

OPTIMAL NETWORKS FOR MASS TRANSPORTATION PROBLEMS

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Abstract. In the framework of transport theory, we are interested in the following optimization problem: given the distributions μ^+ of working people and μ^- of their working places in an urban area, build a transportation network (such as a railway or an underground system) which minimizes a functional depending on the geometry of the network through a particular cost function. The functional is defined as the Wasserstein distance of μ^+ from μ^- with respect to a metric which depends on the transportation network.

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1. INTRODUCTION

Optimal Transportation Theory was first developed by Monge in 1781 in [12] where he raised the following question: given two mass distributions f^+ and f^- , minimize the transport cost

$$\int_{\mathbb{R}^N} |x - t(x)| f^+(x) \, dx$$

among all *transport maps* t , *i.e.* measurable maps such that the mass balance condition

$$\int_{t^{-1}(B)} f^+(x) \, dx = \int_B f^-(y) \, dy$$

holds for every Borel set B . Because of its strong non-linearity, Monge's formulation did not lead to significant advances up to 1940, when Kantorovich proposed his own formulation (see [10, 11]).

In modern notation, given two finite positive Borel measures μ^+ and μ^- on \mathbb{R}^N such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$, Kantorovich was interested to minimize

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| \, d\mu(x, y)$$

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among all *transport plans* μ , *i.e.* positive Borel measures on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\pi_{\#}^+ \mu = \mu^+$ and $\pi_{\#}^- \mu = \mu^-$, where by $\#$ we denoted the *push-forward* operator (*i.e.* $h_{\#} \mu(E) = \mu(h^{-1}(E))$). It is easy to see that if t is a transport map between $\mu^+ = f^+ \mathcal{L}^N$ and $\mu^- = f^- \mathcal{L}^N$, then $(\text{Id} \times t)_{\#} \mu^+$ is a transport plan. So, Kantorovich's problem is a weak formulation of Monge's one.

Of course, one can take, instead of \mathbb{R}^N and the cost function given by the Euclidean modulus, a generic pair of metric spaces X and Y and a positive lower semicontinuous cost function $c : X \times Y \rightarrow \mathbb{R}$, so that the Kantorovich problem reads:

$$\min \left\{ \int_{X \times Y} c(x, y) \, d\mu(x, y) : \pi_{\#}^+ \mu = \mu^+, \pi_{\#}^- \mu = \mu^- \right\}. \quad (1.1)$$

We stress the fact that μ^+ and μ^- must have the same mass, otherwise there are no transport plans.

If we set $X = Y$ and take as cost function the distance d in X , then the minimal value in (1.1) is called *Wasserstein distance* (of power 1) between μ^+ and μ^- . In this case, we shall write $W_d(\mu^+, \mu^-)$.

For other details on transportation problems on networks we refer the interested reader to [2, 3, 5–7, 13].

2. THE OPTIMAL NETWORK PROBLEM

We consider a bounded connected open subset Ω with Lipschitz boundary of \mathbb{R}^N (the urban area) with $N > 1$ and two positive finite measures μ^+ and μ^- on $K := \overline{\Omega}$ (the distributions of working people and of working places). We assume that μ^+ and μ^- have the same mass that we normalize both equal 1, that is μ^+ and μ^- are probability measures on K .

In this section we introduce the optimization problem for transportation networks: to every “urban network” Σ we may associate a suitable “cost function” d_{Σ} which takes into account the geometry of Σ as well as the costs for customers to move with their own means and by means of the network. The cost functional will be then

$$T(\Sigma) = W_{d_{\Sigma}}(\mu^+, \mu^-)$$

so that the optimization problem we deal with is

$$\min \{T(\Sigma) : \Sigma \text{ “admissible network”}\}. \quad (2.2)$$

The main result of this paper is to prove that, under suitable and very mild assumptions, and taking as admissible networks all connected, compact one-dimensional subsets Σ of K , the optimization problem (2.2) admits a solution. The tools we use to obtain the existence result are a suitable relaxation procedure to define the function d_{Σ} (Th. 4.2) and a generalization of the Gołab theorem (Th. 3.3), also obtained by Dal Maso and Toader in [8].

In order to introduce the distance d_{Σ} we consider a function $J : [0, +\infty]^3 \rightarrow [0, +\infty]$. For a given path γ in K the parameter a in $J(a, b, c)$ measures the length of γ outside Σ , b measures the length of γ inside Σ , while c represents the total length of Σ . The cost $J(a, b, c)$ is then the cost of a customer who travels for a length a by his own means and for a length b on the network, being c the length of the latter. For instance we could take $J(a, b, c) = A(a) + B(b) + C(c)$ and then the function $A(t)$ is the cost for traveling a length t by one's own means, $B(t)$ is the price of a ticket to cover the length t on Σ and $C(t)$ represents the cost of a network of length t .

For every closed connected subset Σ in K , we define the cost function d_{Σ} as

$$d_{\Sigma}(x, y) := \inf \{J(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)) : \gamma \in \mathcal{C}_{x,y}\},$$

where $\mathcal{C}_{x,y}$ is the class of all closed connected subsets of K containing x and y . The optimization problem we consider is then (2.2) where we take as *admissible networks* all closed connected subsets Σ of K with

$\mathcal{H}^1(\Sigma) < +\infty$. We also define, for every closed connected subset γ of K

$$L_\Sigma(\gamma) := J(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

We assume that J satisfies the following conditions:

- J is lower semicontinuous;
- J is non-decreasing, *i.e.*

$$a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \implies J(a_1, b_1, c_1) \leq J(a_2, b_2, c_2);$$

- $J(a, b, c) \geq G(c)$ with $G(c) \rightarrow +\infty$ when $c \rightarrow +\infty$;
- J is continuous in its first variable.

A *curve* joining two points $x, y \in K$ is an element of the set

$$\mathcal{C}_{x,y} := \{\gamma \text{ closed connected, } \{x, y\} \subseteq \gamma \subseteq K\}$$

while an element of \mathcal{C} will be, by definition, a closed connected set in K :

$$\mathcal{C} := \{\gamma \text{ closed connected, } \gamma \subseteq K\}.$$

We associate to every admissible network $\Sigma \in \mathcal{C}$ the cost function

$$d_\Sigma(x, y) = \inf\{L_\Sigma(\gamma) : \gamma \in \mathcal{C}_{x,y}\}.$$

We are interested in the functional T given by

$$\Sigma \mapsto T(\Sigma) := W_{d_\Sigma}(\mu^+, \mu^-)$$

which is defined on the class \mathcal{C} , where the Wasserstein distance is defined in the introduction.

Finally by $\overline{L}_\Sigma^{x,y}$ we denote the lower semicontinuous envelope of L_Σ with respect to the Hausdorff convergence on $\mathcal{C}_{x,y}$ (see Sect. 3 for the main definitions). In other words, for every $\gamma \in \mathcal{C}_{x,y}$ we set

$$\overline{L}_\Sigma^{x,y}(\gamma) = \begin{cases} \min\{\liminf_n L_\Sigma(\gamma_n) : \gamma_n \rightarrow \gamma, \gamma_n \in \mathcal{C}_{x,y}\} & \text{if } \gamma \in \mathcal{C}_{x,y} \\ +\infty & \text{if } \gamma \notin \mathcal{C}_{x,y}, \end{cases}$$

where we fix the condition $x, y \in \gamma$. Moreover, we define \overline{L}_Σ as

$$\overline{L}_\Sigma(\gamma) = \min\left\{\liminf_{n \rightarrow +\infty} L_\Sigma(\gamma_n) : \gamma_n \rightarrow \gamma, \gamma_n \in \mathcal{C}\right\},$$

that is to say, the lower semicontinuous envelope of L_Σ with respect to the Hausdorff convergence on the class of closed connected sets of K .

3. THE GOLAB THEOREM AND ITS EXTENSIONS

In this section X will be a set endowed with a distance function d , *i.e.* (X, d) is a metric space. We assume for simplicity X to be *compact*. By $\mathcal{C}(X)$ we indicate the class of all closed subsets of X .

Given two closed subsets C and D , the *Hausdorff distance* between them is defined by

$$d_{\mathcal{H}}(C, D) := 1 \wedge \inf\{r \in [0, +\infty[: C \subseteq D_r, D \subseteq C_r\}$$

where

$$C_r := \{x \in X : d(x, C) < r\}.$$

It is easy to see that $d_{\mathcal{H}}$ is a distance on $\mathcal{C}(X)$, so $(\mathcal{C}(X), d_{\mathcal{H}})$ is a metric space. We remark the following well-known facts (see for example [1]):

- (X, d) compact $\implies (\mathcal{C}(X), d_{\mathcal{H}})$ compact;
- (X, d) complete $\implies (\mathcal{C}(X), d_{\mathcal{H}})$ complete.

In the rest of the paper we will use the notation $C_n \rightarrow C$ to indicate the convergence of a sequence $\{C_n\}_{n \in \mathbb{N}}$ to C with respect to the distance $d_{\mathcal{H}}$.

Proposition 3.1. *Let $\{C_n\}_{n \in \mathbb{N}}$ be a sequence of compact connected subsets in X such that $C_n \rightarrow C$ for some compact subset C . Then C is connected.*

Proof. Suppose, on the contrary, that there exist two closed non-void separated subsets F_1 and F_2 such that $C = F_1 \cup F_2$. Since F_1 and F_2 are compact, $d(F_1, F_2) = d > 0$. Let us choose $\varepsilon = d/4$. By the definition of Hausdorff convergence, there exists a positive integer N such that

$$n \geq N \implies C_n \subseteq (C)_{\varepsilon}, \quad C \subseteq (C_n)_{\varepsilon}.$$

Since C_N is connected, we must have either $C_N \subseteq (F_1)_{\varepsilon}$ or $C_N \subseteq (F_2)_{\varepsilon}$. Let us suppose, for example, that $C_N \subseteq (F_1)_{\varepsilon}$. On one side by the Hausdorff convergence it is $F_2 \subseteq (C_N)_{\varepsilon}$, on the other by the choice of ε we have $(C_N)_{\varepsilon} \cap F_2 = \emptyset$, a contradiction. \square

The *Hausdorff 1-dimensional measure* in (X, d) of a Borel set B is defined by

$$\mathcal{H}^1(B) := \lim_{\delta \rightarrow 0^+} \mathcal{H}^{1, \delta}(B),$$

where

$$\mathcal{H}^{1, \delta}(B) := \inf \left\{ \sum_{n \in \mathbb{N}} \text{diam } B_n : \text{diam } B_n < \delta, B \subseteq \bigcup_{n \in \mathbb{N}} B_n \right\}.$$

The measure \mathcal{H}^1 is Borel regular and if (X, d) is the 1-dimensional Euclidean space, then \mathcal{H}^1 is just the Lebesgue measure \mathcal{L}^1 .

The Gołab classical theorem states that in a metric space, the measure \mathcal{H}^1 is sequentially lower semicontinuous with respect to the Hausdorff convergence over the class of all compact connected subsets of X .

Theorem 3.2 (Gołab). *Let X be a metric space. If $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of compact connected subsets of X and $C_n \rightarrow C$ for some compact connected subset C , then*

$$\mathcal{H}^1(C) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(C_n). \quad (3.3)$$

Actually, this result can be strengthened.

Theorem 3.3. *Let X be a metric space, $\{\Gamma_n\}_{n \in \mathbb{N}}$ and $\{\Sigma_n\}_{n \in \mathbb{N}}$ be two sequences of compact subsets such that $\Gamma_n \rightarrow \Gamma$ and $\Sigma_n \rightarrow \Sigma$ for some compact subsets Γ and Σ . Let us also suppose that Γ_n is connected for all $n \in \mathbb{N}$. Then*

$$\mathcal{H}^1(\Gamma \setminus \Sigma) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\Gamma_n \setminus \Sigma_n). \quad (3.4)$$

A proof of this result has been given by Dal Maso and Toader in [8]; for sake of completeness, we include the proof here below. It is in fact based on the following two rectifiability theorems whose proof can be found in [1].

Theorem 3.4. *Let X be a metric space and C a closed connected subset of finite length, i.e. $\mathcal{H}^1(C) < +\infty$. Then C is compact and connected by injective rectifiable curves.*

Theorem 3.5. *Let C be a closed connected subset in a metric space X such that $\mathcal{H}^1(C) < +\infty$. Then there exists a sequence of Lipschitz curves $\{\gamma_n\}_{n \in \mathbb{N}}$, $\gamma_n : [0, 1] \rightarrow C$, such that*

$$\mathcal{H}^1(C \setminus \bigcup_{n \in \mathbb{N}} \gamma_n([0, 1])) = 0.$$

The first step in the proof of Theorem 3.3 is a localized form of the Gołab classical theorem. To this aim we need the following lemma.

Lemma 3.6. *Let C be a closed connected subset of X and let $x \in C$. If $r \in [0, \frac{1}{2} \text{diam } C]$, then*

$$\mathcal{H}^1(C \cap B_r(x)) \geq r.$$

Proof. See for instance Lemma 4.4.2 of [1] or Lemma 3.4 of [9]. □

Remark 3.7. Lemma 3.6 yields the following estimate from below for the upper density:

$$\bar{\theta}(C, x) := \limsup_{r \rightarrow 0^+} \frac{\mathcal{H}^1(C \cap B_r(x))}{2r} \geq \frac{1}{2}.$$

We recall that for every measure μ the *upper density* is defined by

$$\bar{\theta}(\mu, x) := \limsup_{r \rightarrow 0^+} \frac{\mu(B_r(x))}{2r}.$$

We also recall that $\bar{\theta}(\mu, x) \geq t$ for all $x \in X$ implies $\mu(B) \geq t\mathcal{H}^1(B)$ for every Borel set B (see Th. 2.4.1 in [1]).

We are now in a position to obtain the localized version of the Gołab theorem.

Theorem 3.8. *Let X be a metric space. If $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of compact connected subsets of X such that $C_n \rightarrow C$ for some compact connected subset C , then for every open subset U of X*

$$\mathcal{H}^1(C \cap U) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(C_n \cap U).$$

Proof. We can suppose that $L := \lim_n \mathcal{H}^1(C_n \cap U)$ exists, is finite and $\mathcal{H}^1(C_n \cap U) \leq L + 1$. Let $d_n = \text{diam}(C_n \cap U)$. We can suppose up to a subsequence that $d_n \rightarrow d > 0$. Let us consider the sequence of Borel measures defined by

$$\mu_n(B) := \mathcal{H}^1(B \cap C_n \cap U)$$

for every Borel set B . Up to a subsequence we can assume that $\mu_n \rightharpoonup^* \mu$ for a suitable μ . We choose $x \in C \cap U$ and $r' < r < \text{diam}(C \cap U)/2$. Then, by Lemma 3.6,

$$\mu(B_r(x)) \geq \mu(\overline{B}_{r'}(x)) \geq \limsup_{n \rightarrow +\infty} \mu_n(\overline{B}_{r'}(x)) = \limsup_{n \rightarrow +\infty} \mathcal{H}^1(C_n \cap \overline{B}_{r'}(x) \cap U) \geq r'.$$

Since r' was chosen arbitrarily we get

$$\mu(B_r(x)) \geq r$$

for every $x \in C \cap U$ and $r < \text{diam}(C \cap U)/2$. This implies $\bar{\theta}(C, x) \geq 1/2$. By Remark 3.7

$$\mathcal{H}^1(C \cap U) \leq 2\mu(X) \leq 2 \liminf_{n \rightarrow +\infty} \mu_n(X) = 2 \liminf_{n \rightarrow +\infty} \mathcal{H}^1(C_n \cap U) = 2L.$$

By Theorem 3.5 for \mathcal{H}^1 -almost all $x_0 \in C \cap U$ there exists a Lipschitz curve γ whose range is in $C \cap U$ such that $x_0 = \gamma(t_0)$ and $t_0 \in]0, 1[$. We can also suppose that

$$\lim_{h \rightarrow 0^+} \frac{d(\gamma(t_0 + h), \gamma(t_0 - h))}{2|h|} = 1.$$

We choose arbitrarily $\sigma \in]0, 1[$. If h is small, then

$$d(\gamma(t_0 + h), \gamma(t_0 - h)) \geq (2 - \sigma)|h|$$

and

$$(1 - \sigma)|h| \leq d(\gamma(t_0 \pm h), \gamma(t_0)) \leq (1 + \sigma)|h|.$$

Let us also suppose that $|h| < \sigma/(1 + \sigma)$ and put

$$y := \gamma(t_0 - h), \quad z := \gamma(t_0 + h), \quad r := \max\{d(y, x_0), d(z, x_0)\}.$$

We get

$$r < (1 + \sigma)|h| < \sigma, \quad d(y, z) \geq (2 - \sigma)|h| \geq \frac{2 - \sigma}{2 + \sigma}r.$$

Let $r' := (1 + \sigma)r$. Since $C_n \rightarrow C$, then (see Prop. 4.4.3 in [1]) there exist subsequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ such that $y_n, z_n \in C_n \cap U$, $y_n \rightarrow y$ and $z_n \rightarrow z$. One must have $y_n, z_n \in B_{r'}(x_0)$ for n large enough and

$$\mu_n(\overline{B_{r'}(x)}) = \mathcal{H}^1(C_n \cap \overline{B_{r'}(x)} \cap U) \geq d(z, y_n).$$

Taking the limsup

$$\begin{aligned} \mu(\overline{B_{r'}(x)}) &\geq \limsup_{n \rightarrow +\infty} \mathcal{H}^1(C_n \cap \overline{B_{r'}(x)} \cap U) \geq \limsup_{n \rightarrow +\infty} d(z, y_n) \\ &= d(z, y) \geq \frac{2 - \sigma}{2 + \sigma}r = \frac{2 - \sigma}{(2 + \sigma)(1 + \sigma)}r'. \end{aligned}$$

Since σ was arbitrary, we get $\bar{\theta}(\mu, x_0) \geq 1$ for \mathcal{H}^1 -almost all $x_0 \in C \cap U$. Then, by Remark 3.7

$$\mathcal{H}^1(C \cap U) \leq \mu(X) \leq \liminf_{n \rightarrow +\infty} \mu_n(X) = \liminf_{n \rightarrow +\infty} \mathcal{H}^1(C_n \cap U). \quad \square$$

Proof of Theorem 3.3. Let $A = \Gamma \cap \Sigma$. Thanks to the equality

$$\bigcup_{\varepsilon > 0} (\Gamma \setminus \overline{A_\varepsilon}) = \Gamma \setminus \Sigma$$

we have

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{H}^1(\Gamma \setminus \overline{A_\varepsilon}) = \mathcal{H}^1(\Gamma \setminus \Sigma).$$

Recalling that the following inclusion of sets holds for large values of n

$$\Gamma_n \setminus \overline{A_\varepsilon} \subseteq \Gamma_n \setminus A_n \subseteq \Gamma_n \setminus \Sigma_n$$

by the localized form of Gołab theorem (Th. 3.8) we deduce

$$\mathcal{H}^1(\Gamma \setminus \overline{A_\varepsilon}) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\Gamma_n \setminus \overline{A_\varepsilon}) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\Gamma_n \setminus \Sigma_n).$$

Taking the limit as $\varepsilon \rightarrow 0^+$, we obtain

$$\mathcal{H}^1(\Gamma \setminus \Sigma) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\Gamma_n \setminus \Sigma_n). \quad \square$$

Remark 3.9. It is easy to see that if the number of connected components of C_n is bounded from above by a positive integer independent on n , then the localized form of Golab theorem is still valid. All details can be found in [8].

4. RELAXATION OF THE COST FUNCTION

We can give an explicit expression for the lower semicontinuous envelopes \overline{L}_Σ and $\overline{L}_\Sigma^{x,y}$ in terms of J . In order to achieve this result it is useful to introduce the function:

$$\overline{J}(a, b, c) = \inf\{J(a+t, b-t, c) : 0 \leq t \leq b\}.$$

The following lemma is an important step to establish Theorem 4.2.

Lemma 4.1. *Let γ and Σ be closed connected subsets of K . Let also suppose that Σ has a finite length. Then for every $t \in [0, \mathcal{H}^1(\gamma \cap \Sigma)]$ we can find a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in \mathcal{C} such that*

- $\gamma_n \rightarrow \gamma$;
- $\lim_n \mathcal{H}^1(\gamma_n) = \mathcal{H}^1(\gamma)$;
- $\mathcal{H}^1(\gamma_n \cap \Sigma) \nearrow \mathcal{H}^1(\gamma \cap \Sigma) - t$.

Moreover, if $x, y \in \gamma$ then the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ can be chosen in $\mathcal{C}_{x,y}$.

Proof. The set $\gamma \cap \Sigma$ is closed and with a finite length. By the second rectifiability result (Th. 3.5) it follows the existence of a sequence of curves $\sigma_n \in \text{Lip}([0, 1], K)$ such that

$$\mathcal{H}^1\left(\left(\gamma \cap \Sigma\right) \setminus \bigcup_{n \in \mathbb{N}} \sigma_n([0, 1])\right) = 0.$$

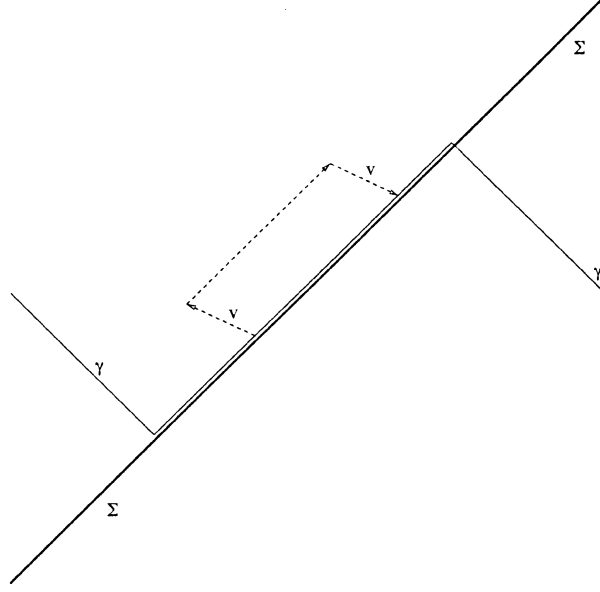
We can also suppose that the subsets $\sigma_n([0, 1])$ are disjoint up to subsets of negligible length. Fix a sufficiently small $\delta > 0$ and choose a sequence of intervals $I_n = [a_n, b_n]$ such that

$$\sum_{n \in \mathbb{N}} \mathcal{H}^1(\sigma_n(I_n)) = t + \delta.$$

For every sequence $\underline{v} = \{v_n\}_{n \in \mathbb{N}}$ of unit vectors of \mathbb{R}^N such that v_n is not tangent to $\gamma \cap \Sigma$ in $\sigma_n(a_n)$ and $\sigma_n(b_n)$, and every sequence $\underline{\varepsilon} = \{\varepsilon_n\}_{n \in \mathbb{N}}$ of positive real numbers, let us consider

$$\begin{aligned} A_{\underline{v}, \underline{\varepsilon}} &= \bigcup_{n \in \mathbb{N}} \sigma_n([0, a_n] \cup [b_n, 1]), \\ B_{\underline{v}, \underline{\varepsilon}} &= \bigcup_{n \in \mathbb{N}} (\sigma_n(a_n) + \varepsilon_n V_n), \\ C_{\underline{v}, \underline{\varepsilon}} &= \bigcup_{n \in \mathbb{N}} (v_n + \sigma_n(I_n)), \\ D_{\underline{v}, \underline{\varepsilon}} &= \bigcup_{n \in \mathbb{N}} (\sigma_n(b_n) + \varepsilon_n V_n) \\ \gamma_{\underline{v}, \underline{\varepsilon}} &= (\gamma \setminus \Sigma) \cup A_{\underline{v}, \underline{\varepsilon}} \cup B_{\underline{v}, \underline{\varepsilon}} \cup C_{\underline{v}, \underline{\varepsilon}} \cup D_{\underline{v}, \underline{\varepsilon}} \end{aligned}$$

where $V_n = \{tv_n : t \in [0, 1]\}$ (see Fig. 1).

FIGURE 1. The approximating curves γ_n .

Since Σ is closed and with a finite length, the class of $\gamma_{\underline{v}, \underline{\varepsilon}}$ that have not \mathcal{H}^1 -negligible intersection with Σ is at most countable. Out of that set we can choose sequences $\delta_m \searrow 0$, and $\{\gamma_{\underline{v}_m, \underline{\varepsilon}_m}\}_{m \in \mathbb{N}}$ such that $\|\underline{\varepsilon}_m\| \searrow 0$, where by $\|\underline{\varepsilon}\|$ we denote the quantity $\sum_n \varepsilon_n$. The sequence $\{\gamma_{\underline{v}_m, \underline{\varepsilon}_m}\}_{m \in \mathbb{N}}$ is the one we were looking for. \square

Theorem 4.2. *For every closed connected subset $\gamma \in \mathcal{C}_{x,y}$ we have*

$$\overline{L}_{\Sigma}^{x,y}(\gamma) = \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

Moreover, if $\gamma \in \mathcal{C}_{x,y}$ then

$$\overline{L}_{\Sigma}^{x,y}(\gamma) = \overline{L}_{\Sigma}(\gamma).$$

Proof. Let γ be a fixed curve in $\mathcal{C}_{x,y}$. First we establish that

$$\overline{L}_{\Sigma}^{x,y}(\gamma) \geq \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

It is enough to show that for every sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}_{x,y}$ converging to γ with respect to the Hausdorff metric, there exists $t \in [0, \mathcal{H}^1(\gamma \cap \Sigma)]$ such that

$$J(\mathcal{H}^1(\gamma \setminus \Sigma) + t, \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma)) \leq \liminf_{n \rightarrow +\infty} L_{\Sigma}(\gamma_n).$$

Up to a subsequence we can suppose the following equalities hold true:

$$\begin{aligned} \liminf_{n \rightarrow +\infty} L_{\Sigma}(\gamma_n) &= \lim_{n \rightarrow +\infty} L_{\Sigma}(\gamma_n), \\ \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n) &= \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n), \\ \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma) &= \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma). \end{aligned}$$

Moreover, by Gołab theorems (Ths. 3.2 and 3.3)

$$\begin{aligned}\mathcal{H}^1(\gamma) &\leq \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n), \\ \mathcal{H}^1(\gamma \setminus \Sigma) &\leq \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma).\end{aligned}$$

Choose $t = \lim_n \mathcal{H}^1(\gamma_n \setminus \Sigma) - \mathcal{H}^1(\gamma \setminus \Sigma)$. Then $\mathcal{H}^1(\gamma \setminus \Sigma) + t = \lim_n \mathcal{H}^1(\gamma_n \setminus \Sigma)$. We have

$$\begin{aligned}\mathcal{H}^1(\gamma_n) &= \mathcal{H}^1(\gamma_n \setminus \Sigma) + \mathcal{H}^1(\gamma_n \cap \Sigma) \\ &= [\mathcal{H}^1(\gamma_n \setminus \Sigma) - t] + [\mathcal{H}^1(\gamma_n \cap \Sigma) + t].\end{aligned}$$

Taking the limit as $n \rightarrow +\infty$ gives

$$\mathcal{H}^1(\gamma) \leq \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n) = [\mathcal{H}^1(\gamma \setminus \Sigma) + t] + \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \cap \Sigma)$$

so that

$$\mathcal{H}^1(\gamma \cap \Sigma) - t \leq \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \cap \Sigma).$$

It follows by the semicontinuity and monotonicity of J in the first two variables

$$J(\mathcal{H}^1(\gamma \setminus \Sigma) + t, \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma)) \leq \liminf_{n \rightarrow +\infty} J(\mathcal{H}^1(\gamma_n \setminus \Sigma), \mathcal{H}^1(\gamma_n \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

Now, we have to establish the opposite inequality:

$$\overline{L}_\Sigma^{x,y}(\gamma) \leq \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

In the same way as before, it is enough to show that for every $t \in [0, \mathcal{H}^1(\gamma \cap \Sigma)]$ we can find a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}_{x,y}$ which converges to γ such that

$$\liminf_{n \rightarrow +\infty} L_\Sigma(\gamma_n) \leq J(\mathcal{H}^1(\gamma \setminus \Sigma) + t, \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma)).$$

Given t , let $\{\gamma_n\}_{n \in \mathbb{N}}$ be the sequence given by Lemma 4.1. Then we get

$$\lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma) = \mathcal{H}^1(\gamma) - \mathcal{H}^1(\gamma \cap \Sigma) + t = \mathcal{H}^1(\gamma \setminus \Sigma) + t.$$

Thanks to $\mathcal{H}^1(\gamma_n \cap \Sigma) \leq \mathcal{H}^1(\gamma \cap \Sigma) - t$, we have

$$J(\mathcal{H}^1(\gamma_n \setminus \Sigma), \mathcal{H}^1(\gamma_n \cap \Sigma), \mathcal{H}^1(\Sigma)) \leq J(\mathcal{H}^1(\gamma_n \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma))$$

and by the continuity of J in the first variable

$$\liminf_{n \rightarrow +\infty} J(\mathcal{H}^1(\gamma_n \setminus \Sigma), \mathcal{H}^1(\gamma_n \cap \Sigma), \mathcal{H}^1(\Sigma)) \leq J(\mathcal{H}^1(\gamma \setminus \Sigma) + t, \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma))$$

which implies the inequality we looked for. The proof of the second statement of the theorem is analogous and hence omitted. \square

The next proposition is a consequence of Theorem 4.2.

Proposition 4.3. *For every $x, y \in K$ we have*

$$d_\Sigma(x, y) = \inf \{ \overline{L}_\Sigma(\gamma) : \gamma \in \mathcal{C}_{x,y} \}.$$

Proof. By a general result of relaxation theory (see for instance [4]), the infimum of a function is the same as the infimum of its lower semicontinuous envelope, so

$$d_{\Sigma}(x, y) = \inf \left\{ \overline{L}_{\Sigma}^{x, y}(\gamma) : \gamma \in \mathcal{C}_{x, y} \right\}.$$

It is then enough to prove that

$$\inf \left\{ \overline{L}_{\Sigma}^{x, y}(\gamma) : \gamma \in \mathcal{C}_{x, y} \right\} = \inf \left\{ \overline{L}_{\Sigma}(\gamma) : \gamma \in \mathcal{C}_{x, y} \right\},$$

which is a consequence of Theorem 4.2. \square

It is more convenient to introduce the function whose variables a, b, c now represent the length $\mathcal{H}^1(\gamma \setminus \Sigma)$ covered by one's own means, the path length $\mathcal{H}^1(\gamma)$, and the length of the network $\mathcal{H}^1(\Sigma)$:

$$\Theta(a, b, c) = \overline{J}(a, b - a, c).$$

Obviously, Θ satisfies

$$\Theta(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma), \mathcal{H}^1(\Sigma)) = \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

We now study some properties of Θ .

Proposition 4.4. *Θ is monotone, non-decreasing with respect to each of its variables.*

Proof. The monotonicity in the third variable is straightforward. The one in the first variable can be obtained observing that

$$\Theta(a, b, c) = \inf_{a \leq s \leq b} J(s, b - s, c) \quad (4.5)$$

and that the right-hand side of (4.5) is a non-decreasing function of a . The monotonicity in the second variable is obtained in a similar way, still relying on (4.5) and paying attention to the sets where the infimum is taken. \square

Proposition 4.5. *Θ is lower semicontinuous.*

Proof. We have to show that

$$\Theta(a, b, c) \leq \liminf_{n \rightarrow +\infty} \Theta(a_n, b_n, c_n)$$

when $a_n \rightarrow a$, $b_n \rightarrow b$ and $c_n \rightarrow c$. Let us consider for every real positive number ε and for every positive integer n a real number s_n such that $a_n \leq s_n \leq b_n$ and

$$J(s_n, b_n - s_n, c_n) \leq \Theta(a_n, b_n, c_n) + \varepsilon.$$

Up to a subsequence, we can suppose that

$$\liminf_{n \rightarrow +\infty} \Theta(a_n, b_n, c_n) = \lim_{n \rightarrow +\infty} \Theta(a_n, b_n, c_n).$$

We can also suppose that $s_n \rightarrow s$, where $a \leq s \leq b$. Thanks to the semicontinuity of J

$$\Theta(a, b, c) \leq J(s, b - s, c) \leq \liminf_{n \rightarrow +\infty} J(s_n, b_n - s_n, c_n) \leq \liminf_{n \rightarrow +\infty} \Theta(a_n, b_n, c_n) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0^+$ yields the desired inequality. \square

5. EXISTENCE THEOREM

In this section we continue to develop the tools we will use to prove Theorem 5.6.

Proposition 5.1. *Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be sequences in K such that $x_n \rightarrow x$ and $y_n \rightarrow y$. If $\{\Sigma_n\}_{n \in \mathbb{N}}$ is a sequence of closed connected sets such that $\Sigma_n \rightarrow \Sigma$, then*

$$d_\Sigma(x, y) \leq \liminf_{n \rightarrow +\infty} d_{\Sigma_n}(x_n, y_n). \quad (5.6)$$

Proof. First, up to a subsequence, we can suppose that

$$\liminf_{n \rightarrow +\infty} d_{\Sigma_n}(x_n, y_n) = \lim_{n \rightarrow +\infty} d_{\Sigma_n}(x_n, y_n).$$

Given $\varepsilon > 0$, we choose a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ such that $\gamma_n \in \mathcal{C}_{x_n, y_n}$ and

$$\Theta(\mathcal{H}^1(\gamma_n \setminus \Sigma_n), \mathcal{H}^1(\gamma_n), \mathcal{H}^1(\Sigma_n)) \leq d_{\Sigma_n}(x_n, y_n) + \varepsilon.$$

Up to a subsequence we can suppose that $\gamma_n \rightarrow \gamma$ (it is easy to check that $x_n \rightarrow x$ and $y_n \rightarrow y$ imply $\gamma \in \mathcal{C}_{x, y}$) and

$$\begin{aligned} \mathcal{H}^1(\gamma \setminus \Sigma) &\leq \lim_n \mathcal{H}^1(\gamma_n \setminus \Sigma_n), \\ \mathcal{H}^1(\gamma) &\leq \lim_n \mathcal{H}^1(\gamma_n), \\ \mathcal{H}^1(\Sigma) &\leq \lim_n \mathcal{H}^1(\Sigma_n). \end{aligned}$$

Using the semicontinuity and monotonicity of Θ (Props. 4.4 and 4.5), we obtain

$$\begin{aligned} d_\Sigma(x, y) &\leq \Theta(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma), \mathcal{H}^1(\Sigma)) \\ &\leq \Theta\left(\lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma_n), \lim_{n \rightarrow +\infty} \mathcal{H}^1(\gamma_n), \lim_{n \rightarrow +\infty} \mathcal{H}^1(\Sigma_n)\right) \\ &\leq \liminf_{n \rightarrow +\infty} \Theta(\mathcal{H}^1(\gamma_n \setminus \Sigma_n), \mathcal{H}^1(\gamma_n), \mathcal{H}^1(\Sigma_n)) \\ &\leq \liminf_{n \rightarrow +\infty} d_{\Sigma_n}(x_n, y_n) + \varepsilon. \end{aligned}$$

The arbitrary choice of ε gives then inequality (5.6). □

As a consequence of Proposition 5.1 we have the following corollary.

Corollary 5.2. *Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be sequences in K such that $x_n \rightarrow x$ and $y_n \rightarrow y$. If Σ is a closed connected set, then*

$$d_\Sigma(x, y) \leq \liminf_{n \rightarrow +\infty} d_\Sigma(x_n, y_n).$$

In other words, d_Σ is a lower semicontinuous function on $K \times K$.

Proposition 5.5 will play a crucial role in the proof of our main existence result. We split its proof in the next two lemmas for convenience.

Lemma 5.3. *Let X be a compact metric space, $\{f_n\}_{n \in \mathbb{N}}$ a sequence of positive real valued functions defined on X . Let also g be a continuous positive real valued function defined on X . Then, the following statements are equivalent:*

- (1) $\forall \varepsilon > 0 \exists N : \forall n \geq N \forall x \in X \quad g(x) \leq f_n(x) + \varepsilon;$
- (2) $\forall x \in X \forall x_n \rightarrow x \quad g(x) \leq \liminf_n f_n(x_n).$

Proof.

- Let $x_n \rightarrow x$. Then

$$g(x_n) = f_n(x_n) + (g(x_n) - f_n(x_n)) \leq f_n(x_n) + \varepsilon.$$

By the continuity of g , taking the lower limit we achieve

$$g(x) \leq \liminf_{n \rightarrow +\infty} f_n(x_n) + \varepsilon. \quad (5.7)$$

Then (1) \Rightarrow (2) is established when $\varepsilon \rightarrow 0^+$.

- Let us now prove that (2) \Rightarrow (1). Suppose on the contrary that there exists a positive ε and an increasing sequence of positive integers $\{n_k\}_k$ such that

$$g(x_{n_k}) \geq f_{n_k}(x_{n_k}) + \varepsilon \quad (5.8)$$

for a suitable x_{n_k} . Thanks to the compactness of X we can suppose up to a subsequence that $x_{n_k} \rightarrow x$. Define

$$x_n = \begin{cases} x_{n_k} & \text{if } n = n_k \text{ for some } k \\ x & \text{otherwise.} \end{cases}$$

Then $x_n \rightarrow x$, and $g(x) \leq \liminf_n f_n(x_n)$. From (5.8) it follows,

$$g(x) \geq \liminf_{k \rightarrow +\infty} f_{n_k}(x_{n_k}) + \varepsilon \geq \liminf_{n \rightarrow +\infty} f_n(x_n) + \varepsilon \geq g(x) + \varepsilon$$

which is false. □

Lemma 5.4. *Let f be a lower semicontinuous function defined on a metric space (X, d) which ranges in $[0, +\infty]$. Then the set of functions $\{g_t : t \geq 0\}$ defined by*

$$g_t(x) = \inf\{f(y) + td(x, y) : y \in X\}$$

satisfies the following properties:

- $g_t \geq 0$;
- g_t is t -Lipschitz continuous;
- $g_t(x) \nearrow f(x)$ as $t \rightarrow +\infty$.

Proof. See Lemma 1.3.1 of [1] or Proposition 1.3.7 of [4]. □

Proposition 5.5. *Let $\{f_n\}_{n \in \mathbb{N}}$ and f be non-negative lower semicontinuous functions, all defined on a compact metric space (X, d) . Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative measures on X such that $\mu_n \rightharpoonup^* \mu$. Suppose that*

$$\forall x \in X \quad \forall x_n \rightarrow x \quad f(x) \leq \liminf_{n \rightarrow +\infty} f_n(x_n).$$

Then

$$\int_X f \, d\mu \leq \liminf_{n \rightarrow +\infty} \int_X f_n \, d\mu_n.$$

Proof. Let ψ be a continuous function with compact support such that $0 \leq \psi \leq 1$. Let g_t be the function of Lemma 5.4; since g_t satisfies the hypothesis of Lemma 5.3 with $g = g_t$, we have $g_t \leq f_n + \varepsilon$ for n large enough and then

$$\int_X g_t \psi \, d\mu = \lim_{n \rightarrow +\infty} \int_X g_t \psi \, d\mu_n \leq \liminf_{n \rightarrow +\infty} \int_X f_n \, d\mu_n.$$

Taking the supremum in t and ψ , we obtain

$$\int_X f \, d\mu \leq \liminf_{n \rightarrow +\infty} \int_X f_n \, d\mu_n. \quad \square$$

We may now state and prove our existence result.

Theorem 5.6. *The problem*

$$\min\{T(\Sigma) : \Sigma \in \mathcal{C}\}$$

admits a solution.

Proof. First, let us prove that for every $l > 0$ the class

$$\mathcal{D}_l := \{\Sigma : \Sigma \in \mathcal{C}, \mathcal{H}^1(\Sigma) \leq l\}$$

is a compact subset of the metric space $(\mathcal{C}(K), d_{\mathcal{H}})$. Since $(\mathcal{C}(K), d_{\mathcal{H}})$ is a compact space, it is enough to show that \mathcal{D}_l is closed. We already know that the Hausdorff limit of a sequence of closed connected set is a closed connected set. If $\{\Sigma_n\}_{n \in \mathbb{N}}$ is a sequence of closed connected sets such that $\mathcal{H}^1(\Sigma_n) \leq l$

$$\Sigma_n \rightarrow \Sigma \implies \mathcal{H}^1(\Sigma) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\Sigma_n) \leq l$$

by Golab theorem (Th. 3.2).

Second, by our assumption on the function J

$$d_{\Sigma}(x, y) \geq G(\mathcal{H}^1(\Sigma))$$

so that

$$T(\Sigma) \geq G(\mathcal{H}^1(\Sigma)).$$

Then, if $\{\Sigma_n\}_{n \in \mathbb{N}}$ is a minimizing sequence, the sequence of 1-dimensional Hausdorff measures $\{\mathcal{H}^1(\Sigma_n)\}_{n \in \mathbb{N}}$ must be bounded, *i.e.* $\mathcal{H}^1(\Sigma_n) \leq l$, for some $l > 0$.

If we prove that the functional $\Sigma \mapsto T(\Sigma)$ is sequentially lower semicontinuous on the class \mathcal{D}_l , then the existence of an optimal Σ will be a consequence of the fact that a sequentially lower semicontinuous function takes a minimum on a compact metric space. Let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{D}_l such that $\Sigma_n \rightarrow \Sigma$. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be an optimal transport plan for the transport problem

$$\min \left\{ \int_{K \times K} d_{\Sigma_n}(x, y) \, d\mu : \pi_{\#}^+ \mu = \mu^+, \pi_{\#}^- \mu = \mu^- \right\}.$$

Up to a subsequence we can suppose $\mu_n \rightharpoonup^* \mu$ for a suitable μ . It is easy to see that μ is a transport plan between μ^+ and μ^- .

Since by Proposition 5.1 $d_{\Sigma}(x, y) \leq \liminf_n d_{\Sigma_n}(x_n, y_n)$ for all $x_n \rightarrow x$ and $y_n \rightarrow y$, by Lemma 5.5 we have

$$\int_{K \times K} d_{\Sigma}(x, y) \, d\mu \leq \liminf_{n \rightarrow +\infty} \int_{K \times K} d_{\Sigma_n}(x, y) \, d\mu_n. \quad (5.9)$$

Then by (5.9) we have

$$T(\Sigma) \leq \int_{K \times K} d_{\Sigma}(x, y) \, d\mu \leq \liminf_{n \rightarrow +\infty} \int_{K \times K} d_{\Sigma_n}(x, y) \, d\mu_n = \liminf_{n \rightarrow +\infty} T(\Sigma_n). \quad \square$$

We end with the following remark.

Remark 5.7. Note that if Σ_n is a minimizing sequence, then the measure μ obtained in the proof of Theorem 5.6 is an optimal transport plan for the transport problem

$$\min \left\{ \int_{K \times K} d_{\Sigma}(x, y) \, d\mu : \pi_{\#}^+ \mu = \mu^+, \pi_{\#}^- \mu = \mu^- \right\}.$$

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