other hand, using again condition (1.7) and (1.24) we get

$$\begin{aligned} \|\varphi\|_{a_{d^0}}^2 &\geq C_1\, \|e_{33}(R^{-1}\varphi)\|_{L^2(\Omega)}^2 \\ &= C_1\, \|\partial_3(\underline{v}_3 - x_1\partial_3\varphi - x_2\partial_3 v_2)\|_{L^2(\Omega)}^2 \\ &= C_1 \left[\int_\Omega (\partial_3\underline{v}_3)^2 \mathrm{d}\Omega + \int_\Omega x_1^2(\partial_{33}\varphi)^2 \mathrm{d}\Omega + \int_\Omega x_2^2(\partial_{33}v_2)^2\mathrm{d}\Omega\right] \\ &\geq C_1 \int_\Omega x_1^2(\partial_{33}\varphi)^2 \mathrm{d}\Omega = C\left(\int_\omega x_1^2\, \mathrm{d}\omega\right) \|\varphi\|_{H_0^2(]0,L[)}^2, \end{aligned} \tag{3.25}$$

where C_1 and C represent different positive constants. Thus the proof of i) is complete.

To prove ii) it suffices to remark that for all $\zeta \in C_0^\infty(]0,L[)$ such that $\zeta \geq 0$ in $]0,L[$ we have

$$\left\{ \begin{aligned} ((\varphi - P_S(\varphi), \zeta)) &= ((\varphi - P_S(\varphi), \zeta + P_S(\varphi) - P_S(\varphi))) \\ &= ((\varphi - P_S(\varphi), \xi - P_S(\varphi))) \leq 0, \end{aligned} \right. \tag{3.26}$$

because of the definition of $P_S(\varphi)$ and the fact that $\xi = \zeta + P_S(\varphi)$ belongs to S.

Due to the Properties i) and ii) and assuming that the set $supp\mu$ is admissible in the sense of (3.18), we finish the proof of this theorem, using exactly the same arguments as in Rao and Sokolowski [11]. □

The verification that the set $supp\mu$ is admissible in the sense of (3.18) is in general difficult (*cf.* also Pierre and Sokolowski [10] for the related subject of differentiability of projections and applications). Nevertheless there is a sufficient condition for which the $supp\mu$ is admissible, as described in the next Proposition 3.4 (*cf.* Rao and Sokolowski [11] for the proof of this statement). Therefore assembling this last comment with Proposition 3.3, we can state the following result concerning the conical differentiability of the projection P_S at a point $\varphi \in S$.

Proposition 3.4. *If the support of the Radon measure μ defined in (3.17) is admissible in the sense of (3.18), then, the set S is polyhedric at the point $\varphi \in S$, and, consequently, P_S is conical differentiable at $\varphi \in S$. In particular, if the C_1-capacity of the compact set $supp\mu$ is zero, that is $C_1(supp\mu) = 0$, where*

$$C_1(supp\mu) = \inf \left\{ \int_0^L |\partial_3 \zeta(x_3)|^2 \,\mathrm{d}x_3 : \quad \zeta \geq 1 \text{ on } supp\mu, \;\; 0 \leq \zeta \in C_0^\infty(]0, L[) \right\}, \tag{3.27}$$

then $supp\mu$ is admissible.

Finally, assuming the hypothesis of the previous proposition 3.4, and using the decomposition (3.11) and (3.16) we conclude that, for each $t \in [0,T]$, the operator $Q(.,t)$ in (0.5), which is the conical derivative of Π at $L_{d^0}(.,t)$, is defined by

$$Q(l)(.,t) = \Upsilon(l)(.,t) + R^{-1} P_{M(.,t)}(\Phi(l))(.,t), \quad \forall l \in V', \tag{3.28}$$

where for each t, the convex cone $M(.,t)$ depends on $L_{d^0}(.,t)$ and the obstacle ψ, and is defined in (3.16) with $\varphi = \Phi(L_{d^0})(.,t)$, that is,

$$M(.,t) = T_S(\Phi(L_{d^0})(.,t)) \cap [\Phi(L_{d^0})(.,t) - \xi]^\perp, \tag{3.29}$$

where $\Phi(L_{d^0})(.,t) = P_S(\xi)$, for some $\xi \in H_0^2(]0,L[)$.

4. Partial Proof of Conditions (0.6) and (0.7)

In this section we prove that conditions (0.6) and (0.7) are satisfied for a sub-family $\{(u^{s_j}, d^{s_j})\}_{j=1}^\infty$ of $\{(u^s, d^s)\}_{s>0}$. In Section 5 we show that these two conditions are still verified for the all family $\{(u^s, d^s)\}_{s>0}$.

By Theorem 2.1 we know that there exists a subsequence that we denote by $(\frac{u^{s_j} - u^0}{s_j}, \frac{d^{s_j} - d^0}{s_j})(.,t)$ that converges weakly to $(\bar{u}, \bar{d})(.,t)$ in $V \times L^2(\Omega)$, when $s_j \to 0^+$. Consequently, by Theorems 2.1 and 2.2, there exist $A' = A'_{\bar{d}}$ and $L' = L'_{\bar{d}}$ that depend explicitly on $\bar{d}$ and implicitly on $\bar{u}$. Using the Theorem 4.14 of Sokolowski and Zolesio [12] p. 178, combined with the expression (3.28) for $Q(.,t)$ (or equivalently Th. 4.30 of Sokolowski and Zolesio [12] p. 210) we conclude that, for all s_j

$$\left[\begin{array}{l} u^{s_j}(.,t) = u^0(.,t) + s_j u'(.,t) + o(s_j), \quad \text{with} \\ u' = Q(L'_{\bar{d}} - A'_{\bar{d}} u^0) = \Upsilon(L'_{\bar{d}} - A'_{\bar{d}} u^0) + R^{-1} P_M(\Phi(L'_{\bar{d}} - A'_{\bar{d}} u^0)), \end{array} \right. \tag{4.1}$$

where $\|o(s_j)/s_j\|_V$ tends to zero when $s_j \to 0^+$, and $\bar{d}$ is the solution of the following ordinary differential equation (*cf.* Th. 2.1)

$$\left[\begin{array}{l} \dot{\bar{d}} = c(d^0)\, e_{33}(\bar{u}) + \bar{d} \left[c'(d^0) e_{33}(u^0) + b'(d^0) \right] - c(d^0)\, \theta_\alpha \,\partial_{33} u_\alpha^0 \\ \bar{d}(x,0) = 0, \quad \text{in} \quad \overline{\Omega}. \end{array} \right. \tag{4.2}$$

Moreover from (4.1) and (2.1) we also conclude that $\bar{u} = u'$. From (4.2) and (2.3) we deduce that $\bar{d} = d'$ and

$$d^{s_j}(.,t) = d^0(.,t) + s_j d'(.,t) + o(s_j), \tag{4.3}$$

where $\frac{\int_\Omega o(s_j) v \mathrm{d}\Omega}{s_j}$ tends to zero when $s_j \to 0^+$, for all $v \in L^2(\Omega)$. So the conditions (0.6) and (0.7) are proved for the subfamily of parameters $s = s_j$.

5. Proof of Theorem 0.1

In this section we prove Theorem 0.1 with the hypotheses of Theorem 2.1 and the sufficient conditions of Proposition 3.3.

Observing (4.1), (4.2) and (4.3), and taking into account the results of Section 3 and also the Theorem 4.14 of Sokolowski and Zolesio [12] p. 178 (or equivalently Th. 4.30 of Sokolowski and Zolesio [12] p. 210), we realize that to prove conditions (0.2)–(0.3) and (0.6)–(0.7), and consequently to prove Theorem 0.1, it only remains to assure that the weak limit $(\bar{u}, \bar{d})(., t)$ is unique. That is, for all $s > 0$, the sequence $(\frac{u^s - u^0}{s}, \frac{d^s - d^0}{s})(., t)$ converges weakly to $(\bar{u}, \bar{d})(., t) \in V \times L^2(\Omega)$. This happens if the system defined by the second equation in (4.1) and (4.2) has a unique solution. In fact this is true, as stated and proved in the next theorem.

Theorem 5.1. *The system*

$$\left[\begin{array}{l} \textit{Find} \quad (u, d)(., t) \in V \times L^2(\Omega) : \\ u = \Upsilon(L'_d - A'_d u^0) + R^{-1} P_M(\Phi(L'_d - A'_d u^0)), \\ \dot{d} = c(d^0)\, e_{33}(u) + d\left[c'(d^0) e_{33}(u^0) + b'(d^0)\right] - c(d^0)\, \theta_\alpha\, \partial_{33} u^0_\alpha, \\ d(x, 0) = 0, \quad \textit{in} \quad \overline{\Omega}, \end{array}\right. \tag{5.1}$$

has a unique solution $(u, d) \in C^1([0, T]; V) \times C^1([0, T]; C^0(\overline{\Omega}))$.

Proof. The proof of existence is analogous to the proof of Theorem 1.1. It relies on Schauder's fixed point theorem together with the Cauchy-Lipschitz-Picard theorem (used to solve the ordinary differential equation for a fixed u) and regularity results, concerning the first equation of (5.1). To prove that the solution of (5.1) is unique let (u, d) and (v, e) be two different solutions of (5.1). Then we have

$$\left[\begin{array}{l} u - v = \Upsilon\big(L'_d - L'_e - (A'_d - A'_e) u^0\big) \\ \qquad\qquad + R^{-1}\Big[P_M\big(\Phi(L'_d - A'_d u^0)\big) + P_M\big(\Phi(L'_e - A'_e u^0)\big)\Big]. \end{array}\right. \tag{5.2}$$

Taking the norm in V and using the continuity of the operators Υ, R^{-1}, P_M, Φ and the linearity of Υ, R^{-1}, Φ, we obtain for each $t \in [0, T]$

$$\|(u - v)(., t)\|_V \leq C \|(d - e)(., t)\|_{L^2(\Omega)}, \tag{5.3}$$

where C is a positive constant. On the other hand, subtracting the two ordinary differential equations, and integrating in time

$$\left[\begin{array}{l} (d - e)(x, t) = \\ \displaystyle\int_0^t \Big(c(d^0)\, e_{33}(u - v) + (d - e)\left[c'(d^0) e_{33}(u^0) + b'(d^0)\right]\Big)(x, r) \mathrm{d}r. \end{array}\right. \tag{5.4}$$

Taking the $L^2(\Omega)$, for each $t \in [0, T]$, and using (5.3) we get

$$\|(d - e)(., t)\|_{L^2(\Omega)} \leq C \int_0^t \|(d - e)(., t)\|_{L^2(\Omega)}(x, r) \mathrm{d}r, \tag{5.5}$$

where C is positive constant independent of t. Applying now to (5.5) the integral Gronwall's inequality we have that $d = e$ and by (5.3) also $u = v$, so the proof is complete. □

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