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GENERIC EXISTENCE RESULT FOR AN EIGENVALUE PROBLEM WITH RAPIDLY GROWING PRINCIPAL OPERATOR

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Abstract. We consider the eigenvalue problem

$$\begin{aligned} -\operatorname{div}(a(|\nabla u|)\nabla u) &= \lambda g(x,u) \ \text{ in } \Omega \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

in the case where the principal operator has rapid growth. By using a variational approach, we show that under certain conditions, almost all $\lambda > 0$ are eigenvalues.

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1. Introduction

This paper is about an existence result for nontrivial solutions (eigenfunctions) to quasilinear elliptic equations of the form

$$-\operatorname{div}(a(|\nabla u|)\nabla u) = \lambda g(x,u) \ \text{ in } \Omega \tag{1.1}$$
$$u = 0 \ \text{ on } \partial\Omega, \tag{1.2}$$

(λ is a positive parameter). We are interested here in the case where the principal operator has very fast growth, that is, the function $\phi(t) = a(t)t$ grows faster than any polynomial (at infinity):

$$t^p = o(\phi(t)) \ \text{ as } t \to \infty, \tag{1.3}$$

for any $p > 0$. Throughout the paper, we assume that ϕ is an increasing, continuous, odd function, and its antiderivative Φ, given by

$$\Phi(t) = \int_0^t \phi(s)\mathrm{d}s, \ t \in \mathbb{R}, \tag{1.4}$$

is a Young function. Furthermore,

$$t^p \ll \Phi(t), \tag{1.5}$$

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for any $p > 1$. Typical examples of the function Φ that we consider here are

$$\Phi(t) = \mathrm{e}^{|t|^p} - 1 \ (p \geq 1) \ \text{ and } \ \Phi(t) = \mathrm{e}^{|t|} - |t| - 1.$$

Note that because of (1.5), Φ does not satisfy a Δ_2 condition (we refer to [1,15] for more details on Δ_2 condition and the ordering "$\ll$" among Young functions).

In this paper, we study the existence of nontrivial solutions of (1.1)–(1.2) by min-max arguments (solutions of mountain pass type). The existence of solutions for problem (1.1)–(1.2) was established in [5] in the case where both Φ and its Hölder conjugate $\overline{\Phi}$ satisfy Δ_2 conditions. In [18], we studied the existence of nontrivial solutions in the case Φ has slow growth (that is when Φ satisfies a Δ_2 condition but $\overline{\Phi}$ does not). We refer to [5], [18], and the references therein for related works in those cases.

We are interested here in the remaining case where Φ has fast growth (*cf.* (1.5)). Note that in the case Φ has fast or slow growth, the potential functional for problem (1.1)–(1.2) is not of class C^1. An approach based on variational inequalities was adopted in [18] to treat problems with slowly growing operators. However, the arguments in [18] are not applicable in our situation here. One of the difficulties for equations with rapidly or slowly growing operators is that although one can show a compactness result leading to the Palais-Smale (PS) condition in the case both Φ and its Hölder conjugate $\overline{\Phi}$ satisfy Δ_2 conditions (*cf.* [17]), there has not been analogous results if either Φ or $\overline{\Phi}$ fails to satisfy this property. In [18], we used a nonsmooth version of the Mountain Pass Theorem to obtain the existence of a (PS) sequence. The arguments there based on the fact that Φ satisfies a Δ_2 condition. The situation is very different when Φ has a rapid growth. In fact, to prove the boundedness of (PS) sequences, one usually needs a "super quadratic" condition on the lower order term, such as

$$\mu \int_0^u g(x,t)\mathrm{d}t \leq g(x,u)u \quad (x \in \Omega,\ u \geq 0 \text{ large}), \tag{1.6}$$

together with a standard condition on the principal term such as

$$u\Phi'(u) \leq \nu\Phi(u) \ (x \in \Omega,\ u \geq 0 \text{ large}), \tag{1.7}$$

with $\nu < \mu$ (*cf.* [5,18]). This latter condition is always satisfied with principal operators having polynomial growths, such as $\Phi(t) = t^p$ (p-Laplacian) and $\nu = p$ in (1.7). In general, when Φ is a Young function, (1.7) holds only if Φ itself satisfies a Δ_2 condition (*cf.* [15]). Therefore, one cannot use standard arguments in problems with the mountain pass geometry in our equation here.

We prove here a generic existence result for the eigenvalue problem (1.1)–(1.2). This approach is motivated by Struwe's monotone arguments (*cf.* [24,25]) which were put in abstract form by Jeanjean and Toland [12,13] for smooth equations. We show that under certain conditions, almost all $\lambda > 0$ are eigenvalues of (1.1)–(1.2). Due to its nature, the formulation of (1.1)–(1.2) seems more suitable in Orlicz-Sobolev spaces than in usual Sobolev spaces. Moreover, working directly in the regular Orlicz-Sobolev space W^1L_Φ (as in [5] or [18]) seems rather difficult here. As shown in the sequel, considering the restriction of our problem to the "small" Orlicz-Sobolev space W^1E_Φ is more convenient. The mountain pass solutions of the original problem in W^1L_Φ is next approximated by the associated Palais-Smale sequences in W^1E_Φ. We also note that because of the lack of conditions such as (1.6)–(1.7) in our problem, we are able to obtain by our approach here only the weaker version of generic rather than universal existence of eigenvalues as in the classical situation of the Mountain Pass Theorem such as the eigenvalue problem

$$-\Delta u = \lambda u^3,$$

where all $\lambda > 0$ are eigenvalues (*cf.* [2,23]).

Eigenvalue problems for equations in Orlicz-Sobolev spaces have been studied before by variational arguments (*cf. e.g.* [8,10,20,21]). However, to our knowledge we are not aware of any previous work in the literature that studied equations with rapid growths by min-max methods (solutions of mountain pass types) as in [5] or [18]. We also consider the more general case where the principal operator is given by a Young function of any growth. In this situation, the problem is generally nonsmooth, even in the "small" Orlicz-Sobolev space. To investigate the problem in this more general setting, we extend Jeanjean-Toland's results to nonsmooth operators, which seems interesting on its own.

The paper is organized as follows. In Section 2, after some preparatory concepts and assumptions, we state and prove our main result about generic existence of nontrivial solutions to equations with rapidly growing principal parts. Some notes and observations in the case of problems with general growths are presented in Section 3.

2. Generic existence for equations with rapidly growing principal operators

We first introduce some notation for the functions spaces that will be needed in the sequel. Assume Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary $\partial\Omega$ and Φ (defined in (1.4)) is a Young function (or N-function, *cf.* [1, 15, 19]). Let $\overline{\Phi}$ be the Hölder conjugate of Φ:

$$\overline{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\}\cdot$$

We denote by L_Φ the Orlicz space and W^1L_Φ the first order Orlicz-Sobolev space associated with Φ and Ω. L_Φ is equipped with the Luxemburg norm

$$\|u\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{u}{\lambda}\right)\mathrm{d}x \leq 1\right\},\ u \in L_\Phi.$$

Its associated norm on W^1L_Φ is defined by:

$$\|u\|_{W^1L_\Phi} = \|u\|_{L_\Phi} + \sum_{j=1}^{N}\|\partial_j u\|_{L_\Phi},\ u \in W^1L_\Phi.$$

Let $X = W_0^1L_\Phi$ be the Orlicz-Sobolev space of functions in W^1L_Φ that vanish on $\partial\Omega$. $W_0^1L_\Phi$ is defined as the closure of $C_c^1(\Omega)$ in W^1L_Φ with respect to the weak* topology (*cf.* [1, 15, 16]). We denote by E_Φ the "small" Orlicz space, which is the closure of $L^\infty(\Omega)$ with respect to the norm topology in L_Φ. Also, the "small" Orlicz-Sobolev space corresponding to E_Φ is denoted by W^1E_Φ:

$$W^1E_\Phi = \{u \in W^1L_\Phi \cap E_\Phi : \partial_j u \in E_\Phi,\ 1 \leq j \leq N\}\cdot$$

The eigenvalue problem (1.1)–(1.2) can be formulated (in the weak sense) as the following variational equation:

$$\begin{cases} \displaystyle\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \mathrm{d}x - \lambda \int_\Omega g(x,u)v\mathrm{d}x = 0,\ \forall v \in W_0^1L_\Phi \\ u \in W_0^1L_\Phi \text{ with } a(|\nabla u|)\nabla u \in (L_{\overline{\Phi}})^N. \end{cases} \tag{2.1}$$

Let us consider some assumptions on the principal term Φ and the lower order term g. Since we are interested here in the case where Φ has rapid growth, our first assumption is:

$$\overline{\Phi} \text{ satisfies a } \Delta_2 \text{ condition (at infinity).} \tag{2.2}$$

Under this assumption, it is proved (*cf.* [8, 9, 15]) that the mapping $u \mapsto \phi(u)$ is continuous from E_Φ to $L_{\overline{\Phi}}$. Let J be the functional defined by

$$J(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x. \tag{2.3}$$

Although J is not differentiable (or even finite) on $W_0^1 L_\Phi$, the above property implies that J is of class C^1 in $W_0^1 E_\Phi$ and its Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \mathrm{d}x, \ \forall u, v \in W_0^1 E_\Phi. \tag{2.4}$$

Furthermore, suppose that there exists $p > N$ and $C_1 > 0$ such that

$$\Phi(u) \geq C_1 |u|^p, \ \forall u \in \mathbb{R}. \tag{2.5}$$

Remark 2.1. Since we consider here Young functions Φ with fast growth rates, Condition (2.5) is natural. We have (2.5) if for some $p > N$,

$$|u|^p = O(\Phi(u)) \ \text{ as } u \to 0.$$

For example, if $\Phi(u) = \mathrm{e}^{|u|^m} - 1$ ($m \geq 1$), then we always have (2.5) with some appropriate $p > N$. In fact, for any $p > N$, (2.5) holds for all $|u|$ sufficiently large. On the other hand,

$$\Phi(u) \approx |u|^m \ \text{ for } |u| \text{ small.}$$

If $m > N$ then we can choose $p = m$. If $m \leq N$, then

$$\Phi(u) \geq |u|^m \geq |u|^N \geq |u|^p,$$

for all $p \geq N$, all $u \in [-1, 1]$. Hence, one can choose any $p > N$ in (2.5) (with some appropriate C_1).

Assume next that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(\cdot, 0) = 0$. Let $G(x, u)$ be the antiderivative of $g(x, u)$ with respect to u:

$$G(x, u) = \int_0^u g(x, t)\mathrm{d}t, \ x \in \Omega, u \in \mathbb{R}. \tag{2.6}$$

Assume that g has the following (rather general) growth condition:

$$|g(x, u)| \leq D_1(x) + \Psi_1(|u|), \tag{2.7}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, where $D_1 \in L^1(\Omega)$, $D_1 \geq 0$ and $\Psi_1 : [0, \infty) \to [0, \infty)$ is increasing and continuous. It follows that

$$|G(x, u)| \leq D_2 + \Psi(|u|), \tag{2.8}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, where $D_2 \in [0, \infty)$ and Ψ is also increasing and continuous from $[0, \infty)$ into itself. (Ψ is an antiderivative of Ψ_1.) Without loss of generality, we can assume that

$$\Psi_1(u), \Psi(u) \to \infty \ \text{ as } \ u \to \infty. \tag{2.9}$$

Also, replacing Ψ by an equivalent function (at ∞) if necessary, we can assume without loss of generality that

$$\lim_{r \to 0} \frac{\Psi(r)}{r^p} = 0. \tag{2.10}$$

We also assume the following behavior of $G(x, u)$ for u small:

$$G(x, u) = o(|u|^p) \ \text{ as } u \to 0,$$

uniformly with respect to almost every x, that is,

$$\lim_{u\to 0}\frac{G(x,u)}{|u|^p}=0, \tag{2.11}$$

uniformly for a.e. $x\in\Omega$. Moreover, suppose that G has more rapid growth than Φ at infinity, *i.e.*, there are $C_2, C_3\geq 0$ and a Young function M such that $M\gg\Phi$ and

$$G(x,u)\geq C_2M(u)-C_3,\quad \text{for a.e. } x\in\Omega, \text{ all } u\in\mathbb{R}. \tag{2.12}$$

We are now ready to state and prove our main result.

Theorem 2.2. *Assume Φ and G satisfy the conditions in (2.2), (2.5), (2.7), (2.11), and (2.12). Then, for almost all $\lambda>0$, equation (2.1) has a nontrivial solution. In other words, almost all $\lambda>0$ are eigenvalues of (2.1).*

Proof. The proof is divided into four steps.

Step 1. Instead of an equation in $W_0^1L_\Phi$, we first study the restriction of equation (2.1) to $W_0^1E_\Phi$:

$$\begin{cases}\displaystyle\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla v\mathrm{d}x-\lambda\int_\Omega g(x,u)v\mathrm{d}x=0,\ \forall v\in W_0^1E_\Phi\\ u\in W_0^1E_\Phi.\end{cases} \tag{2.13}$$

As noted above, the functional J defined in (2.3) is of class C^1 in $W_0^1E_\Phi$ and its derivative is given in (2.4). On the other hand, note that (2.5) implies the following continuous embeddings:

$$W_0^1L_\Phi\hookrightarrow W_0^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega}). \tag{2.14}$$

(2.8), together with these embeddings, imply that the functional $B: W_0^1L_\Phi\to\mathbb{R}$ defined by

$$B(u)=\int_\Omega G(x,u(x))\mathrm{d}x,\ u\in W_0^1L_\Phi, \tag{2.15}$$

is of class C^1 on $W_0^1L_\Phi$ and

$$\langle B'(u),v\rangle=\int_\Omega g(x,u)v\mathrm{d}x,\ \forall u,v\in W_0^1L_\Phi. \tag{2.16}$$

Consequently, the functional

$$I=I_\lambda:=J-\lambda B \tag{2.17}$$

is of class C^1 from $W_0^1E_\Phi$ to $\mathbb{R}$ and critical points of I are solutions of (2.13).

To establish the existence of nontrivial critical points of I_λ, *i.e.*, nontrivial solutions of (2.1), we need the following abstract result in [12, 13] about generic existence of bounded Palais-Smale (PS) sequences.

Theorem 2.3 [13] (Th. 2.1 and Ex. 2.1)**.** *Let $\beta=[\alpha_0,\alpha_1]$ be a closed bounded interval in $(-\infty,\infty)$ and $(X,\|\cdot\|)$ be a Banach space. For $\lambda\in\beta$, consider*

$$I_\lambda(u)=A(u)-\lambda B(u), \tag{2.18}$$

with $A,B\in C^1(X,\mathbb{R})$. Suppose A and B satisfy the following condition: if $\{(\lambda_n,u_n)\}$ is a sequence in $\beta\times X$ such that $\lambda_n\to\lambda_0\in\beta$, $\{\lambda_n\}$ is strictly increasing, $\{I_{\lambda_n}(u_n)\}$ is bounded above, and $\{I_{\lambda_0}(u_n)\}$ is bounded below, then

$$\begin{cases}(i)\quad \text{If } \|u_n\|\to\infty \text{ then } B(u_n)\to\infty\\ (ii)\quad \text{If } \{u_n\} \text{ is bounded then } \{B(u_n)\} \text{ is bounded below.}\end{cases} \tag{2.19}$$

Furthermore, assume that I_λ has the moutain pass geometry condition: there exist $v_0, v_1 \in X$ such that for every $\lambda \in \beta$,

$$c(\lambda) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_0), I_\lambda(v_1)\}, \tag{2.20}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_0, \gamma(1) = v_1\}$. Then, for almost all $\lambda \in \beta$, I_λ has a bounded (PS) sequence $\{u_n\} \subset X$ at level $c(\lambda)$, that is,

$$I_\lambda(u_n) \to c(\lambda) \text{ in } \mathbb{R} \text{ and,} \tag{2.21}$$

$$I'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{2.22}$$

In the next two steps, we check that under the conditions of Theorem 2.2, the restricted equation (2.13) satisfies the assumptions in this abstract theorem.

Step 2. Let $[\alpha_0, \alpha_1]$ be any closed bounded interval in $(0, \infty)$. In this step, we prove that I_λ has the geometric structure of the generic Mountain Pass Theorem 2.3 in $W_0^1E_\Phi$. In fact, we check that if Φ and G satisfy Assumptions (2.5), (2.11), and (2.12), then, the functional I_λ defined in (2.17) satisfies the mountain pass Condition (2.20) for certain v_0 and v_1.

For $r > 0$, consider the following ball in $W_0^1E_\Phi$ with respect to the $W_0^{1,p}(\Omega)$-norm:

$$U_r = \left\{ u \in W_0^1E_\Phi : \|u\|_{W_0^{1,p}} := \left(\int_\Omega |\nabla u|^p\right)^{1/p} < r \right\}.$$

From (2.5), one can verify by straightforward arguments that the functional

$$u \mapsto \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \tag{2.23}$$

is continuous on $W_0^1L_\Phi$ and thus on $W_0^1E_\Phi$. It follows that U_r is open in $W_0^1E_\Phi$ and moreover,

$$\partial U_r = \left\{ u \in W_0^1E_\Phi : \|u\|_{W_0^{1,p}} = r \right\}.$$

Let us show that for $r > 0$ sufficiently small, there exists $\alpha > 0$ such that

$$\inf_{u\in\partial U_r} I_\lambda(u) \geq \alpha, \tag{2.24}$$

for all $\lambda \in J$. For $u \in \partial U_r$, (2.5) implies that

$$J(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \geq C_1 \int_\Omega |\nabla u|^p \mathrm{d}x = C_1 r^p. \tag{2.25}$$

On the other hand, from (2.11), we see that for any $\epsilon > 0$ (to be specified later), there exists $u_1 > 0$ such that for a.e. $x \in \Omega$,

$$|G(x,u)| \leq \epsilon |u|^p \ \text{ if } |u| \leq u_1. \tag{2.26}$$

It follows from (2.8) and (2.9) that there are $D_3 > 0$ and $u_2 > 0$ such that

$$|G(x,u)| \leq D_3\Psi(u),$$

for almost all $x \in \Omega$, all u with $|u| \geq u_2$. Furthermore, by increasing D_3 if necessary, we can also choose $u_2 = u_1$. Thus,

$$|G(x,u)| \leq \epsilon |u|^p + D_3\Psi(u),\ \forall u \in \mathbb{R}. \tag{2.27}$$

By Poincaré's inequality, there exists $C_4 > 0$ such that

$$\int_\Omega |u|^p \mathrm{d}x \le C_4 \int_\Omega |\nabla u|^p \mathrm{d}x, \ \forall u \in W_0^{1,p}(\Omega). \tag{2.28}$$

Also, because of the embedding $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, there is $C_5 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_5 \|u\|_{W_0^{1,p}(\Omega)}, \ \forall u \in W_0^{1,p}(\Omega). \tag{2.29}$$

For $u \in \partial U_r$, one has

$$\begin{aligned} \int_\Omega G(x,u)\mathrm{d}x &\le \epsilon \int_\Omega |u|^p \mathrm{d}x + D_3 \int_\Omega \Psi(u)\mathrm{d}x \\ &\le \epsilon C_4 \int_\Omega |\nabla u|^p \mathrm{d}x + D_3 \int_\Omega \Psi(\|u\|_{L^\infty(\Omega)})\mathrm{d}x \\ &\le \epsilon C_4 r^p + D_3 |\Omega| \Psi(C_5 r). \end{aligned} \tag{2.30}$$

Since $\alpha_0 \le \lambda \le \alpha_1$, it follows from (2.25) and (2.30) that

$$\begin{aligned} I_\lambda(u) &= J(u) - \lambda \int_\Omega G(x,u)\mathrm{d}x \\ &\ge C_1 r^p - \lambda[\epsilon C_4 r^p + D_3|\Omega|\Psi(C_5 r)] \\ &\ge C_1 r^p - \alpha_1[\epsilon C_4 r^p + D_3|\Omega|\Psi(C_5 r)]. \end{aligned}$$

Choosing $\epsilon = \frac{C_1}{2\alpha_1 C_4}$, we obtain

$$\begin{aligned} I_\lambda(u) &\ge \frac{C_1}{2} r^p - \alpha_1 D_3 |\Omega| \Psi(C_5 r) \\ &= r^p \left[\frac{C_1}{2} - \alpha_1 D_2 |\Omega| C_5^p \frac{\Psi(C_5 r)}{(C_5 r)^p} \right], \ \forall u \in \partial U_r. \end{aligned} \tag{2.31}$$

According to (2.10), there exists $r_0 \in (0,1)$ such that

$$\frac{\Psi(C_5 r)}{(C_5 r)^p} \le (\alpha_1 D_2 |\Omega| C_5^p)^{-1} \frac{C_1}{4},$$

for all $r \in (0, r_0)$. For such r, we have

$$r^p \left[\frac{C_1}{2} - \alpha_1 D_2 |\Omega| C_5^p \frac{\Psi(C_5 r)}{(C_5 r)^p} \right] \ge \frac{C_1}{4} r^p.$$

This estimate and (2.31) yield (2.24).

Next, we choose a function ϕ_0 in $C_c^\infty(\Omega)$ such that

$$\phi_0 \not\equiv 0, \ \phi_0(x) \ge 0, \ \forall x \in \Omega, \tag{2.32}$$

and

$$|\nabla \phi_0(x)| \le 1, \ \forall x \in \Omega. \tag{2.33}$$

Choose $a \in \Omega$ and $\rho > 0$ such that $\overline{B_\rho(a)} \subset \Omega$ and

$$m_0 := \min_{x \in \overline{B_\rho(a)}} \phi_0(x) > 0. \tag{2.34}$$

Considering $u = s\phi$ $(s > 0)$, one has from (2.33) that

$$\int_\Omega \Phi(s|\nabla \phi_0|)\mathrm{d}x \leq |\Omega|\Phi(s).$$

On the other hand, it follows from (2.12) and (2.34) that

$$\begin{aligned}\lambda \int_\Omega G(x, s\phi_0)\mathrm{d}x &\geq \int_\Omega [C_2 M(s\phi_0) - C_3]\mathrm{d}x \\ &\geq \alpha_0 C_2 \int_{B_\rho(a)} M(s\phi_0)\mathrm{d}x - \alpha_1 C_3 |\Omega| \\ &\geq \alpha_0 C_2 |B_\rho(a)| M(m_0 s) - \alpha_1 C_3 |\Omega|.\end{aligned}$$

Combining these estimates, we get

$$\begin{aligned}I_\lambda(s\phi_0) &= \int_\Omega \Phi(s|\nabla\phi_0|)\mathrm{d}x - \lambda \int_\Omega G(x, s\phi_0)\mathrm{d}x \\ &\leq M(m_0 s)\left[\frac{|\Omega|(\Phi(s) + \alpha_1 C_3)}{M(m_0 s)} - \alpha_0 C_2 |B_\rho(a)|\right].\end{aligned}$$

Since

$$\lim_{s\to\infty} \frac{|\Omega|\Phi(s)}{M(m_0 s)} = \lim_{s\to\infty} \frac{\alpha_1 C_3 |\Omega|}{M(m_0 s)} = 0,$$

by choosing s_1 sufficiently large, we have for all $s \geq s_1$,

$$I_\lambda(s\phi_0) \leq -\frac{1}{2}\alpha_0 C_2 |B_\rho(a)| (< 0). \tag{2.35}$$

Also, for s_1 large, $\|s\phi_0\|_{W_0^{1,p}} > r$. As usual, we choose $v_0 = 0$ and $v_1 = s_1\phi_0 \in W_0^1 E_\Phi$. Assume $\gamma \in C([0,1], W_0^1 E_\Phi)$ is such that $\gamma(0) = v_0$ and $\gamma(1) = v_1$. The mapping $u \mapsto \|u\|_{W_0^{1,p}}$ is continuous on $W_0^1 E_\Phi$ (*cf.* (2.23)), implying the continuity of the function $t \mapsto \|\gamma(t)\|_{W_0^{1,p}}$ (from $[0,1]$ to $\mathbb{R}$). Since

$$\|\gamma(0)\|_{W_0^{1,p}} < r < \|\gamma(1)\|_{W_0^{1,p}},$$

there is $t_0 \in (0,1)$ such that $\|\gamma(t_0)\|_{W_0^{1,p}} = r$. According to (2.24),

$$\sup_{t\in[0,1]} I_\lambda(\gamma(t)) \geq I_\lambda(\gamma(t_0)) \geq \alpha.$$

Because this holds for all $\gamma \in \Gamma$, all $\lambda \in [\alpha_0, \alpha_1]$, we have (2.20).

Step 3. In this step, we verify that I_λ satisfies Condition (2.19) in Theorem 2.3. Let $\{(\lambda_n, u_n)\}$ be a sequence in $[\alpha_0, \alpha_1] \times W_0^1 E_\Phi$ such that $\lambda_n \nearrow \lambda_0 \in [\alpha_0, \alpha_1]$,

$$J(u_n) - \lambda_n B(u_n) \leq c_1, \tag{2.36}$$

and

$$J(u_n) - \lambda_0 B(u_n) \geq c_2, \tag{2.37}$$

for all n, with some $c_1, c_2 \in \mathbb{R}$.

To check (*i*) of (2.19), we assume that $\|u_n\| \to \infty$. From a Poincaré type inequality for Orlicz-Sobolev functions (*cf.* Cor. 5.8, [9]), one has $\||\nabla u_n|\|_{L_\Phi} \to \infty$ and thus $J(u_n) \to \infty$ (*cf.* Th. 9.5, [15]). It follows from (2.36) that $\lambda_n B(u_n) \to \infty$. Since $\lambda_n \in [\alpha_0, \alpha_1] \subset (0, \infty)$, we have $B(u_n) \to \infty$.

Let us verify (2.19)(*ii*). Suppose that $\{u_n\}$ is bounded in $W_0^1 E_\Phi$. From (2.14), we see that $\{u_n\}$ is also bounded in $C(\overline{\Omega})$ (with the sup-norm). It follows from (2.8) and (2.15) that $\{B(u_n)\}$ is bounded. This prove (2.19)(*ii*) and completes the verification of (2.19).

Step 4. We have checked all assumptions of Theorem 2.3. According to that theorem, for almost all $\lambda \in [\alpha_0, \alpha_1]$, there exists a sequence $\{u_n\}$ in $W_0^1 E_\Phi$ (depending on λ) such that

$$\|u_n\|_{W_0^1 L_\Phi} \leq C, \ \forall n, \tag{2.38}$$

$$\int_\Omega \Phi(|\nabla u_n|)\mathrm{d}x - \lambda \int_\Omega G(x, u_n)\mathrm{d}x \to c(\lambda)(> 0), \tag{2.39}$$

and

$$\tilde{\chi_n} := I'_\lambda(u_n) \to 0 \ \text{ in } (W_0^1 E_\Phi)^*(= W^{-1} L_{\overline{\Phi}}). \tag{2.40}$$

We have

$$\langle \tilde{\chi_n}, v\rangle = \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot \nabla v \mathrm{d}x - \lambda \int_\Omega g(x, u_n) v \mathrm{d}x, \ \forall v \in W_0^1 E_\Phi. \tag{2.41}$$

From (2.38), by passing to a subsequence if necessary, one can assume that

$$u_n \rightharpoonup^* u \ \text{ in } W_0^1 L_\Phi, \tag{2.42}$$

for some $u \in W_0^1 L_\Phi$. From (2.2), we also have

$$\phi(|\nabla u_n|) \in L_\Phi, \ \forall n. \tag{2.43}$$

Let $\chi_n \in W^{-1} L_\Phi$ be defined by

$$\langle \chi_n, v\rangle = \langle \tilde{\chi_n}, v\rangle + \lambda \int_\Omega g(x, u_n) v \mathrm{d}x, \ \forall v \in W_0^1 E_\Phi.$$

One has

$$\langle \chi_n, v\rangle = \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot \nabla v \mathrm{d}x, \ \forall v \in W_0^1 E_\Phi. \tag{2.44}$$

From (2.42) and the embeddings in (2.14), we have $u_n \to u$ in $C(\overline{\Omega})$. Together with the growth Condition (2.7), this implies that

$$g(\cdot, u_n) \to g(\cdot, u) \ \text{ in } L^1(\Omega), \tag{2.45}$$

and thus,

$$\int_\Omega g(x, u_n) v \mathrm{d}x \to \int_\Omega g(x, u) v \mathrm{d}x, \ \forall v \in W_0^1 L_\Phi. \tag{2.46}$$

(2.40) and (2.46) yield

$$\chi_n \to \chi \ \text{ in } W^{-1} L_{\overline{\Phi}}, \tag{2.47}$$

with $\chi \in W^{-1} L_{\overline{\Phi}}$ being defined by

$$\langle \chi, v\rangle = \lambda \int_\Omega g(x, u) v \mathrm{d}x, \ \forall v \in W_0^1 L_\Phi.$$

Finally, because of (2.45) and (2.42), we have

$$\lim_{n\to\infty}\int_\Omega a(|\nabla u_n|)\nabla u_n\cdot\nabla u_n \mathrm{d}x = \lim_{n\to\infty}\langle\chi_n,u_n\rangle \;=\; \lim_{n\to\infty}\lambda\int_\Omega g(x,u_n)u_n\mathrm{d}x$$
$$= \lambda\int_\Omega g(x,u)u\mathrm{d}x \;=\; \langle\chi,u\rangle. \tag{2.48}$$

From (2.42), (2.43), (2.44), (2.47), (2.48), and the pseudo-monotonicity of the Φ-Laplacian (*cf.* [9, 11, 26]), we obtain

$$\phi(|\nabla u|) = a(|\nabla u|)|\nabla u| \in L_{\overline{\Phi}}, \tag{2.49}$$

and

$$\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla v\mathrm{d}x = \langle\chi,v\rangle = \lambda\int_\Omega g(x,u)v\mathrm{d}x, \tag{2.50}$$

for all $v\in W_0^1E_\Phi$. Because $W_0^1E_\Phi$ is dense in $W_0^1L_\Phi$ with respect to the weak* topology $\sigma(\prod L_\Phi,\prod E_{\overline{\Phi}})$ and also the topology $\sigma(\prod L_\Phi,\prod L_{\overline{\Phi}})$ (*cf.* [9]), it follows from (2.49), (2.7), (2.45), and (2.46) that (2.50) also holds for all $v\in W_0^1L_\Phi$. Together with this fact, routine arguments show that u is a solution of (2.1).

Finally, let us prove that u is a nontrivial solution. In fact, assume otherwise that $u=0$, *i.e.*, $u_n \rightharpoonup^* 0$ in (2.42). We have, as above, $g(\cdot,u_n)\to g(\cdot,0)=0$ in $L^1(\Omega)$ and $u_n\to 0$ uniformly on $\overline{\Omega}$. Consequently, $\int_\Omega g(x,u_n)u_n\mathrm{d}x\to 0$ and also

$$\int_\Omega G(x,u_n)\mathrm{d}x\to 0. \tag{2.51}$$

It follows from (2.48) that

$$\lim_{n\to\infty}\int_\Omega a(|\nabla u_n|)|\nabla u_n|^2\mathrm{d}x = 0.$$

Because $\Phi(s)\le\phi(s)s$, $\forall s\in\mathbb{R}$, this implies that

$$\lim_{n\to\infty}\int_\Omega \Phi(|\nabla u_n|)\mathrm{d}x = 0. \tag{2.52}$$

As a consequence of (2.51) and (2.52), one gets $I_\lambda(u_n)\to 0$, contradicting (2.39). Hence, u is a nontrivial solution of (2.1). We have shown that for almost all $\lambda\in[\alpha_0,\alpha_1]$, (2.1) has a nontrivial solution. Choosing $[\alpha_0,\alpha_1]=[k^{-1},k](k\in\mathbb{N})$, we see that it has nontrivial solutions for almost all $\lambda>0$. □

We conclude this section with an example illustrating the above conditions and assumptions.

Example. Consider the following eigenvalue problem:

$$\begin{cases} -\mathrm{div}\Big(\mathrm{e}^{|\nabla u|^2}\nabla u\Big) = \lambda g(u) \ \text{ in } \ \Omega \\ u=0 \ \text{ on } \ \partial\Omega. \end{cases}$$

In this case the potential function Φ is given by

$$\Phi(t) = \frac{1}{2}\left(\mathrm{e}^{t^2}-1\right),\ t\in\mathbb{R}.$$

Although Φ does not satisfies a Δ_2 condition, its Hölder conjugate $\overline{\Phi}$ does (*cf.* [15]). Since

$$\Phi(t)\approx\frac{1}{2}t^2$$

for $|t|$ small, there is $C_1 > 0$ such that

$$\Phi(t) \geq C_1 |t|^{N+2},\ \forall t \in \mathbb{R},$$

that is, (2.5) holds with $p = N + 2$. Let $G(u) = \int_0^u g(s)\mathrm{d}s$. If $g(u) = o(|u|^{N+1})$ as $u \to 0$, *i.e.*,

$$\lim_{u\to 0} \frac{g(u)}{|u|^{N+1}} = 0,$$

then, $\lim_{u\to 0} \frac{G(u)}{|u|^{N+2}} = 0$ and we have (2.11). Also, Condition (2.12) is satisfied, for example, if there exist $q > 2$ and $C > 0$ such that $G(u) \geq C\mathrm{e}^{|u|^q}$ for $|u|$ sufficiently large. In fact, we can choose in this case

$$M(u) = \mathrm{e}^{|u|^q} - 1.$$

Then, $\Phi \ll M$ and (2.12) holds.

3. Notes on the case of general Young functions

In this section, we make some observations about the case of equations with general Young functions Φ, where no Δ_2 condition is imposed on either Φ or its Hölder conjugate $\overline{\Phi}$. In this general case, the mapping $u \mapsto \phi(u)$ may not be continuous from E_Φ to $L_{\overline{\Phi}}$ and therefore the functional J defined in (2.3) is not of class C^1 anymore. This means that one cannot apply Theorem 2.1 in [13] (see Th. 2.3 above). A nonsmooth version of that result would therefore be more suitable for our problem.

We first have a simple property of J on the "small" Orlicz-Sobolev space $W_0^1 E_\Phi$.

Lemma 3.1. *The functional J defined in (2.3) is locally Lipschitz continuous on $W_0^1 E_\Phi$.*

Proof. First, we note that the effective domain of the functional

$$L_\Phi \to \mathbb{R} \cup \{\infty\},\ f \mapsto \int_\Omega \Phi(f)\mathrm{d}x,$$

is the Orlicz class $\tilde{L}_\Phi$ and that

$$\{f \in L_\Phi : \mathrm{dist}(f, E_\Phi) < 1\} \subset \tilde{L}_\Phi \subset \{f \in L_\Phi : \mathrm{dist}(f, E_\Phi) \leq 1\},$$

(*cf.* Th. 10.1, [15]). Therefore, the effective domain $D(J)$ of the functional J in (2.3) satisfies

$$\{f \in W_0^1 L_\Phi : \mathrm{dist}(|\nabla f|, E_\Phi) < 1\} \subset D(J) \subset \{f \in W_0^1 L_\Phi : \mathrm{dist}(|\nabla f|, E_\Phi) \leq 1\}\cdot$$

Because the set $\{f \in W_0^1 L_\Phi : \mathrm{dist}(|\nabla f|, E_\Phi) < 1\}$ is open, we have

$$\{f \in W_0^1 L_\Phi : |\nabla f| \in E_\Phi\} \subset [D(J)]^\circ. \tag{3.1}$$

Furthermore, we observe that:

$$\text{If } f \in W_0^1 E_\Phi \text{ then } |\nabla f| \in E_\Phi. \tag{3.2}$$

In fact, if $f \in W_0^1 E_\Phi$ then $\partial_i f \in E_\Phi$, $\forall i \in \{1, \ldots, N\}$ and there exist sequences $\{g_{ij}\} \subset L^\infty(\Omega)$ such that

$$\|g_{ij} - \partial_i f\|_{L_\Phi} \to 0 \text{ as } j \to \infty. \tag{3.3}$$

Cauchy-Schwarz' inequality implies that

$$\left| \left(\sum_{i=1}^N g_{ij}^2 \right)^{1/2} - \left(\sum_{i=1}^N (\partial_i f)^2 \right)^{1/2} \right| \leq \sum_{i=1}^N |g_{ij} - \partial_i f| \text{ a.e. on } \Omega. \tag{3.4}$$

It is easy to check that the Luxemburg norm $\|\cdot\|_{L_\Phi}$ is monotone, in the sense that, if $0 \leq f \leq g$ a.e. in Ω then $\|f\|_{L_\Phi} \leq \|g\|_{L_\Phi}$. From (3.4), we have

$$\left\| \left(\sum_{i=1}^N g_{ij}^2 \right)^{1/2} - |\nabla f| \right\|_{L_\Phi} \leq \left\| \sum_{i=1}^N |g_{ij} - \partial_i f| \right\|_{L_\Phi} \leq \sum_{i=1}^N \|g_{ij} - \partial_i f\|_{L_\Phi}.$$

This and (3.3) give

$$\left\| \left(\sum_{i=1}^N g_{ij}^2 \right)^{1/2} - |\nabla f| \right\|_{L_\Phi} \to 0.$$

Since $(\sum_{i=1}^N g_{ij}^2)^{1/2} \in L^\infty(\Omega)$ for all j, this shows that $|\nabla f| \in E_\Phi$. (3.2) is proved.

From (3.1) and (3.2), it follows that

$$W_0^1 E_\Phi \subset [D(J)]^\circ. \tag{3.5}$$

Since convex functionals are locally Lipschitz in the interiors of their effective domains (*cf. e.g.* [4]), J is locally Lipschitz on $W_0^1 E_\Phi$ (and $J(W_0^1 E_\Phi) \subset \mathbb{R}$). ☐

Lemma 3.1 allows us to use Clarke's generalized gradient instead of the Fréchet derivative. For completeness, some basic concepts related to Clarke's gradient for locally Lipschitz functionals are given here, more details can be found, for example, in [3, 4, 22]. Assume X is a real Banach space with dual X^* and $f : X \to \mathbb{R}$ is a locally Lipschitz functional. For $x, v \in X$, the generalized directional derivative of f at x in the direction v is defined as

$$f^o(x; v) = \limsup_{h \to 0, \lambda \searrow 0} \frac{1}{\lambda}[f(x + h + \lambda v) - f(x + h)].$$

It can be proved that the functional

$$v \mapsto f^o(x; v)$$

is subadditive, continuous, convex, and positively homogeneous on X. The generalized gradient of f at x is the subdifferential of the convex function $f^o(x; \cdot)$ at $v = 0$, that is,

$$\partial f(x) = \{w \in X^* : \langle w, v \rangle \leq f^o(x; v), \forall v \in X\}\cdot \tag{3.6}$$

The following properties of ∂f are proved *e.g.* in [4] (see also [3]):

(i) For all x in X, $\partial f(x)$ is a nonempty convex subset of X^* which is compact with respect to the weak* topology.

(ii) If f is convex then $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis, *i.e.*,

$$\partial f(x) = \{w \in X^* : f(y) - f(x) \geq \langle w, y - x \rangle, \forall y \in X\}.$$

(iii) The set-valued mapping $x \mapsto \partial f(x)$ is upper semicontinuous.

(iv) The function $\lambda(x) = \min\{\|w\|_X : w \in \partial f(x)\}$ exists and is lower semicontinuous (in x).

(v) If f is Fréchet differentiable at x then $\partial f(x) = \{f'(x)\}\cdot$

(vi) If f local Lipschitz and g is of class C^1 on X, then $\partial(f + g)(x) = \partial f(x) + g'(x)$.

These concepts and properties lead to the general concept of critical points for locally Lipschitz functionals (*cf. e.g.* [3,4] and the references therein): a point $x_0 \in X$ is called a critical point of f if $0 \in \partial f(x)$.

Note that if f is differentiable or convex then the concept of critical points in the above definition coincides with the corresponding concepts that were defined previously for differentiable or convex functionals.

We are now ready to state the following generic Mountain Pass theorem for locally Lipschitz functionals, which is a nonsmooth version of the results in [12,13].

Theorem 3.2. *Assume $(X, \|\cdot\|)$ is a Banach space and β is a compact interval in $\mathbb{R}$. For $\lambda \in \beta$, consider the functional $I_\lambda(u)$ given by (2.18) with $B \in C^1(X,\mathbb{R})$ and $A : X \to \mathbb{R}$ is locally Lipschitz. Assume A and B satisfy Condition (2.19) and the mountain pass geometry Condition (2.20) in Theorem 2.3.*

Then, for almost all $\lambda \in \beta$, I_λ has a bounded (PS) sequence $\{u_n\} \subset X$ at level $c(\lambda)$ in the following sense:

$$I_\lambda(u_n) \to c(\lambda) \text{ in } \mathbb{R}, \tag{3.7}$$

and

$$\min\{\|w\|_{X^*} : w \in \partial I_\lambda(u_n)\} \to 0. \tag{3.8}$$

Note that (3.8) is equivalent to the following condition:

$$\exists \{w_n\} \subset X^* : w_n \to 0 \text{ in } X^* \text{ and } w_n \in \partial A(u_n) + \lambda B'(u_n). \tag{3.9}$$

The proof of Theorem 3.2 is based on the same arguments in the proof of Theorem 2.1 and Lemmas 2.1, 2.2 in [13] (Th. 1.1 and Props. 2.1, 2.2 in [12]). The only modification needed here is a deformation lemma for locally Lipschitz functionals instead of C^1 functionals as used in [12,13]. This necessary deformation lemma for locally Lipschitz functionals was already proved in [3] (*cf.* Lem. 3.3 and Th. 3.1, [3], see also [14]).

Now, let us consider equation (2.13) with the principal operator containing a general Young function Φ. The functional J given in (2.3) is always Gâteaux differentiable on $W_0^1E_\Phi$ and its (Gâteaux) derivative $J'(u)$ is still given by (2.4). As noted above, J is both convex and locally Lipschitz on $W_0^1E_\Phi$. Hence, the subgradient $\partial J(u)$ in the sense of convex analysis or as Clarke's generalized gradient in (3.6) is the same as $\{J'(u)\}$ for every $u \in W_0^1E_\Phi$.

(2.13) is equivalent to the inclusion:

$$0 \in \partial J(u) - \lambda B'(u),\ u \in W_0^1E_\Phi. \tag{3.10}$$

In the general case, we assume that g satisfies the following growth condition (instead of (2.7)):

$$|g(x,u)| \le D_1(x) + \Psi'(u), \tag{3.11}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, where $D_1 \ge 0$, $D_1 \in L^\infty(\Omega)$ and Ψ is a smooth Young function such that

$$\Psi \ll \Phi^*, \tag{3.12}$$

(Φ^* is the Sobolev conjugate of Φ, *cf.* [1,6,7]). It follows from (2.6)–(2.7) that

$$|G(x,u)| \le D_1|u| + D_2\Psi(u), \tag{3.13}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$.

Let us check that if g satisfies (3.11)–(3.12), then $I_\lambda = J - \lambda B$ still satisfies Condition (2.19). In fact, the proof of (2.19)(i) is the same as above. We just note that for $u \in L_\Phi$, $\|u\|_{L_\Phi} > 1$, we always have

$$\int_\Omega \Phi(u)\mathrm{d}x \ge \|u\|_{L_\Phi}.$$

Hence, again by Poincaré's inequality in Orlicz-Sobolev spaces,

$$J(u) = \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \to \infty \ \text{ as } \ \|u\|_{W_0^1 L_\Phi} \to \infty.$$

To prove (2.19)(ii), we assume that $\{u_n\}$ is bounded in $W_0^1 E_\Phi$ and thus in $W_0^1 L_\Phi$. From (3.12), we can choose a Young function Ψ_1 such that

$$\Psi \ll \Psi_1 \ll \Phi^*. \tag{3.14}$$

Since the embedding $W_0^1 L_\Phi \hookrightarrow L_{\Psi_1}$ is continuous, $\{u_n\}$ is bounded in L_{Ψ_1}, that is, $c := \sup\{\|u_n\|_{L_{\Psi_1}} : n \in \mathbb{N}\} < \infty$. Thus, $\int_\Omega \Psi_1\left(\frac{u_n}{c}\right)\mathrm{d}x \le 1\ \forall n$. From (3.14), there exist constants $c_1, c_2 > 0$ such that

$$\Psi(u) \le c_1 \Psi_1\left(\frac{u_n}{c}\right) + c_2,\ \forall u \in \mathbb{R}.$$

We have

$$\int_\Omega \Psi_1(u_n)\mathrm{d}x \le \int_\Omega \Psi_1\left(\frac{u_n}{c}\right)\mathrm{d}x + c_2|\Omega| \le c_1 + c_2|\Omega|,\ \forall n. \tag{3.15}$$

From (3.13), there are $\overline{D}_1, \overline{D}_2 > 0$ such that

$$|G(x,u)| \le \overline{D}_1 + \overline{D}_2\Psi(u),$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$. This, together with (3.15), shows that $\{B(u_n)\}$ is bounded.

Moreover, if conditions such as (2.5), (2.10), (2.11), (2.12) are imposed on Φ and G then the functional I_λ has the mountain pass geometry structure (2.20) (note that here we do not assume the Δ_2 Conditions (2.2) and (2.7), (2.8), (2.9) are replaced by Conditions (3.11), (3.12) above).

As a consequence of the above arguments, we have the following generic version of the Mountain Pass theorem for equation (2.1) in this case.

Theorem 3.3. *Assume g satisfies (3.11)–(3.12) and Conditions (2.5), (2.10), (2.11), and (2.12) hold.*

Then, for almost all $\lambda > 0$, equation (2.1) has nontrivial solutions.

Proof. For any compact interval $\beta = [\alpha_0, \alpha_1] \subset (0,\infty)$, by using arguments as in Step 2 in the proof of Theorem 2.2, we see that the functional I_λ defined by (2.17) satisfies the mountain pass geometry Condition (2.20). Moreover, as noted above, I_λ also satisfied (2.19). It follows from Theorem 3.2 that for almost all $\lambda \in \beta$, I_λ has a bounded (PS) sequence $\{u_n\}$. We have $\{u_n\}$ satisfies (3.7)–(3.9) and also $\{\|u_n\|_{W_0^1 L_\Phi}\}$ is bounded.

Since ∂J is understood in the sense of convex analysis, (3.9) is equivalent to the following variational inequality:

$$J(v) - J(u_n) - \lambda \int_\Omega g(\cdot, u_n)(v-u_n)\mathrm{d}x \ge \langle w_n, v-u_n\rangle_{W_0^1 E_\Phi, [W_0^1 E_\Phi]^*},\ \forall v \in W_0^1 E_\Phi. \tag{3.16}$$

On the other hand, because J is Gâteaux differentiable on $W_0^1 E_\Phi$ and J' is given by (2.4), we immediately have from (3.16) that u_n also satisfies

$$\int_\Omega a(|\nabla u_n|)\nabla u_n \cdot \nabla v \mathrm{d}x - \lambda \int_\Omega g(\cdot, u_n) v\mathrm{d}x = \langle w_n, v\rangle,\ \forall v \in W_0^1 E_\Phi. \tag{3.17}$$

In particular, since $u_n \in W_0^1 E_\Phi$, one obtain from (3.2) that $|\nabla u_n| \in E_\Phi$ and thus

$$\phi(|\nabla u_n|) \in L_{\overline{\Phi}}. \tag{3.18}$$

Because $w_n \to 0$ in $W^{-1}L_{\overline{\Phi}}$, we are in the same situation as the last part of the proof of Theorem 2.2, with w_n instead of $\tilde{\chi_n}$ (*cf.* (3.17), and (2.41), (3.18), (2.40)). By using the same arguments, we get (2.42), with u being a nontrivial solution of (2.1). $\square$

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