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CARLEMAN ESTIMATES FOR THE NON-STATIONARY LAMÉ SYSTEM AND THE APPLICATION TO AN INVERSE PROBLEM

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Abstract. In this paper, we establish Carleman estimates for the two dimensional isotropic non-stationary Lamé system with the zero Dirichlet boundary conditions. Using this estimate, we prove the uniqueness and the stability in determining spatially varying density and two Lamé coefficients by a single measurement of solution over $(0,T)\times\omega$, where $T>0$ is a sufficiently large time interval and a subdomain ω satisfies a non-trapping condition.

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1. Introduction

This paper is concerned with Carleman estimates for the two dimensional non-stationary isotropic Lamé system with the zero Dirichlet boundary condition and an application to an inverse problem of determining spatially varying density and the Lamé coefficients by a single interior measurement of the solution. The Carleman estimate is a weighted L^2-estimate of the solution to a partial differential equation and it has been fundamental for proving the uniqueness in a Cauchy problem for the partial differential equation or the unique continuation.

More precisely, we consider the two dimensional isotropic non-stationary Lamé system:

$$\begin{aligned}(P\mathbf{u})(x_0,x') \equiv \rho(x')\partial_{x_0}^2\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x'),\\ x\equiv(x_0,x')\in Q\equiv(0,T)\times\Omega,\end{aligned}\tag{1.1}$$

where

$$\begin{aligned}(L_{\lambda,\mu}\mathbf{v})(x') \equiv\ & \mu(x')\Delta\mathbf{v}(x') + (\mu(x')+\lambda(x'))\nabla_{x'}\mathrm{div}\,\mathbf{v}(x')\\ &+ (\mathrm{div}\,\mathbf{v}(x'))\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{v} + (\nabla_{x'}\mathbf{v})^T)\nabla_{x'}\mu(x'), \qquad x'\in\Omega.\end{aligned}\tag{1.2}$$

Throughout this paper, $\Omega\subset\mathbb{R}^2$ is a bounded domain whose boundary $\partial\Omega$ is of class C^3, x_0 and $x'=(x_1,x_2)$ denote the time variable and the spatial variable respectively, and $\mathbf{u}=(u_1,u_2)^T$ where $\cdot^T$ denotes the transpose

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of matrices, E_k is the identity matrix of the size $k \times k$,

$$\partial_{x_j}\varphi = \varphi_{x_j} = \frac{\partial \varphi}{\partial x_j}, \quad j = 0, 1, 2.$$

We set $\nabla_{x'}\mathbf{v} = (\partial_{x_k} v_j)_{1\le j,k\le 2}$ for a vector function $\mathbf{v} = (v_1, v_2)^T$ and $\nabla_{x'}\phi = (\partial_{x_1}\phi, \partial_{x_2}\phi)^T$ for a scalar function ϕ. Henceforth ∇ means $\nabla_x = (\partial_{x_0}, \partial_{x_1}, \partial_{x_2})$ if we do not specify.

Moreover the coefficients ρ, λ, μ satisfy

$$\rho, \lambda, \mu \in C^2(\overline{\Omega}), \quad \rho(x') > 0, \ \mu(x') > 0, \ \lambda(x') + \mu(x') > 0 \quad \text{for } x' \in \overline{\Omega}. \tag{1.3}$$

As for more details for the Lamé system, see for example, Chapter III of Duvaut and Lions [11] or Gurtin [14].

The Carleman estimate is an essential technique not only for the unique continuation, but also for solving the exact controllability and stabilizability (*e.g.*, Bellassoued [2–4], Imanuvilov [17], Imanuvilov and Yamamoto [25], Kazemi and Klibanov [32], Tataru [44], Zhang [51], and Lasiecka and Triggiani [37] as a related book) and the inverse problems (*e.g.*, Bukhgeim [6], Bukhgeim and Klibanov [8], Klibanov [35]). Thus the first main purpose of this paper is to establish Carleman estimates for system (1.1). Our method works, in principle, also for the three dimensional case but the arguments are more complicated and independent consideration is required. Thus in this paper, we will exclusively discuss the spatially two dimensional case. In a forthcoming paper, we will treat the three dimensional case.

Since the pioneering work [9] by Carleman, the theory of inequalities of Carleman's type has been rapidly developed and now many general results are available for a single partial differential equation (see [12, 15, 29, 30, 44]), while for strongly coupled systems of partial differential equations, the situation is more complicated and much less understood. To our best knowledge, the most general result for systems of partial differential equations is Calderon's uniqueness theorem (see *e.g.*, [12, 52]). The technique developed by Calderon, reduces the system of partial differential equations to a system of pseudo-differential operators of the first order:

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}x_2} = \mathbf{M}(x, D_{x_0}, D_{x_1})\mathbf{U} + \mathbf{F},$$

where $\mathbf{M}(x, D_{x_0}, D_{x_1})$ is a matrix pseudo-differential operator. Then by some change of variables $\mathbf{U} = \mathbf{S}(x, D_{x_0}, D_{x_1})\widetilde{\mathbf{U}}$, this matrix pseudo-differential operator $\mathbf{M}$ is reduced to $\mathbf{S}^{-1}\mathbf{M}\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{M}\mathbf{S}$ consists of blocks of a small size located on the main diagonal and that in each block the principal symbols of all the operators located below the main diagonal are zero. In order to construct the matrix $\mathbf{S}$, the eigenvalues and eigenvectors of the matrix $\mathbf{M}(x, \xi_0, \xi_1)$ should be smooth functions of the variables x and $\xi_0, \xi_1 \in \mathbb{R}^1$ and each eigenvalue should not change the multiplicity. This condition is restrictive, especially in the case where we are looking for a Carleman estimate near boundary, and therefore the choice for a variable x_2 is limited. For example the non-stationary Lamé system does not satisfy this condition, in general. On the other hand, for the stationary Lamé system, this method works well and produces the unique continuation result from an arbitrary open subset (see [10]). See also Imanuvilov and Yamamoto [27] as for a Carleman estimate for the stationary Lamé system.

As long as the non-stationary Lamé system is concerned, it is known that thanks to the special structure of the system, the functions $\operatorname{div}\mathbf{u}$ and $\operatorname{rot}\mathbf{u}$ satisfy scalar wave equations (modulo lower order terms). The system of partial differential equations for the functions $\mathbf{u}$, $\operatorname{div}\mathbf{u}$, $\operatorname{rot}\mathbf{u}$, is coupled *via* only first order terms. This allows us to apply the Carleman estimate for a scalar hyperbolic equation in the case where the function $\mathbf{u}$ has a compact support (see [13, 16, 19]).

The structure of our proof is in principle similar to Yamamoto [49]. That is, we work mainly with two hyperbolic equations depending on a parameter $s > 0$ for the functions $z_{\lambda+2\mu} \equiv \mathrm{e}^{s\phi}\operatorname{div}\mathbf{u}$ and $z_\mu \equiv \mathrm{e}^{s\phi}\operatorname{rot}\mathbf{u}$: $P_{\lambda+2\mu}(x, D, s)z_{\lambda+2\mu} = (\operatorname{div}\mathbf{f})e^{s\phi}$ and $P_\mu(x, D, s)z_\mu = (\operatorname{rot}\mathbf{f})\mathrm{e}^{s\phi}$. The main difficulty one should overcome, is that there are no boundary conditions for these functions. This problem is solved in the following way: outside an exceptional set in the contangent bundle $T^*(Q)$, the operators $P_{\lambda+2\mu}$ and P_μ can be microlocally factorized as the product of some function $\widetilde{\beta}(x)$ and two pseudo-differential operators of the first order:

$$P_\beta(x,D,s) = \widetilde{\beta} P_{-,\beta}(x,D,s) P_{+,\beta}(x,D,s),$$

where $\beta = \lambda + 2\mu$ or $= \mu$, $P_{\pm,\beta} = D_{x_2} - \Gamma^{\pm}_\beta(x, D_{x_0}, D_{x_1}, s)$, and x_2 is normal to the boundary $\partial\Omega$. Since the principal symbol of the operator $\Gamma^-_\beta(x,\xi_0,\xi_1,s)$ satisfies the inequality

$$-\mathrm{Im}\, \Gamma^-_\beta(x,\xi_0,\xi_1,s) \ge C|s|$$

with a constant $C > 0$, we have *a priori* estimates for $P_{+,\beta}(x,D,s) z_\beta|_{x_2=0}$ in an L^2-space. These estimates and the zero Dirichlet boundary condition yield the H^1-boundary estimates for z_β. The set on which we cannot factorize both the operators $P_\beta(x,D,s)$ into a product of the first order operators, has to be discussed separately.

Next we will prove a Carleman estimate with the $H^{-1}(Q)$ norm of the force $\mathbf{f}$ in the right hand side. The Carleman estimate with right hand side in $H^{-1}(Q)$-space was proved by Imanuvilov [18], Ruiz [43], for a scalar hyperbolic equation and by Imanuvilov and Yamamoto [26] for a parabolic equation. In this paper, by a method in [26], we will derive an $H^{-1}(Q)$-Carleman estimate (Th. 2.3) for (1.1) from a Carleman estimate (Th. 2.1) with H^1-norm.

Finally we consider an inverse problem of determining the coefficients λ, μ and ρ from one single measurement of the solution $\mathbf{u}$ in $(0,T)\times\omega$, where $\omega \subset \Omega$ is a suitable subdomain and $T > 0$ is sufficiently large. By our $H^{-1}(Q)$-Carleman estimate for the Lamé system, we will establish the uniqueness and the stability result for the inverse problem.

This paper is composed of nine sections and two appendices. In Section 2, we state Carleman estimates (Ths. 2.1–2.3) for functions which do not have compact supports but satisfy the zero Dirichlet boundary condition on $(0,T)\times\partial\Omega$. Theorem 2.1 is a Carleman estimate whose right hand side is estimated in H^1-space. Theorems 2.2 and 2.3 are Carleman estimates respectively with right hand sides in L^2-space and in H^{-1}-space. In Section 3, we will apply the H^{-1}-Carleman estimate (Th. 2.3), and prove the uniqueness and the conditional stability in the inverse problem with a single interior measurement. In Sections 4–8, we prove Theorem 2.1; In Section 4, we will reduce Theorem 2.1 to Lemma 4.1, and in Section 5, we further localize Lemma 4.1 by means of pseudo-differential operators. Dividing all the possible cases into three cases, in Sections 6–8, we will complete the proof of the localized estimate separately in those three cases. Finally Theorems 2.2 and 2.3 are proved in Section 9.

2. Carleman estimates for the two dimensional non-stationary Lamé system

Let us consider the two dimensional Lamé system

$$P\mathbf{u}(x_0,x') \equiv \rho(x')\partial^2_{x_0}\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x') \quad \text{in } Q, \tag{2.1}$$

$$\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{u}(T,x') = \partial_{x_0}\mathbf{u}(T,x') = \mathbf{u}(0,x') = \partial_{x_0}\mathbf{u}(0,x') = 0, \tag{2.2}$$

where $\mathbf{u} = (u_1,u_2)^T, \mathbf{f} = (f_1,f_2)^T$ are vector-valued functions, and the partial differential operator $L_{\lambda,\mu}$ is defined by (1.2). The coefficients ρ, λ, $\mu \in C^2(\overline{\Omega})$ are assumed to satisfy (1.3).

Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain (not necessary connected). Denote by $\vec{n}(x') = (n_1(x'), n_2(x'))$ and $\vec{\tau}(x')$ the outward unit normal vector and a unit tangential vector to $\partial\Omega$ at x' respectively, and set $\frac{\partial v}{\partial \vec{n}} = \nabla_{x'} v \cdot \vec{n}$ and $\frac{\partial v}{\partial \vec{\tau}} = \nabla_{x'} v \cdot \vec{\tau}$.

We set

$$Q_\omega = (0,T)\times\omega.$$

Let $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \xi_2)$. We set

$$\begin{cases} p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(|\xi_1|^2 + |\xi_2|^2), \\ p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x') + 2\mu(x'))(|\xi_1|^2 + |\xi_2|^2) \end{cases} \tag{2.3}$$

and $\nabla_\xi = (\partial_{\xi_0}, \partial_{\xi_1}, \partial_{\xi_2})$. For arbitrary smooth functions $\varphi(x,\xi)$ and $\psi(x,\xi)$, we define the Poisson bracket by the formula

$$\{\varphi, \psi\} = \sum_{j=0}^{2} (\partial_{\xi_j}\varphi)(\partial_{x_j}\psi) - (\partial_{\xi_j}\psi)(\partial_{x_j}\varphi).$$

We set $i = \sqrt{-1}$ and $\langle a, b\rangle = \sum_{k=1}^{3} a_k \overline{b}_k$ for $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{C}^3$.

We assume that the density ρ, the Lamé coefficients λ, μ and the domains Ω, ω satisfy the following condition (*cf.* [15]).

Condition 2.1. *There exists a function $\psi \in C^3(\overline{Q})$ such that $|\nabla'_x\psi| \neq 0$ on $\overline{Q \setminus Q_\omega}$ and*

$$\{p_k, \{p_k, \psi\}\}(x,\xi) > 0, \quad \forall k \in \{1,2\} \tag{2.4}$$

if $(x,\xi) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^3 \setminus \{0\})$ satisfies $p_k(x,\xi) = \{p_k, \psi\}(x,\xi) = 0$,

$$\{p_k(x, \xi - is\nabla\psi(x)), p_k(x, \xi + is\nabla\psi(x))\}/2is > 0, \quad \forall k \in \{1,2\} \tag{2.5}$$

if $(x,\xi,s) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^3 \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$p_k(x, \xi + is\nabla\psi(x)) = \langle \nabla_\xi p_k(x, \xi + is\nabla\psi(x)), \nabla\psi(x)\rangle = 0.$$

On the lateral boundary, we assume

$$\sqrt{\rho}\left|\psi_{x_0}\right| < \frac{\mu}{\sqrt{\lambda + 2\mu}}\left|\frac{\partial\psi}{\partial\vec{\tau}}\right| + \frac{\sqrt{\mu}\sqrt{\lambda+\mu}}{\sqrt{\lambda+2\mu}}\left|\frac{\partial\psi}{\partial\vec{n}}\right|, \quad p_1(x, \nabla\psi) < 0, \quad \forall x \in \overline{(0,T) \times \partial\Omega},$$

$$\text{and } \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{\overline{(0,T)\times(\partial\Omega\setminus\partial\omega)}} < 0. \tag{2.6}$$

Let $\psi(x)$ be the weight function in Condition 2.1. Using this function, we introduce the function $\phi(x)$ by

$$\phi(x) = \mathrm{e}^{\tau\psi(x)}, \quad \tau > 1, \tag{2.7}$$

where the parameter $\tau > 0$ will be fixed below. Denote

$$\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} \equiv \int_Q \Bigg(\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 + s|\nabla \mathrm{rot}\,\mathbf{u}|^2 + s^3|\mathrm{rot}\,\mathbf{u}|^2$$
$$+ s|\nabla \mathrm{div}\,\mathbf{u}|^2 + s^3|\mathrm{div}\,\mathbf{u}|^2 \Bigg) \mathrm{e}^{2s\phi}\mathrm{d}x, \tag{2.8}$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $\alpha_j \in \mathbb{N}_+ \cup \{0\}$, $\partial_x^\alpha = \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$.

Now we formulate our Carleman estimates as main results.

Theorem 2.1. *Let $\mathbf{f} \in (H^1(Q))^2$, and let the function ψ satisfy Condition 2.1. Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in (H^1(Q))^2 \cap L^2(0,T;(H^2(\Omega)^2)$ to problem (2.1)–(2.2), the following estimate holds true:*

$$\|\mathbf{u}\|^2_{Y(\phi,Q)} \triangleq \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} + s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$\le C_1(s^2\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\phi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s \ge s_0(\tau), \tag{2.9}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

Remark. In Carleman estimate (2.9), the weights which correspond to $\operatorname{rot}\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ are better than the weights which correspond to $\nabla\mathbf{u}$. This is a result of the special structure of the Lamé system which allows us to decouple into two wave equations for $\operatorname{rot}\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ (see (4.1)).

Next we formulate other two Carleman estimates where norms of the function $\mathbf{f}$ are taken in $(L^2(Q))^2$ and $H^{-1}(Q)$. In particular, the second of these two Carleman estimate is essential for obtaining our sharp uniqueness result in the inverse problem.

In addition to Condition 2.1, we assume

$$\partial_{x_0}\psi(T,x') < 0, \quad \partial_{x_0}\psi(0,x') > 0, \qquad \forall x' \in \overline{\Omega}. \tag{2.10}$$

Theorem 2.2. *Let $\mathbf{f} \in (L^2(Q))^2$ and let the function ψ satisfy (2.10) and Condition 2.1 and let function ϕ be given by (2.7). Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in (H^1(Q))^2$ to problem (2.1)–(2.2), the following estimate holds true:*

$$\int_Q (|\nabla\mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx \le C_1\left(\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}(|\nabla\mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx\right), \quad \forall s \ge s_0(\tau), \tag{2.11}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

Theorem 2.3. *Let $\mathbf{f} = \mathbf{f}_{-1} + \sum_{j=0}^2 \partial_{x_j}\mathbf{f}_j$ with $\mathbf{f}_{-1} \in (H^{-1}(Q))^2$ and $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2 \in (L^2(Q))^2$, and let the function ψ satisfy (2.10) and Condition 2.1 and let the function ϕ be given by (2.7). Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in (L^2(Q))^2$ to problem (2.1)–(2.2), the following estimate holds true:*

$$\int_Q |\mathbf{u}|^2 e^{2s\phi}dx \quad \le \quad C_1\left(\|\mathbf{f}_{-1}e^{s\phi}\|^2_{(H^{-1}(Q))^2} + \sum_{j=0}^2\|\mathbf{f}_j e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}|\mathbf{u}|^2 e^{2s\phi}dx\right), \quad \forall s \quad \ge \quad s_0(\tau), \tag{2.12}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

3. Determination of the density and the Lamé coefficients by a single measurement

Recall that the differential operator $L_{\lambda,\mu}$ is defined by (1.2). We assume (1.3) for ρ, λ, μ. By $\mathbf{u} = \mathbf{u}(\lambda,\mu,\rho,\mathbf{p},\mathbf{q},\eta)(x)$, we denote the sufficiently smooth solution to

$$\rho(x')(\partial_{x_0}^2 \mathbf{u})(x) = (L_{\lambda,\mu}\mathbf{u})(x), \quad x \in Q, \tag{3.1}$$

$$\mathbf{u}(x) = \eta(x), \qquad x \in (0,T) \times \partial\Omega, \tag{3.2}$$

$$\mathbf{u}(T/2, x') = \mathbf{p}(x'), \quad (\partial_{x_0}\mathbf{u})(T/2, x') = \mathbf{q}(x'), \quad x' \in \Omega, \tag{3.3}$$

where η, $\mathbf{p}$ and $\mathbf{q}$ are suitably given functions.

Let $\omega \subset \Omega$ be a suitably given subdomain. We consider

Inverse Problem. Let $\mathbf{p}_j, \mathbf{q}_j, \eta_j$, $1 \le j \le \mathcal{N}$, be appropriately given. Then determine $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)(x), \quad x \in Q_\omega \equiv (0,T) \times \omega. \tag{3.4}$$

Our formulation of the inverse problem is one with a finite number of observations (*i.e.*, $\mathcal{N} < \infty$). For inverse problems for the non-stationary Lamé equation by infinitely many boundary observations (*i.e.*, Dirichlet-to-Neumann map), we refer to Rachele [42], for example. A monograph of Yahkno [48] is concerned with the inverse problems for the Lamé system.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [8] proposed a remarkable method based on a Carleman estimate and established the uniqueness for similar inverse problems for scalar partial differential equations. As works after [8], see:

(1) Baudouin and Puel [5], Bukhgeim [6] for an inverse problem of determining potentials in Schrödinger equations;
(2) Imanuvilov and Yamamoto [21], Isakov [29, 30], Klibanov [35] for the corresponding inverse problems for parabolic equations;
(3) Bukhgeim, Cheng, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [22–24], Isakov [28–30], Isakov and Yamamoto [31], Khaĭdarov [33, 34], Klibanov [35], Puel and Yamamoto [40, 41], Yamamoto [50] for inverse problems of determining potentials, damping coefficients or the principal terms in scalar hyperbolic equations.

In particular, for inverse hyperbolic equations, we have to assume that the observation subdomain ω should satisfy a geometric condition and the observation time T has to be sufficiently large, which is a natural consequence of the hyperbolicity of the governing partial differential equations. Such situations are similar for our inverse problem for the Lamé system.

The Carleman estimate for the non-stationary Lamé equation was obtained for functions with compact supports, by Eller, Isakov, Nakamura and Tataru [13], Ikehata, Nakamura and Yamamoto [16], Imanuvilov, Isakov and Yamamoto [19], Isakov [28], and, by the methodology by [8] or [22], several uniqueness results are available for the inverse problem for Lamé system (3.1)–(3.3): [28] established the uniqueness in determining a single coefficient $\rho(x')$, using four measurements (*i.e.*, $\mathcal{N} = 4$).
Later [16] reduced the number of measurements to three (*i.e.*, $\mathcal{N} = 3$) for determining ρ.
Recently [19] proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, with only two measurements (*i.e.*, $\mathcal{N} = 2$).

In all the papers [16, 19, 28], the authors have to assume that $\partial\omega \supset \partial\Omega$ because the basic Carleman estimates require that solutions under consideration have compact supports in Q.

In [28] and [16], the key is a Carleman estimate where the right hand side is estimated in an L^2-space with the divergence and the estimate is proved *via* a system of hyperbolic equations of $\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ with the same principal terms. On the other hand, in [19], the key is a Carleman estimate with L^2-right hand side where $\|e^{s\phi}\operatorname{div}\mathbf{u}\|^2_{L^2(Q)}$ is reduced to $\|\mathbf{u}e^{s\phi}\|^2_{L^2(Q)}$ by means of an H^{-1}-Carleman estimate for a scalar hyperbolic equation. In [19], as its consequence, we can reduce $\mathcal{N}$ to take $\mathcal{N} = 2$ for simultaneous determination of all the three functions λ, μ, ρ.

In this section, we will further apply a Carleman estimate (Th. 2.3) whose right hand side is estimated in H^{-1} space to prove the conditional stability and the uniqueness with a single measurement: $\mathcal{N} = 1$. Thus the main achievements are

(1) the reduction of the number of observations, *i.e.*, $\mathcal{N} = 1$. The previous paper [19] requires $\mathcal{N} = 2$;
(2) the relaxation of the assumptions on the observation subdomain ω.

We will be able to prove similar results on the uniqueness and the stability in the three dimensional case on the basis of the corresponding Carleman estimate, and in a forthcoming paper, we will discuss the details.

In order to formulate our main result, we will introduce notations and an admissible set of unknown parameters λ, μ, ρ. Henceforth we set $(x', y') = \sum_{j=1}^{2} x_j y_j$ for $x' = (x_1, x_2)$ and $y' = (y_1, y_2)$. Let a subdomain $\omega \subset \Omega$ satisfy

$$\partial\omega \supset \{x' \in \partial\Omega; ((x' - y'), \vec{n}(x')) \geq 0\} \equiv \Gamma \tag{3.5}$$

with some $y' \notin \overline{\Omega}$.

Remark. Under Condition (3.5) on ω, we can prove the observability inequality for the wave equation $\partial_{x_0}^2 - \Delta$ if the observation time T is larger than $2\sup_{x'\in\Omega} |x' - y'|$ (*e.g.*, [39]). If (3.5) holds and $T > 0$ is sufficiently large, then ω and T satisfy the geometric optics condition in [1], so that we can prove observability inequalites. On the other hand, for solving inverse problems, a Carleman estimate is essential and observability inequalities are not directly applicable. If for other ω and $T > 0$, we will be able to verify Condition 2.1 similarly to Lemma 3.1 or [24], then we can establish similar results to Theorem 3.1 below. However searches for other ω and T are omitted here because those are lengthy.

Denote

$$d = \left(\sup_{x'\in\Omega} |x' - y'|^2 - \inf_{x'\in\Omega} |x' - y'|^2 \right)^{\frac{1}{2}}. \tag{3.6}$$

Next we define an admissible set of unknown coefficients λ, μ, ρ. Let $M_0 \geq 0$, $0 < \theta_0 \leq 1$ and $\theta_1 > 0$ be arbitrarily fixed and let us introduce the conditions on a function β:

$$\begin{cases} \beta(x') \geq \theta_1 > 0, \quad x' \in \overline{\Omega}, \\ \|\beta\|_{C^3(\overline{\Omega})} \leq M_0, \quad \frac{(\nabla_{x'}\beta(x'), (x'-y'))}{2\beta(x')} \leq 1 - \theta_0, \quad x' \in \overline{\Omega \setminus \omega}. \end{cases} \tag{3.7}$$

For fixed functions a, b, η on $\partial\Omega$ and $\mathbf{p}$, $\mathbf{q}$ in Ω, we set

$$\mathcal{W} = \mathcal{W}_{M_0,M_1,\theta_0,\theta_1,a,b} = \Bigg\{ (\lambda, \mu, \rho) \in (C^3(\overline{\Omega}))^3; \lambda = a, \mu = b \quad \text{on } \partial\Omega,$$
$$\frac{\lambda + 2\mu}{\rho}, \ \frac{\mu}{\rho} \text{ satisfy (3.7)}, \frac{\min\{\mu^2(x'), \mu(x')(\lambda+\mu)(x')\}}{\rho(x')(\lambda + 2\mu)(x')} \geq \theta_1 > 0, \ x' \in \overline{\Omega}, \ \|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)\|_{W^{7,\infty}(Q)} \leq M_1 \Bigg\} \tag{3.8}$$

where the constant M_1 is given.

Remark. The admissible set $\mathcal{W}$ is restrictive, but contains sufficiently many (λ, μ, ρ). We here give a subset of $\mathcal{W}$ which suggests that the set $\mathcal{W}$ is not very small. Let $\mathbf{p}, \mathbf{q} \in C^\infty(\overline{\Omega})$ be given arbitrarily and let us choose arbitrary positive constants a, b, ρ_0. Then, for the Dirichlet boundary data $\eta \in C^\infty([0, T] \times \partial\Omega)$, we assume

$$(\partial_{x_0}^{2j}\eta)(T/2, x') = \left(\frac{1}{\rho_0} L_{a,b}\right)^j \mathbf{p}(x'), \quad (\partial_{x_0}^{2j+1}\eta)(T/2, x') = \left(\frac{1}{\rho_0} L_{a,b}\right)^j \mathbf{q}(x'),$$
$$x' \in \partial\Omega, \ 0 \leq j \leq N_0.$$

Here N_0 is a sufficiently large natural number.

We set

$$\mathcal{W}_0 = \Bigg\{ (\lambda,\mu,\rho) \in (C^\infty(\overline{\Omega}))^3; \lambda = a, \mu = b, \rho = \rho_0 \quad \text{in a neighbourhood of } \partial\Omega,$$

$$\left(\frac{\lambda+2\mu}{\rho}\right)(x') > \theta_1, \quad \left(\frac{\mu}{\rho}\right)(x') > \theta_1, \quad \frac{\min\{\mu^2(x'), \mu(x')(\lambda+\mu)(x')\}}{\rho(x')(\lambda+2\mu)(x')} > \theta_1, \qquad x' \in \overline{\Omega},$$

$$\|\rho\|_{C^\infty(\overline{\Omega})}, \|\lambda\|_{C^\infty(\overline{\Omega})}, \|\mu\|_{C^\infty(\overline{\Omega})} < M_0,$$

$$\left\| \frac{\rho}{2(\lambda+2\mu)} \nabla \left(\frac{\lambda+2\mu}{\rho}\right)\right\|_{C(\overline{\Omega})}, \left\| \frac{\rho}{2\mu}\nabla\left(\frac{\mu}{\rho}\right)\right\|_{C(\overline{\Omega})} < \frac{1-\theta_0}{\sup_{x'\in\Omega\setminus\omega}|x'-y'|} \Bigg\}.$$

The set $\mathcal{W}_0$ is not empty and is not "thin". Then, since the conditions on η yield compatibility conditions of sufficient orders with $\mathbf{p}, \mathbf{q}$ at $\{T/2\}\times\partial\Omega$ for any $(\lambda,\mu,\rho) \in \mathcal{W}_0$, we can prove by an argument similar to [pp. 1369–1370, 19] that $\mathbf{u}(\lambda,\mu,\rho,\mathbf{p},\mathbf{q},\eta) \in C^7(\overline{Q})$ and there exists a constant $M_1 = M_1(a,b,\rho_0,\theta_1,M_0,\mathbf{p},\mathbf{q},\eta) > 0$ such that

$$\|\mathbf{u}(\lambda,\mu,\rho,\mathbf{p},\mathbf{q},\eta)\|_{C^7(\overline{Q})} \le M_1$$

for all $(\lambda,\mu,\rho) \in \mathcal{W}_0$. Therefore we see that $\mathcal{W}_0$ is a subset of $\mathcal{W} = \mathcal{W}_{M_0,M_1,\theta_0,\theta_1,a,b}$ defined by (3.8). Thus, after a suitable choice of η, we can conclude that the admissible set $\mathcal{W}$ can contain sufficiently many elements.

It is rather restrictive that $\frac{\lambda+2\mu}{\rho}$ and $\frac{\mu}{\rho}$ should satisfy (3.7), which is one possible sufficient condition for the pseudoconvexity (*i.e.*, Condition (2.1)). We can relax Condition (3.7) to a more generous condition which can be related with a necessary condition enabling us to establish a Carleman estimate. See Imanuvilov, Isakov and Yamamoto [20], where a scalar hyperbolic equation is discussed but the modification to the Lamé system is straightforward. Such a relaxed condition guarantees that the geodesics which are generated by the hyperbolic equations defined by (2.3), cannot remain on the level sets given by the weight function ϕ. In particular, by [20], we can replace the condition that $\frac{\lambda+2\mu}{\rho}$ and $\frac{\mu}{\rho}$ satisfy (3.7) by one that the Hessians

$$\left(\partial_{x_j}\partial_{x_k}\left(\frac{\rho}{\mu}\right)^{\frac12}\right)_{1\le j,k\le 2}, \quad \left(\partial_{x_j}\partial_{x_k}\left(\frac{\rho}{\lambda+2\mu}\right)^{\frac12}\right)_{1\le j,k\le 2}$$

are non-negative and $\left|\nabla\left(\frac{\rho}{\mu}\right)\right| \ne 0$ and $\left|\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\right| \ne 0$ on $\overline{\Omega}$.

We choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \quad \theta_1 \inf_{x'\in\Omega}|x'-y'|^2 - \theta d^2 > 0. \tag{3.9}$$

Here we note that since $y' \notin \overline{\Omega}$, such $\theta > 0$ exists.

Let $[\cdot]_1$ denote the first component of the vector under consideration and let E_2 the 2×2 identity matrix. We note that $(L_{\lambda,\mu}\mathbf{p})(x')$, etc., are 2-column vectors for 2-column vectors $\mathbf{p}$. Let (λ,μ,ρ) be an arbitrary element of $\mathcal{W}$.

Now we are ready to state

Theorem 3.1. *We assume that*

$$\Omega = \{(x_1,x_2);\ \gamma_0(x_2) < x_1 < \gamma_1(x_2),\ x_2 \in I\} \tag{3.10}$$

with some open interval I and $\gamma_0, \gamma_1 \in C(\overline{I})$. Moreover we assume that the functions $\mathbf{p} = (p_1,p_2)^T$ and $\mathbf{q} = (q_1,q_2)^T$ satisfy

$$\det\begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & (\mathrm{div}\mathbf{p}(x'))E_2 & (\nabla_{x'}\mathbf{p}(x') + (\nabla_{x'}\mathbf{p}(x'))^T)(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & (\mathrm{div}\mathbf{q}(x'))E_2 & (\nabla_{x'}\mathbf{q}(x') + (\nabla_{x'}\mathbf{q}(x'))^T)(x'-y') \end{pmatrix} \ne 0,\ \forall x' \in \overline{\Omega}, \tag{3.11}$$

$$\det \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & \nabla_{x'}\mathbf{p}(x') + (\nabla_{x'}\mathbf{p}(x'))^T & (\mathrm{div}\mathbf{p})(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & \nabla_{x'}\mathbf{q}(x') + (\nabla_{x'}\mathbf{q}(x'))^T & (\mathrm{div}\mathbf{q})(x'-y') \end{pmatrix} \ne 0, \, \forall x' \in \overline{\Omega}, \tag{3.12}$$

$$\begin{aligned} & x_1 - y_1 \ne 0, \\ & [L_{\lambda,\mu}\mathbf{q}]_1(\partial_{x_1}p_2 + \partial_{x_2}p_1)(x') \ne [L_{\lambda,\mu}\mathbf{p}]_1(\partial_{x_1}q_2 + \partial_{x_2}q_1)(x'), \quad \forall x' \in \overline{\Omega} \end{aligned} \tag{3.13}$$

and that

$$T > \frac{2}{\sqrt{\theta}} d. \tag{3.14}$$

Then there exist constants $\kappa = \kappa(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) \in (0,1)$ and $C_1 = C_1(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ such that

$$\begin{aligned} & \|\widetilde{\lambda} - \lambda\|_{L^2(\Omega)} + \|\widetilde{\mu} - \mu\|_{L^2(\Omega)} + \|\widetilde{\rho} - \rho\|_{H^{-1}(\Omega)} \\ \le & C_1 \|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta) - \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)\|^{\kappa}_{H^4(0,T;(L^2(\omega))^2)} \end{aligned}$$

for any $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}) \in \mathcal{W}$.

As for the corresponding results on the stability for inverse problems for scalar hyperbolic equations, we refer to [22–24] for example.

Our stability and uniqueness result requires only one measurement: $\mathcal{N} = 1$. For the determination of the three coefficients by a single measurement, we have to choose initial data which satisfy stronger Conditions (3.11)–(3.13) than in the case of $\mathcal{N} \ge 2$. Thus Conditions (3.11)–(3.13) are not generic properties and should be realized in a non-physical way by us. Moreover, as the following example shows, we can take $\mathbf{p}$ and $\mathbf{q}$ satisfying those.

Example of Ω, p, q meeting (3.11)–(3.13). We assume that λ, μ are positive constants and that $\{(x_1, x_2) \in \overline{\Omega};\, x_2 = y_2\}$ and $\{(x_1, x_2) \in \overline{\Omega};\, x_1 = y_1\}$ are empty. Noting that the fourth columns of the matrices in (3.11) and (3.12) have $x' - y'$ as factors, we will take quadratic functions in x'. For example, we take

$$\mathbf{p}(x') = \begin{pmatrix} 0 \\ (x_1 - y_1)(x_2 - y_2) \end{pmatrix}, \quad \mathbf{q}(x') = \begin{pmatrix} (x_2 - y_2)^2 \\ 0 \end{pmatrix}.$$

Then we can verify that (3.11)–(3.13) are all satisfied.

Remark 3.1. In place of (3.10), let us assume

$$\Omega = \left\{ (x_1, x_2);\, \widetilde{\gamma}_0(x_1) < x_2 < \widetilde{\gamma}_1(x_1),\, x_1 \in \widetilde{I} \right\} \tag{3.10'}$$

with some open interval $\widetilde{I}$. Then, after replacing (3.13) by

$$\begin{aligned} & x_2 - y_2 \ne 0, \\ & [L_{\lambda,\mu}\mathbf{q}]_2(\partial_{x_1}p_2 + \partial_{x_2}p_1)(x') \ne [L_{\lambda,\mu}\mathbf{p}]_2(\partial_{x_1}q_2 + \partial_{x_2}q_1)(x'), \quad x' \in \overline{\Omega}, \end{aligned} \tag{3.13'}$$

the conclusion of Theorem 3.1 holds under Conditions (3.11), (3.12) and (3.14). Moreover in the case when Ω is a more general smooth domain, we can prove the conditional stability in our inverse problem under other conditions on $\omega \subset \Omega$. We will omit the details, for the sake of compact description of the proof.

We set

$$\psi(x) = |x' - y'|^2 - \theta \left(x_0 - \frac{T}{2} \right)^2, \quad \phi(x) = \mathrm{e}^{\tau\psi(x)}, \quad x = (x_0, x') \in Q. \tag{3.15}$$

First we show

Lemma 3.1. *Let $(\lambda, \mu, \rho) \in \mathcal{W}$, and let us assume (3.9) and (3.14). Then, for sufficiently large $\tau > 0$, the function ψ given by (3.15) satisfies Conditions 2.1 and (2.10). Therefore the conclusion of Theorem 2.3 holds and the constants $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ in (2.12) can be taken independently of $(\lambda, \mu, \rho) \in \mathcal{W}$.*

Proof. Conditions (2.10) and the third condition in (2.6) are directly verified by means of (3.5). Conditions (2.4) and (2.5) can be verified by the same way as in Imanuvilov and Yamamoto [24], for example. Finally we have to verify the first and second conditions in (2.6). Without loss of generality, we may assume that $T = \frac{2d}{\sqrt{\theta}} + \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Because if Theorem 3.1 is proved for this value of T, then the conclusion is true for any $\widetilde{T} > T$. Then, by noting that

$$\left(\left| \frac{\partial \psi}{\partial \vec{\tau}} \right|^2 + \left| \frac{\partial \psi}{\partial \vec{n}} \right|^2 \right)^{\frac{1}{2}} = |\nabla_{x'} \psi|$$

and that the right hand side of the first inequality in (2.6) is greater than or equal to

$$\min \left\{ \frac{\mu(x')}{\sqrt{(\lambda + 2\mu)(x')}}, \frac{\sqrt{\mu(\lambda + \mu)(x')}}{\sqrt{(\lambda + 2\mu)(x')}} \right\} \left(\left| \frac{\partial \psi}{\partial \vec{\tau}} \right|^2 + \left| \frac{\partial \psi}{\partial \vec{n}} \right|^2 \right)^{\frac{1}{2}}$$

in terms of (3.8), it suffices to verify

$$-(\theta(x_0 - T/2))^2 + \theta_1 |x' - y'|^2 > 0$$

for $x \in [0, T] \times \partial\Omega$. In fact, by means of the second inequality in (3.9), we have

$$\begin{aligned} 4\theta_1 |x' - y'|^2 - 4\theta^2 \left(x_0 - \frac{T}{2} \right)^2 &\geq 4\theta_1 \inf_{x' \in \Omega} |x' - y'|^2 - \theta(\theta T^2) \\ &\geq 4\theta_1 \inf_{x' \in \Omega} |x' - y'|^2 - \theta(2d + \varepsilon\sqrt{\theta})^2 \\ &> 0 \end{aligned}$$

because $\varepsilon > 0$ is sufficiently small. The uniformity of the constants $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ follows similarly to [19]. Thus the proof of Lemma 3.1 is complete. □

Next we prove a Carleman estimate for a first order partial differential operator

$$(P_0 g)(x') = \sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} g(x'),$$

where $p_{0,j} \in C^1(\overline{\Omega})$, $j = 1, 2$.

Lemma 3.2. *We assume*

$$\sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} \phi(T/2, x') > 0, \quad x' \in \overline{\Omega}. \tag{3.16}$$

Then there exists a constant $\tau_0 > 0$ such that for all $\tau > \tau_0$, there exist $s_0 = s_0(\tau) > 0$ and $C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0$ such that

$$\int_\Omega s^2 |g|^2 \mathrm{e}^{2s\phi(T/2, x')} \mathrm{d}x' \leq C_2 \int_\Omega |P_0 g|^2 \mathrm{e}^{2s\phi(T/2, x')} \mathrm{d}x'$$

for all $s > s_0$ and $g \in H^1(\Omega)$ satisfying

$$g = 0 \quad on \left\{ x' \in \partial\Omega; \sum_{j=1}^{2} p_{0,j}(x')n_j(x') \geq 0 \right\}.$$

Lemma 3.3. *We assume*

$$\sum_{j=1}^{2} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') \neq 0, \quad x' \in \overline{\Omega}.$$

Then the conclusion of Lemma 3.2 is true for all $s > s_0$ and $g \in H_0^1(\Omega)$.

Proof of Lemma 3.2. For simplicity, we set $\phi_0(x') = \phi(T/2, x')$ and $w = \mathrm{e}^{s\phi_0}g$, $Q_0 w = \mathrm{e}^{s\phi_0}P_0(\mathrm{e}^{-s\phi_0}w)$. Then

$$\int_\Omega |P_0 g|^2 \mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' = \int_\Omega |Q_0 w|^2 \mathrm{d}x'.$$

We have

$$Q_0 w = P_0 w - s q_0 w,$$

where $q_0(x') = \sum_{j=1}^{2} p_{0,j}(x')\partial_{x_j}\phi_0(x')$. Therefore, by (3.16) and integration by parts, we obtain

$$\begin{aligned}
\|Q_0 w\|^2_{L^2(\Omega)} &= \|P_0 w\|^2_{L^2(\Omega)} + s^2\|q_0 w\|^2_{L^2(\Omega)} - 2s\int_\Omega \sum_{j=1}^{2} p_{0,j}(\partial_{x_j} w) q_0 w \mathrm{d}x' \\
&\geq s^2 \int_\Omega q_0(x')^2 w^2(x')\mathrm{d}x' - s\int_\Omega \sum_{j=1}^{2} p_{0,j} q_0 \partial_{x_j}(w^2)\mathrm{d}x' \\
&\geq C_0 s^2 \int_\Omega w^2(x')\mathrm{d}x' - s\int_{\partial\Omega} \sum_{j=1}^{2} p_{0,j} q_0 w^2 n_j \mathrm{d}S + s\int_\Omega \sum_{j=1}^{2} \partial_{x_j}(p_{0,j} q_0) w^2 \mathrm{d}x' \\
&\geq (C_2 s^2 - C_3 s)\int_\Omega w^2 \mathrm{d}x' - s\int_{\partial\Omega \cap \{\sum_{j=1}^{2} p_{0,j} n_j \leq 0\}} \left(\sum_{j=1}^{2} p_{0,j} n_j \right) q_0 w^2 \mathrm{d}S.
\end{aligned}$$

By (3.16), we have $q_0 > 0$ on $\partial\Omega$, so that the right hand side is greater than or equal to $(C_2 s^2 - C_3 s)\int_\Omega w^2 \mathrm{d}x'$. Thus by taking $s > 0$ sufficiently large, the proof of Lemma 3.2 is complete. □

The proof of Lemma 3.3 is similar, thanks to the fact that the integral on $\partial\Omega$ vanishes for $g \in H_0^1(\Omega)$.

Now we proceed to

Proof of Theorem 3.1. The proof is similar to Isakov, Imanuvilov and Yamamoto [19], Imanuvilov and Yamamoto [22–24] and the new ingredient is an H^{-1}-Carleman estimate (Lem. 3.1) . Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}.$$

In (3.13), without loss of generality, we may assume that

$$x_1 - y_1 > 0, \qquad (x_1, x_2) \in \overline{\Omega}.$$

Then we set

$$F(x_1,x_2)=\int_{\gamma_1(x_2)}^{x_1} f(\xi,x_2)\mathrm{d}\xi,\quad (x_1,x_2)\in\Omega. \tag{3.17}$$

If $x_1-y_1<0$ for $(x_1,x_2)\in\overline{\Omega}$, then it is sufficient to replace (3.17) by $F(x_1,x_2)=\int_{\gamma_0(x_2)}^{x_1} f(\xi,x_2)\mathrm{d}\xi$, $(x_1,x_2)\in\Omega$. Then

$$\widetilde{\rho}\partial_{x_0}^2\mathbf{y}=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{y}+G\mathbf{u}\quad\text{in } Q \tag{3.18}$$

and

$$\mathbf{y}\left(\frac{T}{2},x'\right)=\partial_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)=0,\qquad x'\in\Omega \tag{3.19}$$

and

$$\mathbf{y}=0\qquad\text{on } (0,T)\times\partial\Omega. \tag{3.20}$$

Here we set

$$\begin{aligned}G\mathbf{u}(x)=-\partial_{x_1}F(x')\partial_{x_0}^2\mathbf{u}(x)+(g+h)(x')\nabla_{x'}(\operatorname{div}\mathbf{u})(x)+h(x')\Delta\mathbf{u}(x)\\ +(\operatorname{div}\mathbf{u})(x)\nabla_{x'}g(x')+(\nabla_{x'}\mathbf{u}(x)+(\nabla_{x'}\mathbf{u}(x))^T)\nabla_{x'}h(x').\end{aligned} \tag{3.21}$$

By (3.14), we have the inequality $\frac{\theta T^2}{4}>d^2$. Therefore, by (3.6) and Definition (3.15) of the function ϕ, we have

$$\phi(T/2,x')\ge d_1,\quad \phi(0,x')=\phi(T,x')<d_1,\qquad x'\in\overline{\Omega}$$

with $d_1=\exp(\tau\inf_{x'\in\Omega}|x'-y'|^2)$. Thus, for given $\varepsilon>0$, we can choose a sufficiently small $\delta=\delta(\varepsilon)>0$ such that

$$\phi(x)\ge d_1-\varepsilon,\quad x\in\left[\frac{T}{2}-\delta,\frac{T}{2}+\delta\right]\times\overline{\Omega} \tag{3.22}$$

and

$$\phi(x)\le d_1-2\varepsilon,\quad x\in([0,2\delta]\cup[T-2\delta,T])\times\overline{\Omega}. \tag{3.23}$$

In order to apply Lemma 3.1, it is necessary to introduce a cut-off function χ satisfying $0\le\chi\le1$, $\chi\in C^\infty(\mathbb{R})$ and

$$\chi=\begin{cases}0 & \text{on } [0,\delta]\cup[T-\delta,T],\\ 1 & \text{on } [2\delta,T-2\delta].\end{cases} \tag{3.24}$$

In the sequel, $C_j>0$ denote generic constants depending on s_0, τ, M_0, M_1, θ_0, θ_1, η, Ω, T, y', ω, χ and $\mathbf{p}$, $\mathbf{q}$, ε, δ, but independent of $s>s_0$.

Setting $\mathbf{z}_1=\chi\partial_{x_0}^2\mathbf{y}$, $\mathbf{z}_2=\chi\partial_{x_0}^3\mathbf{y}$ and $\mathbf{z}_3=\chi\partial_{x_0}^4\mathbf{y}$, we have

$$\left\{\begin{aligned}&\widetilde{\rho}\partial_{x_0}^2\mathbf{z}_1=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_1+\chi G(\partial_{x_0}^2\mathbf{u})+2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^3\mathbf{y}+\widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^2\mathbf{y},\\ &\widetilde{\rho}\partial_{x_0}^2\mathbf{z}_2=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_2+\chi G(\partial_{x_0}^3\mathbf{u})+2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^4\mathbf{y}+\widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^3\mathbf{y},\\ &\widetilde{\rho}\partial_{x_0}^2\mathbf{z}_3=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_3+\chi G(\partial_{x_0}^4\mathbf{u})+2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^5\mathbf{y}+\widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^4\mathbf{y}\quad\text{in } Q.\end{aligned}\right. \tag{3.25}$$

Henceforth we set

$$\mathcal{E}=\int_{Q_\omega}(|\partial_{x_0}^2\mathbf{y}|^2+|\partial_{x_0}^3\mathbf{y}|^2+|\partial_{x_0}^4\mathbf{y}|^2)\mathrm{e}^{2s\phi}\mathrm{d}x.$$

Noting that $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (3.24) and Lemma 3.1, we can apply Theorem 2.3 to (3.25), so that

$$\begin{aligned}
&\sum_{j=2}^{4} \int_Q |\partial_{x_0}^j \mathbf{y}|^2 \chi^2 \mathrm{e}^{2s\phi} \mathrm{d}x \le C_5(\|F\mathrm{e}^{s\phi}\|^2_{L^2(Q)} + \|g\mathrm{e}^{s\phi}\|^2_{L^2(Q)} + \|h\mathrm{e}^{s\phi}\|^2_{L^2(Q)}) \\
&+ C_5 \sum_{j=3}^{5} \|(\partial_{x_0}\chi)(\partial_{x_0}^j \mathbf{y})\mathrm{e}^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} \\
&+ C_5 \sum_{j=2}^{4} \|(\partial_{x_0}^2\chi)(\partial_{x_0}^j \mathbf{y})\mathrm{e}^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + C_5\mathcal{E} \\
&\le C_6(\|F\mathrm{e}^{s\phi}\|^2_{L^2(Q)} + \|g\mathrm{e}^{s\phi}\|^2_{L^2(Q)} + \|h\mathrm{e}^{s\phi}\|^2_{L^2(Q)}) + C_6\mathrm{e}^{2s(d_1-2\varepsilon)} + C_7\mathcal{E}
\end{aligned} \tag{3.26}$$

for all large $s > 0$.

On the other hand,

$$\begin{aligned}
&\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(T/2, x')|^2 \mathrm{e}^{2s\phi(T/2,x')} \mathrm{d}x' \\
&= \int_0^{T/2} \frac{\partial}{\partial x_0} \left(\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(x_0, x')|^2 \chi(x_0)^2 \mathrm{e}^{2s\phi} \mathrm{d}x' \right) \mathrm{d}x_0 \\
&= \int_0^{T/2} \int_\Omega 2((\partial_{x_0}^3 \mathbf{y}) \cdot (\partial_{x_0}^2 \mathbf{y}))\chi^2 \mathrm{e}^{2s\phi} \mathrm{d}x \\
&\quad + 2s \int_0^{T/2} \int_\Omega |\partial_{x_0}^2 \mathbf{y}|^2 \chi^2 (\partial_{x_0}\phi) \mathrm{e}^{2s\phi} \mathrm{d}x + \int_0^{T/2} \int_\Omega |\partial_{x_0}^2 \mathbf{y}|^2 (\partial_{x_0}(\chi^2)) \mathrm{e}^{2s\phi} \mathrm{d}x \\
&\le C_7 \int_Q s\chi^2 (|\partial_{x_0}^3 \mathbf{y}|^2 + |\partial_{x_0}^2 \mathbf{y}|^2) \mathrm{e}^{2s\phi} \mathrm{d}x + C_7 \mathrm{e}^{2s(d_1-2\varepsilon)}.
\end{aligned}$$

Therefore (3.26) yields

$$\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(T/2, x')|^2 \mathrm{e}^{2s\phi(T/2,x')} \mathrm{d}x' \le C_8 s \int_Q (|F|^2 + |g|^2 + |h|^2) \mathrm{e}^{2s\phi} \mathrm{d}x + C_8 s \mathrm{e}^{2s(d_1-2\varepsilon)} + C_8 s\mathcal{E} \tag{3.27}$$

for all large $s > 0$. Similarly we can estimate $\int_\Omega |(\partial_{x_0}^3 \mathbf{y})(T/2, x')|^2 \mathrm{e}^{2s\phi(T/2,x')} \mathrm{d}x'$ to obtain

$$\begin{aligned}
&\int_\Omega (|(\partial_{x_0}^2 \mathbf{y})(T/2, x')|^2 + |(\partial_{x_0}^3 \mathbf{y})(T/2, x')|^2) \mathrm{e}^{2s\phi(T/2,x')} \mathrm{d}x' \\
&\le C_9 s \int_Q (|F|^2 + |g|^2 + |h|^2) \mathrm{e}^{2s\phi} \mathrm{d}x + C_9 s \mathrm{e}^{2s(d_1-2\varepsilon)} + C_9 s\mathcal{E}
\end{aligned} \tag{3.28}$$

for all large $s > 0$.

Now first order partial differential equations satisfied by h, g and F are going to be considered. That is, by (3.18), (3.19) and $\mathbf{u}, \mathbf{v} \in W^{7,\infty}(Q)$, we have

$$\widetilde{\rho}\partial_{x_0}^2 \mathbf{y} \left(\frac{T}{2}, x' \right) = G\mathbf{u} \left(\frac{T}{2}, x' \right), \quad \widetilde{\rho}\partial_{x_0}^3 \mathbf{y} \left(\frac{T}{2}, x' \right) = G\partial_{x_0}\mathbf{u} \left(\frac{T}{2}, x' \right). \tag{3.29}$$

Then, setting

$$
\begin{cases}
-\frac{1}{\rho}L_{\lambda,\mu}\mathbf{p} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad -\frac{1}{\rho}L_{\lambda,\mu}\mathbf{q} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \\
\operatorname{div}\mathbf{p} = b_1, \qquad \operatorname{div}\mathbf{q} = b_2, \\
\nabla_{x'}\mathbf{p} + (\nabla_{x'}\mathbf{p})^T = \begin{pmatrix} c_1 & d_1 \\ d_1 & e_1 \end{pmatrix}, \quad \nabla_{x'}\mathbf{q} + (\nabla_{x'}\mathbf{q})^T = \begin{pmatrix} c_2 & d_2 \\ d_2 & e_2 \end{pmatrix}, \\
\widetilde{\rho}\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{p}) - h\Delta\mathbf{p} = \begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix}, \\
\widetilde{\rho}\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{q}) - h\Delta\mathbf{q} = \begin{pmatrix} G_{12} \\ G_{22} \end{pmatrix},
\end{cases}
\tag{3.30}
$$

we rewrite (3.29) as

$$
\begin{pmatrix} a_{11} & b_1 & 0 \\ a_{21} & 0 & b_1 \\ a_{12} & b_2 & 0 \\ a_{22} & 0 & b_2 \end{pmatrix}
\begin{pmatrix} \partial_{x_1}F \\ \partial_{x_1}g \\ \partial_{x_2}g \end{pmatrix}
=
\begin{pmatrix} G_{11} - c_1\partial_{x_1}h - d_1\partial_{x_2}h \\ G_{21} - d_1\partial_{x_1}h - e_1\partial_{x_2}h \\ G_{12} - c_2\partial_{x_1}h - d_2\partial_{x_2}h \\ G_{22} - d_2\partial_{x_1}h - e_2\partial_{x_2}h \end{pmatrix}.
\tag{3.31}
$$

Because linear system (3.31) possesses a solution $(\partial_{x_1}F, \partial_{x_1}g, \partial_{x_2}g)$, the coefficient matrix must satisfy

$$
\det \begin{pmatrix} a_{11} & b_1 & 0 & G_{11} - c_1\partial_{x_1}h - d_1\partial_{x_2}h \\ a_{21} & 0 & b_1 & G_{21} - d_1\partial_{x_1}h - e_1\partial_{x_2}h \\ a_{12} & b_2 & 0 & G_{12} - c_2\partial_{x_1}h - d_2\partial_{x_2}h \\ a_{22} & 0 & b_2 & G_{22} - d_2\partial_{x_1}h - e_2\partial_{x_2}h \end{pmatrix} = 0,
$$

that is,

$$
(\partial_{x_1}h)\det \begin{pmatrix} a_{11} & b_1 & 0 & c_1 \\ a_{21} & 0 & b_1 & d_1 \\ a_{12} & b_2 & 0 & c_2 \\ a_{22} & 0 & b_2 & d_2 \end{pmatrix} + (\partial_{x_2}h)\det \begin{pmatrix} a_{11} & b_1 & 0 & d_1 \\ a_{21} & 0 & b_1 & e_1 \\ a_{12} & b_2 & 0 & d_2 \\ a_{22} & 0 & b_2 & e_2 \end{pmatrix} = \det \begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix},
\tag{3.32}
$$

by the linearity of the determinant. Under Condition (3.11), taking into consideration $h = \mu - \widetilde{\mu} = 0$ on $\partial\Omega$ and considering (3.32) as a first order partial differential operator in h, we apply Lemma 3.3, so that

$$
\begin{aligned}
s^2 \int_\Omega |h|^2 e^{2s\phi(T/2,x')}\mathrm{d}x' &\le C_{10} \left\| \det \begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix} \mathrm{e}^{s\phi(T/2,\cdot)} \right\|^2_{L^2(\Omega)} \\
&\le C_{11} \int_\Omega \left(\left| \partial_{x_0}^2\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left| \partial_{x_0}^3\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right) \mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \\
&\qquad + C_{11} \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')}\mathrm{d}x',
\end{aligned}
\tag{3.33}
$$

in view of (3.30). We rewrite (3.29) as

$$
\begin{pmatrix} a_{11} & c_1 & d_1 \\ a_{21} & d_1 & e_1 \\ a_{12} & c_2 & d_2 \\ a_{22} & d_2 & e_2 \end{pmatrix}
\begin{pmatrix} \partial_{x_1}F \\ \partial_{x_1}h \\ \partial_{x_2}h \end{pmatrix}
=
\begin{pmatrix} G_{11} - b_1\partial_{x_1}g \\ G_{21} - b_1\partial_{x_2}g \\ G_{12} - b_2\partial_{x_1}g \\ G_{22} - b_2\partial_{x_2}g \end{pmatrix}
$$

and, using (3.12), we can similarly deduce

$$s^2\int_\Omega |g|^2 \mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \le C_{12}\int_\Omega \left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' + C_{12}\int_\Omega (|g|^2+|h|^2)e^{2s\phi(T/2,x')}\mathrm{d}x' \tag{3.34}$$

for all large $s>0$. By (3.33) and (3.34), for sufficiently large $s>0$, we have

$$s^2\int_\Omega (|g|^2+|h|^2)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \le C_{13}\int_\Omega \left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x'. \tag{3.35}$$

Moreover, eliminating $\partial_{x_2}h$ in the first and the third rows in (3.31) and using (3.13), we have

$$\partial_{x_1}\left(F+\frac{d_2b_1-d_1b_2}{d_2a_{11}-d_1a_{12}}g+\frac{d_2c_1-d_1c_2}{d_2a_{11}-d_1a_{12}}h\right) = \frac{d_2G_{11}-d_1G_{12}}{d_2a_{11}-d_1a_{12}}+g\partial_{x_1}\left(\frac{d_2b_1-d_1b_2}{d_2a_{11}-d_1a_{12}}\right)+h\partial_{x_1}\left(\frac{d_2c_1-d_1c_2}{d_2a_{11}-d_1a_{12}}\right).$$

By (3.10) and (3.17), if $n_1(x')\ge 0$, then $x_1=\gamma_1(x_2)$, that is, we have: $F(x_1,x_2)=0$ for $n_1(x')\ge 0$. Therefore, noting $g=h=0$ on $\partial\Omega$ and setting $p_{0,1}=1$, $p_{0,2}=0$ in Lemma 3.2, we can apply the lemma. Thus, in view of (3.35) and (3.30), we obtain

$$s^2\int_\Omega |F|^2 \mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \le C_{14}\int_\Omega \left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \tag{3.36}$$

for all large $s>0$. Consequently, substituting (3.35) and (3.36) into (3.28) and using $\phi(T/2,x')\ge\phi(x_0,x')$ for $(x_0,x')\in Q$, we obtain

$$\int_\Omega (|F|^2+|g|^2+|h|^2)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' \le \frac{C_{15}T}{s}\int_\Omega (|F|^2+|g|^2+|h|^2)\mathrm{e}^{2s\phi(T/2,x')}\mathrm{d}x' + \frac{C_{15}}{s}\mathrm{e}^{2s(d_1-2\varepsilon)}+\frac{C_{15}}{s}\mathcal{E}$$

for all large $s>0$. Taking $s>0$ sufficiently large and noting $\mathrm{e}^{2s\phi(T/2,x')}\ge \mathrm{e}^{2sd_1}$ for $x'\in\overline{\Omega}$, we obtain

$$\int_\Omega (|F|^2+|g|^2+|h|^2)\mathrm{d}x' \le C_{16}\mathrm{e}^{-4s\varepsilon}+C_{17}\mathrm{e}^{2sC_{18}}\mathcal{E} \tag{3.37}$$

for all large $s>s_0$: a constant which is dependent on τ, but independent of s. Next we take in (3.37) instead of the constant C_{17} the constant $C_{17}e^{2s_0C_{18}}$. Now this inequality holds true for all $s>0$.

Now we choose $s>0$ such that

$$\mathrm{e}^{2sC_{16}}\mathcal{E}=\mathrm{e}^{-4s\varepsilon},$$

that is,

$$s=-\frac{1}{4\varepsilon+2C_{16}}\ln\mathcal{E}.$$

Here we may assume that $\mathcal{E}<1$ and so $s>0$. Then it follows from (3.37) that

$$\int_\Omega (|F|^2+|g|^2+|h|^2)\mathrm{d}x' \le 2C\mathcal{E}^{\frac{4\varepsilon}{4\varepsilon+2C}}.$$

By Definition (3.17) of F, we have

$$\int_\Omega fr\mathrm{d}x_1\mathrm{d}x_2 = \int_\Omega (\partial_{x_1}F)r\mathrm{d}x_1\mathrm{d}x_2 = \int_\Omega F(\partial_{x_1}r)\mathrm{d}x_1\mathrm{d}x_2$$

for all $r \in H_0^1(\Omega)$ by integration by parts. Hence we can directly verify that $\|f\|_{H^{-1}(\Omega)} \le C\|F\|_{L^2(\Omega)}$, so that the proof of Theorem 3.1 is complete. $\square$

4. Proof of Theorem 2.1

Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$ to argue similarly. We can directly verify that the functions $\mathrm{rot}\,\mathbf{u} \equiv \partial_{x_1}u_2 - \partial_{x_2}u_1$ and $\mathrm{div}\,\mathbf{u}$ satisfy the equations

$$\partial_{x_0}^2\mathrm{rot}\,\mathbf{u} - \mu\Delta\mathrm{rot}\,\mathbf{u} = m_1, \quad \partial_{x_0}^2\mathrm{div}\,\mathbf{u} - (\lambda+2\mu)\Delta\mathrm{div}\,\mathbf{u} = m_2 \qquad \text{in}\, Q, \tag{4.1}$$

where

$$m_1 = K_1\mathrm{rot}\,\mathbf{u} + K_2\mathrm{div}\,\mathbf{u} + \mathcal{K}_1\mathbf{u} + \mathrm{rot}\,\mathbf{f}, \quad m_2 = K_3\mathrm{rot}\,\mathbf{u} + K_4\mathrm{div}\,\mathbf{u} + \mathcal{K}_2\mathbf{u} + \mathrm{div}\,\mathbf{f}$$

and K_j, $\mathcal{K}_k$ are first order differential operators with L^∞ coefficients.

Thanks to Condition 2.1 on the weight function ψ, there exists $\widehat{\tau}$ such that for all $\tau > \widehat{\tau}$, the Carleman estimate for equations (4.1) (see *e.g.*, [45]) yields the inequality

$$\begin{aligned}
&s\|(\nabla\mathrm{rot}\,\mathbf{u})\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + s\|(\nabla\mathrm{div}\,\mathbf{u})\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + s^3\|(\mathrm{rot}\,\mathbf{u})\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + s^3\|(\mathrm{div}\,\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}\\
&\qquad \le C_1\left(s^2\|\mathbf{f}\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}\mathrm{e}^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2}\right.\\
&\qquad\left. + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}\mathrm{e}^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}\mathrm{e}^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\right), \quad \forall s \ge s_0(\tau),
\end{aligned} \tag{4.2}$$

where the constant C_1 is independent of s.

In order to estimate the $H^1(Q)$-norm of the function $\mathbf{u}$, we need the following proposition.

Proposition 4.1. *There exists $\widehat{\tau} > 1$ such that for any $\tau > \widehat{\tau}$, there exists $s_0(\tau)$ such that*

$$\begin{aligned}
&\int_Q\left(\frac{1}{s}\sum_{j,k=1}^2|\partial_{x_j}\partial_{x_k}\mathbf{u}|^2 + s|\nabla_{x'}\mathbf{u}|^2 + s^3|\mathbf{u}|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x\\
&\le C_2\left(\|(\mathrm{rot}\mathbf{u})\mathrm{e}^{s\phi}\|^2_{H^1(Q)} + \|(\mathrm{div}\mathbf{u})\mathrm{e}^{s\phi}\|^2_{H^1(Q)} + \int_{Q_\omega}(s|\nabla\mathbf{u}|^2 + s^3|\mathbf{u}|^2)\mathrm{e}^{2s\phi}\mathrm{d}x\right),\\
&\qquad \forall s \ge s_0(\tau),\ \mathbf{u} \in (H_0^1(Q))^2.
\end{aligned} \tag{4.3}$$

Proof of Proposition 4.1. Denote $\mathrm{rot}\,\mathbf{u} = \mathbf{y}$ and $\mathrm{div}\,\mathbf{u} = \mathbf{w}$ and let $\mathrm{rot}^*v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)$. Using a well-known formula: $\mathrm{rot}^*\mathrm{rot} = -\Delta_{x'} + \nabla_{x'}\mathrm{div}$, we obtain

$$-\Delta_{x'}\mathbf{u} = -\mathrm{rot}^*\,\mathbf{y} - \nabla_{x'}\mathbf{w} \quad \text{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = 0.$$

Then (4.3) follows from the Carleman estimate for an elliptic equations obtained by the first author in [17]. $\square$

By (4.2) and (4.3), we estimate $\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^2 \|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ *via* the right hand side of inequality (4.2). Next using this estimate and equation (1.1), we obtain the estimate for the norm $\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ *via* the right hand side of (4.2). Finally we obtain the estimate for $\|(\partial_{x_0}\partial_{x_j}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ and $s^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ by the interpolation argument. Therefore, combining these estimates with (4.2), we have

$$\begin{aligned}\|\mathbf{u}\|^2_{Y(\phi,Q)} \le C_3\Bigg(& s^2\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\phi}\|^2_{(L^2(Q))^2} \\ & + s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} \\ & + s^3\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\Bigg), \quad \forall s \ge s_0(\tau), \end{aligned} \tag{4.4}$$

where the constant C_3 is independent of s. Here we recall definition (2.8) of $\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}$ and the definition of $\|\mathbf{u}\|^2_{Y(\phi,Q)}$ in (2.9).

Now we need to estimate the boundary integrals at the right hand side of (4.4). In order to do that, it is convenient to use another weight function φ such that $\varphi|_{\partial\Omega} = \phi|_{\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all x in small neighbourhood of $(0,T)\times\partial\Omega$. We introduce the function φ by formulae:

$$\varphi(x) = e^{\tau\widetilde{\psi}(x)}, \quad \widetilde{\psi}(x) = \psi(x) - \widehat{\epsilon}\ell_1(x') + N\ell_1^2(x'),$$

where $\widehat{\epsilon} > 0$ is a small positive parameter, $N > 0$ is a large positive parameter, and $\ell_1 \in C^3(\overline{\Omega})$ is a function such that

$$\ell_1(x') > 0, \quad \forall x' \in \Omega, \quad \ell_1|_{\partial\Omega} = 0, \ \nabla_{x'}\ell_1|_{\partial\Omega} \ne 0.$$

Denote $\Omega_{1/N^2} = \{x' \in \Omega; \operatorname{dist}(x', \partial\Omega) \le \frac{1}{N^2}\}$. Obviously for any fixed $\widehat{\epsilon} > 0$, there exists $N_0(\widehat{\epsilon})$ such that

$$\varphi(x) < \phi(x), \quad \forall x \in [0,T]\times\Omega_{1/N^2}, \ N \in (N_0(\widehat{\epsilon}), \infty).$$

Now we will prove the following estimate:

Lemma 4.1. *Under conditions of Theorem 2.1, there exist $\widehat{\tau} > 0$ and $N_0 > 1$ such that for all $\tau > \widehat{\tau}$, there exists $s_0(\tau, N)$ such that*

$$\begin{aligned}\|\mathbf{u}\|^2_{Y(\varphi,Q)} + N\sum_{|\alpha|=0}^2 s^{4-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2} \le C_4\Big(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2} \\ + \|(\nabla\mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\varphi,Q_\omega)}\Big), \quad \forall s \ge s_0(\tau,N), \ N > N_0, \ \operatorname{supp}\mathbf{u} \subset [0,T]\times\Omega_{1/N^2}, \end{aligned} \tag{4.5}$$

where the constant C_4 is independent of s and N.

The proof of Lemma 4.1 is given in Sections 5–8. Now, using the result of this lemma, we finish the proof of Theorem 2.1. Let us fix the parameter N such that (4.5) holds true. We take $\widetilde{\delta} \in \left(0, \frac{1}{N^2}\right)$ sufficiently small such that

$$\phi(x) > \varphi(x), \qquad \forall x \in \overline{\Omega_{\widetilde{\delta}} \setminus \Omega_{\widetilde{\delta}/2}}. \tag{4.6}$$

We consider a cut off function $\widetilde{\theta} \in C^3(\overline{\Omega}_{\widetilde{\delta}})$ such that $\widetilde{\theta}|_{\Omega_{\frac{\widetilde{\delta}}{2}}} = 1$ and $\widetilde{\theta}|_{\Omega_{\widetilde{\delta}}\setminus\Omega_{\frac{3\widetilde{\delta}}{4}}} = 0$. The function $\widetilde{\theta}\mathbf{u}$ satisfies the equation

$$P(\widetilde{\theta}\mathbf{u}) = \widetilde{\theta}\mathbf{f} + [P,\widetilde{\theta}]\mathbf{u}, \quad \mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_{x_0}(0,\cdot) = \mathbf{u}(T,\cdot) = \mathbf{u}_{x_0}(T,\cdot) = 0.$$

Applying Carleman estimate (4.5) to this equation, we obtain

$$\begin{aligned} & s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} \\ & \le C_8(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2} + s^2\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{(L^2(Q))^2} \\ & \quad + \|\nabla([P,\widetilde{\theta}]\mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s \ge s_0(\tau). \end{aligned} \tag{4.7}$$

Since the supports of the coefficients of the commutator $[P,\widetilde{\theta}]$ are in $\overline{\Omega_{\widetilde{\delta}}\setminus\Omega_{\widetilde{\delta}/2}}$ by (4.6), we have

$$\begin{aligned} & s^2\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{(L^2(Q))^2} + \|\nabla([P,\widetilde{\theta}]\mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\varphi,Q_\omega)} \\ & \qquad \le C_9\left(\sum_{|\alpha|=0}^{2} s^{3-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right). \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\begin{aligned} & s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} \\ & \le C_{10}\left(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2} + \sum_{|\alpha|=0}^{2} s^{3-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right), \quad \forall s \ge s_0(\tau). \end{aligned} \tag{4.9}$$

Finally we will estimate the surface integrals at the right hand side of (4.4) by the right hand side of (4.9). In the new inequality, the term

$$\sum_{|\alpha|=0}^{2} s^{3-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$$

which appears at the right hand side, can be absorbed by $\|\mathbf{u}\|^2_{Y(\phi,Q)}$. Thus the proof of Theorem 2.1 is complete. □

5. Proof of Lemma 4.1

In this section, we will prove Lemma 4.1. Following the standard technique, we reduce the proof of estimate (4.5) to subelliptic estimate (5.13) for the operator $\mathbb{P}_\sigma$. Next show that we can act microlocally in this case. Namely we reduce estimate (5.13) to estimate (5.15). In the situation with the Lamé system this reduction is not trivial, since we have the subelliptic estimate with loss of one derivative. This difficulty is overcome with the help of the second large parameter N inserted into the function φ. Finally we formulate several lemmata on factorization of pseudo-differential operators, *a priori* estimates of Cauchy problem for pseudo-differential operators, and Carleman estimate for a second order scalar hyperbolic equation, which are used in Sections 6–8.

Proof of Lemma 4.1. First we note that, thanks to the large parameter N, it suffices to prove (4.5) only locally by assuming

$$\operatorname{supp}\mathbf{u} \subset B_\delta \cap ([0,T]\times\Omega_{1/N^2}),$$

where B_δ is the ball of the radius $\delta > 0$ centered at some point y^*. In the case of $B_\delta \cap ((0,T)\times\partial\Omega) = \emptyset$, we can prove (4.5) in a usual way for a function with compact support (see *e.g.*, [15]). Without loss of generality, we may assume that $y^* = (y_0^*, 0, 0)$. Moreover the parameter $\delta > 0$ can be chosen arbitrarily small. Assume that near $(0,0)$, the boundary $\partial\Omega$ is locally given by the equation $x_2 - \ell(x_1) = 0$. Furthermore, since the function

$\widetilde{\mathbf{u}} = \mathcal{O}\mathbf{u}(x_0, \mathcal{O}^{-1}x')$ satisfies system (2.1) and (2.2) with $\widetilde{\mathbf{f}} = \mathcal{O}\mathbf{f}(x_0, \mathcal{O}^{-1}x')$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\ell'(0) \equiv \frac{\mathrm{d}\ell}{\mathrm{d}x_1}(0) = 0. \tag{5.1}$$

Making the change of variables $y_1 = x_1$ and $y_2 = x_2 - \ell(x_1)$, we reduce equation (2.1) to the form

$$\begin{cases} \mathbb{P}_1\mathbf{u} = \frac{\partial^2 u_1}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 u_1}{\partial y_2^2}\right) + \mu\ell''(y_1)\frac{\partial u_1}{\partial y_2} \\ -(\lambda+\mu)\frac{\partial}{\partial y_1}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + (\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right)\ell' + \widetilde{K}_1\mathbf{u} = f_1, \\ \mathbb{P}_2\mathbf{u} = \frac{\partial^2 u_2}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 u_2}{\partial y_2^2}\right) + \mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} \\ -(\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + \widetilde{K}_2\mathbf{u} = f_2, \end{cases} \tag{5.2}$$

where we use the same notations $\mathbf{u}, \mathbf{f}$ after the change of variables and $\widetilde{K}_1, \widetilde{K}_2$ are partial differential operators of the first order. We set $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2)$ and

$$z_1 = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2}\ell'(y_1) - \frac{\partial u_1}{\partial y_2}, \quad z_2 = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2}\ell'(y_1).$$

After the change of variables, equations (4.1) have the form

$$\begin{aligned} P_\mu z_1 &= \frac{\partial^2 z_1}{\partial y_0^2} - \mu\left(\frac{\partial^2 z_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_1}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 z_1}{\partial y_2^2}\right) + \mu\ell''(y_1)\frac{\partial z_1}{\partial y_2} \\ &= m_1 \quad \text{in } \mathcal{G}_N \triangleq \mathbb{R}^2 \times \left[0, \frac{\widehat{\kappa}}{N^2}\right], \end{aligned} \tag{5.3}$$

$$\begin{aligned} P_{\lambda+2\mu} z_2 &= \frac{\partial^2 z_2}{\partial y_0^2} - (\lambda+2\mu)\left(\frac{\partial^2 z_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_2}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 z_2}{\partial y_2^2}\right) + (\lambda+2\mu)\ell''(y_1)\frac{\partial z_2}{\partial y_2} \\ &= m_2 \quad \text{in } \mathcal{G}_N. \end{aligned} \tag{5.4}$$

Here we use the same notations m_1, m_2 after the change of variables and the constant $\widehat{\kappa} > 0$ is chosen sufficiently large such that the image of $([0,T] \times \Omega_{1/N^2}) \cap B_\delta(y^*)$ belongs to $\mathcal{G}_N$. Henceforth we write $(z_1, z_2) = R(y, D)\mathbf{u}$, where

$$D = (D_{y_0}, D_{y_1}, D_{y_2}), \quad D_{y_j} = \frac{1}{i}\partial_{y_j}, \qquad j = 0, 1, 2, \text{ etc.},$$

and $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$.

Now we claim that in order to prove Lemma 4.1, it suffices to establish the following estimate for the function $\mathbf{w} = (w_1, w_2) = \mathrm{e}^{s\varphi}(z_1, z_2) = \mathrm{e}^{s\varphi}R(y, D)\mathbf{u}$:

$$\begin{aligned} \|\mathbf{w}\|_*^2 \equiv s\|\mathbf{w}\|^2_{(H^1(\mathcal{G}_N))^2} + s^3\|\mathbf{w}\|^2_{(L^2(\mathcal{G}_N))^2} + s\left\|\frac{\partial \mathbf{w}}{\partial y_2}\right\|^2_{(L^2(\partial\mathcal{G}_N))^2} + s\|\mathbf{w}\|^2_{(H^1(\partial\mathcal{G}_N))^2} \\ + s^3\|\mathbf{w}\|^2_{(L^2(\partial\mathcal{G}_N))^2} \le C_5(\|\mathbb{P}\mathbf{u}\mathrm{e}^{s\varphi}\|^2_{(H^1(\mathcal{G}_N))^2} + s^2\|\mathbb{P}\mathbf{u}\mathrm{e}^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + s\|\mathbf{g}\|^2_{(L^2(\partial\mathcal{G}_N))^2} \\ + \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|(\partial^\alpha_{y'}\mathbf{u})\mathrm{e}^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}), \quad \forall s \ge s_0(\tau, N), \end{aligned} \tag{5.5}$$

for all $\mathbf{u} \in (H^2(\mathcal{G}_N))^2$ satisfying $\mathbf{u}|_{\partial\mathcal{G}_N} = 0$ and supp$\mathbf{u} \subset B_\delta \cap \mathcal{G}_N$. Obviously the function $\mathbf{w}$ satisfies the boundary condition

$$\frac{\partial w_1}{\partial y_2} = \frac{\lambda+2\mu}{\mu}\frac{\partial w_2}{\partial y_1} + s\varphi_{y_2}(y^*)w_1 - s\frac{\lambda+2\mu}{\mu}\varphi_{y_1}(y^*)w_2 + g_1, \quad \text{on } \partial\mathcal{G}_N, \tag{5.6}$$

$$\frac{\partial w_2}{\partial y_2} = -\frac{\mu}{\lambda+2\mu}\frac{\partial w_1}{\partial y_1} + s\varphi_{y_2}(y^*)w_2 + s\frac{\mu}{\lambda+2\mu}\varphi_{y_1}(y^*)w_1 + g_2, \quad \text{on } \partial\mathcal{G}_N, \tag{5.7}$$

where the function $\mathbf{g} = (g_1, g_2)$ satisfies the estimate

$$s\|\mathbf{g}\|^2_{(L^2(\partial\mathcal{G}_N))^2} \le \epsilon(\delta)\left(s\left\|\frac{\partial\mathbf{w}}{\partial y_2}\right\|^2_{(L^2(\partial\mathcal{G}_N))^2} + s\|\mathbf{w}\|^2_{(H^1(\partial\mathcal{G}_N))^2} + s^3\|\mathbf{w}\|^2_{(L^2(\partial\mathcal{G}_N))^2}\right) + C_6 s\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\partial\mathcal{G}_N))^2}, \tag{5.8}$$

and $\lim_{\delta\to 0}\epsilon(\delta) = 0$.

Boundary Conditions (5.6) and (5.7) with property (5.8) follow from equation (5.2) and the zero Dirichlet boundary condition for $\mathbf{u}$.

In order to deduce (4.5) from estimate (5.5), it suffices to show

$$\|\mathbf{u}\|^2_{Y(\varphi,\mathcal{G}_N)} \le C_7(\|\mathbf{w}\|^2_* + \|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(H^1(\mathcal{G}_N))^2} + s^2\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}), \quad \forall s \ge s_0(\tau, N). \tag{5.9}$$

For the proof of (5.9), we need

Proposition 5.1. *There exist $\widehat{\tau} > 1$ and $N_0 > 1$ such that for any $\tau > \widehat{\tau}$ and $N > N_0(\tau)$, there exists $s_0(\tau, N)$ such that*

$$N\int_{\mathcal{G}_N}\left(\frac{1}{s\varphi}\sum_{j,k=1}^{2}|\partial_{y_j}\partial_{y_k}\mathbf{u}|^2 + s\varphi|\partial_{y_j}\mathbf{u}|^2 + s^3\varphi^3|\mathbf{u}|^2\right)e^{2s\varphi}dy$$
$$\le C_8(\|z_1 e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)} + \|z_2 e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)}), \quad \forall\mathbf{u} \in (H^1_0(\mathcal{G}_N))^2, \operatorname{supp}\mathbf{u} \subset B_\delta\cap\mathcal{G}_N, \forall s \ge s_0(\tau, N),$$

where the constant C_8 is independent of N.

We give the proof of Proposition 5.1 in Appendix I.

Thanks to Proposition 5.1 and equations (5.2), we obtain

$$N\|(\partial^2_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + \sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} Ns^{4-2|\alpha|}\|(\partial^\alpha_{y'}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}$$
$$\le C_9(\|w\|^2_* + N\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}) \quad \forall s \ge s_0(\tau, N). \tag{5.10}$$

By (5.5) and (5.8)–(5.10), we obtain

$$N\|(\partial^2_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + \sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} Ns^{4-2|\alpha|}\|(\partial^\alpha_{y'}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + \|\mathbf{u}\|^2_{Y(\varphi,\mathcal{G}_N)}$$
$$\le C_{10}(\|\nabla(\mathbb{P}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + s^2\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}) \quad \forall s \ge \max\{s_0(\tau, N), N\}\cdot \tag{5.11}$$

Finally, combining (5.11) with the estimates

$$s^2\|(\partial_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} \le C_{11}\left(\|(\partial^2_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + s^4\|\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}\right)$$

and

$$\|(\partial_{y_0}\partial_{y_k}\mathbf{u})\mathrm{e}^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} \le C_{11}\sum_{j=0}^{2}\|(\partial^2_{y_j}\mathbf{u})\mathrm{e}^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}, \quad k\in\{1,2\},$$

we obtain (5.9).

Now we will proceed to the proof of (5.5). We set $P_{\mu,s}=\mathrm{e}^{|s|\varphi}P_\mu\mathrm{e}^{-|s|\varphi}$ and $P_{\lambda+2\mu,s}=\mathrm{e}^{|s|\varphi}P_{\lambda+2\mu}\mathrm{e}^{-|s|\varphi}$. By $\mathbf{p}(y,\xi_0,\xi_1,\xi_2)$ and $p_\beta(y,\xi_0,\xi_1,\xi_2)$ with $\beta=\mu$ or $\lambda+2\mu$, we denote the principal symbols of the operators $\mathbb{P}$ and P_β respectively. In order to prove Carleman estimate (5.5), it is convenient for us to introduce a new variable σ and consider s as a dual variable to σ. Following [46], Chapter 14, we consider the pseudo-differential operators defined by

$$\mathbf{P}_\beta(y,D_\sigma,D_{y_0},D_{y_1},D_{y_2})v=\int_{\mathbb{R}^3}p_\beta(y,\xi_0+i|s|\varphi_{y_0},\xi_1+i|s|\varphi_{y_1},D_{y_2}+i|s|\varphi_{y_2})\widehat{v}(s,\xi_0,\xi_1,y_2)\mathrm{e}^{i(<y',\xi'>+\sigma s)}\mathrm{d}\sigma\mathrm{d}\xi',$$

$$\mathbb{P}_\sigma(y,D_\sigma,D_{y_0},D_{y_1},D_{y_2})v=\int_{\mathbb{R}^3}\mathbf{p}(y,\xi_0+i|s|\varphi_{y_0},\xi_1+i|s|\varphi_{y_1},D_{y_2}+i|s|\varphi_{y_2})\widehat{v}(s,\xi_0,\xi_1,y_2)\mathrm{e}^{i(<y',\xi'>+\sigma s)}\mathrm{d}\sigma\mathrm{d}\xi',$$

where $\xi'=(\xi_0,\xi_1), y'=(y_0,y_1)$ and $\widehat{v}(s,\xi_0,\xi_1,y_2)$ is the Fourier transform of $v(\sigma,y_0,y_1,y_2)$ with respect to σ,y_0,y_1. Let $\mathbf{v}(\sigma,y)=(v_1(\sigma,y),v_2(\sigma,y))$ be a function with the domain $\mathcal{Q}=\mathbb{R}^3\times\mathbb{R}^1_+$. Henceforth $\mathcal{F}_\sigma$ denotes the Fourier transform with respect to the variable σ. Let $h(s)=(1+s^2)^{\frac14}$, $\Sigma=\partial\mathcal{Q}$. Moreover we set $\mathbf{g}=(g_1,g_2)$,

$$R_s(y,D)\mathcal{U}=\mathrm{e}^{|s|\varphi}R(y,D)\mathrm{e}^{-|s|\varphi}\mathcal{U}, \tag{5.12}$$

and

$$\begin{cases} B_1\mathbf{w}\triangleq-\dfrac{\partial w_1}{\partial y_2}+\dfrac{\lambda+2\mu}{\mu}\dfrac{\partial w_2}{\partial y_1}+|s|\varphi_{y_2}(y^*)w_1-|s|\dfrac{\lambda+2\mu}{\mu}\varphi_{y_1}(y^*)w_2,\\ B_2\mathbf{w}\triangleq-\dfrac{\partial w_2}{\partial y_2}-\dfrac{\mu}{\lambda+2\mu}\dfrac{\partial w_1}{\partial y_1}+|s|\varphi_{y_2}(y^*)w_2+|s|\dfrac{\mu}{\lambda+2\mu}\varphi_{y_1}(y^*)w_1 \quad \text{on } \Sigma\end{cases}$$

for $\mathbf{w}=(w_1,w_2)$, provided that the right hand sides are well-defined.

Then we claim that in order to prove (5.5), it suffices to establish the following estimate

$$\begin{aligned}|||\mathbf{v}|||^2 &\triangleq\sum_{j=0}^{1}\|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{L^2(\mathbb{R}^1;(H^j(\mathcal{G}_N))^2)}+\|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{(H^j(\Sigma))^2}+\left\|h(D_\sigma)\frac{\partial\mathbf{v}}{\partial y_2}\right\|^2_{(L^2(\Sigma))^2}\\ &\le C_{12}\left(\|\mathbb{P}_\sigma(y,D)\mathcal{F}^{-1}_\sigma\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}+\|h(D_\sigma)\mathcal{F}^{-1}_\sigma\mathbf{g}\|^2_{(L^2(\Sigma))^2}+\|\mathcal{F}^{-1}_\sigma\mathcal{U}\|^2_{(H^2(\mathcal{Q}))^2}\right),\end{aligned} \tag{5.13}$$

if $\mathcal{U}$ and $\mathbf{v}$ satisfy $\operatorname{supp}\mathcal{U}\subset\mathbb{R}^1\times(B_\delta\cap\mathcal{G}_N)$, $\operatorname{supp}\mathcal{F}^{-1}_\sigma\mathcal{U}\subset(-\sigma_0,\sigma_0)\times(B_\delta\cap\mathcal{G}_N)$ with arbitrarily small parameter $\sigma_0>0$, and

$$\begin{cases}R_s(y,D)\mathcal{U}=\mathcal{F}_\sigma\mathbf{v}, \quad \mathcal{U}|_\Sigma=0\\ B_1(\mathcal{F}_\sigma\mathbf{v})=g_1, \quad B_2(\mathcal{F}_\sigma\mathbf{v})=g_2 \quad \text{on } \Sigma.\end{cases}$$

We set

$$\mathcal{F}_\sigma\mathbf{v}=\mathbf{w}.$$

Then

$$(B_1\mathbf{w},\, B_2\mathbf{w})=(g_1,g_2)\equiv\mathbf{g}. \tag{5.14}$$

This fact can be proved exactly in the same way as in [46], Chapter 14, Section 2.

Consider the finite covering of the unit sphere $\mathbb{S}^2\equiv\{(s,\xi_0,\xi_1);\ s^2+\xi_0^2+\xi_1^2=1\}$: $\mathbb{S}^2\subset\cup_{\zeta^*\in S^2}\{\zeta=(s,\xi_0,\xi_1)\in\mathbb{S}^2;\ |\zeta-\zeta^*|<\delta_1\}$ and the partition of unity $\chi_\nu(\zeta)$: $\sum_{\nu=1}^{K(\delta_1)}\chi_\nu(\zeta)=1$ for any $\zeta\in\mathbb{S}^2$ and $\operatorname{supp}\chi_\nu\subset\{\zeta\in\mathbb{S}^2;\ |\zeta-\zeta^*_\nu|<\delta_1\}$.

We extend the function χ_ν on the set $|\zeta|>1$ as the homogeneous function of the order zero in such a way that

$$\operatorname{supp}\chi_\nu\subset\mathcal{O}(\delta_1)\equiv\left\{\zeta;\,\left|\frac{\zeta}{|\zeta|}-\zeta^*\right|<\delta_1\right\},$$

and continue χ_ν on the set $|\zeta|<1$ up to a C^∞ function.

We set $D'=(D_\sigma,D_{y_0},D_{y_1})$, and consider the pseudo-differential operator $\chi_\nu(D')$ and the function $\chi_\nu(D')\mathbf{v}$. Obviously equalities (5.14) hold true with $\mathbf{w}$ and $\mathbf{g}$ replaced by $\mathbf{w}_\nu=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D')\mathbf{v}e^{-is\sigma}d\sigma$ and $\mathbf{g}_\nu=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathbf{g}e^{-is\sigma}d\sigma$.

Moreover we claim that instead of (5.13), it suffices to prove the following estimate

$$|||\chi_\nu(D')\mathbf{v}|||\le C_{13}\left(\|\mathbb{P}_\sigma\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^1(\mathcal{Q}))^2}+\|h(D_\sigma)\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathbf{g}\|_{(L^2(\Sigma))^2}+\|\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^2(\mathcal{Q}))^2}\right),\tag{5.15}$$

where

$$R_s(y,D')\mathcal{U}=\mathcal{F}_\sigma\mathbf{v},\quad \mathcal{U}|_\Sigma=0,\quad \operatorname{supp}\mathcal{F}_\sigma^{-1}\mathcal{U}\subset(-\sigma_0,\sigma_0)\times(B_\delta\cap\mathcal{G}_N),$$
$$B_1(w_{1,\nu},w_{2,\nu})=g_{1,\nu},\quad B_2(w_{1,\nu},w_{2,\nu})=g_{2,\nu}\tag{5.16}$$

and C_{13} is independent of N. In fact, assume that estimate (5.15) is already proved. Then

$$\begin{aligned}
|||\mathbf{v}|||^2&\le\sum_{\nu=1}^{K(\delta_1)}|||\chi_\nu(D')\mathbf{v}|||^2\\
&\le C_{14}\sum_{\nu=1}^{K}\left(\|\mathbb{P}_\sigma(y,D)\chi_\nu\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}+\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}+\|\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^2(\mathcal{Q}))^2}\right)\\
&\le C_{15}\sum_{\nu=1}^{K}\Big(\|\chi_\nu(D')\mathbb{P}_\sigma(y,D)\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}+\|[\chi_\nu(D'),\mathbb{P}_\sigma(y,D')]\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}\\
&\quad+\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}+\|\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}\Big)\\
&\le C_{16}\left(\|\mathbb{P}_\sigma(y,D)\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2}+\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}+\|\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^2(\mathcal{Q}))^2}\right),
\end{aligned}$$

where $K=K(\delta_1)$ and C_{16} are independent of N.

Estimate (5.15) follows from Lemmas 6.1, 7.1 and 8.1 which are proved in Sections 6–8. □

Now we formulate some results and introduce some definitions which will be used in the proof of estimate (5.15).

The principal symbol of the operator $P_{\beta,s}$ has the form

$$p_\beta(y,s,\xi_0,\xi_1)=-(\xi_0+i|s|\varphi_{y_0})^2+\beta[(\xi_1+i|s|\varphi_{y_1})^2-2\ell'(\xi_1+i|s|\varphi_{y_1})(\xi_2+i|s|\varphi_{y_2})+(\xi_2+i|s|\varphi_{y_2})^2|G|^2],\tag{5.17}$$

where $|G|^2=1+(\ell'(y_1))^2$. The roots of this polynomial with respect to the variable ξ_2, are

$$\Gamma^{\pm}_\beta(y,s,\xi_0,\xi_1)=-i|s|\varphi_{y_2}(y)+\alpha^{\pm}_\beta(y,s,\xi_0,\xi_1),\tag{5.18}$$

$$\alpha^{\pm}_\beta(y,s,\xi_0,\xi_1)=\frac{(\xi_1+i|s|\varphi_{y_1}(y))\ell'(y_1)}{|G|^2}\pm\sqrt{r_\beta(y,s,\xi_0,\xi_1)},\tag{5.19}$$

$$r_\beta(y,\zeta)=\frac{((\xi_0+i|s|\varphi_{y_0}(y))^2-\beta(\xi_1+i|s|\varphi_{y_1}(y))^2)|G|^2+\beta(\xi_1+i|s|\varphi_{y_1})^2(\ell')^2}{\beta|G|^4},\tag{5.20}$$

where the function $\sqrt{r_\beta}$ is defined below.

Denote $\gamma = (y^*, \zeta^*) = (y^*, s^*, \xi_0^*, \xi_1^*)$.

Proposition 5.2. *Suppose that* $|r_\beta(\gamma)| \ge 2\widehat{\delta} > 0$. *Then there exists* $\delta_0 = \delta_0(\widehat{\delta}) > 0$ *such that for all* $\delta, \delta_1 \in (0, \delta_0)$, *there exists a constant* $C_{20} > 0$, *independent of* s, *such that for one of the roots of polynomial (5.17), which we denote by* Γ_β^-, *we have*

$$-\mathrm{Im}\,\Gamma_\beta^-(y, s, \xi_0, \xi_1) \ge C_{20}|s|, \quad \forall (y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1). \tag{5.21}$$

Proof of Proposition 5.2. If $\mathrm{Im}\sqrt{r_\beta(\gamma)} \neq 0$, then statement (5.21) is trivial. So it suffices to consider the case $\mathrm{Im}\sqrt{r_\beta(\gamma)} = 0$. Let $\theta \in \left(0, \frac{1}{8}\right)$ be a constant. Obviously there exists $\widetilde{\delta}(\theta)$ such that for all $\delta, \delta_1 \in (0, \widetilde{\delta}(\theta))$,

$$\mathrm{Re}\, r_\beta(y, \zeta) \ge (1 - 2\theta)|r_\beta(y, \zeta)|, \quad \forall (y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

Then

$$|\mathrm{Im}\, r_\beta(y, \zeta)| \le \frac{2\theta}{1 - 2\theta} \mathrm{Re}\, r_\beta(y, \zeta), \quad \forall (y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

We denote $b(y, \zeta) = \mathrm{Im}\, r_\beta(y, \zeta)$ and $a(y, \zeta) = \mathrm{Re}\, r_\beta(y, \zeta)$ with $\zeta = (s, \xi_0, \xi_1)$. First, if $\mathrm{Im}\sqrt{r_\beta(\gamma)} = 0$, then we have $a(\gamma) > 0$ and $b(\gamma) = 0$. In that case we define the function $\sqrt{r_\beta(y, \zeta)}$ by the infinite series

$$(1 + x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < 1,$$

where $c_n = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-(n-1)\right)}{n!}$.

That is, assuming that $|\frac{b}{a}| < \frac{2\theta}{1-2\theta} < \frac{1}{2}$ for all $(y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$, we set

$$\sqrt{r_\beta(y, \zeta)} = \sqrt{a} \sum_{n=0}^{\infty} c_n \left(\frac{ib}{a}\right)^n = \sqrt{a} + \frac{i}{2}|s| \left(\frac{b}{|s|\sqrt{a}}\right) - |s| \left(\frac{b}{a}\right) \frac{b}{|s|\sqrt{a}} \sum_{n=0}^{\infty} c_{n+2} \left(\frac{ib}{a}\right)^n. \tag{5.22}$$

The first term in infinite series (5.22) is real, and the absolute value of the third term is $\left||s|\frac{b}{|s|\sqrt{a}}\right| O(\theta)$. The function $\frac{b}{|s|\sqrt{a}}$ is a continuous homogeneous function of the order zero in the variable ζ.

If $\frac{b(\gamma)}{|s^*|\sqrt{a(\gamma)}} \le 0$, then we take $\Gamma_\beta^-(y, \zeta) = -i|s|\frac{\partial \varphi}{\partial y_2} + \alpha_\beta^-(y, \zeta)$ where $\alpha_\beta^-(y, \zeta)$ equals the right hand side of (5.22) plus $(\xi_1 + i|s|\varphi_{y_1})\ell'(y_1)/|G|^2$. Otherwise $\Gamma_\beta^-(y, \zeta) = -i|s|\frac{\partial \varphi}{\partial y_2} + \alpha_\beta^+(y, \zeta)$ where $\alpha_\beta^+(y, \zeta)$ equals the right hand side of (5.22) multiplied by -1 plus $(\xi_1 + i|s|\varphi_{y_1})\ell'(y_1)/|G|^2$.

For $\frac{b}{|s^*|\sqrt{a}}(\gamma) \le 0$, we obtain that $\frac{b}{|s|\sqrt{a}}(\gamma) - \frac{1}{2}\varphi_{y_2}(y) < 0$ for all $(y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$ and for $\frac{b}{|s^*|\sqrt{a}}(\gamma) \ge 0$ we obtain that $-\frac{b}{|s|\sqrt{a}}(\gamma) - \frac{1}{2}\varphi_{y_2}(y) < 0$ for all $(y, s, \xi_0, \xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$. These inequalities imply (5.21) provided that δ_1 is taken sufficiently small. The proof of Proposition 5.2 is finished. □

Under some conditions, we can see that the operator $\mathbf{P}_\beta$ can be factorized as a product of two first order pseudo-differential operators:

Proposition 5.3. *Let* $\beta \in \{\mu, \lambda + 2\mu\}$ *and* $|r_\beta(y, \zeta)| \ge \widehat{\delta} > 0$ *for all* $(y, \zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. *Then we can factorize the operator* $\mathbf{P}_\beta$ *as the product of two first order pseudo-differential operators:*

$$\mathbf{P}_\beta \chi_\nu(D')V = \beta|G|^2(D_{y_2} - \Gamma_\beta^-(y, D'))(D_{y_2} - \Gamma_\beta^+(y, D'))\chi_\nu(D')V + T_\beta V, \tag{5.23}$$

where $\mathrm{supp}\, V \subset B_\delta \cap \mathcal{G}_N$ *and* T_β *is a continuous operator:*

$$T_\beta : L^2(0, 1; H^1(\mathbb{R}^3)) \to L^2(0, 1; L^2(\mathbb{R}^3)).$$

Let us consider the equation

$$(D_{y_2} - \Gamma^-_\beta(y, D'))\chi_\nu(D')V = q, \quad V|_{y_2=\frac{\hat\kappa}{N^2}} = 0, \quad \operatorname{supp} V \subset B_\delta \cap \mathcal{G}_N.$$

For the solutions to this problem, we have an *a priori* estimate:

Proposition 5.4. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. Then there exists a constant $C_{22} > 0$, which is independent of N, such that*

$$\|h(D_\sigma)\chi_\nu(D')V|_{y_2=0}\|_{L^2(\mathbb{R}^3)} \le C_{22}\|q\|_{L^2(\mathcal{Q})}. \tag{5.24}$$

Proof of Proposition 5.4. Taking the scalar product of q and $h^2(D_\sigma)\chi_\nu(D')V$ for fixed y_2, we obtain

$$\begin{aligned}2\mathrm{Re}\,(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)}\mathrm{e}^{2\widetilde{\kappa}y_2} &= \frac{\partial}{\partial y_2}\left(\mathrm{e}^{2\widetilde{\kappa}y_2}\|h(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}\right)\\ &\quad - 2\mathrm{Re}\,(i\Gamma^-_\beta(y,D')\chi_\nu(D')V + \widetilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)}\mathrm{e}^{2\widetilde{\kappa}y_2}.\end{aligned}$$

By (5.21) and Proposition 2.4.A in [47], for sufficiently large positive $\widetilde{\kappa}$, we have

$$\mathrm{Re}\,(i\Gamma^-_\beta(y,D')h^{-2}(D_\sigma)h^2(D_\sigma)\chi_\nu(D')V + \widetilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)} \ge C_{23}\|h^2(D_\sigma)\chi_\nu(D')V\|^2_{L^2(\Sigma)}.$$

Thus

$$\begin{aligned}&2\mathrm{Re}\,(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)}\mathrm{e}^{2\widetilde{\kappa}y_2}\\ &\qquad\le \frac{\partial}{\partial y_2}\left(\mathrm{e}^{2\widetilde{\kappa}y_2}\|h(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}\right) - C_{23}\|h^2(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}\mathrm{e}^{2\widetilde{\kappa}y_2},\end{aligned}$$

and (5.24) follows from Gronwall's inequality. □

Let $\widetilde{w}(s,y)$ satisfy a scalar second order hyperbolic equation

$$P_{\beta,s}\widetilde{w} = q \quad \text{in } \mathcal{G}_N, \quad \frac{\partial \widetilde{w}}{\partial y_2}|_{y_2=1} = \widetilde{w}|_{y_2=1} = 0, \quad \operatorname{supp}\widetilde{w} \subset \mathbb{R}^1 \times (B_\delta \cap \mathcal{G}_N)$$

for almost all $s \in \mathbb{R}^1$. Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{\beta,s}$, where $\beta \in \{\mu, \lambda+2\mu\}$. Set

$$L_{+,\beta} = \frac{P_{\beta,s} + P^*_{\beta,s}}{2}, \quad L_{-,\beta} = \frac{P_{\beta,s} - P^*_{\beta,s}}{2}.$$

One can easily check that the principal part operator $L_{-,\beta}$ is given by formula

$$L_{-,\beta}\widetilde{w} = -2|s|\varphi_{y_0}\frac{\partial\widetilde{w}}{\partial y_0} + \beta\left(2|s|\varphi_{y_1}\frac{\partial\widetilde{w}}{\partial y_1} - 2|s|\ell'(y_1)\left(\varphi_{y_2}\frac{\partial\widetilde{w}}{\partial y_1} + \varphi_{y_1}\frac{\partial\widetilde{w}}{\partial y_2}\right) + 2|s|(1+(\ell'(y_1))^2)\varphi_{y_2}\frac{\partial\widetilde{w}}{\partial y_2}\right).$$

Obviously $L_{+,\beta}\widetilde{w} + L_{-,\beta}\widetilde{w} = q$. For almost all $s \in \mathbb{R}^1$, the following equality holds true:

$$B_\beta + \|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \mathrm{Re}\int_{\mathcal{G}_N}([L_{+,\beta}, L_{-,\beta}]\widetilde{w}, \overline{\widetilde{w}})\mathrm{d}y = \|q\|^2_{L^2(\mathcal{G}_N)}, \tag{5.25}$$

where

$$B_\beta = \mathrm{Re}\int_{\partial\mathcal{G}_N} \widetilde{p}_\beta(y,\nabla\varphi,-\vec{e}_3)(|s|\widetilde{p}_\beta(y,\nabla\widetilde{w}) - |s|^3\widetilde{p}_\beta(y,\nabla\varphi,\nabla\varphi)\widetilde{w}^2)\mathrm{d}y_0\mathrm{d}y_1 + \mathrm{Re}\int_{\partial\mathcal{G}_N}\widetilde{p}_\beta(y,\nabla\widetilde{w},-\vec{e}_3)\overline{L_{-,\beta}\widetilde{w}}\mathrm{d}y_0\mathrm{d}y_1, \tag{5.26}$$

$\vec{e}_3 = (0,0,1)$ and

$$\widetilde{p}_\beta(y,\xi,\widetilde{\xi}) = \xi_0\widetilde{\xi}_0 - \beta(\xi_1\widetilde{\xi}_1 - \ell'(y_1)(\xi_1\widetilde{\xi}_2 + \xi_2\widetilde{\xi}_1) + (1+|\ell'(y_1)|^2)\xi_2\widetilde{\xi}_2).$$

We note that $\phi_{y_k}|_\Sigma = \varphi_{y_k}|_\Sigma$ for $k\in\{0,1\}$ and $\varphi_{y_2}|_\Sigma = (\phi_{y_2} - \widehat{\epsilon}\tau(\partial_{y_2}\ell_1)\phi)|_\Sigma$. Therefore on Σ the function $\nabla\varphi$ is independent of N and $|\nabla\phi(y) - \nabla\varphi(y)| \le C_{25}\widehat{\epsilon}$ for all $y\in\Sigma$ where $C_{25}>0$ is independent of $\widehat{\epsilon}$ and N. In particular, taking $\widehat{\epsilon}$ sufficiently small, we have (2.6) for the function φ. It is convenient for us to rewrite (5.26) in the form

$$B_\beta = B_\beta^{(1)} + B_\beta^{(2)},$$

$$\begin{aligned} B_\beta^{(1)} \equiv \mathrm{Re}\int_{y_2=0} 2|s|\beta\frac{\partial\widetilde{w}}{\partial y_2}&\overline{\left(\beta\frac{\partial\widetilde{w}}{\partial y_1}\varphi_{y_1}(y^*) + \beta\frac{\partial\widetilde{w}}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial\widetilde{w}}{\partial y_0}\varphi_{y_0}(y^*)\right)}\mathrm{d}y_0\mathrm{d}y_1\\ &+\int_{y_2=0}|s|\beta\varphi_{y_2}(y^*)\Bigg\{\left|\frac{\partial\widetilde{w}}{\partial y_0}\right|^2 - \beta\left(\left|\frac{\partial\widetilde{w}}{\partial y_1}\right|^2 + \left|\frac{\partial\widetilde{w}}{\partial y_2}\right|^2\right)\\ &-|s|^2(\varphi_{y_0}^2(y^*) - \beta(\varphi_{y_1}^2(y^*) + \varphi_{y_2}^2(y^*)))|\widetilde{w}|^2\Bigg\}\mathrm{d}y_0\mathrm{d}y_1. \end{aligned}$$

Then

$$|B_\beta^{(2)}| \le \epsilon_0\left(|s|\left\|\frac{\partial\widetilde{w}}{\partial y_2}\right\|^2_{L^2(\partial\mathcal{G}_N)} + |s|\|\widetilde{w}\|^2_{H^1(\partial\mathcal{G}_N)} + |s|^3\|\widetilde{w}\|^2_{L^2(\partial\mathcal{G}_N)}\right), \tag{5.27}$$

where $\epsilon_0 = \epsilon_0(\delta)\to 0$ as $|\delta|\to 0$. It is known (see *e.g.*, [18]) that there exists a parameter $\widehat{\tau}>1$ such that for any $\tau>\widehat{\tau}$, there exists $s_0(\tau)$ such that

$$\begin{aligned} \|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \mathrm{Re}\int_{\mathcal{G}_N}\left([L_{+,\beta},L_{-,\beta}]\widetilde{w},\overline{\widetilde{w}}\right)\mathrm{d}y\\ + C'_{26}|s|\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)} \ge C_{26}\left(|s|\|\widetilde{w}\|^2_{H^1(\mathcal{G}_N)} + |s|^3\|\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}\right), \quad \forall|s|\ge s_0(\tau), \end{aligned} \tag{5.28}$$

where $C_{26}>0$ is independent of s. We also claim that the constant C_{26} is independent of N. The proof of estimate (5.28) is given in Appendix II.

Set

$$\varXi_\beta = \int_{-\infty}^{\infty} B_\beta\mathrm{d}s, \quad \varXi_\beta^{(j)} = \int_{-\infty}^{\infty} B_\beta^{(j)}\mathrm{d}s, \quad j=1,2.$$

Therefore, integrating (5.28) with respect to s in $\mathbb{R}^1$, we have

$$\begin{aligned} C_{27}(\|h(s)\widetilde{w}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)\widetilde{w}\|^2_{L^2(\mathcal{Q})}) + \varXi_\beta \le C_{26}|s|\int_{-\infty}^{\infty}\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\mathrm{d}s\\ + \|q\|^2_{L^2(\mathcal{Q})} + \|\widetilde{w}\|^2_{H^1(\mathcal{Q})}, \quad \forall|s|\ge s_0(\tau) \end{aligned} \tag{5.29}$$

with some constant $C_{27}>0$ and by (5.27)

$$|\varXi_{\widetilde{\beta}}^{(2)}| + |s|\int_{-\infty}^{\infty}\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\mathrm{d}s \le \epsilon(\delta)\left\|\left(\frac{\partial\widetilde{w}}{\partial y_2},\widetilde{w}\right)\right\|^2_X, \tag{5.30}$$

where we set

$$\left\|\left(\frac{\partial\widetilde{w}}{\partial y_2},\widetilde{w}\right)\right\|_X^2=\left\|h(s)\frac{\partial\widetilde{w}}{\partial y_2}\right\|^2_{L^2(\Sigma)}+\|h(s)\widetilde{w}\|^2_{L^2(\mathbb{R}^1;H^1(\mathbb{R}^2))}+\|h^3(s)\widetilde{w}\|^2_{L^2(\Sigma)}$$

and the parameter $\epsilon(\delta)\to+0$ as $\delta\to+0$.

We set

$$w_{1,\nu}=\mathcal{F}_\sigma\chi_\nu(D')v_1,\quad w_{2,\nu}=\mathcal{F}_\sigma\chi_\nu(D')v_2.$$

Later we will need to apply (5.29) and (5.30) to the functions $w_{1,\nu}$ and $w_{2,\nu}$, since we would like to take the advantage of (5.23). However it is directly impossible because the condition $\operatorname{supp}\chi_\nu(D')\mathbf{v}\subset B_\delta\times\mathbb{R}^1$ does not hold true, in general. On the other hand, using the fact that

$$\int_{\mathbb{R}^2\setminus B_{2\delta}}\int_{\mathbb{R}^1}h^4(s)\sum_{|\alpha|\le 2}|D^\alpha w_{j,\nu}|^2\mathrm{d}y_0\mathrm{d}y_1\mathrm{d}s\le C_{28}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2},$$

we can modify (5.29) and (5.30):

$$\begin{aligned}&C_{29}(\|h(s)w_{j(\beta),\nu}\|^2_{H^1(\mathcal{Q})}+\|h^3(s)w_{j(\beta),\nu}\|^2_{L^2(\mathcal{Q})})+\varXi_\beta\\ \le&\|P_{\beta,s}w_{j(\beta),\nu}\|^2_{L^2(\mathcal{Q})}+C_{30}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}+C_{30}|s|\int_{-\infty}^{\infty}\|w_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}w_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\mathrm{d}s,\end{aligned}\tag{5.31}$$

where $C_{29}>0$ is independent of s,N and we set $j(\beta)=1$ if $\beta=\mu$ and $j(\beta)=2$ if $\beta=\lambda+2\mu$, and

$$|\varXi^{(2)}_\beta|+|s|\int_{-\infty}^{\infty}\|w_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}w_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\mathrm{d}s\le\epsilon\left\|\left(\frac{\partial w_{j(\beta),\nu}}{\partial y_2},w_{j(\beta),\nu}\right)\right\|^2_X+C_{31}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}.\tag{5.32}$$

Now we will prove (5.15) separately in the cases: $r_\mu(\gamma)=0$ (Sect. 6), $r_{\lambda+2\mu}(\gamma)=0$ (Sect. 7) and $r_\mu(\gamma)\ne0$, $r_{\lambda+2\mu}(\gamma)\ne0$ (Sect. 8).

6. The case $\mathsf{r}_\mu(\gamma)=0$

In this section, we treat the case where $r_\mu(\gamma)=0$ with $\gamma=(y^*,\zeta^*)\equiv(y^*,s^*,\xi_0^*,\xi_1^*)\in\Sigma\times\mathbb{S}^2$. Let χ_ν be a member of the partition of unity such that

$$\operatorname{supp}\chi_\nu\subset\mathcal{O}(\delta_1)\equiv\left\{\zeta=(s,\zeta_0,\zeta_1);\ \left|\frac{\zeta}{|\zeta|}-\zeta^*\right|<\delta_1\right\}.$$

We note that by (5.31) and (5.32), there exist $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}&C_1\left(\|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})}+\|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})}\right)+\varXi^{(1)}_\mu\\ &\qquad\le C_2\left(\|\mathbf{P}_\mu v_{1,\nu}\|^2_{L^2(\mathcal{Q})}+\|w_{1,\nu}\|^2_{H^1(\mathcal{Q})}\right)+\epsilon(\delta)\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2},w_{1,\nu}\right)\right\|^2_X,\end{aligned}\tag{6.1}$$

and the parameter ϵ can be taken sufficiently small, if we decrease δ. Note that $\Xi^{(1)}_\mu$ can be written in the form

$$
\begin{aligned}
\Xi^{(1)}_\mu = &\int_\Sigma \left(|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + |s|^3\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2\right) \mathrm{d}\Sigma \\
&+ \mathrm{Re}\int_\Sigma 2|s|\mu\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left(\mu\varphi_{y_1}(y^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(y^*)\frac{\partial w_{1,\nu}}{\partial y_0}\right)}\mathrm{d}\Sigma \\
&+ \int_\Sigma |s|\mu\varphi_{y_2}(y^*)(\xi_0^2 - \mu\xi_1^2 - s^2\varphi^2_{y_0}(y^*) + s^2\mu\varphi^2_{y_1}(y^*))|\widehat{v}_{1,\nu}|^2\mathrm{d}\Sigma \\
\equiv &\, J_1 + J_2 + J_3.
\end{aligned}
\tag{6.2}
$$

Let us introduce the set $\mathcal{M}$ by formula

$$
\mathcal{M} = \left\{\zeta = (s,\xi_0,\xi_1) \in \mathbb{S}^2; \frac{\mu}{2}\varphi_{y_2}(y^*)\widehat{C}s^2 > 4\mu^2\frac{\varphi^2_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\varphi^2_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_0^2 + 2\mu^2\varphi_{y_2}(y^*)(|\xi_0|^2+|\xi_1|^2)\right\}, \tag{6.3}
$$

where $\widehat{C} = -p_\mu(y^*,\nabla\varphi(y^*))$. By (2.6), it follows that $\widehat{C}$ is positive.

Next we introduce the set $\widetilde{\mathcal{M}}$ by the formula

$$
\widetilde{\mathcal{M}} = \left\{\zeta = (s,\xi_0,\xi_1) \in \mathbb{S}^2;\ \frac{\mu}{4}\varphi_{y_2}(y^*)\widehat{C}s^2 < 4\mu^2\frac{\varphi^2_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\varphi^2_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_0^2 + 2\mu^2\varphi_{y_2}(y^*)(|\xi_0|^2+|\xi_1|^2)\right\}.
$$

Then we can see that $\mathbb{S}^2 \subset \mathcal{M}\cup\widetilde{\mathcal{M}}$. Therefore, taking the parameter δ_1 sufficiently small, we obtain either $\mathcal{O}(\delta_1)\subset\mathcal{M}$ or $\mathcal{O}(\delta_1)\subset\widetilde{M}$. The main purpose of this section is the proof of the following lemma.

Lemma 6.1. *If $\gamma = (y^*,\zeta^*)$ is a point on $\Sigma\times\mathbb{S}^2$ such that $r_\mu(\gamma) = 0$ and $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)\subset\widetilde{\mathcal{M}}$, then estimate (5.15) holds true. If $\gamma = (y^*,\zeta^*)\in\mathcal{M}$, then estimate (5.15) holds true also.*

Proof. We consider two cases.
Case A. Assume that $\operatorname{supp}\widehat{\mathbf{v}}_\nu \subset \mathcal{O}(\delta_1)\subset\mathcal{M}$.

Applying the Cauchy-Bunyakovskii inequality and using (6.3) and (2.6), we see that there exists a constant $C_3 > 0$ such that

$$
\begin{aligned}
\Xi^{(1)}_\mu \geq &\int_\Sigma \left(|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 - |s|^3\mu\varphi_{y_2}(y^*)p_\mu(y^*,\nabla\varphi(y^*))|w_{1,\nu}|^2\right)\mathrm{d}\Sigma \\
&- \int_\Sigma\left(\frac{1}{2}|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + 4|s|\mu^2\frac{\varphi^2_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|}\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + 4|s|\frac{\varphi^2_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|}\left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right)\mathrm{d}\Sigma \\
&- \int_\Sigma |s|\mu^2\varphi_{y_2}(y^*)\xi_1^2|\widehat{v}_{1,\nu}|^2\mathrm{d}\Sigma \\
\geq &\, C_3\int_\Sigma\left(\frac{1}{2}|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + |s|\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + |s|\left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2 + \frac{1}{2}|s|^3\mu\varphi_{y_2}(y^*)\widehat{C}|w_{1,\nu}|^2\right)\mathrm{d}\Sigma.
\end{aligned}
\tag{6.4}
$$

Similary we have

$$
\Xi^{(1)}_{\lambda+2\mu} \geq C_4\int_\Sigma\left\{|s|\left(\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + \left|\frac{\partial w_{2,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{2,\nu}}{\partial y_0}\right|^2\right) + |s|^3|w_{2,\nu}|^2\right\}\mathrm{d}\Sigma. \tag{6.5}
$$

Combining (6.4) and (6.5), we obtain

$$\Xi_\mu^{(1)} + \Xi_{\lambda+2\mu}^{(1)} \ge C_5 \left\| \left(\frac{\partial w_\nu}{\partial y_2}, w_\nu \right) \right\|_X^2. \tag{6.6}$$

If we apply (5.31) with $\beta = \lambda + 2\mu$, then (6.1), (6.4) and (6.6) imply (5.15).

Case B. Assume that $\operatorname{supp} \widehat{\mathbf{v}}_\nu \subset \widetilde{\mathcal{M}}$.

By (5.18)–(5.20), there exists $C_6 > 0$ such that

$$|\xi_0^2 - s^2\varphi_{y_0}^2(y^*) - \mu\xi_1^2 + \mu s^2 \varphi_{y_1}^2(y^*)| + |\xi_0 s\varphi_{y_0}(y^*) - \mu s\xi_1\varphi_{y_1}(y^*)| \\ \le \delta_1 C_6(|\xi_1|^2 + |\xi_0|^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{6.7}$$

Now we suppose that the parameter δ_1 is sufficiently small such that there exists a constant $C_7 > 0$ such that

$$|\xi_0|^2 \le C_7(|\xi_1|^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{6.8}$$

Then, by (6.7), we have

$$|J_3| \le \delta_1 \mu \varphi_{y_2}(y^*) \left\| \left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_X^2. \tag{6.9}$$

Moreover we claim that there exists $\delta_0 > 0$ such that if $\delta_1 \in (0, \delta_0)$, then there exists $C_8 > 0$ such that

$$|\xi_0| \le C_8|\xi_1|, \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{6.10}$$

Our proof is by contradiction. Suppose that (6.10) is not true. Then for the sequence $\delta_1(n) = \frac{1}{n}$, there exists a sequence $(\xi_0(n), \xi_1(n)) \to (\xi_0^*, \xi_1^*)$ such that $\xi_1(n)/\xi_0(n) \to 0$. Hence for ζ^* we have $r_\mu(y^*, \zeta^*) = 0$, and $\xi_1^* = 0, \xi_0^* \ne 0$ by the definition of the set $\widetilde{\mathcal{M}}$. Therefore $s^*\varphi_{y_0}(y^*) = 0$. If $s^* = 0$, then we obtain $(\xi_0^*)^2 = 0$ and if $\varphi_{y_0}(y^*) = 0$, then $(\xi_0^*)^2 + \mu\varphi_{y_1}^2(y^*)(s^*)^2 = 0$ by (5.19), (5.20). Therefore in the both cases, we have the equality $\xi_0^* = 0$ which leads us to a contradiction.

Note that if $r_{\lambda+2\mu}(\gamma) = 0$, then

$$\varphi_{y_0}(y^*) = 0, \quad \varphi_{y_1}(y^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \;\; s^* = 1$$

and the conic neighbourhood of ζ^* is in the set $\mathcal{M}$ provided that the parameter δ_1 is chosen sufficiently small. Therefore if $\gamma \in \widetilde{\mathcal{M}}$ and $r_\mu(\gamma) = 0$, then we have $r_{\lambda+2\mu}(\gamma) \ne 0$ and by Proposition 5.4, decomposition (5.23) holds true. We set $V_{\lambda+2\mu}^+ = (D_{y_2} - \Gamma_{\lambda+2\mu}^+(y, D'))v_{2,\nu}$. Then

$$\mathbf{P}_{\lambda+2\mu} v_{2,\nu} = (\lambda + 2\mu)|G|^2(D_{y_2} - \Gamma_{\lambda+2\mu}^-(y, D'))V_{\lambda+2\mu}^+ + T_{\lambda+2\mu}v_{2,\nu},$$

where $T_{\lambda+2\mu} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and Proposition 5.4 immediately imply

$$\|h(D_\sigma)(D_{y_2} - \Gamma_{\lambda+2\mu}^+(y, D'))v_{2,\nu}|_{y_2=0}\|_{L^2(\Sigma)} \le C_9(\|P_{\lambda+2\mu,s} w_{2,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}). \tag{6.11}$$

Now we need again obtain the estimate of $\Xi_\mu^{(1)}$. We start from the term J_2. By (5.16), we have

$$J_2 = \mathrm{Re} \int_\Sigma 2|s|(\lambda + 2\mu) \left(\frac{\partial w_{2,\nu}}{\partial y_1} - |s|\varphi_{y_1}(y^*) w_{2,\nu} \right) \times \overline{\left(\mu \frac{\partial w_{1,\nu}}{\partial y_1} \varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma \\ + \mathrm{Re} \int_\Sigma 2|s|\mu(|s|\varphi_{y_2}(y^*) w_{1,\nu} - g_{1,\nu}) \overline{\left(\mu \frac{\partial w_{1,\nu}}{\partial y_1} \varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma \tag{6.12}$$

and denoting

$$\tilde{\alpha}^+_{\lambda+2\mu}(y',D) = \alpha^+_{\lambda+2\mu}(y',D) + i|D_\sigma|(\varphi_{y_2} - \varphi_{y_2}(y^*)),$$

$$-\frac{\mu}{\lambda+2\mu}\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) - i\tilde{\alpha}^+_{\lambda+2\mu}(y,D')v_{2,\nu} = iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu}\mathcal{F}^{-1}_\sigma g_{2,\nu}. \quad (6.13)$$

Here and henceforth $|D_\sigma|$ is the pseudo-differential operator with the symbol $|s|$.

First assume that $s^* = 0$. Then we can see by $|s^*|^2 + |\xi_0^*|^2 + |\xi_1^*|^2 = 1$ that $|\tilde{\alpha}^+_{\lambda+2\mu}(\gamma)| = |r_{\lambda+2\mu}(\gamma)| \neq 0$. Therefore, by Proposition 5.2.A from [47], p. 105, there exists a parametrix of the operator $\tilde{\alpha}^+_{\lambda+2\mu}(y,D')$ which we denote by $(\tilde{\alpha}^+_{\lambda+2\mu}(y,D'))^{-1}$. From (6.13) we obtain

$$v_{2,\nu} = -\frac{1}{i}(\tilde{\alpha}^+_{\lambda+2\mu}(y,D'))^{-1}\left(\frac{\mu}{\lambda+2\mu}\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) + iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu}\mathcal{F}^{-1}_\sigma g_{2,\nu}\right) + T_0 v_{2,\nu}, \quad (6.14)$$

where $T_0 \in \mathcal{L}(L^2(\Sigma), H^1(\Sigma))$. Using (6.14), we transform (6.12) to obtain

$$J_2 = \mathrm{Re}\int_\Sigma -\frac{2|D_\sigma|\mu}{i}\left(\frac{\partial}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)\right)(\tilde{\alpha}^+_{\lambda+2\mu}(y,D'))^{-1} \left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right)\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \kappa_3, \quad (6.15)$$

where

$$\kappa_3 = \mathrm{Re}\int_\Sigma 2|D_\sigma|\mu(|D_\sigma|\varphi_{y_2}(y^*)v_{1,\nu} + \mathcal{F}^{-1}_\sigma g_{1,\nu})\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \mathrm{Re}\int_\Sigma 2|D_\sigma|(\lambda+2\mu)\left(\frac{\partial}{\partial y_1} - |s|\varphi_{y_1}(y^*)\right)$$
$$\times\left[-\frac{1}{i}(\tilde{\alpha}^+_{\lambda+2\mu}(y,D'))^{-1}\left(iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu}\mathcal{F}^{-1}_\sigma g_{2,\nu}\right) + T_0 v_{2,\nu}\right] \times \overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma.$$

Then we have

$$|\kappa_3| \le \epsilon\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X + C_{10}\left(\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})}\right) \quad (6.16)$$

and ϵ can be chosen arbitrarily small by taking δ small enough.

Let us consider the pseudo-differential operator

$$b(y,D') \equiv \frac{1}{i}\left(\frac{\partial}{\partial y_1} - |s|\varphi_{y_1}(y^*)\right)(\tilde{\alpha}^+_{\lambda+2\mu}(y,D'))^{-1}.$$

By (6.7), for the principal symbol of this operator, we have

$$\begin{aligned} b(y^*,\zeta) &= \frac{1}{i}(i\xi_1 - |s|\varphi_{y_1}(y^*))(\tilde{\alpha}^+_{\lambda+2\mu}(y^*,\zeta))^{-1} \\ &\equiv -\mathrm{sign}(\xi_1^*)\sqrt{\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)(y^*)}\frac{(i\xi_1 - |s|\varphi_{y_1}(y^*))}{\xi_1 + i|s|\varphi_{y_1}(y^*)} + \widetilde{b}(y^*,\zeta) \\ &= \frac{1}{i}\sqrt{\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)(y^*)} + \widetilde{b}(y^*,\zeta), \end{aligned} \quad (6.17)$$

where $\widetilde{b}(y^*, \xi^*) = 0$. Therefore the operator $b(y, D')$ can be represented in the form

$$b(y, D') = \frac{1}{i}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}(y)} + \widetilde{b}(y, D'),$$

where $\widetilde{b}(y, D') \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma))$ and

$$\|\widetilde{b}(y, D')\|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))} \le \epsilon. \tag{6.18}$$

Using (6.17) in (6.15), we obtain

$$\begin{aligned}
J_2 = \mathrm{Re}\int_\Sigma -2|D_\sigma|\mu\left(\frac{\mathrm{sign}(\xi_1^*)}{i}\sqrt{\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)(y^*)} + \widetilde{b}(y, D')\right)\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right)\\
\times\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \kappa_3\\
= \mathrm{Re}\int_\Sigma -2|D_\sigma|\mu\Big(\widetilde{b}(y, D') + \frac{\mathrm{sign}(\xi_1^*)}{i}\sqrt{\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)(y^*)}\Big)\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right)\\
\times\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}\mathrm{d}\Sigma + \mathrm{Re}\,\kappa_3.
\end{aligned}$$

By (6.7), (6.16) and (6.18), taking the parameters δ, δ_1 sufficiently small, we obtain

$$|J_2| \le \epsilon(\delta, \delta_1)\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|_X^2 + C_{11}(\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}), \tag{6.19}$$

and $\epsilon(\delta, \delta_1) \to 0$ as $|\delta| + |\delta_1| \to 0$.

Next assume that $s^* \ne 0$. Then by (6.7) we have

$$|\mu(y^*)\varphi_{y_1}(y^*)\xi_1 - \varphi_{y_0}(y^*)\xi_0| \le C\delta_1|\zeta|, \quad \forall\zeta \in \mathcal{O}(\delta_1)$$

and (6.19) follows immediately. Therefore, for any $s^* \in \mathbb{R}^1$, by (6.1), (6.2), (6.9) and (6.19), we have

$$\begin{aligned}
\int_\Sigma\left(h^2(s)\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + h^6(s)\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2\right)\mathrm{d}\Sigma + C_{12}(\|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})})\\
\le C_{13}(\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}) + \epsilon\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|_X^2.
\end{aligned} \tag{6.20}$$

From (5.16), we obtain

$$\begin{aligned}
\int_\Sigma\left(|s|\left|\frac{\partial w_{2,\nu}}{\partial y_1}\right|^2 + |s|^3\mu^2\varphi^2_{y_1}(y^*)|w_{2,\nu}|^2\right)\mathrm{d}\Sigma\\
\le C_{14}\int_\Sigma\left(|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + |s|^3\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2\right)\mathrm{d}\Sigma + C_{14}\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}.
\end{aligned} \tag{6.21}$$

Using (6.10), (6.21) and the definition of the set $\widetilde{\mathcal{M}}$, we obtain

$$\int_\Sigma \left(h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s)|w_{2,\nu}|^2 \right) \mathrm{d}\Sigma$$
$$\le C_{15} \left\{ \int_\Sigma \left(|s|\mu^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + |s|^3 \mu^2 \varphi^3_{y_2}(y^*) |w_{1,\nu}|^2 \right) \mathrm{d}\Sigma + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \right\}. \tag{6.22}$$

From (6.11) and (6.22), we have

$$\begin{aligned}
&\int_\Sigma h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 \mathrm{d}\Sigma \\
&\le C_{16} \left\{ \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s)|w_{2,\nu}|^2 \right) \mathrm{d}\Sigma \right. \\
&\quad \left. + \|V^+_{\lambda+2\mu}(\cdot, 0)\|^2_{L^2(\Sigma)} + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \right\} \\
&\le C_{17} \left\{ \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s)|w_{1,\nu}|^2 \right) \mathrm{d}\Sigma + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \right. \\
&\quad \left. + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X \right\}.
\end{aligned} \tag{6.23}$$

Finally (5.16), (6.10), (6.20) and (6.23) imply

$$\begin{aligned}
&\int_\Sigma h^2(s) \left(\left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 + \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 \right) \mathrm{d}\Sigma \\
&\le C_{18} \left\{ \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s)|w_{1,\nu}|^2 \right) \mathrm{d}\Sigma + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \right. \\
&\quad \left. + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X \right).
\end{aligned} \tag{6.24}$$

Inequalities (6.1), (6.20)–(6.24) imply

$$\begin{aligned}
&\left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X + \|h(s) w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s) w_{1,\nu}\|^2_{L^2(\mathcal{Q})} \le \epsilon \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X \\
&\qquad + C_{19} \left(\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} + \|P_{\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} \right).
\end{aligned}$$

From this inequality and (5.31), (5.32) with $\beta = \lambda + 2\mu$, we obtain (5.15). Thus the proof of Lemma 6.1 is complete. □

7. The case $\mathsf{r}_{\lambda+2\mu}(\gamma) = 0$

In this section, we will prove

Lemma 7.1. *Let $\gamma = (y^*, \zeta^*)$ be a point on $\Sigma \times \mathbb{S}^2$ such that $r_{\lambda+2\mu}(\gamma) = 0$. If $\mathrm{supp}\chi_\nu \subset \mathcal{O}(\delta_1) \subset \widetilde{\mathcal{M}}$, then estimate (5.15) holds true.*

Proof. We note that if $r_\mu(\gamma) = 0$, then $s^* \neq 0$ and $\xi_0^* = \xi_1^* = \varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = 0$. Consequently $\zeta^* \in \mathcal{M}$ and this case was treated in the previous section. Therefore, taking the parameters δ and δ_1 sufficiently small, we may assume that there exists a constant $\widehat{C} > 0$ such that

$$|r_\mu(y,\zeta)| \geq \widehat{C}|\zeta|^2, \quad \forall (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1),\ |\zeta| \geq 1.$$

By (5.19) and (5.20), there exist $\delta_0 > 0$ and $C_1 > 0$ such that for all $\delta_1 \in (0, \delta_0)$ we have

$$|\xi_0|^2 \leq C_1(\xi_1^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{7.1}$$

We consider the following three cases.
Case A. Assume that $s^* = 0$ and

$$\mu(y^*)\varphi_{y_2}(y^*) > \frac{|\mu(y^*)\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|\xi_1^*|}.$$

In that case, there exists a constant $C_2 > 0$ such that

$$-\mathrm{Im}\,\Gamma_\mu^{\pm}(y,\zeta) \geq C_2|s|, \quad \forall (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1),$$

provided that $|\delta| + |\delta_1|$ is sufficiently small. Since $s^* = 0$, we may assume that for some constant $C_3 > 0$,

$$|\xi_0|^2 + s^2 \leq C_3\xi_1^2, \quad \forall \zeta \in \mathcal{O}(\delta_1), \tag{7.2}$$

taking a sufficiently small δ_1. We set $V_\mu^{\pm} = (D_{y_2} - \Gamma_\mu^{\pm}(y, D'))v_{1,\nu}$. Then, by Proposition 5.3,

$$\mathbf{P}_\mu v_{1,\nu} = |G|^2\mu(D_{y_2} - \Gamma_\mu^{\mp}(y,D'))V_\mu^{\pm} + T_\mu^{\pm}v_{1,\nu}, \tag{7.3}$$

where $T_\mu^{\pm} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and Proposition 5.4 imply

$$\|h(D_\sigma)(D_{y_2} - \Gamma_\mu^{\pm}(y,D'))v_{1,\nu}|_{y_2=0}\|_{L^2(\Sigma)} \leq C_4(\|\mathbf{P}_\mu v_{1,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}). \tag{7.4}$$

We have

$$-V_\mu^{+}(\cdot,0) + V_\mu^{-}(\cdot,0) = (\alpha_\mu^{+}(y,D') - \alpha_\mu^{-}(y,D'))v_{1,\nu} \quad \text{on } \Sigma. \tag{7.5}$$

Since $\alpha_\mu^{+}(y^*,\zeta^*) - \alpha_\mu^{-}(y^*,\zeta^*) = 2\sqrt{r_\mu(y^*,\zeta^*)} \neq 0$, by (7.4), (7.5) and Gårding's inequality, we have

$$\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2 \right) + h^6(s)|w_{1,\nu}|^2 \right) \mathrm{d}\Sigma \leq C_5(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{7.6}$$

By (7.6) and (7.4), we obtain

$$\int_\Sigma h^2(s)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 \mathrm{d}\Sigma \leq C_6\left(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right). \tag{7.7}$$

Finally, by (7.6), (7.7) combined with (5.16), we obtain

$$\left\| \left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2 \le C_7 \left(\|P_{\mu,s} w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} \right). \tag{7.8}$$

Since (7.6)–(7.8), (5.31) and (5.32), we obtain (5.15).

Case B. Assume that $s^* = 0$ and

$$\mu(y^*)\varphi_{y_2}(y^*) \le \frac{|\mu(y^*)\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|\xi_1^*|}. \tag{7.9}$$

Then $\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s| \ne 0$. Since $s^* = 0$, we note that $\operatorname{Re} r_\mu(y^*,\zeta^*) > 0$. Set $I = \operatorname{sign} \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s|$. Then we have

$$\Gamma^+_\mu(y^*,\zeta^*) = I\sqrt{\operatorname{Re} r_\mu(y^*,\zeta^*)}.$$

Therefore for some $C_8 > 0$ we have

$$-\Gamma^+_\mu(y^*,\zeta^*)(\mu\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*) > C_8.$$

Taking the parameters $\delta > 0$ and $\delta_1 > 0$ sufficiently small, we obtain

$$-\operatorname{Re}\Gamma^+_\mu(y,\zeta)(\mu\varphi_{y_1}(y)\xi_1 - \varphi_{y_0}(y)\xi_0) > 0, \quad \forall (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1). \tag{7.10}$$

Let us consider estimate (6.1). Let us recall that J_1, J_2, J_3 are defined in (6.2). We have

$$\begin{aligned}
J_2 &= \operatorname{Re} \int_\Sigma 2|s|\mu \frac{\partial w_{1,\nu}}{\partial y_2} \overline{\left(\mu \frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma \\
&= \operatorname{Re} \int_\Sigma 2|D_\sigma|\mu i \Gamma^+_\mu(y,D')v_{1,\nu} \overline{\left(\mu \frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma \\
&\quad + \operatorname{Re} \int_\Sigma 2|D_\sigma|\mu i V^+_\mu(\cdot,0) \overline{\left(\mu \frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma \\
&= \operatorname{Re} \int_\Sigma 2\mu(D_{y_1}\varphi_{y_1}(y^*) - D_{y_0}\varphi_{y_0}(y^*))\Gamma^+_\mu(y,D')|D_\sigma|^{\frac12}\widehat{v}_{1,\nu}\overline{|D_\sigma|^{\frac12}\widehat{v}_{1,\nu}}\mathrm{d}\Sigma \\
&\quad + \operatorname{Re} \int_\Sigma 2|D_\sigma|\mu i V^+_\mu(\cdot,0) \overline{\left(\mu \frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*) \right)} \mathrm{d}\Sigma.
\end{aligned} \tag{7.11}$$

By (7.10) we obtain from Gårding's inequality that the first integral in the right hand side of (7.11) is negative. Consider two cases. First let

$$\varphi_{y_1}(y^*)\xi_1^*\Gamma^+_\mu(y^*,\zeta^*) > 0.$$

This inequality and (7.10) yield $|\xi_0^*\varphi_{y_0}(y^*)| > |\xi_1^*\mu(y^*)\varphi_{y_1}(y^*)|$. If $\xi_0^*\varphi_{y_0}(y^*) > 0$ then $\Gamma^+_\mu(y^*,\zeta^*) = |\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) > 0$. Hence $\varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*)\xi_1^* - \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*|}{\left(\frac{\lambda+\mu}{\mu}(y^*)\right)^{\frac12}|\xi_1^*|} = \frac{-\varphi_{y_1}(y^*)\xi_1^* + \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\left(\frac{\lambda+\mu}{\mu}(y^*)\right)^{\frac12}|\xi_1^*|}$. By the first equation in (2.6), this contradicts (7.9).

If $\xi_0\varphi_{y_0}(y^*) < 0$ then $\Gamma^+_\mu(y^*,\zeta^*) = -|\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) < 0$. Therefore $\varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*)\xi_1^* - \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*|}{\left(\frac{\lambda+\mu}{\mu}(y^*)\right)^{\frac12}|\xi_1^*|} = \frac{\varphi_{y_1}(y^*)\xi_1^* - \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\left(\frac{\lambda+\mu}{\mu}(y^*)\right)^{\frac12}|\xi_1^*|}$. By (2.6) this again contradicts (7.9).

In the second case one have to consider $\varphi_{y_1}(y^*)\xi_1^*\Gamma_\mu^+(y^*,\zeta^*)<0$. By Gårding's inequality we have

$$\mathrm{Re}\int_\Sigma 2|D_\sigma|\mu i\Gamma^+_\mu(y,D')v_{1,\nu}\overline{\mu(y^*)\varphi_{y_1}(y^*)\frac{\partial v_{1,\nu}}{\partial y_1}}\mathrm{d}\Sigma<0.$$

This inequality and the fact that the second integral in the right hand side of J_2 is negative, imply that

$$-\mathrm{Re}\int_\Sigma 2|D_\sigma|\mu i\Gamma^+_\mu(y,D')v_{1,\nu}\overline{\left((\lambda+2\mu)\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*)-\frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}\mathrm{d}\Sigma>0. \tag{7.12}$$

Note that

$$\begin{aligned}\Xi^{(1)}_{\lambda+2\mu}&=\int_\Sigma\left(|s|(\lambda+2\mu)^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2+|s|^3(\lambda+2\mu)^2\varphi^3_{y_2}(y^*)|w_{2,\nu}|^2\right)\mathrm{d}\Sigma\\
&+\mathrm{Re}\int_\Sigma 2|s|(\lambda+2\mu)\frac{\partial w_{2,\nu}}{\partial y_2}\overline{\left((\lambda+2\mu)\varphi_{y_1}(y^*)\frac{\partial w_{2,\nu}}{\partial y_1}-\varphi_{y_0}(y^*)\frac{\partial w_{2,\nu}}{\partial y_0}\right)}\mathrm{d}\Sigma\\
&+\int_\Sigma|s|(\lambda+2\mu)\varphi_{y_2}(y^*)(\xi_0^2-(\lambda+2\mu)\xi_1^2-s^2\varphi^2_{y_0}(y^*)+s^2(\lambda+2\mu)\varphi^2_{y_1}(y^*))|\widehat{v}_{2,\nu}|^2\mathrm{d}\Sigma\\
&=\widetilde{J}_1+\widetilde{J}_2+\widetilde{J}_3.\end{aligned} \tag{7.13}$$

Using equalities (5.14) we can transform $\widetilde{J}_2$ as

$$\widetilde{J}_2=-\mathrm{Re}\int_\Sigma 2|s|\frac{\mu}{\lambda+2\mu}\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left((\lambda+2\mu)\varphi_{y_1}(y^*)\frac{\partial w_{1,\nu}}{\partial y_1}-\varphi_{y_0}(y^*)\frac{\partial w_{1,\nu}}{\partial y_0}\right)}\mathrm{d}\Sigma+I,$$

where

$$|I|\le\epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|^2_X+C_9\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}.$$

Then by (7.12) there exists $C_{10}>0$ such that

$$\widetilde{J}_2>C_{10}\int_\Sigma\left(|s|\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2+|s|^3|w_{1,\nu}|^2\right)\mathrm{d}\Sigma. \tag{7.14}$$

Since $r_{\lambda+2\mu}(\gamma)=0$, we have

$$|\widetilde{J}_3|\le C'_{11}\delta_1\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2},w_{2,\nu}\right)\right\|^2_X.$$

This inequality and (7.14) imply

$$\begin{aligned}\Xi^{(1)}_{\lambda+2\mu}\ge C_{11}\int_\Sigma&\left(|s|\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2+|s|^3|w_{2,\nu}|^2+|s|\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2+|s|^3|w_{1,\nu}|^2\right)\mathrm{d}\Sigma\\
&-\epsilon(\delta)\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|^2_X+C_9\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}.\end{aligned} \tag{7.15}$$

Now we will estimate J_3. By (5.18) and (5.19), there exists a constant $C'_{12}>0$ such that

$$\begin{aligned}&|\xi_0^2-s^2\varphi^2_{y_0}(y^*)-(\lambda+2\mu)\xi_1^2+(\lambda+2\mu)s^2\varphi^2_{y_1}(y^*)|\\
\le&C'_{12}\delta_1(|\xi_0|^2+|\xi_1|^2+s^2),\quad\forall\zeta\in\mathcal{O}(\delta_1).\end{aligned} \tag{7.16}$$

Using this inequality we obtain

$$\begin{aligned}&\xi_0^2-\mu\xi_1^2-s^2\varphi_{y_0}^2(y^*)+s^2\mu\varphi_{y_1}^2(y^*)\\=&(\lambda+\mu)(\xi_1^2-s^2\varphi_{y_1}^2(y^*))+(\xi_0^2-(\lambda+2\mu)\xi_1^2-s^2\varphi_{y_0}^2(y^*)+s^2(\lambda+2\mu)\varphi_{y_1}^2(y^*))\\\geq&(\lambda+\mu)(\xi_1^2-s^2\varphi_{y_1}^2(y^*))-C_{12}\delta_1(|\xi_0|^2+|\xi_1|^2+s^2).\end{aligned}$$

Therefore, for all sufficiently small δ_1, there exists $C_{13}>0$ such that

$$\xi_0^2-\mu\xi_1^2-s^2\varphi_{y_0}^2(y^*)+s^2\mu\varphi_{y_1}^2(y^*)\geq C_{13}(|\xi_0|^2+|\xi_1|^2+s^2). \tag{7.17}$$

By (7.17), we see that $J_3\geq 0$. Therefore by (7.15) and (6.1), there exist constants $C_{13}'>0$, $C_{14}>0$ such that

$$\begin{aligned}\varXi_\mu^{(1)}+C_{13}'\varXi_{\lambda+2\mu}^{(1)}\geq C_{14}\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2},w_{1,\nu}\right)\right\|_X^2-C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})}+\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2})\\-\epsilon(\delta)\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_2},\mathbf{w}_{\nu}\right)\right\|_X^2+C_9\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}.\end{aligned}$$

This inequality, (5.16) and (7.4) with the sign +, imply

$$\begin{aligned}\varXi_\mu^{(1)}\geq C_{15}\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_2},\mathbf{w}_{\nu}\right)\right\|_X^2\\-C_{16}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})}+\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}+\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}).\end{aligned} \tag{7.18}$$

By (7.18), (5.31) and (5.32), we obtain (5.15).

Case C. Assume that $s^*\neq 0$. If $\delta_1>0$ is small enough, then there exists a constant $C_{17}>0$ such that

$$|\xi_0\varphi_{y_1}(y^*)-(\lambda+2\mu)\xi_1\varphi_{y_1}(y^*)|^2\leq\delta_1^2C_{17}(|\xi_1|^2+s^2). \tag{7.19}$$

By (5.31), there exists $C_{18}>0$ such that

$$\begin{aligned}\varXi_{\lambda+2\mu}^{(1)}+C_{18}\left(\|h(s)w_{2,\nu}\|^2_{H^1(\mathcal{Q})}+\|h^3(s)w_{2,\nu}\|^2_{L^2(\mathcal{Q})}\right)\\\leq C_{18}\left(\|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})}+\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right)+\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2},w_{2,\nu}\right)\right\|_X^2.\end{aligned} \tag{7.20}$$

By (7.16) and (7.19), we have

$$|\widetilde{J}_2+\widetilde{J}_3|\leq C_{19}\delta_1\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2},w_{2,\nu}\right)\right\|_X^2. \tag{7.21}$$

By (7.21) we obtain from (7.13) that there exists a constant $C_{20}>0$ such that

$$\begin{aligned}\varXi_{\lambda+2\mu}^{(1)}\geq-\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2},w_{2,\nu}\right)\right\|_X^2\\+C_{20}\int_\Sigma\left(h^2(s)(\lambda+2\mu)^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2+h^6(s)(\lambda+2\mu)^2\varphi_{y_2}^3(y^*)|w_{2,\nu}|^2\right)\mathrm{d}\Sigma.\end{aligned} \tag{7.22}$$

From (5.16), we easily obtain

$$\left\| h(s)\left(\frac{\partial w_{2,\nu}}{\partial y_2} - |s|\varphi_{y_2}(y^*)w_{2,\nu} + g_{2,\nu}\right)\right\|^2_{L^2(\Sigma)} = \frac{\mu^2}{(\lambda+2\mu)^2}\left(\left\| h(s)\frac{\partial w_{1,\nu}}{\partial y_1}\right\|^2_{L^2(\Sigma)} + \varphi^2_{y_1}(y^*)\|h(s)|s|w_{1,\nu}\|^2_{L^2(\Sigma)}\right).$$

Hence (7.22) and this equality imply

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{21}\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2\right) + h^6(s)|w_{2,\nu}|^2\right) d\Sigma - \epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X - C_{22}\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}. \tag{7.23}$$

Now we claim that inequality (7.2) holds true for all sufficiently small δ_1. First we may assume that for all $\zeta \in \mathcal{O}(\delta_1)$ we have $s^2 \le C_{23}(\xi_0^2+\xi_1^2)$. In fact, if the last inequality is not true, then $\zeta^* \in \mathcal{M}$ and the case was treated in the previous section. Suppose that (7.2) is not true. In that case $\xi_1^* = 0$ and $\xi_0^* \ne 0, s^* \ne 0$. Therefore $\varphi_{y_0}(y^*) = 0$ by (5.18). However, this implies $(\xi_0^*)^2 + (\lambda(y^*)+2\mu(y^*))\varphi^2_{y_1}(y^*)(s^*)^2 = 0$. Hence we arrived at a contradiction and the verification of (7.2) is complete.

Inequalities (7.2) and (7.23) imply that there exists a constant $C_{24} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{24}\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right) + h^6(s)|\mathbf{w}_\nu|^2\right) d\Sigma$$
$$- \epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X - C_{22}\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}. \tag{7.24}$$

By inequality (7.4) for $V^+_\mu(\cdot,0)$, we obtain the estimate

$$\left\| h(s)\frac{\partial w_{1,\nu}}{\partial y_2}\right\|^2_{L^2(\Sigma)} \le C_{25}\Bigg\{\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right) + h^6(s)|w_{1,\nu}|^2\right) d\Sigma$$
$$+ \quad \|\mathbf{P}_\mu v_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\Bigg\}. \tag{7.25}$$

Inequalities (7.24) and (7.25) imply that there exists a constant $C_{26} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{26}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X - C_{27}(\delta,\delta_1)\left(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right). \tag{7.26}$$

By (7.26), (5.31) and (5.32), we obtain (5.15). The proof of Lemma 7.1 is finished. □

8. The case $\mathsf{r}_\mu(\gamma) \ne 0$ and $\mathsf{r}_{\lambda+2\mu}(\gamma) \ne 0$

In this section, we will prove

Lemma 8.1. *Let $\gamma = (y^*,\zeta^*) \in \Sigma \times \mathbb{S}^2$ be a point such that*

$$|r_\mu(y^*,\zeta^*)| \ne 0 \quad \textit{and} \quad |r_{\lambda+2\mu}(y^*,\zeta^*)| \ne 0. \tag{8.1}$$

If $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$ *and* $\delta_1 > 0$ *is sufficiently small, then estimate (5.15) holds true.*

Proof. Thanks to (8.1) and Proposition 5.3, decomposition (5.23) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Therefore we have

$$(D_{y_2} - \Gamma^+_\mu(y, D'))v_{1,\nu}|_{y_2=0} = V^+_\mu(\cdot, 0), \tag{8.2}$$
$$(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, D'))v_{2,\nu}|_{y_2=0} = V^+_{\lambda+2\mu}(\cdot, 0). \tag{8.3}$$

By Proposition 5.4, we have an *a priori* estimate

$$\|h(D_\sigma)V^+_\mu(\cdot, 0)\|^2_{L^2(\Sigma)} + \|h(D_\sigma)V^+_{\lambda+2\mu}(\cdot, 0)\|^2_{L^2(\Sigma)} \le C_1 \left(\|\mathbf{P}_{\lambda+2\mu} v_2\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right). \tag{8.4}$$

Denote

$$\tilde{\alpha}^+_\mu(y', D) = \alpha^+_\mu(y', D) + i|D_\sigma|(\varphi_{y_2} - \varphi_{y_2}(y^*)).$$

Using (5.16), we may rewrite (8.2) and (8.3) as

$$\frac{\lambda + 2\mu}{\mu}\left(\frac{\partial v_{2,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{2,\nu}\right) - i\tilde{\alpha}^+_\mu(y, D')v_{1,\nu} = iV^+_\mu(\cdot, 0) - i\mathcal{F}^{-1}_\sigma g_{1,\nu}, \tag{8.5}$$

$$\frac{\mu}{\lambda + 2\mu}\left(-\frac{\partial v_{1,\nu}}{\partial y_1} + |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) - i\tilde{\alpha}^+_{\lambda+2\mu}(y, D')v_{2,\nu} = iV^+_{\lambda+2\mu}(\cdot, 0) - i\mathcal{F}^{-1}_\sigma g_{2,\nu}. \tag{8.6}$$

Let $\mathbf{B}(y, D')$ be the matrix pseudo-differential operator with the symbol

$$\mathbf{B}(y, \zeta) = \begin{pmatrix} -i\tilde{\alpha}^+_\mu(y, \zeta) & \frac{\lambda+2\mu}{\mu}(i\xi_1 - |s|\varphi_{y_1}(y)) \\ \frac{\mu}{\lambda+2\mu}(-i\xi_1 + |s|\varphi_{y_1}(y)) & -i\tilde{\alpha}^+_{\lambda+2\mu}(y, \zeta) \end{pmatrix}.$$

By (5.19) and (5.20), we see: if $\det \mathbf{B}(y^*, \zeta^*) = 0$, then either $\xi^*_0 + is^*\varphi_{y_0}(y^*) = 0$ or

$$\zeta^* \in \left\{\zeta \in \mathbb{R}^3;\ (\xi_1 + i|s|\varphi_{y_1}(y^*))^2 = \frac{(\xi_0 + i|s|\varphi_{y_0}(y^*))^2}{(\lambda + 3\mu)(y^*)}\right\}. \tag{8.7}$$

In this case of (8.7), we have $\varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = \xi^*_0 = \xi^*_1 = 0,\ s^* = 1$.
Now we consider two cases

Case A. $\det \mathbf{B}(\gamma) \ne 0$.
In this case, there exists a parametrix of the operator $\mathbf{B}(y, D')$, which we denote by $\mathbf{B}^{-1}(y, D')$, such that

$$(v_{1,\nu}, v_{2,\nu}) = \mathbf{B}^{-1}(y, D')(V^+_\mu(\cdot, 0) - \mathcal{F}^{-1}_\sigma g_{1,\nu}, V^+_{\lambda+2\mu}(\cdot, 0) - \mathcal{F}^{-1}_\sigma g_{2,\nu})^T + K(v_{1,\nu}, v_{2,\nu}), \tag{8.8}$$

where $K : (L^2(\mathcal{Q}))^2 \to (H^1(\mathcal{Q}))^2$. By (8.4) and (8.8),

$$|\varXi_\mu| + |\varXi_{\lambda+2\mu}| \le C_2 \left(\|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_{\lambda+2\mu} v_2\|^2_{L^2(\mathcal{Q})} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right). \tag{8.9}$$

(Here and henceforth, for simplicity, we do not distinguish $\mathbf{a}^T$ from a vector $\mathbf{a}$.) By (8.9), (5.30) and (5.31), we obtain (5.15).

Case B. $\det \mathbf{B}(\gamma) = 0$.

We claim that this situation is possible in the two cases:

$$\text{(i)}\quad \varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = \xi^*_0 = \xi^*_1 = 0, \quad s^* = 1;$$
$$\text{(ii)}\quad \xi^*_0 = 0, \quad s^*\varphi_{y_0}(y^*) = 0. \tag{8.10}$$

The first subcase was treated in Section 6. Let us consider the second subcase (8.10). Next we may assume that

$$\zeta^* \in \widetilde{\mathcal{M}}.$$

Otherwise, $\zeta^* \in \mathcal{M}$, so that the case was treated in Section 6. Next we may assume that

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = \operatorname{Im}\Gamma^+_{\lambda+2\mu}(\gamma) \ge 0. \tag{8.11}$$

Really if

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = \operatorname{Im}\Gamma^+_{\lambda+2\mu}(\gamma) < 0, \tag{8.12}$$

then the situation is simple since we have the decomposition

$$\mathbf{P}_\beta v_{j(\beta),\nu} = \beta|G|^2(D_{y_2} - \Gamma^\mp_\beta(y, D'))V^\pm_\beta + T^\pm_\mu v_{j(\beta),\nu},$$

where $T^\pm_\beta \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$, $\beta \in \{\mu, \lambda + 2\mu\}$, $j(\beta) = 1$ for $\beta = \mu$ and $j(\beta) = 2$ for $\beta = \lambda + 2\mu$. This decomposition, (8.12) and Proposition 5.3 imply

$$\|h(D_\sigma)(D_{y_2} - \Gamma^\pm_\beta(y, D'))v_{j(\beta),\nu}|_{y_2=0}\|_{L^2(\Sigma)} \le C_3\left(\|\mathbf{P}_\beta v_{j(\beta),\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^2(\mathcal{Q}))^2}\right). \tag{8.13}$$

Obviously

$$-V^+_\beta(\cdot, 0) + V^-_\beta(\cdot, 0) = (\alpha^+_\beta(y, D') - \alpha^-_\beta(y, D'))v_{1,\nu} \quad \text{on } \Sigma.$$

Since $\alpha^+_\mu(y^*, \zeta^*) - \alpha^-_\mu(y^*, \zeta^*) = 2\sqrt{r_\mu(y^*, \zeta^*)} \ne 0$, we have

$$\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X \le C_4\left(\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}\right) \tag{8.14}$$

by (8.13) and Gårding's inequality.

By (8.14), (5.30) and (5.31), we obtain (5.15) under Condition (8.12).

In order to treat (8.10) under (8.11), we will use Calderon's method. First we introduce the new variables $U = (U_1, U_2, U_3, U_4)$ with four components, where

$$(U_1, U_2) = \Lambda(D')\mathcal{F}^{-1}_\sigma \mathcal{U}, \quad (U_3, U_4) = (D_2 + i|D_\sigma|\varphi_{y_2})\mathcal{F}^{-1}_\sigma \mathcal{U},$$

and Λ is the pseudo-differential operator with the symbol $(s^2 + \xi_1^2 + \xi_0^2 + 1)^{\frac{1}{2}}$. In the new notations, problem (5.1) and (5.2) can be written in the form

$$D_{y_2}U = M(y, D')U + F \quad \text{in } \mathbb{R}^3 \times [0, 1], \quad (U_1, U_2)(y)|_{y_2=0} = 0, \tag{8.15}$$

where $F = (0, \mathbb{P}_\sigma \mathcal{F}^{-1}_\sigma \mathcal{U})$. Here $M(y, D')$ is the matrix pseudo-differential operator whose principal symbol $M_1(y, \zeta)$ is given by

$$M_1(y, \zeta) = \begin{pmatrix} 0 & \Lambda_1 E_2 \\ A^{-1}M_{21}\Lambda_1^{-1} & A^{-1}M_{22} \end{pmatrix} - i|s|\varphi_{y_2}E_4$$

(see [49]). Here we set $\vec{\theta} = (\xi_1 + i|s|\varphi_{y_1}, 0)$, $G(y_1) = (-\mathrm{d}\ell(y_1)/\mathrm{d}y_1, 1)$, $\Lambda_1 = |\zeta|$, $M_{21}(y, \xi' + i|s|\nabla_{y'}\varphi(y)) = ((\xi_0 + i|s|\varphi_{y_0}(y))^2 - \mu(\xi' + i|s|\varphi_{y_1}(y))^2)E_2 - (\lambda + \mu)(y)\vec{\theta}^T\vec{\theta}$, $M_{22}(y, \xi') = -(\lambda + \mu)(y)\vec{\theta}^T G + G^T\vec{\theta} - 2\mu(\vec{\theta}, G)E_2$, $A = (\lambda + \mu)(y)G^T G + \mu(y)|G|^2 E_2$. The matrix $M_1(\gamma)$ has only two eigenvalues $M^\pm$ given by (5.18)–(5.20). Moreover it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form

$$M^\pm = \begin{pmatrix} \Gamma^\pm_\mu(\gamma) & 1 \\ 0 & \Gamma^\pm_\mu(\gamma) \end{pmatrix}.$$

Following [46] and using the change of variables $W = S^{-1}(y, D')U$ which is constructed below, we can reduce system (8.15) to the form

$$D_{y_2} W = \widetilde{M}(y, D')W + T(y, D')W + \widetilde{F}, \tag{8.16}$$

where the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} M_+(y,\zeta) & 0 \\ 0 & M_-(y,\zeta) \end{pmatrix}, \quad M_\pm = \begin{pmatrix} \Gamma^\pm_{\lambda+2\mu}(y,\zeta) & m^\pm_{12}(y,\zeta) \\ 0 & \Gamma^\pm_\mu(y,\zeta) \end{pmatrix},$$

the operator T is in $L^\infty(0,1;\mathcal{L}((H^1(\Sigma))^4,(H^1(\Sigma))^4))$, $m^\pm_{12}(y,D')$ are first order operators and

$$\|\widetilde{F}\|_{L^2(\mathbb{R}^1;(H^1(\Sigma))^2)} \le C_5 \left(\|\mathbb{P}_\sigma \mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^1(\mathcal{Q}))^2} + \|\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{L^2(\mathbb{R}^1;(H^1(\Sigma))^2)} \right).$$

Now we describe the construction of the pseudo-differential operator S. We take the symbol S in the form $S = (s_1^+, s_2^+, s_1^-, s_2^-)$. Here

$$s_1^\pm = \left((\vec{\theta} + \alpha^\pm_{\lambda+2\mu} G)\Lambda_1^{-1},\ \alpha^\pm_{\lambda+2\mu}(\vec{\theta} + \alpha^\pm_{\lambda+2\mu} G)\Lambda_1^{-2} \right)$$

are the eigenvectors of the matrix $M_1(y,\zeta)$ on the sphere $\zeta \in \mathbb{S}^2$ which corresponds to the eigenvalue $\Gamma^\pm_{\lambda+2\mu}$ and the vectors $s_2^\pm$ are given by the formula

$$s_2^\pm = E_\pm s^\pm, \quad E_\pm = \frac{1}{2\pi i} \int_{C^\pm} (z - M_1(y,\zeta))^{-1} \mathrm{d}z,$$

where $C^\pm$ are small circles centered at $\Gamma^\pm_\mu(\gamma)$ and $s^\pm$ solves the equation $M_1(\gamma)s^\pm - \Gamma^\pm_\mu(\gamma)s^\pm = s_1^\pm(\gamma)$. Since $\zeta^* \in \widetilde{\mathcal{M}}$ and $\xi_0^* = 0$, we have $\xi_1^* \neq 0$. Therefore the circles $C^\pm$ may be taken such that the disks bounded by these circles do not intersect, provided that δ_1, δ are taken sufficiently small. Note that the vectors $s_j^\pm \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in (s, ξ_0, ξ_1). Now using a standard argument (see [36], p. 241), we can estimate the last two components of W as follows

$$\|(W_3, W_4)\|_{(H^{\frac{3}{2}}(\Sigma))^2} \le C_6 \left(\|\mathbb{P}_\sigma \mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^1(\mathcal{Q}))^2} + \|\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^2(\mathcal{Q}))^2} \right),$$

where the constant C_6 is independent of N.

Now we need to estimate the first two components of the vector function W on Σ. Thanks to the zero boundary conditions for U_3 and U_4, we have

$$S_{11}(y_0,y_1,0,D')(W_1,W_2) = -S_{12}(y_0,y_1,0,D')(W_3,W_4) + T_{-1}(y_0,y_1,0,D')\mathcal{F}_\sigma^{-1}\mathcal{U}, \tag{8.17}$$

where we set

$$S(y,\zeta) = \begin{pmatrix} S_{11}(y,\zeta) & S_{12}(y,\zeta) \\ S_{21}(y,\zeta) & S_{22}(y,\zeta) \end{pmatrix}, \quad T_{-1} : (H^1(\Sigma))^2 \to (H^2(\Sigma))^2.$$

The principal symbol of the pseudo-differential operator S_{11} is the 2×2 matrix such that the first column equals the last two coordinates of the vector s_1^+ and the second column equals the last two coordinates of the vector s_2^+. At the point γ, these vectors are given by the formulae

$$\vec{\eta} = (\xi_1^* + i|s^*|\varphi_{y_1}(y^*), i\mathrm{sign}(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{y_1}(y^*)))$$

$$s_1^+(\gamma) = \left(\vec{\eta},\ i\frac{\mathrm{sign}(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{y_1}(y^*))}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}\vec{\eta} \right),$$

$$\vec{\varsigma} = \frac{-1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}} \frac{\lambda + 3\mu}{2(\lambda + \mu)}(y^*)(i\mathrm{sign}(\xi_1^*), 1),$$

$$s_2^+(\gamma) = \left(\vec{\varsigma}, \frac{1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}} \{i\mathrm{sign}(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{y_1}(y^*))\vec{\varsigma} + \vec{\eta}\} \right).$$

Therefore $\det S_{11}(\gamma) \neq 0$. From (8.15), (8.16) and Gårding's inequality, we obtain

$$\left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|_X \leq C_7(\|\mathbb{P}_\sigma \mathcal{F}_\sigma^{-1} \mathcal{U}\|_{(H^1(\mathcal{Q}))^2} + \|\mathcal{F}_\sigma^{-1} \mathcal{U}\|_{(H^2(\mathcal{Q}))^2}), \tag{8.18}$$

where the constant C_7 is independent of N. By (8.9), (5.30) and (5.31), we obtain (5.15). The proof of Lemma 8.1 is finished. □

9. Proofs of Theorems 2.2 and 2.3

In this section we prove Theorems 2.2 and 2.3. The proof is based on the duality argument and the scenario is described as follows. In view of the fact that observability implies controllability and *vice versa*, we will introduce an extremal problem, and, using Carleman estimate (2.9), we show that there exists a solution to this problem which solves the control problem for the operator P^* and minimizes weighted $L^2(Q)$-norm. At the next step, we obtain an estimate of this solution in the weighted H^1-norm. This estimate implies (2.11) and (2.12).

We introduce the Banach space $\mathcal{X} = (H^1(Q))^2$ with the norm

$$\|\mathbf{w}\|_{\mathcal{X}}^2 = \int_Q (|\nabla \mathbf{w}|^2 + s^2|\mathbf{w}|^2)\mathrm{d}x.$$

In order to prove the theorems, we consider the following extremal problem

$$J(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2) = \frac{1}{2}\|\mathbf{z}\mathrm{e}^{-s\phi}\|_{(L^2(Q))^2}^2 + \frac{1}{2}\|\mathbf{v}_1\mathrm{e}^{-s\phi}\|_{(L^2(Q_\omega))^2}^2 + \frac{1}{2s^2}\|\mathbf{v}_2\mathrm{e}^{-s\phi}\|_{(L^2(Q_\omega))^2}^2 \longrightarrow \inf, \tag{9.1}$$

$$P\mathbf{z} = \mathbf{u}e^{2s\phi} + \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 \quad \text{in } Q, \tag{9.2}$$

$$\mathrm{supp}\, \mathbf{v}_j \subset \overline{Q_\omega}, \quad j = 1, 2, \quad \mathbf{z}|_{(0,T)\times\partial\Omega} = 0, \quad \frac{\partial \mathbf{z}}{\partial x_0}(0, \cdot) = \frac{\partial \mathbf{z}}{\partial x_0}(T, \cdot) = 0. \tag{9.3}$$

Denote by $(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$ the solution to extremal problem (9.1)–(9.3).

We have

Lemma 9.1. *Under the conditions of Theorem 2.2 for all $\mathbf{u} \in (L^2(Q))^2$, there exists a unique solution $(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2) \in (H^1(Q))^2 \times (H^1(0, T; L^2(\Omega)))^2 \times (H^1(Q))^2$ to problem (9.1)–(9.3). Moreover this solution satisfies the optimality system*

$$P\mathbf{p} + \mathbf{z}\mathrm{e}^{-2s\phi} = 0 \quad \text{in } Q, \tag{9.4}$$

$$\mathbf{p}|_{(0,T)\times\partial\Omega} = 0, \quad \frac{\partial \mathbf{p}}{\partial x_0}(0, \cdot) = \frac{\partial \mathbf{p}}{\partial x_0}(T, \cdot) = 0, \tag{9.5}$$

$$\mathbf{p} = \frac{1}{s^2}\mathbf{v}_2\mathrm{e}^{-2s\phi} \quad \text{in } Q_\omega, \quad \frac{\partial \mathbf{p}}{\partial x_0} = -\mathbf{v}_1\mathrm{e}^{-2s\phi} \quad \text{in } Q_\omega, \tag{9.6}$$

$$P\mathbf{z} = \mathbf{u}\mathrm{e}^{2s\phi} + \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 \quad \text{in } Q, \ \mathrm{supp}\, \mathbf{v}_j \subset \overline{Q_\omega}, \quad j \in \{1, 2\}, \tag{9.7}$$

$$\mathbf{z}|_{(0,T)\times\partial\Omega} = 0, \quad \frac{\partial \mathbf{z}}{\partial x_0}(0, \cdot) = \frac{\partial \mathbf{z}}{\partial x_0}(T, \cdot) = 0, \tag{9.8}$$

and the following estimate holds true:

$$\|\mathbf{z}e^{-s\phi}\|^2_{\mathcal{X}} + \left\|\frac{\partial \mathbf{v}_1}{\partial x_0}e^{-s\phi}\right\|^2_{(L^2(Q_\omega))^2} + s^2\|\mathbf{v}_1 e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} + \|\mathbf{v}_2 e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} \le C_1\|\mathbf{u}e^{s\phi}\|^2_{(L^2(Q))^2}. \tag{9.9}$$

Proof of Lemma 9.1. The proof is done along the standard argument (*e.g.*, [38]) and for completeness we will give it. For any $\varepsilon \in (0,1)$, we consider the following extremal problem

$$J_\varepsilon(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}) = \frac{1}{2}\int_Q |\mathbf{z}|^2 e^{-2s\phi}dx + \frac{1}{2}\int_Q m_\varepsilon\left(|\mathbf{v}_1|^2 + \frac{|\mathbf{v}_2|^2}{s^2}\right)e^{-2s\phi}dx + \frac{1}{2\varepsilon}\int_Q |\mathbf{w}|^2 dx \longrightarrow \inf, \tag{9.10}$$

$$P\mathbf{z} = \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 + \mathbf{u}e^{2s\phi} + \mathbf{w} \quad \text{in } Q, \tag{9.11}$$

$$\mathbf{z}|_{(0,T)\times\partial\Omega} = 0, \quad \frac{\partial \mathbf{z}}{\partial x_0}(0,x') = \frac{\partial \mathbf{z}}{\partial x_0}(T,x') = 0, \tag{9.12}$$

where $m_\varepsilon \in C^2(\overline{\Omega})$, $m_\varepsilon(x') > 0$ on $\overline{\Omega}$,

$$m_\varepsilon(x') = \begin{cases} 1, & \text{for } x \in \omega \\ \frac{1}{\varepsilon}, & \text{for dist}\,(x,\omega) \ge \frac{1}{ln\frac{1}{\varepsilon}}. \end{cases}$$

Denote by $(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon)$ a solution to extremal problem (9.10)–(9.12).

Remark. We understand equation (9.11) and the boundary Conditions (9.12) in the sense of the equality:

$$(\mathbf{z}, P\delta)_{(L^2(Q))^2} = -(\mathbf{v}_1, \partial_{x_0}\delta)_{(L^2(Q))^2} + (\mathbf{v}_2 + \mathbf{u}^{2s\phi} + \mathbf{w}, \delta)_{(L^2(Q))^2}$$

for any $\delta \in (H^1(Q))^2$ satisfying $P\delta \in (L^2(Q))^2, \delta|_{(0,T)\times\partial\Omega} = 0$, $\frac{\partial\delta}{\partial x_0}(0,.) = \frac{\partial\delta}{\partial x_0}(T,.) = 0$. If $\mathbf{z}$, $\mathbf{v}_1$ are regular, then $\frac{\partial \mathbf{z}}{\partial x_0}(0,.) - \mathbf{v}_1(0,.) = 0$ and $\frac{\partial \mathbf{z}}{\partial x_0}(T,.) - \mathbf{v}_1(T,.) = 0$.

We have

Proposition 9.1. *Under conditions of Theorem 2.2 for all* $\mathbf{u} \in (L^2(Q))^2$*, there exists a unique solution* $(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon) \in (H^1(Q))^2 \times (H^1(0,T;L^2(\Omega)))^2 \times (H^2(Q))^2 \times (H^2(Q))^2$ *to problem (9.10)–(9.12). Moreover this solution satisfies the optimality system:*

$$\mathbf{p}_\varepsilon(x) = \frac{\widehat{\mathbf{w}}_\varepsilon(x)}{\varepsilon} \quad \text{in } Q, \tag{9.13}$$

$$P\mathbf{p}_\varepsilon + e^{-2s\phi}\widehat{\mathbf{z}}_\varepsilon = 0 \quad \text{in } Q, \tag{9.14}$$

$$\begin{gathered}\mathbf{p}_\varepsilon|_{(0,T)\times\partial\Omega} = \widehat{\mathbf{z}}_\varepsilon|_{(0,T)\times\partial\Omega} = 0,\\ \frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}(0,\cdot) = \frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}(T,\cdot) = \frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}(0,\cdot) = \frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}(T,\cdot) = 0,\end{gathered} \tag{9.15}$$

$$P\widehat{\mathbf{z}}_\varepsilon = \frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0} + \widehat{\mathbf{v}}_{2,\varepsilon} + \mathbf{u}e^{2s\phi} + \widehat{\mathbf{w}}_\varepsilon \quad \text{in } Q, \tag{9.16}$$

$$\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0} + m_\varepsilon \widehat{\mathbf{v}}_{1,\varepsilon}e^{-2s\phi} = 0 \quad \text{in } Q, \tag{9.17}$$

$$\mathbf{p}_\varepsilon - m_\varepsilon\frac{\widehat{\mathbf{v}}_{2,\varepsilon}}{s^2}e^{-2s\phi} = 0 \quad \text{in } Q, \tag{9.18}$$

and the following estimate holds:

$$\|\widehat{\mathbf{z}}_\varepsilon e^{-s\phi}\|^2_{\mathcal{X}} + \left\|\frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0}e^{-s\phi}\right\|^2_{(L^2(Q_\omega))^2} + s^2\|\widehat{\mathbf{v}}_{1,\varepsilon}e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} + \|\widehat{\mathbf{v}}_{2,\varepsilon}e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} \le C_2\|\mathbf{u}e^{s\phi}\|^2_{(L^2(Q))^2}. \quad (9.19)$$

Proof of Proposition 9.1. Since the functional J_ε is strictly convex and the set of admissible elements is a linear space, problem (9.10)–(9.12) has at most one solution. First let us prove that there exists a solution to (9.10)–(9.12): an element $(\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{w}})$ in the space $(L^2(Q))^8$. Obviously $(0,0,0,-\mathbf{u}e^{-2s\phi})$ is an admissible element and so the set of an admissible elements is not empty. Hence there exists a minimizing sequence $\{(\widehat{\mathbf{z}}_{j,\varepsilon}, \widehat{\mathbf{v}}_{1,j,\varepsilon}, \widehat{\mathbf{v}}_{2,j,\varepsilon}, \widehat{\mathbf{w}}_{j,\varepsilon})\}_{j=1}^\infty$ such that

$$(\widehat{\mathbf{z}}_{j,\varepsilon}, \widehat{\mathbf{v}}_{1,j,\varepsilon}, \widehat{\mathbf{v}}_{2,j,\varepsilon}, \widehat{\mathbf{w}}_{j,\varepsilon}) \to (\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon) \quad \text{weakly in } (L^2(Q))^8. \quad (9.20)$$

Passing to the limit in (9.11), (9.12) and using (9.20), we obtain that $(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon)$ is an admissible element. On the other hand, since the functional J_ε is lower semi-continuous with respect to the weak convergence in $(L^2(Q))^8$, this element is a solution to problem (9.10)–(9.12).

In order to obtain optimality system (9.13)–(9.18), we introduce the function $\mathbf{q}(\delta_1,\delta_2,\delta_3) = J_\varepsilon(\widehat{\mathbf{z}}_\varepsilon + \delta_1 d_1, \widehat{\mathbf{v}}_{1,\varepsilon} + \delta_2 d_2, \widehat{\mathbf{v}}_{2,\varepsilon} + \delta_3 d_3, r(\delta_1,\delta_2,\delta_3))$, where $d_1 \in (L^2(Q))^2$ with $Pd_1 \in (L^2(Q))^2$, $d_2 \in (H^1(Q))^2$, $d_3 \in (L^2(Q))^2$,

$$r(\delta_1,\delta_2,\delta_3) = P(\widehat{\mathbf{z}}_\varepsilon + \delta_1 d_1) - \left(\frac{\partial}{\partial x_0}(\widehat{\mathbf{v}}_{1,\varepsilon} + \delta_2 d_2) + \widehat{\mathbf{v}}_{2,\varepsilon} + \delta_3 d_3\right) - \mathbf{u}e^{2s\phi}.$$

Obviously the function $\mathbf{q}$ attains the minimum in $\mathbb{R}^3$ at $(0,0,0)$ if the variation is admissible. Thus $\nabla\mathbf{q}(0,0,0) = 0$. Moreover the equalities $\frac{\partial \mathbf{q}}{\partial \delta_2}(0,0,0) = \frac{\partial \mathbf{q}}{\partial \delta_3}(0,0,0) = 0$ imply

$$-\frac{1}{\varepsilon}\int_Q \widehat{\mathbf{w}}_\varepsilon \frac{\partial d_2}{\partial x_0}dx + \int_Q m_\varepsilon \widehat{\mathbf{v}}_{1,\varepsilon} d_2 e^{-2s\phi}dx = 0, \forall d_2 \in (H^1(Q))^2 \text{ such that } d_2(0,\cdot) = d_2(T,\cdot) = 0,$$

$$-\frac{1}{\varepsilon}\int_Q \widehat{\mathbf{w}}_\varepsilon d_3 dx + \int_Q m_\varepsilon \frac{\widehat{\mathbf{v}}_{2,\varepsilon} d_3}{s^2}e^{-2s\phi}dx = 0, \quad \forall d_3 \in (L^2(Q))^2.$$

On the other hand, these equalities are equivalent to

$$\frac{1}{\varepsilon}\frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0} + m_\varepsilon \widehat{\mathbf{v}}_{1,\varepsilon}e^{-2s\phi} = 0 \quad \text{in } Q, \quad (9.21)$$

$$\frac{\widehat{\mathbf{w}}_\varepsilon}{\varepsilon} - m_\varepsilon \frac{\widehat{\mathbf{v}}_{2,\varepsilon}}{s^2}e^{-2s\phi} = 0 \quad \text{in } Q. \quad (9.22)$$

By the equality $\frac{\partial \mathbf{q}}{\partial \delta_1}(0,0,0) = 0$, we obtain

$$\left(\frac{\widehat{\mathbf{w}}_\varepsilon}{\varepsilon}, Pd_1\right)_{(L^2(Q))^2} + \int_Q \widehat{\mathbf{z}}_\varepsilon d_1 e^{-2s\phi}dx = 0, \quad \forall d_1 \in X, \quad (9.23)$$

where $X = \{d_1 \in L^2(0,T;(H^2(\Omega))^2);\ Pd_1 \in (L^2(Q))^2,\ d_1|_{(0,T)\times\partial\Omega} = 0,\ \frac{\partial d_1}{\partial x_0}(0,\cdot) = \frac{\partial d_1}{\partial x_0}(T,\cdot) = 0\}$.

Since $\widehat{\mathbf{v}}_{1,\varepsilon} \in (L^2(Q))^2$, we obtain immediately from (9.21) that $\frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0} \in (L^2(Q))^2$. Since $d_1(0,\cdot)$ and $d_1(T,\cdot)$ can be chosen arbitrarily, it follows from (9.23) that

$$\frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0}(0,\cdot) = \frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0}(T,\cdot) = 0, \quad \widehat{\mathbf{w}}_\varepsilon|_{(0,T)\times\partial\Omega} = 0.$$

Introducing the function $\mathbf{p}_\varepsilon$ by formula (9.13), in terms of (9.21)–(9.23), we immediately obtain equalities (9.17), (9.18) and (9.14), (9.15). Equation (9.17) implies $\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0} \in (L^2(Q))^2$. From (9.14), (9.15) we obtain $\mathbf{p}_\varepsilon \in (H^1(Q))^2$.

Next we will show that $\mathbf{p}_\varepsilon \in (H^2(Q))^2$. We extend $\mathbf{p}_\varepsilon$ on the set $[-T, 2T] \times \Omega$ by the formula: $\mathbf{p}_\varepsilon(x_0, x') = \mathbf{p}_\varepsilon(-x_0, x')$ for $x \in [-T, 0] \times \Omega$ and $\mathbf{p}_\varepsilon(x_0, x') = \mathbf{p}_\varepsilon(2T - x_0, x')$ for $(x_0, x') \in [T, 2T] \times \Omega$. In the same way, we extend $-\widehat{\mathbf{z}}_\varepsilon \mathrm{e}^{-2s\phi}$ on the domain $[-T, 2T] \times \Omega$ and denote the extended function by $\widetilde{\mathbf{f}}$. Since $\frac{\partial \phi}{\partial x_0}(T, x') < 0$ for all $x' \in \overline{\Omega}$ and $\frac{\partial \phi}{\partial x_0}(0, x') > 0$ for all $x' \in \overline{\Omega}$ by (2.10), there exists $\delta > 0$ such that we can continue the function ϕ on $[-\delta, T + \delta] \times \Omega$ up to a C^3-function such that $\frac{\partial \phi}{\partial x_0}(x) < 0$ for all $x \in [T, T + \delta] \times \overline{\Omega}$ and $\frac{\partial \phi}{\partial x_0}(x) > 0$ for all $x \in [-\delta, 0] \times \overline{\Omega}$. By (9.14), we have

$$P\mathbf{p}_\varepsilon = \widetilde{\mathbf{f}} \quad \text{in } \widetilde{Q} \equiv [-\delta, T + \delta] \times \Omega. \tag{9.24}$$

Also Condition 2.1 for the function ϕ holds true if we replace the domains Q, Q_ω by $\widetilde{Q}$, $[-\delta, T+\delta]\times\omega$ respectively.

Let $D_h f = \frac{f(x_0+h,x')-f(x)}{h}$ and $D_{\overline{h}} f = \frac{f(x)-f(x_0-h,x')}{h}$. For the function $D_h D_{\overline{h}} \mathbf{p}_\varepsilon$, we have

$$\frac{\partial}{\partial x_0} D_h D_{\overline{h}} \mathbf{p}_\varepsilon |_{x_0=0} = \frac{\partial}{\partial x_0} D_h D_{\overline{h}} \mathbf{p}_\varepsilon |_{x_0=T} = 0.$$

Note that $PD_h D_{\overline{h}} \mathbf{p}_\varepsilon = D_h D_{\overline{h}} \widetilde{\mathbf{f}}$. Hence

$$(\widehat{\mathbf{z}}_\varepsilon, D_h D_{\overline{h}} \widetilde{\mathbf{f}})_{(L^2(Q))^2} = -(\widehat{\mathbf{v}}_{1,\varepsilon}, \partial_{x_0} D_h D_{\overline{h}} \mathbf{p}_\varepsilon)_{(L^2(Q))^2} + (\widehat{\mathbf{v}}_{2,\varepsilon} + \mathbf{u} e^{2s\phi} + \widehat{\mathbf{w}}_\varepsilon, D_h D_{\overline{h}} \mathbf{p}_\varepsilon)_{(L^2(Q))^2}.$$

Using (9.17), (9.18) and the definition of the function $\widetilde{\mathbf{f}}$, we have

$$\begin{aligned}
&\frac{1}{2}(D_h \widehat{\mathbf{z}}_\varepsilon, D_h(e^{-2s\phi}\widehat{\mathbf{z}}_\varepsilon))_{(L^2(Q))^2} + \frac{1}{2}(D_{\overline{h}} \widehat{\mathbf{z}}_\varepsilon, D_{\overline{h}}(e^{-2s\phi}\widehat{\mathbf{z}}_\varepsilon))_{(L^2(Q))^2} \\
+&\frac{1}{2}(D_h \widehat{\mathbf{v}}_{1,\varepsilon}, D_h(m_\varepsilon e^{-2s\phi}\widehat{\mathbf{v}}_{1,\varepsilon}))_{(L^2(Q))^2} + \frac{1}{2}(D_{\overline{h}} \widehat{\mathbf{v}}_{1,\varepsilon}, D_{\overline{h}}(m_\varepsilon e^{-2s\phi}\widehat{\mathbf{v}}_{1,\varepsilon}))_{(L^2(Q))^2} \\
+&\frac{1}{2}(D_h \widehat{\mathbf{v}}_{2,\varepsilon}, D_h(s^{-2} m_\varepsilon e^{-2s\phi}\widehat{\mathbf{v}}_{2,\varepsilon}))_{(L^2(Q))^2} + \frac{1}{2}(D_{\overline{h}} \widehat{\mathbf{v}}_{2,\varepsilon}, D_{\overline{h}}(s^{-2} m_\varepsilon e^{-2s\phi}\widehat{\mathbf{v}}_{2,\varepsilon}))_{(L^2(Q))^2} \\
+&\frac{1}{2\varepsilon}(D_h \mathbf{w}_\varepsilon, D_h \mathbf{w}_\varepsilon)_{(L^2(Q))^2} + \frac{1}{2\varepsilon}(D_{\overline{h}} \mathbf{w}_\varepsilon, D_{\overline{h}} \mathbf{w}_\varepsilon)_{(L^2(Q))^2} \\
=&(\mathbf{u} e^{2s\phi}, D_h D_{\overline{h}} \mathbf{p}_\varepsilon)_{(L^2(Q))^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|D_h \widehat{\mathbf{z}}_\varepsilon\|_{(L^2(Q))^2} + \|D_h \widehat{\mathbf{v}}_{1,\varepsilon}\|_{(L^2(Q))^2} + \|D_h \widehat{\mathbf{v}}_{2,\varepsilon}\|_{(L^2(Q))^2} \\
\le& C_2'(\|D_h \mathbf{u}\|_{(L^2(Q))^2} + \|(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon})\|_{(L^2(Q))^6}),
\end{aligned}$$

where the constant $C_2' > 0$ is independent of h. Therefore

$$(\partial_{x_0} \widehat{\mathbf{z}}_\varepsilon, \partial_{x_0} \widehat{\mathbf{v}}_{1,\varepsilon}, \partial_{x_0} \widehat{\mathbf{v}}_{2,\varepsilon}) \in (L^2(Q))^6.$$

Equations (9.11) - (9.18) imply that $\widehat{\mathbf{z}}_\varepsilon \in (H^1(Q))^2$ and $\mathbf{p}_\varepsilon \in (H^2(Q))^2$.

Let $\chi_1 \in C_0^\infty(-\delta, T + \delta)$ be a cut-off function such that $\chi_1|_{[-\frac{\delta}{2}, T+\frac{\delta}{2}]} = 1$. Then

$$P(\chi_1 \mathbf{p}_\varepsilon) = \chi_1 \widetilde{\mathbf{f}} - [\chi_1, P]\mathbf{p}_\varepsilon \quad \text{in } \widetilde{Q}, \tag{9.25}$$

where $\mathrm{supp}\,[\chi_1, P]\mathbf{p}_\varepsilon \subset ([T + \frac{\delta}{2}, T + \delta] \times \overline{\Omega}) \cup ([-\delta, -\frac{\delta}{2}] \times \overline{\Omega})$. We will apply Carleman estimate (2.9) to equation (9.25).

For this, we observe that

$$\begin{aligned}&\|\widetilde{\mathbf{f}}e^{s\phi}\|_{L^2(-\delta,T+\delta;(L^2(\Omega))^2)} \le C_3\|\mathbf{z}_\varepsilon \mathrm{e}^{-s\phi}\|_{(L^2(Q))^2},\\ &\|([\chi_1,P]\mathbf{p}_\varepsilon)\mathrm{e}^{s\phi}\|_{L^2(-\delta,T+\delta;(L^2(\Omega))^2)} \le \frac{C_4}{s}\|\mathbf{p}_\varepsilon \mathrm{e}^{s\phi}\|_{\mathcal{X}}.\end{aligned} \tag{9.26}$$

Moreover, by a way similar to Appendix II in [26] (*i.e.*, the final step of the proof of Lem. 2.3 in [26]), we can prove that at the right hand side of (2.9), we can replace the integral over Q_ω by the following integral

$$\int_{Q_\omega}\left(\left|\frac{\partial^2\mathbf{u}}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{u}}{\partial x_0}\right|^2 + s^4|\mathbf{u}|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x.$$

Note that thanks to the choice of the extension of the function ϕ, we have

$$\begin{aligned}&\int_{(-\delta,T+\delta)\times\omega}\left(\left|\frac{\partial^2(\chi_1\mathbf{p}_\varepsilon)}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial(\chi_1\mathbf{p}_\varepsilon)}{\partial x_0}\right|^2 + s^4|\chi_1\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x\\ &\qquad\le C_5\int_{Q_\omega}\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x.\end{aligned} \tag{9.27}$$

In fact, let us denote the left and the right hand sides of (9.27) respectively by I_1 and I_2. First we can easily see

$$I_1 \le C_5'\int_{(-\delta,T+\delta)\times\omega}\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x.$$

On the other hand, since $\mathbf{p}_\varepsilon(x_0,x') = \mathbf{p}_\varepsilon(-x_0,x')$, $-\delta \le x' \le 0$ by the extension, we have

$$\begin{aligned}&\int_{-\delta}^0\int_\omega\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi(x_0,x')}\mathrm{d}x_0\mathrm{d}x' =\\ &\qquad\int_0^\delta\int_\omega\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi(-x_0,x')}\mathrm{d}x_0\mathrm{d}x'.\end{aligned}$$

By (2.10), we have $\partial_{x_0}\phi(0,x') > 0$. Therefore, for all sufficiently small $\delta > 0$, we obtain $\partial_{x_0}\phi(x) > 0$ for all $x_0 \in [-\delta,\delta]$. This implies $e^{2s\phi(-x_0,x')} \le e^{2s\phi(x_0,x')}$ for $0 < x_0 < \delta$. Hence

$$\begin{aligned}&\int_0^\delta\int_\omega\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi(-x_0,x')}\mathrm{d}x_0\mathrm{d}x'\\ &\qquad\le\int_0^\delta\int_\omega\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi(x_0,x')}\mathrm{d}x_0\mathrm{d}x' \le I_2.\end{aligned}$$

We can similarly estimate

$$\int_T^{T+\delta}\int_\omega\left(\left|\frac{\partial^2\mathbf{p}_\varepsilon}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}\right|^2 + s^4|\mathbf{p}_\varepsilon|^2\right)\mathrm{e}^{2s\phi}\mathrm{d}x_0\mathrm{d}x'.$$

Thus the verification of (9.27) is complete.

Using equations (9.17), (9.18), (9.24) and estimate (9.27), by Theorem 2.1, we obtain

$$\sum_{|\alpha|=2} \|(\partial_x^\alpha \mathbf{p}_\varepsilon)\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + s^2\|\mathbf{p}_\varepsilon \mathrm{e}^{s\phi}\|^2_{\mathcal{X}} \le C_6 M(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}), \tag{9.28}$$

where we set

$$M(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}) = \|\widehat{\mathbf{z}}_\varepsilon \mathrm{e}^{-s\phi}\|^2_{\mathcal{X}} + \int_{Q_\omega} \left(\left| \frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0} \right|^2 + s^2 |\widehat{\mathbf{v}}_{1,\varepsilon}|^2 + |\widehat{\mathbf{v}}_{2,\varepsilon}|^2 \right) \mathrm{e}^{-2s\phi} \mathrm{d}x.$$

By (9.14)–(9.18) and integration by parts, we have

$$\left(\frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0} + \widehat{\mathbf{v}}_{2,\varepsilon} + \mathbf{u}e^{2s\phi} + \widehat{\mathbf{w}}_\varepsilon,\, \mathbf{p}_\varepsilon \right)_{(L^2(Q))^2} = (P\widehat{\mathbf{z}}_\varepsilon, \mathbf{p}_\varepsilon)_{(L^2(Q))^2} = (\widehat{\mathbf{z}}_\varepsilon, P\mathbf{p}_\varepsilon)_{(L^2(Q))^2} = -(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon \mathrm{e}^{-2s\phi})_{(L^2(Q))^2}.$$

Therefore, taking the scalar product of (9.16) and $\mathbf{p}_\varepsilon$ in $(L^2(Q))^2$ and using (9.17) and (9.18), we obtain

$$2J_\varepsilon(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon) = -\frac{1}{2}\int_Q (\mathbf{u}\mathrm{e}^{2s\phi}, \mathbf{p}_\varepsilon)\mathrm{d}x.$$

By (9.28), we obtain from this inequality that

$$s^2 J_\varepsilon(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon) \le C_7 \|\mathbf{u}e^{s\phi}\|_{(L^2(Q))^2} M(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon})^{\frac{1}{2}}. \tag{9.29}$$

Next we differentiate equations (9.14) and (9.16) with respect to the variable x_0:

$$P\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0} = \frac{\partial}{\partial x_0}\widetilde{\mathbf{f}} \quad \text{in } Q, \tag{9.30}$$

$$P\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0} = \frac{\partial^2 \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0^2} + \frac{\partial \widehat{\mathbf{v}}_{2,\varepsilon}}{\partial x_0} + \frac{\partial(\mathbf{u}e^{2s\phi})}{\partial x_0} + \frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0} \quad \text{in } Q. \tag{9.31}$$

Taking the scalar product of (9.31) and $\frac{\partial \mathbf{p}_\varepsilon}{\partial x_0}$ in $(L^2(Q))^2$ and integrating by parts, in terms of (9.14)–(9.18), we similarly obtain

$$\begin{aligned} 2J_\varepsilon\left(\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}, \frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0}, \frac{\partial \widehat{\mathbf{v}}_{2,\varepsilon}}{\partial x_0}, \frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0}\right) = \int_Q \Bigg\{ &\left(\mathbf{u}\mathrm{e}^{2s\phi}, \frac{\partial^2 \mathbf{p}_\varepsilon}{\partial x_0^2}\right) + 2s\frac{\partial \phi}{\partial x_0}\left(\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}, \widehat{\mathbf{z}}_\varepsilon\right)\mathrm{e}^{-2s\phi} \\ &+ 2s\frac{\partial \phi}{\partial x_0} m_\varepsilon \left(\frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0}, \widehat{\mathbf{v}}_{1,\varepsilon}\right)\mathrm{e}^{-2s\phi} + \frac{2m_\varepsilon}{s}\frac{\partial \phi}{\partial x_0}\left(\frac{\partial \widehat{\mathbf{v}}_{2,\varepsilon}}{\partial x_0}, \widehat{\mathbf{v}}_{2,\varepsilon}\right)\mathrm{e}^{-2s\phi}\Bigg\}\mathrm{d}x. \end{aligned}$$

This equality and (9.28), (9.29) imply

$$J_\varepsilon\left(\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}, \frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0}, \frac{\partial \widehat{\mathbf{v}}_{2,\varepsilon}}{\partial x_0}, \frac{\partial \widehat{\mathbf{w}}_\varepsilon}{\partial x_0}\right) \le C_8 \|\mathbf{u}\mathrm{e}^{s\phi}\|_{(L^2(Q))^2} M(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon})^{\frac{1}{2}}. \tag{9.32}$$

Let $\widetilde{L}$ denote the part of first order of $L_{\lambda,\mu}$, that is, $(\widetilde{L}\mathbf{v})(x') = \operatorname{div}\mathbf{v}(x')\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{v} + (\nabla_{x'}\mathbf{v})^T)\nabla_{x'}\mu(x')$. Taking the scalar product of (9.16) with $\widehat{\mathbf{z}}_\varepsilon e^{-2s\phi}$ in $(L^2(Q))^2$, we obtain

$$
\begin{aligned}
&\int_Q (\mu|\nabla_{x'}\widehat{\mathbf{z}}_\varepsilon|^2 + (\lambda+\mu)(\operatorname{div}\widehat{\mathbf{z}}_\varepsilon)^2)\mathrm{e}^{-2s\phi}\mathrm{d}x - \int_Q (\widetilde{L}\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon \mathrm{e}^{-2s\phi})\mathrm{d}x \\
&= \int_Q \left(\left|\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}\right|^2 - 2s\partial_{x_0}\phi\left(\frac{\partial \widehat{\mathbf{z}}_\varepsilon}{\partial x_0}, \widehat{\mathbf{z}}_\varepsilon\right)\right)\mathrm{e}^{-2s\phi}\mathrm{d}x \\
&+ \int_Q \left(2\mu s\sum_{k=1}^2 (\partial_{x_k}\widehat{\mathbf{z}}_\varepsilon, (\partial_{x_k}\phi)\widehat{\mathbf{z}}_\varepsilon) + 2(\lambda+\mu)s(\operatorname{div}\widehat{\mathbf{z}}_\varepsilon)(\nabla_{x'}\phi, \widehat{\mathbf{z}}_\varepsilon)\right)\mathrm{e}^{-2s\phi}\mathrm{d}x \\
&- \int_Q \sum_{k=1}^2 (\widehat{\mathbf{z}}_\varepsilon, \partial_{x_k}\widehat{\mathbf{z}}_\varepsilon)(\partial_{x_k}\mu)\mathrm{e}^{-2s\phi}\mathrm{d}x - \int_Q (\operatorname{div}\widehat{\mathbf{z}}_\varepsilon)(\nabla_{x'}(\lambda+\mu), \widehat{\mathbf{z}}_\varepsilon)\mathrm{e}^{-2s\phi}\mathrm{d}x \\
&+ \int_Q (\mathbf{u}\mathrm{e}^{2s\phi} + \widehat{\mathbf{w}}_\varepsilon, \widehat{\mathbf{z}}_\varepsilon)\mathrm{e}^{-2s\phi}\mathrm{d}x + \int_Q \left(\frac{\partial \widehat{\mathbf{v}}_{1,\varepsilon}}{\partial x_0} + \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{z}}_\varepsilon \mathrm{e}^{-2s\phi}\right)\mathrm{d}x.
\end{aligned}
$$

We note that $|\partial_{x_j} z_k||z_\ell| \le \frac{\delta}{2}|\partial_{x_j} z_k|^2 + \frac{1}{2\delta}|z_\ell|^2$ for any $\delta > 0$. Therefore if we take sufficiently small $\delta > 0$ and sufficiently large $s > 0$, then by (9.28), (9.29) and (9.32), we obtain (9.19). The proof of Proposition 9.1 is complete. □

Now we finish the proof of Lemma 9.1. Obviously $\widehat{\mathbf{w}}_\varepsilon \to 0$ in $(L^2(Q))^2$ and $\widehat{\mathbf{v}}_{1,\varepsilon_k}, \widehat{\mathbf{v}}_{2,\varepsilon_k} \to 0$ in $(L^2(Q \setminus Q_\omega))^2$ as $\varepsilon \to +0$. In terms of (9.19), from the sequence $\{(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \mathbf{p}_\varepsilon)\}$, one can extract a subsequence $\{(\widehat{\mathbf{z}}_{\varepsilon_k}, \widehat{\mathbf{v}}_{1,\varepsilon_k}, \widehat{\mathbf{v}}_{2,\varepsilon_k}, \mathbf{p}_{\varepsilon_k})\}$ such that

$$
(\widehat{\mathbf{z}}_{\varepsilon_k}, \widehat{\mathbf{v}}_{1,\varepsilon_k}, \widehat{\mathbf{v}}_{2,\varepsilon_k}, \mathbf{p}_{\varepsilon_k}) \to (\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \mathbf{p}) \text{ weakly in } \mathcal{X} \times (H^1(0,T;L^2(\Omega)))^2 \times (L^2(Q))^4. \tag{9.33}
$$

Thanks to (9.33), we can pass to the limit in (9.14)-(9.18), so that the element $(\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \mathbf{p})$ satisfies the equations

$$
P\mathbf{p} + \mathrm{e}^{-2s\phi}\widehat{\mathbf{z}} = 0 \quad \text{in } Q, \tag{9.34}
$$

$$
\begin{gathered}
\mathbf{p}|_{(0,T)\times\partial\Omega} = \widehat{\mathbf{z}}|_{(0,T)\times\partial\Omega} = 0, \\
\frac{\partial \mathbf{p}}{\partial x_0}(0,\cdot) = \frac{\partial \mathbf{p}}{\partial x_0}(T,\cdot) = \frac{\partial \widehat{\mathbf{z}}}{\partial x_0}(0,\cdot) = \frac{\partial \widehat{\mathbf{z}}}{\partial x_0}(T,\cdot) = 0,
\end{gathered} \tag{9.35}
$$

$$
P\widehat{\mathbf{z}} = \frac{\partial \widehat{\mathbf{v}}_1}{\partial x_0} + \widehat{\mathbf{v}}_2 + \mathbf{u}\mathrm{e}^{2s\phi} \quad \text{in } Q, \tag{9.36}
$$

$$
\frac{\partial \mathbf{p}}{\partial x_0} + \widehat{\mathbf{v}}_1 \mathrm{e}^{-2s\phi} = 0 \quad \text{in } Q, \tag{9.37}
$$

$$
\mathbf{p} - \frac{\widehat{\mathbf{v}}_2}{s^2}\mathrm{e}^{-2s\phi} = 0 \quad \text{in } Q, \quad \operatorname{supp}\widehat{\mathbf{v}}_j \subset \overline{Q}_\omega, \quad j = 1,2. \tag{9.38}
$$

Estimate (9.9) follows from (9.19). Finally we note that $J_\varepsilon(\widehat{\mathbf{z}}_\varepsilon, \widehat{\mathbf{v}}_{1,\varepsilon}, \widehat{\mathbf{v}}_{2,\varepsilon}, \widehat{\mathbf{w}}_\varepsilon) \le J(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$ for all $\varepsilon \in (0,1)$. Hence $J(\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2) \le J(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$, the element $(\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2)$ is a solution to extremal problem (9.1)–(9.3). Since a solution to this problem is unique, we have $(\widehat{\mathbf{z}}, \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2) = (\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$. The proof of Lemma 9.1 is complete. □

Proof of Theorem 2.2. Taking the scalar product of (2.1) with $\mathbf{z}$ in $(L^2(Q))^2$ and integrating by parts, in terms of (2.1), (2.2), (9.7) and (9.8), we obtain the equality

$$
\|\mathbf{u}e^{s\phi}\|^2_{(L^2(Q))^2} = \int_Q (\mathbf{f}, \mathbf{z})\mathrm{d}x - \int_{Q_\omega}\left(\mathbf{u}, \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2\right)\mathrm{d}x. \tag{9.39}
$$

Applying (9.9) to this equality and using again an inequality $|ab| \le \frac{\delta}{2}|a|^2 + \frac{1}{2\delta}|b|^2$ for any $\delta > 0$, we obtain

$$\int_Q s^2|\mathbf{u}|^2 \mathrm{e}^{2s\phi} \mathrm{d}x \le C_9 \left(\|\mathbf{f}\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega} (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)\mathrm{e}^{2s\phi}\mathrm{d}x \right), \quad \forall s \ge s_0(\tau). \tag{9.40}$$

In order to estimate the derivatives of first order for the function $\mathbf{u}$, replacing $\mathbf{u}$ by $\frac{\partial \mathbf{u}}{\partial x_0}$, we consider extremal problem (9.1)–(9.3). Let $(\widetilde{\mathbf{z}}, \widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2)$ be the corresponding solution. Then Lemma 9.1 yields

$$\|\widetilde{\mathbf{z}}\mathrm{e}^{-s\phi}\|^2_{\mathcal{X}} + \left\| \frac{\partial \widetilde{\mathbf{v}}_1}{\partial x_0} \mathrm{e}^{-s\phi} \right\|^2_{(L^2(Q_\omega))^2} + \|\widetilde{\mathbf{v}}_2 \mathrm{e}^{-s\phi}\|^2_{(L^2(Q_\omega))^2} \le C_{10} \left\| \frac{\partial \mathbf{u}}{\partial x_0} \mathrm{e}^{s\phi} \right\|^2_{(L^2(Q))^2}. \tag{9.41}$$

Since the Lamé coefficients are independent of x_0, we have

$$P\frac{\partial \mathbf{u}}{\partial x_0} = \frac{\partial \mathbf{f}}{\partial x_0} \quad \text{in } Q, \quad \frac{\partial \mathbf{u}}{\partial x_0}|_{(0,T)\times\partial\Omega} = 0, \ \frac{\partial \mathbf{u}}{\partial x_0}(T,\cdot) = \frac{\partial \mathbf{u}}{\partial x_0}(0,\cdot) = 0. \tag{9.42}$$

Taking the scalar product of (9.42) with $\widetilde{\mathbf{z}}$ in $(L^2(Q))^2$ and integrating by parts, we obtain the equality

$$\left\| \frac{\partial \mathbf{u}}{\partial x_0}\mathrm{e}^{s\phi} \right\|^2_{(L^2(Q))^2} = \int_Q \left(\frac{\partial \mathbf{f}}{\partial x_0}, \widetilde{\mathbf{z}} \right) \mathrm{d}x - \int_{Q_\omega} \left(\frac{\partial \mathbf{u}}{\partial x_0}, \frac{\partial \widetilde{\mathbf{v}}_1}{\partial x_0} + \widetilde{\mathbf{v}}_2 \right) \mathrm{d}x.$$

Applying the inequality $2|ab| \le \delta|a|^2 + \frac{1}{\delta}|b|^2$ to the second term at the right hand side of this equality, by means of (9.41), we obtain

$$\int_Q \left(\left| \frac{\partial \mathbf{u}}{\partial x_0} \right|^2 + s^2|\mathbf{u}|^2 \right) \mathrm{e}^{2s\phi}\mathrm{d}x \le C_{11} \left\{ \|\mathbf{f}\mathrm{e}^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega} (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)\mathrm{e}^{2s\phi}\mathrm{d}x \right\}, \quad \forall s \ge s_0(\tau). \tag{9.43}$$

Finally, taking the scalar product of (2.1) with $\mathbf{u}\mathrm{e}^{2s\phi}$ in $(L^2(Q))^2$, we obtain

$$\begin{aligned}
\int_Q (\mu|\nabla_{x'}\mathbf{u}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{u})^2)\mathrm{e}^{2s\phi}\mathrm{d}x &= \int_Q \left(\left| \frac{\partial \mathbf{u}}{\partial x_0} \right|^2 + 2s\partial_{x_0}\phi \left(\frac{\partial \mathbf{u}}{\partial x_0}, \mathbf{u} \right) \right) \mathrm{e}^{2s\phi}\mathrm{d}x \\
- \int_Q \left(2\mu s \sum_{k=1}^2 (\partial_{x_k}\mathbf{u}, (\partial_{x_k}\phi)\mathbf{u}) + 2(\lambda+\mu)s(\operatorname{div}\mathbf{u})(\nabla_{x'}\phi, \mathbf{u}) \right) &\mathrm{e}^{2s\phi}\mathrm{d}x - \int_Q \sum_{k=1}^2 (\mathbf{u}, \partial_{x_k}\mathbf{u})(\partial_{x_k}\mu)\mathrm{e}^{2s\phi}\mathrm{d}x \\
- \int_Q (\operatorname{div}\mathbf{u})(\nabla_{x'}(\lambda+\mu), \mathbf{u})&\mathrm{e}^{2s\phi}\mathrm{d}x + \int_Q (\widetilde{L}\mathbf{u}, \mathbf{u}\mathrm{e}^{2s\phi})\mathrm{d}x + \int_Q (\mathbf{f}, \mathbf{u})\mathrm{e}^{2s\phi}\mathrm{d}x.
\end{aligned}$$

This equality and (9.43) imply (2.11), the conclusion of Theorem 2.2. □

Proof of Theorem 2.3. In order to complete the proof, it is sufficient to estimate $\int_Q (\mathbf{f}, \mathbf{z})\mathrm{d}x$ in (9.39) as follows:

$$\left| \int_Q (\mathbf{f}_{-1}, \mathbf{z})\mathrm{d}x \right| \le \|\mathbf{f}_{-1}\mathrm{e}^{s\phi}\|_{(H^{-1}(Q))^2} \|\mathbf{z}\mathrm{e}^{-s\phi}\|_{(H^1(Q))^2} \le \|\mathbf{f}_{-1}\mathrm{e}^{s\phi}\|_{(H^{-1}(Q))^2} \|\mathbf{z}\mathrm{e}^{-s\phi}\|_{\mathcal{X}}$$

and

$$\begin{aligned}
\left| \int_Q (\partial_{x_j}\mathbf{f}_j, \mathbf{z})\mathrm{d}x \right| = \left| \int_Q (\mathbf{f}_j, \partial_{x_j}\mathbf{z})\mathrm{d}x \right| &\le \|\mathbf{f}_j\mathrm{e}^{s\phi}\|_{(L^2(Q))^2} \|(\partial_{x_j}\mathbf{z})\mathrm{e}^{-s\phi}\|_{(L^2(Q))^2} \\
&\le C_{12}\|\mathbf{f}_j\mathrm{e}^{s\phi}\|_{(L^2(Q))^2} (\|\nabla(\mathbf{z}\mathrm{e}^{-s\phi})\|_{(L^2(Q))^2} + s\|\mathbf{z}\mathrm{e}^{-s\phi}\|_{(L^2(Q))^2}) \\
&\le C_{13}\|\mathbf{f}_j\mathrm{e}^{s\phi}\|_{(L^2(Q))^2}\|\mathbf{z}\mathrm{e}^{-s\phi}\|_{\mathcal{X}}.
\end{aligned}$$

Therefore

$$\left|\int_Q \left(\left(\mathbf{f}_{-1}+\sum_{j=0}^{2}\partial_{x_j}\mathbf{f}_j\right),\mathbf{z}\right)\mathrm{d}x\right| \le C_{14}\left(\|\mathbf{f}_{-1}\mathrm{e}^{s\phi}\|_{(H^{-1}(Q))^2}+\sum_{j=0}^{2}\|\mathbf{f}_j\mathrm{e}^{s\phi}\|_{(L^2(Q))^2}\right)\|\mathbf{z}\mathrm{e}^{-s\phi}\|_{\mathcal{X}}.$$

Then, again by using the inequality $|ab|\le \frac{\delta}{2}|a|^2+\frac{1}{2\delta}|b|^2$ for $\delta>0$, this inequality and estimates (9.9), (9.39) imply (2.12). □

Appendix I. Proof of Proposition 5.1

In order to prove the proposition, it is convenient to use the coordinate x instead of y. Moreover it suffices to prove the estimate for an arbitrary but fixed $x_0\in[0,T]$. Therefore we should establish the estimate: there exist $\widehat{\tau}>1$ and $N_0>1$ such that for any $\tau>\widehat{\tau}$ and $N>N_0$, there exists $s_0(\tau,N)$ such that

$$N\int_{\Omega_{1/N^2}}\left(\frac{1}{s\varphi}\sum_{j,k=1}^{2}|\partial_{x_j}\partial_{x_k}\mathbf{u}|^2+s\varphi|\nabla_{x'}\mathbf{u}|^2+s^3\varphi^3|\mathbf{u}|^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x'\le C_0(\|\mathrm{rot}\,\mathbf{u}\,\mathrm{e}^{s\varphi}\|^2_{H^1(\Omega_{1/N^2})}+\|\mathrm{div}\,\mathbf{u}\mathrm{e}^{s\varphi}\|^2_{H^1(\Omega_{1/N^2})}),$$
$$\forall\mathbf{u}\in(H^1_0(\Omega_{1/N^2}))^2,\quad\forall s\ge s_0(\tau,N),\quad \mathrm{supp}\,\mathbf{u}\subset B_\delta\cap\Omega_{1/N^2},\quad (1)$$

where the constant C_0 is independent of N. Recall that $\Omega_{1/N^2}=\left\{x'\in\Omega;\mathrm{dist}\,(x',\partial\Omega)\le\frac{1}{N^2}\right\}$.

First we choose $N_0>0$ sufficiently large such that

$$\nabla_{x'}\psi(x)\ne 0,\quad\forall x'\in\Omega_{1/N^2},\ \forall x_0\in(0,T).$$

The existence of such N_0 follows from (2.6).

Denote $\mathrm{rot}\,\mathbf{u}\equiv\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=\mathbf{y}$ and $\mathrm{div}\,\mathbf{u}\equiv\mathbf{w}$. Let $\mathrm{rot}^*v=(\frac{\partial v}{\partial x_2},-\frac{\partial v}{\partial x_1})$. Using a formula $\mathrm{rot}^*\mathrm{rot}=-\Delta_{x'}+\nabla_{x'}\mathrm{div}$, we obtain

$$-\Delta_{x'}\mathbf{u}=-\mathrm{rot}^*\,\mathbf{y}-\nabla_{x'}\mathbf{w}\quad\text{in }\Omega_{1/N^2},\quad \mathbf{u}|_{\partial\Omega_{1/N^2}}=0.$$

The function $\widetilde{\mathbf{u}}=\mathbf{u}\mathrm{e}^{s\varphi}$ satisfies the equation

$$L_1\widetilde{\mathbf{u}}+L_2\widetilde{\mathbf{u}}=\mathbf{q}_s\quad\text{in }\Omega_{1/N^2},\ \widetilde{\mathbf{u}}|_{\partial\Omega_{1/N^2}}=0,\qquad(2)$$

where $L_1\widetilde{\mathbf{u}}=-\Delta_{x'}\widetilde{\mathbf{u}}-s^2|\nabla_{x'}\varphi|^2\widetilde{\mathbf{u}}$, $L_2\widetilde{\mathbf{u}}=2s\sum_{k=1}^2(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_k}+s(\Delta_{x'}\varphi)\widetilde{\mathbf{u}}$ and $\mathbf{q}_s=(-\mathrm{rot}^*\,\mathbf{y}-\nabla_{x'}\mathbf{w})\mathrm{e}^{s\varphi}$. Taking the L^2 norms of the right and the left hand sides of equation (2), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2}+\|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2}+2(L_1\widetilde{\mathbf{u}},L_2\widetilde{\mathbf{u}})_{(L^2(\Omega_{1/N^2}))^2}=\|\mathbf{q}_s\|^2_{(L^2(\Omega_{1/N^2}v))^2}.$$

Therefore we can obtain the formula

$$(L_1\widetilde{\mathbf{u}},L_2\widetilde{\mathbf{u}})_{(L^2(\Omega_{1/N^2}))^2}=\int_{\Omega_{1/N^2}}\Bigg(2s\sum_{k,j=1}^{2}(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_jx_k}+s^3(\mathrm{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi)$$
$$-|\nabla_{x'}\varphi|^2\Delta_{x'}\varphi)|\widetilde{\mathbf{u}}|^2-\frac{s}{2}\sum_{j=1}^{2}\frac{\partial^2\Delta_{x'}\varphi}{\partial x_j^2}|\widetilde{\mathbf{u}}|^2\Bigg)\mathrm{d}x'-s\int_{\partial\Omega_{1/N^2}}\left|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right|^2(\nabla_{x'}\varphi,\vec{n})\mathrm{d}\sigma.\quad(3)$$

By (2.6), the last integral in (3) is nonnegative. Denote $\psi_1(x) = \psi(x) - \widehat{\varepsilon}\ell_1(x)$. Then

$$
\begin{aligned}
\operatorname{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi) - |\nabla_{x'}\varphi|^2\Delta_{x'}\varphi &= 2\sum_{k,j=1}^{2}\varphi_{x_k}\varphi_{x_j}\varphi_{x_kx_j} \\
&= 2\varphi^3\sum_{k,j=1}^{2}\tau^4(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)^2(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)^2 \\
&\quad + \tau^3(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)(\partial_{x_j}\partial_{x_k}\psi_1 + 2N\partial_{x_k}\ell_1\partial_{x_j}\ell_1 + 2N\ell_1\partial_{x_k}\partial_{x_j}\ell_1).
\end{aligned}
$$

Since $(\nabla_{x'}\psi_1, \nabla_{x'}\ell_1) > 0$ on $\partial\Omega$, there exists a constant $C_1 > 0$ which is independent of N, τ, s such that

$$
\operatorname{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi) - |\nabla_{x'}\varphi|^2\Delta_{x'}\varphi \ge 2\varphi^3\tau^4|\nabla_{x'}\psi_1|^4 + C_1N\tau^3\varphi^3 + \varphi^2O(\tau^3). \tag{4}
$$

On the other hand, by the definition of $\widetilde{\psi} = \psi - \widehat{\varepsilon}\ell_1 + N\ell_1^2 = \psi_1 + N\ell_1^2$,

$$
\begin{aligned}
\sum_{k,j=1}^{2}(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_jx_k} &= \tau^2(\nabla_{x'}\widetilde{\mathbf{u}}, \nabla_{x'}\widetilde{\psi})^2\varphi \\
&\quad + \tau\sum_{k,j=1}^{2}(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})(\partial_{x_j}\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_j}\partial_{x_k}\ell_1)\varphi + 2N\tau(\nabla_{x'}\widetilde{\mathbf{u}}, \nabla_{x'}\ell_1)^2\varphi.
\end{aligned} \tag{5}
$$

Note that there exists a constant $C_2 > 0$, independent of N, such that

$$
\|N\ell_1\partial^2_{x_ix_j}\ell_1\|_{C^0(\overline{\Omega_{1/N^2}})} \le C_2/N. \tag{6}
$$

By (3)–(6), we obtain

$$
\begin{aligned}
\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2} + \|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2} &+ \int_{\Omega_{1/N^2}}(2\varphi^3\tau^4|\nabla_{x'}\psi_1|^4 + C_1N\tau^3\varphi^3)|\widetilde{\mathbf{u}}|^2\mathrm{d}x' \\
&- s\tau C_3\int_{\Omega_{1/N^2}}\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2\mathrm{d}x' \le \|\mathbf{q}_s\|^2_{(L^2(\Omega_{1/N^2}))^2}.
\end{aligned} \tag{7}
$$

Multiplying equation (2) by $sN\varphi\widetilde{\mathbf{u}}$ and integrating by parts, we obtain

$$
\int_{\Omega_{1/N^2}}(sN\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2 + s^2N(\Delta_{x'}\varphi)\varphi|\widetilde{\mathbf{u}}|^2 - s^3\varphi^3|\nabla_{x'}\varphi|^2|\widetilde{\mathbf{u}}|^2 - \frac{sN}{2}\operatorname{div}\varphi|\widetilde{\mathbf{u}}|^2)\mathrm{d}x' = \int_{\Omega_{1/N^2}}\mathbf{q}_s sN\varphi\widetilde{\mathbf{u}}\mathrm{d}x'. \tag{8}
$$

Next we note that

$$
\Delta_{x'}\varphi = (|\nabla_{x'}\widetilde{\psi}|^2\tau^2 + \tau\Delta_{x'}\psi_1 + 2\tau N|\nabla_{x'}\ell_1|^2 + 2\tau N\ell_1\Delta_{x'}\ell_1)\varphi \ge C_4\tau N\varphi.
$$

This inequality and (8) imply

$$
\int_{\Omega_{1/N^2}}\left\{sN\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2 + \frac{1}{2}s^2N(\Delta_{x'}\varphi)\varphi|\widetilde{\mathbf{u}}|^2 - s^3\varphi^3|\nabla_{x'}\varphi|^2|\widetilde{\mathbf{u}}|^2\right\}\mathrm{d}x' \le C_4\|\mathbf{q}_s\|^2_{(L^2(\Omega_{1/N^2}))^2}. \tag{9}
$$

By (7) and (9), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2} + \|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_{1/N^2}))^2} + \int_{\Omega_{1/N^2}} \left(\frac{1}{2}\varphi^3\tau^4|\nabla_{x'}\psi_1|^4 + C_1N\tau^3\varphi^3\right)|\widetilde{\mathbf{u}}|^2\mathrm{d}x' \\ + sN\int_{\Omega_{1/N^2}} \varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2\mathrm{d}x' \le C_5\|\mathbf{q}_s\|^2_{(L^2(\Omega_{1/N^2}))^2}. \quad (10)$$

Let $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_2$ where the functions $\widetilde{\mathbf{u}}_j$ are solutions to the boundary value problems

$$-\Delta_{x'}\widetilde{\mathbf{u}}_1 = L_1\widetilde{\mathbf{u}} \quad \text{in} \quad \Omega_{1/N_0^2}, \quad \widetilde{\mathbf{u}}_1|_{\partial\Omega_{1/N_0^2}} = 0, \quad -\Delta_{x'}\widetilde{\mathbf{u}}_2 = s^2|\nabla_{x'}\varphi|^2\widetilde{\mathbf{u}} \text{ in}\,\Omega_{1/N_0^2},\ \widetilde{\mathbf{u}}_2|_{\partial\Omega_{1/N_0^2}} = 0.$$

By means of a standard *a priori* estimate for the Laplace operator, we have

$$\|\widetilde{\mathbf{u}}_1\|_{(H^2(\Omega_{1/N^2}))^2} \le C_6\|L_1\widetilde{\mathbf{u}}\|_{(L^2(\Omega_{1/N^2}))^2}, \quad (11)$$

$$\frac{\sqrt{N}}{\sqrt{s}}\|\widetilde{\mathbf{u}}_2\|_{(H^2(\Omega_{1/N^2}))^2} \le C_7\sqrt{N}\|s^{\frac{3}{2}}|\nabla_{x'}\varphi|^2\widetilde{\mathbf{u}}\|_{(L^2(\Omega_{1/N^2}))^2}, \quad (12)$$

where the constants C_6 and C_7 are independent of N. Taking $s_0(\tau, N) \ge N$, we obtain (1) from (9)–(12). The proof of Proposition 5.1 is finished. □

Appendix II. Proof of estimate (5.28)

We prove (5.28) for a more general hyperbolic operator. Denote $y = (y_0, y') = (y_0, y_1, ..., y_n)$, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, ..., \xi_n)$ and $\mathcal{G}_N = \mathbb{R}^n \times [0, \frac{1}{N^2}]$.

Let a function $w \in H^1(\mathcal{G}_N)$ satisfy the equations:

$$R(y', D)w = \frac{\partial^2 w}{\partial y_0^2} - \sum_{j,k=1}^n \frac{\partial}{\partial y_j}\left(a_{jk}(y')\frac{\partial w}{\partial y_k}\right) + \sum_{j=0}^n b_j(y')\frac{\partial w}{\partial y_j} + c(y')w = g \text{ in } \mathcal{G}_N, \quad (1)$$

$$w|_{y_n=\frac{1}{N^2}} = \frac{\partial w}{\partial y_n}|_{y_n=\frac{1}{N^2}} = 0, \quad \operatorname{supp} w \subset B_\delta(x^*), \quad (2)$$

where x^* is an arbitrary point on $\partial\mathcal{G}_N$ and $B_\delta(x^*)$ is a ball of radius δ centered at x^*.

We assume that the coefficients of the linear operator R satisfy the conditions

$$a_{jk} \in C^1(\overline{\mathcal{G}_N}),\ a_{jk} = a_{kj}, \quad 1 \le j, k \le n, \quad b_\ell \in L^\infty(\mathcal{G}_N), \quad 0 \le \ell \le n, \quad c \in L^\infty(\mathcal{G}_N) \quad (3)$$

and the uniform ellipticity: there exists $\delta > 0$ such that

$$a(y', \xi, \xi) \equiv \sum_{j,k=1}^n a_{jk}(y')\xi_j\xi_k \ge \delta|\xi|^2, \quad \forall\xi \in \mathbb{R}^{n+1}, \quad \forall y \in \overline{\mathcal{G}_N}. \quad (4)$$

By $R(y', \xi)$, we denote the principal symbol of the operator R:

$$R(y', \xi) = -\xi_0^2 + \sum_{j,k=1}^n a_{jk}(y')\xi_j\xi_k,$$

and by $\widetilde{R}(y', \xi^1, \xi^2)$ the quadratic form

$$\widetilde{R}(y', \xi^1, \xi^2) = \xi_0^1 \xi_0^2 - \sum_{j,k=1}^{n} a_{jk}(y')\xi_j^1 \xi_k^2$$

with $\xi^1 = (\xi_0^1, ..., \xi_n^1)$ and $\xi^2 = (\xi_0^2, ..., \xi_n^2)$. Following [15], we introduce the notations:

$$R^{(j)}(y', \xi) = \frac{\partial R(y', \xi)}{\partial \xi_j}, \quad R^{(j,k)}(y', \xi) = \frac{\partial^2 R(y', \xi)}{\partial \xi_j \partial \xi_k}, \quad R_{(j)}(y', \xi) = \frac{\partial R(y', \xi)}{\partial y_j}\cdot$$

We assume that there exists a function $\psi_1 \in C^2(\overline{\mathcal{G}_N})$ such that

$$\{R, \{R, \psi_1\}\}(y, \xi) > 0 \tag{5}$$

if $(y, \xi) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfies

$$R(y', \xi) = \langle \nabla_\xi R(y', \xi), \nabla \psi_1(y) \rangle = 0,$$

and

$$\{R(y', \xi - is\nabla\psi_1(y)), R(y', \xi + is\nabla\psi_1(y))\}/2is > 0 \tag{6}$$

if $(y, \xi, s) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$R(y', \xi + is\nabla\psi_1(y)) = \langle \nabla_\xi R(y', \xi + is\nabla\psi_1(y)), \nabla\psi_1(y) \rangle = 0,$$

$$R(y, \nabla\psi_1) < 0.$$

Using the function ψ_1 and following [15], we construct the function ϕ by

$$\phi(y) = \mathrm{e}^{\tau\psi_1(y)}, \quad \tau > 1. \tag{7}$$

It is known (see *e.g.*, Th. 8.6.2, p. 205 [15]) that provided that the parameter τ is sufficiently large,

$$\{R, \{R, \phi\}\}(y, \xi) > 0 \tag{8}$$

if $(y, \xi) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfies

$$R(y', \xi) = 0, \tag{9}$$

and

$$\{R(y', \xi - is\nabla\phi(y)), R(y', \xi + is\nabla\phi(y))\}/2is > 0$$

if $(y, \xi, s) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$R(y', \xi + is\nabla\phi(x)) = 0.$$

Now we fix the parameter τ such that inequalities (8) and (9) hold true. Let $\ell_1 \in C^2(\mathcal{G}_N)$ be a function such that $\ell_1|_{y_n=0} = 0$. Let $\widetilde{\psi}(y) = \psi_1(y) + N\ell_1^2(y)$ and $\varphi = \mathrm{e}^{\tau\widetilde{\psi}}$. Since $\varphi(y) = \phi(y)\mathrm{e}^{\tau N \ell_1^2(y)}$, using $\ell_1|_{y_n=0} = 0$, we have

$$\varphi \to \phi \quad \text{in } C^1(\overline{\mathcal{G}_N}) \text{ as } N \to +\infty. \tag{10}$$

Moreover

$$
\begin{aligned}
&\{R(y',\xi-is\nabla\varphi(y)),R(y',\xi+is\nabla\varphi(y))\}/2is\\
&\qquad -2N\tau\sum_{j,k=1}^{n}(\partial_{y_j}\ell_1(y))(\partial_{y_k}\ell_1(y))(R^{(j)}(y',\xi)R^{(k)}(y',\xi)+s^2R^{(j)}(y',\nabla\varphi)R^{(k)}(y',\nabla\varphi))\\
&\qquad\qquad \longrightarrow \{R(y',\xi-is\nabla\phi(y)),R(y',\xi+is\nabla\phi(y))\}/2is \text{ in } C(\mathcal{G}_N\times\mathbb{S}^n) \quad \text{as} \quad N\to+\infty. \qquad (11)
\end{aligned}
$$

Here we set $\mathbb{S}^n=\{\xi\in\mathbb{R}^{n+1};|\xi|=1\}$. By (8)–(11), there exists $N_0>0$ such that for any $N>N_0$, the following inequalities hold true:

$$
\{R,\{R,\varphi\}\}(y,\xi)>0 \tag{12}
$$

if $(y,\xi)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})$ satisfies $R(y,\xi)=0$, and

$$
\{R(y',\xi-is\nabla\varphi(y)),R(y',\xi+is\nabla\varphi(y))\}/2is>C_1(|\xi|^2+Ns^2) \tag{13}
$$

if $(y,\xi,s)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}\setminus\{0\})$ satisfies $R(y',\xi+is\nabla\varphi(y))=0$, where the constant $C_1>0$ is independent of ξ,s,N.

Denote $\widetilde{w}(y)=w(y)e^{s\varphi}$. By (11), the following equality holds:

$$
\mathrm{e}^{s\phi}R(y',D)(\mathrm{e}^{-s\varphi}\widetilde{w})=g\mathrm{e}^{s\varphi} \quad \text{in } \mathcal{G}_N. \tag{14}
$$

The short calculations give the equation

$$
L_{2,\varphi}\widetilde{w}+L_{1,\varphi}\widetilde{w}=g_s \text{ in } \mathcal{G}_N, \tag{15}
$$

where

$$
\begin{aligned}
&L_{1,\varphi}\widetilde{w}=-\sum_{j=0}^{n}s\varphi_{y_j}R^{(j)}(y',\nabla\widetilde{w}), \quad L_{2,\varphi}\widetilde{w}=R\widetilde{w}+s^2R(y',\nabla\varphi)\widetilde{w},\\
&g_s(y)=ge^{s\varphi}+\widetilde{w}R\varphi. \qquad (16)
\end{aligned}
$$

Taking the L_2-norms of the both sides of (15), we obtain

$$
\|g_s\|^2_{L^2(\mathcal{G}_N)}=\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+2(L_{1,\varphi}\widetilde{w},L_{2,\varphi}\widetilde{w})_{L^2(\mathcal{G}_N)}. \tag{17}
$$

Denote

$$
\begin{aligned}
G_\phi(y,s,\widetilde{w})=&\{R,\{R,\phi\}\}(y',\nabla\widetilde{w})+s^2\sum_{j,k=0}^{n}R_{(k)}(y',\nabla\phi)R^{(j)}(y',\nabla\phi)\widetilde{w}^2\\
&+s^2\sum_{j,k=0}^{n}\phi_{y_jy_k}R^{(j)}(y',\nabla\phi)R^{(k)}(y',\nabla\phi)\widetilde{w}^2 \qquad (18)
\end{aligned}
$$

and $G_\varphi(y,s,\widetilde{w})$ is defined similarly.

Let us transform the last term at the right side of (17). In [18], one can find the following identity:

$$(L_{1,\varphi}\widetilde{w}, L_{2,\varphi}\widetilde{w})_{L^2(\mathcal{G}_N)} = \int_{\partial\mathcal{G}_N} \widetilde{R}(y', \vec{n}, \nabla\widetilde{w}) L_{1,\varphi}\widetilde{w}\,\mathrm{d}\Sigma + s\int_{\partial\mathcal{G}_N} \widetilde{R}(y', \nabla\varphi, \vec{n}) R(y', \nabla\widetilde{w})\mathrm{d}\Sigma$$
$$- s^3 \int_{\partial\mathcal{G}_N} R(y', \nabla\varphi)\widetilde{R}(y', \vec{n}, \nabla\varphi)\widetilde{w}^2\mathrm{d}\Sigma + \int_{\mathcal{G}_N} sG_\varphi(y, s, \widetilde{w})\,\mathrm{d}x$$
$$+ \int_{\mathcal{G}_N} \frac{s}{2}\left(\sum_{j,k=0}^{n} R^{(k)}_{(k)}(y', \nabla\widetilde{w})\varphi_{y_j} R^{(j)}(y', \nabla\widetilde{w}) - \theta(R(y', \nabla\widetilde{w}) - s^2 R(y', \nabla\varphi)\widetilde{w}^2)\right)\mathrm{d}y, \tag{19}$$

where $\vec{n}$ is the unit outward normal vector to $\partial\mathcal{G}_N$ and

$$\theta(y) = \sum_{l,m=0}^{n} (\varphi_{y_l y_m} R^{(l,m)}(y', \nabla\widetilde{w}) + \varphi_{y_l} R^{(l,m)}_{(m)}(y', \nabla\widetilde{w})).$$

Now we need the following Lemma proved in [18].

Lemma 1. *Let $w \in H^1(\mathcal{G}_N)$ be a solution to (1) and (2).*

$$s\int_{\mathcal{G}_N} (|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)\mathrm{d}y \le C_2 \int_{\mathcal{G}_N} sG_\phi(y, s, \widetilde{w})\mathrm{d}y$$
$$+ C_3\left(\frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right), \quad \forall s \ge s_0(\tau), \tag{20}$$

where the constants C_2 and C_3 are independent of s, N.

We claim :

$$\left|\int_{\mathcal{G}_N} \frac{s}{2}\left(\sum_{j,k=0}^{n} R^{(k)}_{(k)}(y', \nabla\widetilde{w})\varphi_{y_j} R^{(j)}(y', \nabla\widetilde{w}) - \theta\{R(y', \nabla\widetilde{w}) - s^2 R(y', \nabla\varphi)\widetilde{w}^2\}\right)\mathrm{d}y\right|$$
$$\le \left|\frac{s}{2}\int_{\mathcal{G}_N} \sum_{j,k=0}^{n} R^{(k)}_{(k)}(y', \nabla\widetilde{w})\varphi_{x_j} R^{(j)}(y', \nabla\widetilde{w})\,dy\right| + \left|s\int_{\mathcal{G}_N} \theta(R(y', \nabla\widetilde{w}) - s^2 R(y', \nabla\varphi)\widetilde{w}^2)\mathrm{d}y\right|$$
$$\le \frac{\varepsilon s}{2}\int_{\mathcal{G}_N} (|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)\mathrm{d}y + C_4\left(\frac{1}{s\varepsilon}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s\varepsilon}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} \quad + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right). \tag{21}$$

In fact, by the Cauchy-Bunyakovskii inequality,

$$\left|\int_{\mathcal{G}_N} s\sum_{j,k=0}^{n} R^{(k)}_{(k)}(y', \nabla\widetilde{w})\varphi_{y_j} R^{(j)}(y', \nabla\widetilde{w})\mathrm{d}y\right| \le \frac{\varepsilon s}{4}\|\widetilde{w}\|^2_{H^1(\mathcal{G}_N)} + \frac{C_5}{s\varepsilon}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}. \tag{22}$$

Since the function θ is continuous, there exists $\theta_\varepsilon \in C^2(\overline{\mathcal{G}_N})$ such that $\|\theta-\theta_\varepsilon\|_{C(\overline{\mathcal{G}_N})} \le \frac{\varepsilon}{8}$. Taking the scalar product in $L^2(\mathcal{G}_N)$ of the functions $\theta_\varepsilon\widetilde{w}$ and $L_{2,\varphi}\widetilde{w}$, we obtain the equality

$$\int_{\mathcal{G}_N}\theta_\varepsilon(sR(y',\nabla\widetilde{w})-s^3R(y',\nabla\varphi)\widetilde{w}^2)\mathrm{d}y=-s\int_{\mathcal{G}_N}(L_{2,\varphi}\widetilde{w})\theta_\varepsilon\widetilde{w}\mathrm{d}y$$
$$+s\int_{\mathcal{G}_N}\sum_{j,k=1}^n\left(\frac{\partial a_{jk}}{\partial y_j}\frac{\partial\widetilde{w}}{\partial y_k}\theta_\varepsilon\widetilde{w}-\widetilde{R}(y',\nabla\widetilde{w},\nabla\theta_\varepsilon)\widetilde{w}\right)\mathrm{d}y+\int_{\partial\mathcal{G}_N}a(y,\vec{n},\nabla\widetilde{w})\theta_\varepsilon\widetilde{w}\mathrm{d}\Sigma.$$

Thus

$$\left|\int_{\mathcal{G}_N}\theta(sR(y',\nabla\widetilde{w})-s^3R(y',\nabla\varphi)\widetilde{w}^2)\mathrm{d}y\right|$$
$$\le\left|\int_{\mathcal{G}_N}(\theta-\theta_\varepsilon)(sR(y',\nabla\widetilde{w})-s^3R(y',\nabla\varphi)\widetilde{w}^2)\mathrm{d}y\right|+\left|\int_{\mathcal{G}_N}\theta_\varepsilon(sR(x',\nabla\widetilde{w})-s^3R(x',\nabla\varphi)\widetilde{w}^2)\mathrm{d}y\right|$$
$$\le\frac{\varepsilon s}{4}\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2+s^2\widetilde{w}^2)\mathrm{d}y+C_6\left(\frac{1}{s}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+\frac{1}{s}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}\quad+s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right).\tag{23}$$

Inequalities (22) and (23) imply (21).

By Lemma 1, we have

$$s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2+s^2\widetilde{w}^2)\mathrm{d}y+\int_{\mathcal{G}_N}4N\tau\sum_{j,k=1}^n\partial_{y_j}\ell_1(y')\partial_{y_k}\ell_1(y')\{R^{(j)}(y',\nabla\widetilde{w})R^{(k)}(y',\nabla\widetilde{w})$$
$$+s^2R^{(j)}(y,\nabla\varphi)R^{(k)}(y',\nabla\varphi)\}\mathrm{d}y\le\int_{\mathcal{G}_N}2sG_\varphi(y,s,\widetilde{w})\mathrm{d}y+\int_{\mathcal{G}_N}\Bigg\{2sG_\phi(y,s,\widetilde{w})-2sG_\varphi(y,s,\widetilde{w})$$
$$+4N\tau\sum_{j,k=1}^n\partial_{y_j}\ell_1(y')\partial_{y_k}\ell_1(y')\{R^{(j)}(y',\nabla\widetilde{w})R^{(k)}(y',\nabla\widetilde{w})+s^2R^{(j)}(y',\nabla\varphi)R^{(k)}(y',\nabla\varphi)\}\Bigg\}\mathrm{d}y$$
$$+C_8\left(\frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+\frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right).\tag{24}$$

Note that there exists a constant $C_9>0$, independent of N, such that

$$\int_{\mathcal{G}_N}4N\tau\sum_{j,k=1}^n\partial_{y_j}\ell_1(y')\partial_{y_k}\ell_1(y')\{R^{(j)}(y',\nabla\widetilde{w})R^{(k)}(y',\nabla\widetilde{w})+s^2R^{(j)}(y',\nabla\varphi)R^{(k)}(y',\nabla\varphi)\}\mathrm{d}y\ge C_9N\int_{\mathcal{G}_N}\widetilde{w}^2\mathrm{d}y\tag{25}$$

for all sufficiently large N.

By (11), we have

$$\int_{\mathcal{G}_N}\Bigg(2sG_\varphi(y,s,\widetilde{w})-2sG_\phi(y,s,\widetilde{w})$$
$$-4N\tau\sum_{j,k=1}^n\partial_{y_j}\ell_1(y')\partial_{y_k}\ell_1(y')\{R^{(j)}(y',\nabla\widetilde{w})R^{(k)}(y',\nabla\widetilde{w})+s^2R^{(j)}(y',\nabla\varphi)R^{(k)}(y',\nabla\varphi)\}\Bigg)\mathrm{d}y$$
$$\le C_{10}(N)s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2+s^2\widetilde{w}^2)\mathrm{d}y,\tag{26}$$

where $C_{10}(N) \to 0$ as $N \to +\infty$. By (10), we obtain

$$\left| \frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} - \frac{1}{s}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} - \frac{1}{s}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} \right|$$
$$\leq C_{11}(N)s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)\mathrm{d}y, \tag{27}$$

where $C_{11}(N) \to 0$ as $N \to +\infty$. Using (25)–(27), from (24) we obtain

$$\frac{1}{C_7}s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)\mathrm{d}y \leq \frac{1}{4}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{4}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}$$
$$+\int_{\mathcal{G}_N} 2sG_\phi(y,s,\widetilde{w})\mathrm{d}y + sC_9\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}, \qquad \forall s \geq s_0(\tau). \tag{28}$$

Inequalities (21), (28) imply (5.28). The proof is finished. □

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