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REGULARITY OF OPTIMAL SHAPES FOR THE DIRICHLET'S ENERGY WITH VOLUME CONSTRAINT *

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Abstract. In this paper, we prove some regularity results for the boundary of an open subset of $\mathbb{R}^d$ which minimizes the Dirichlet's energy among all open subsets with prescribed volume. In particular we show that, when the volume constraint is "saturated", the reduced boundary of the optimal shape (and even the whole boundary in dimension 2) is regular if the state function is nonnegative.

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Introduction

In this paper, we study the regularity of the boundary of an open subset of $\mathbb{R}^d$ which minimizes Dirichlet's energy among all open subsets with prescribed Lebesgue's measure and included in a fixed open subset of $\mathbb{R}^d$.

More precisely, let $D \subset \mathbb{R}^d$ be open, $f \in L^2(D) \cap L^\infty(D)$ and $a > 0$. For each $\Omega \subset D$ open with $|\Omega| = a$ (where $|\Omega|$ is the Lebesgue's measure of Ω), we define u_Ω as the solution of,

$$u_\Omega \in \mathrm{H}^1_0(\Omega), -\Delta u_\Omega = f \text{ on } \Omega.$$

We know that u_Ω minimizes the Dirichlet energy,

$$J(u) = \int_D \frac{1}{2}|\nabla u|^2 \mathrm{d}x - \int_D fu,$$

among all $u \in \mathrm{H}^1_0(\Omega)$ (recall that $\mathrm{H}^1_0(\Omega) \subset \mathrm{H}^1_0(D)$). Our goal is to study the regularity of $\partial\Omega^*$ where Ω^* is a solution of:

$$J(u_{\Omega^*}) = \min\{J(u_\Omega), \Omega \subset D \text{ open }, |\Omega| = a\}. \tag{1}$$

This problem has been studied, for example, in [5, 14] and in [11]. We mainly prove that, when the constraint $|\Omega^*| = a$ is "saturated", then the reduced boundary $\partial^*\Omega$ is regular in regions where u does not change its sign

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(see *e.g.* [7,9] or [6] for a definition of $\partial^*\Omega^*$). Moreover we prove that, in dimension 2, the full boundary $\partial\Omega^*$ is regular if u is nonnegative.

Let us make some comments on the result. Recall that, if $\partial\Omega^*$ is regular, the Euler-Lagrange equation of the problem (1) may be written as:

$$\left\{\begin{array}{llcl} (a) & -\Delta u^* & = & f \text{ in } \Omega^*, \\ (b) & u^* & = & 0 \text{ on } \partial\Omega^*, \\ (c) & |\nabla u^*| & = & \Lambda \text{ on } \partial\Omega^*. \end{array}\right. \tag{2}$$

This free boundary system appears in several places in the literature. Regularity for the corresponding various minimization problems has been also considered. Our approach is widely inspired by those in [2,3] or [10]. In these papers, penalized version of (1) are considered like, for instance, minimizing expressions of the kind

$$J(u_\Omega) + \lambda|\Omega|.$$

Actually, we prove that our constraint problem may be also written, in the saturated case, as a "pseudo-penalized" problem in the following way: the solution $u^* = u_{\Omega^*}$ satisfies:

i) $\exists\, \lambda^* > 0$ such that $J(u^*) + \lambda^*|\Omega^*| \leq J(v) + \lambda^*|\{v \neq 0\}|,$ (3)

for all v such that $|\Omega^*| \leq |\{v \neq 0\}|$;

ii) $\exists\, \mu(h) > 0$ such that $J(u^*) + \mu(h)|\Omega^*| \leq J(v) + \mu(h)|\{v \neq 0\}|,$ (4)

for all v such that $|\Omega^*| - h \leq |\{v \neq 0\}| \leq |\Omega^*|$.

Then we are able to prove a weaker version of the Euler-Lagrange's equation (2), namely

$$\Delta u_{\Omega^*} + f\chi_{\Omega^*} = \Lambda\mathcal{H}^{d-1}\lfloor\partial\Omega^*, \tag{5}$$

in the sense of distribution in D, where $\mathcal{H}^{d-1}\lfloor\partial\Omega^*$ is the restriction to $\partial\Omega^*$ of the $d-1$ dimensional Hausdorff measure and χ_{Ω^*} is the characteristic function of Ω^*. For doing this, and in particular for the existence of a positive $\mu(h)$, we need one hypothesis saying that the constraint is saturated (see Rem. 1.4). As proved in the last section, this hypothesis is always true if $f \geq 0, f \not\equiv 0$. Note that we do not need this hypothesis for the existence of λ^*.

Our goal is to prove that, nevertheless, after some work, the regularity question may fit into the approach in [2,10], at least in the saturated case. After reaching this main step, we are able to directly use regularity results proved in [2,10] for what they called weak solutions. We deduce that, in regions where the sign of u_{Ω^*} does not change, then $\partial^*\Omega^*$ is regular. We also reach full regularity of $\partial\Omega^*$ in dimension 2 in the positive case. Note that cusps may occur for the boundary at points where u changes its sign, even in dimension 2.

Although different, this may be compared with [1] where pure constraints are also considered and also reduced to a "penalized" version. But our problem and our apprach are different. First, here the nonhomogeneity is inside the domain and not on the boundary. This makes the situation slightly different. For instance, an assumption of "saturation" is necessary to obtain regularity. Next, in [1], the authors first deduce the regularity of the boundary for penalized problem from the results in [2]. Then, they use this regularity to prove that it is equivalent to the problem with volume constraint. Here we prove directly the equivalence between constrained and penalized problems. This is valid without sign condition and without a priori knowledge of regularity.

Our paper is organized as follows.

In Section 1, we recall the known existence and regularity results for our problem and we state our main results. We know that one first step in proving regularity of the boundary $\partial\Omega^*$ is to prove regularity of the solution u_{Ω^*}, for instance that u_{Ω^*} is Lipschitz continuous when $u_{\Omega^*} \geq 0$. This will be assumed here and references will be given, in particular [4], [13] or also [2,10].

We will show in Section 2 that $\Delta u_{\Omega^*} + f\chi_{\Omega^*}$ is in D the difference of two Radon measures absolutely continuous with respect to $\mathcal{H}^{d-1}\lfloor\partial\Omega^*$ (which is not yet, at this step, a Radon measure). This may be seen as the easier half of (5). Moreover we will see (under some hypothesis, see Th. 2.4), that Ω^* has (locally) finite perimeter (for the definitions and properties of sets with finite perimeter see for example [6, 9] or [7]).

In Section 3 we study the blow-up of u_{Ω^*} (*i.e.* limits of $u_{\Omega^*}(x_0 + rx)/r$ as r goes to 0) and in particular around $x_0 \in \partial^*\Omega^*$ where we can identify the limit function.

In Section 4, we want to control $J(u_\Omega) - J(u_{\Omega^*})$ in term of $|\Omega| - |\Omega^*|$, with $|\Omega|$ close to a. We essentially show (3) and (4). Moreover if we define, for $h > 0$, $\mu(h)$ as the biggest μ such that (4) is true for every Ω with $|\Omega^*| - h \leq |\Omega| \leq |\Omega^*|$ and $\lambda^*(h)$ as the smallest λ^* such that (3) is true for all Ω with $|\Omega| \leq |\Omega^*| + h$, then we prove that $\lim_{h\to 0}\mu(h) = \lim_{h\to 0}\lambda^*(h)$. This proves that "asymptotically", our problem is equivalent to a penalized version with this common limit.

This will allow us, in Section 5, to use some methods from [2] and [10], under the hypothesis $u \geq 0$. First we show that $\mathcal{H}^{d-1}(\partial\Omega^* \setminus \partial^*\Omega^*) = 0$, then the absolute continuity of $\mathcal{H}^{d-1}\lfloor\partial\Omega^*$ (now a Radon measure) with respect to $\Delta u_{\Omega^*} + f\chi_{\Omega^*}$. Finally, with the result of blow-up near $\partial^*\Omega^*$, we are able to derive their exact relationship: it is exactly given by (5). With this, we get that u_{Ω^*} is a weak solution in the sense of [3] and [10]: so we may deduce the C^1 regularity of $\partial^*\Omega^*$. Moreover, using the precise behavior of $\mu(h)$ and λ^*, we get (as in [3]) the regularity of the whole boundary $\partial\Omega^*$ in dimension 2.

Finally, in Section 6, we discuss the hypothesis of saturation. We show that this hypothesis is always true if $f \geq 0, f \neq 0$. After that we also remark that this hypothesis is true for, at least, a dense open subset of a.

1. The main results

1.1. **Existence, first results**

We recall here the necessary existence results for our problem.

Let D be any open subset of $\mathbb{R}^d$, $f \in L^2(D) \cap L^\infty(D)$ and $0 < a < |D|$ (we denote by $|E|$ the Lebesgue's measure of any measurable subset E of $\mathbb{R}^d$). For every $u \in \mathrm{H}^1_0(D)$, we define

$$J(u) = \int_D \frac{1}{2}|\nabla u|^2 \mathrm{d}x - \int_D fu,$$

and $\Omega_u = \{x \in D; u(x) \neq 0\} = \{u \neq 0\}$. We are interested in the regularity of the solution of the following shape optimization problem:

$$(\mathcal{P}_f)\ J(u_{\Omega^*}) = \inf\left\{J(u_\Omega), \Omega \subset D \text{ open }, |\Omega| = a\right\}, \tag{6}$$

where u_Ω is defined by:

$$J(u_\Omega) = \min\{J(v), v \in \mathrm{H}^1_0(\Omega)\},$$

so that u_Ω is the solution of:

$$-\Delta u_\Omega = f, u \in \mathrm{H}^1_0(\Omega).$$

It is easy to see that, if there exists $w \in \mathrm{H}^1_0(D)$ such that $-\Delta w = f$ in D with $|\Omega_w| < a$, then **any** open set Ω^* such that $\Omega_w \subset \Omega^* \subset D$ and $|\Omega_w| = a$ is a solution of $(\mathcal{P}_f)$. Similarly, if there exists $w \in \mathrm{H}^1_0(D)$ such that $-\Delta w = f$ in D and $|\Omega_w| = a$, then $\Omega^* = \Omega_w$ is the only possible solution of $(\mathcal{P}_f)$. As proved in [11, 12] Ω_w, may even not be open! In fact, one can expect regularity of Ω^* only if the volume constraint $|\Omega_w| \leq a$ is effective. Therefore, since we are interested in proving regularity, in the rest of this paper, we will naturally assume that:

$$\left.\begin{array}{c}\textbf{There does not exist any } v \in \mathrm{H}^1_0(D) \textbf{ with } |\Omega_v| \leq a\\ \textbf{such that } -\Delta v = f \text{ in } D.\end{array}\right\} \tag{7}$$

It turns out that this condition implies (see [5]) that the following problem has a solution,

$$(\mathcal{P})\left\{\begin{array}{c} u\in \mathrm{H}_0^1(D), |\Omega_u| = a \\ J(u)\leq J(v), \forall v\in \mathbb{V}_a^0, \end{array}\right.$$

where $\mathbb{V}_a^0$ is defined by

$$\mathbb{V}_a^0 = \left\{v\in \mathrm{H}_0^1(D), |\Omega_v|\leq a\right\}.$$

And it is clear that, if a solution u of $(\mathcal{P})$ is continuous in D, then the open set Ω_u is a solution of the shape optimization problem $(\mathcal{P}_f)$. Using (7), we see that, if $u\in\mathbb{V}_a^0$ is such that $J(u)\leq J(v)$ for all $v\in\mathbb{V}_a^0$ then, $|\Omega_u| = a$ and u is a solution of $(\mathcal{P})$.

Remark 1.1. If $f\geq 0, f\neq 0$ by the (strict) maximum principle (7) holds. Moreover if u is a solution of $(\mathcal{P})$ then $u\geq 0$.

In the following we will only consider solutions of $(\mathcal{P})$.

As explained above; to obtain regularity, we will have to assume that the volume constraint $|\Omega_u| = a$ (or $|\Omega_u|\leq a$) does play its role. Part of this is contained in assumption (5). But we will also assume that the Lagrange- multiplier λ in the Euler-Lagrange's equation of the minimization problem is strictly positive.

Proposition 1.2 (Euler Lagrange's equation)**.** *Let u be a solution of $(\mathcal{P})$. Then there exists $\lambda\geq 0$ such that for all $\Phi\in \mathrm{C}_0^\infty(D,\mathbb{R}^d)$ we have:*

$$\int_D \langle D\Phi\nabla u\cdot\nabla u\rangle - \frac{1}{2}\int_D |\nabla u|^2 \mathrm{div}\Phi = \int_D f\langle\nabla u\cdot\Phi\rangle + \lambda\int_{\Omega_u}\mathrm{div}\Phi. \tag{8}$$

Remark 1.3. This is proved in [5]. The idea is to write that the derivative of $t\to J(u(I+t\Phi))$ vanishes at $t=0$ for Φ satisfying $\int_{\Omega_u}\mathrm{div}\Phi\geq 0$. Then, the Lagrange multiplier λ appears for general Φ.

Notation: In the rest of this paper, we will be mainly interested in solutions of $(\mathcal{P})$ verifying the Euler-Lagrange equation in D with $\lambda>0$: **we will simply write that u is a solution of $(\mathcal{P})$ with $\lambda>0$.**

Remark 1.4. As explained above, if the constraint $|\Omega_u|\leq a$ is not "saturated" (*i.e.* $\lambda=0$) the optimal form may not be regular. For instance, we can construct a solution u of $(\mathcal{P})$ with $\lambda=0$ and $\partial\Omega_u$ very unregular as follows: take $D=B(0,1), u\in\mathrm{C}_0^\infty(D)$, such that $\partial\Omega_u$ is not regular, $f=-\Delta u$ and $a=|\Omega_u|$. It is obvious that u is a solution of $(\mathcal{P})$ since $J(u)\leq J(v)$ for all $v\in\mathrm{H}_0^1(D)$. In Section 6 we will discuss the saturation hypothesis $\lambda>0$. We show that if $f\geq 0, f\not\equiv 0$ this is always true. If f changes its sign, this happen at least for a dense open subset of a.

1.2. **The regularity result**

Our main regularity result is the following. As usual, if Ω has finite perimeter, we denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\mathcal{H}^{d-1}$ the Hausdorff measure of dimension $d-1$. (see [6,9] or [7]).

Theorem 1.5. *Let u be a nonnegative solution of $(\mathcal{P})$ with $\lambda>0$. Then Ω_u has locally finite perimeter in D and,*

$$u \textit{ is locally Lipschitz continuous in } D \tag{9}$$

$$\textit{for every compact } K\subset D, \mathcal{H}^{d-1}((\partial\Omega_u\setminus\partial^*\Omega_u)\cap K)=0. \tag{10}$$

Moreover:

$$\Delta u + f\chi_{\Omega_u} = \sqrt{2\lambda}\mathcal{H}^{d-1}\lfloor\partial\Omega_u, \textit{ in } \mathcal{D}'(D), \tag{11}$$

$$\partial^*\Omega_u\cap D \textit{ is a } C^{1,\alpha} \textit{ hypersurface with } \alpha>0. \tag{12}$$

Finally, if $d = 2$, we have $\partial^*\Omega_u \cap D = \partial\Omega_u \cap D$ and so $\partial\Omega_u$ is regular in D. If f is analytic, $\partial^*\Omega_u$ (or $\partial\Omega_u$ if $d = 2$) is analytic.

Notation: Here $\mathcal{H}^{d-1}\lfloor\partial\Omega_u$ denotes the restriction of the measure $\mathcal{H}^{d-1}$ to $\partial\Omega_u$.

Extensions: In the case where the sign of u changes in D, we also have the same kind of regularity in open subsets of D where the sign of u is constant (see Th. 5.1). This means that, in dimension 2, we can have singularities for $\partial\Omega_u$ only in points where u changes its sign (and we know by simple examples that this indeed happens!).

For the proof of this theorem, we use tools from [2, 10]. In these papers, the authors study the regularity of minima of functionals like:

$$G(u) = \int_{D_1} \frac{1}{2}|\nabla u|^2 \mathrm{d}x - \int_{D_1} fu + \lambda|\{u > 0\}|, \tag{13}$$

with $u \geq 0, u = u_0$ on ∂D_1 ($f = 0$ in [2]). The above problem looks like "penalized" versions of our problem. Indeed, in [2] and in [10] there is no constraint such as $|\Omega_u| \leq a$, but there is the extra term $''\lambda|\{u > 0\}|''$ in the functional. This may be viewed as a penalization term for our problem.

Our strategy (and main task) will actually consist in showing that, for our problem, there are, in general, two constants μ and λ^* with $0 < \mu \leq \lambda \leq \lambda^*$ such that (4) and (3) hold. Then, using technics from [2, 10] we are able to prove (10) and (11).

More precisely, we show the following.

Proposition 1.6. *Let u be a solution of $(\mathcal{P})$ with $\lambda > 0$ (see Prop. 1.2). Let $B(x_0, r) \subset D$ be such that $|B(x_0, r)| < |D| - a$,*

$$0 < \frac{|B(x_0, r) \cap \Omega_u|}{\omega_d r^d} < 1, \tag{14}$$

and u not identically 0 on $\partial B(x_0, r)$. Let

$$\mathcal{F}_0 = \left\{v \in \mathrm{H}_0^1(D); u - v \in \mathrm{H}_0^1(B(x, r))\right\}.$$

For $h > 0$ we define

$$\mu(h) = \sup\left\{\mu \geq 0; J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v|, \forall v \in \mathcal{F}_0, a - h \leq |\Omega_v| \leq a\right\},$$

$$\lambda^*(h) = \inf\left\{\lambda^* \geq 0; J(u) \leq J(v) + \lambda^*(|\Omega_v| - a)^+, \forall v \in \mathcal{F}_0, |\Omega_v| \leq a + h\right\}.$$

Then

$$\mu(h) \leq \lambda \leq \lambda^*(h) < +\infty,$$

and

$$\lim_{h\to 0} \mu(h) = \lim_{h\to 0} \lambda^*(h) = \lambda.$$

In particular, $\mu(h) > 0$ for h small enough.

Remark 1.7. The fact that $\lambda^*(h)$ exists and is finite may be found essentially in [13]. It means that:

$$J(u) + \lambda^*(h)|\Omega_u| \leq J(v) + \lambda^*(h)|\Omega_v|, \tag{15}$$

for all $v \in \mathcal{F}_0$ with $a \leq |\Omega_u| \leq a + h$.

The definition of $\mu(h)$ means that

$$J(u) + \mu(h)|\Omega_u| \leq J(v) + \mu(h)|\Omega_v|, \tag{16}$$

for all $v \in \mathcal{F}_0$ with $a - h \leq |\Omega_v| \leq |\Omega_u|$, The most important point is that $\mu(h) > 0$ (at least for h small enough). The precise value of the limit of $\mu(h)$ as $h \to 0$ will only be used in dimension 2.

If $x_0 \in \partial^*\Omega_u$, we have $\lim_{r\to 0} \frac{|B(x_0,r)\cap\Omega_u|}{\omega_d r^d} = \frac{1}{2}$, and so, condition (14) is true for r small enough. Moreover, since $x_0 \in \partial\Omega_u$, we can find r such that u is not equal to 0 on $\partial B(x_0, r)$. Therefore, Proposition 1.6 may be applied to all $x_0 \in \partial\Omega^*$.

If we have $\mu(h) = \lambda^*(h)$, we get that u is exactly a minimum for (13). But, in general, we have $\mu(h) < \lambda^*(h)$ and the two problems are different. For instance, if we take $D = B(0,1)$ in $\mathbb{R}^2$ and $f = 1$, it is easy to compute exactly μ and λ and to show that $\mu(h) < \lambda^*(h)$ (see [4] for details).

The main point is that although the problem is different from those considered in [2] and in [10], we will reach the same kind of regularity for the boundary.

It is well-known that the first step in proving regularity for the boundary of optimal shapes is to prove regularity of the state function. We will not do it here, but rather add it in our assumptions when needed and refer to corresponding previous results in the literature. For instance we have.

Theorem 1.8. *Let u be a solution of $(\mathcal{P})$; then u is Hölder-continuous with power α for every $0 < \alpha < 1$. Moreover, for every open D_1 such that $D_1 \subset \{u \geq 0\}$ (or $D_1 \subset \{u \leq 0\}$) u is locally Lipschitz on D_1. Finally, if $D = \mathbb{R}^d$ and $u \geq 0$, u is globally Lipschitz on $\mathbb{R}^d$.*

The first part of this theorem is in [13] and in [3]. The proof of the second part may be found in [13] or also in the proof of Lemma 2.5 and Corollary 2.6 in [10] (see also [2] 3.2 and 3.3). It is also proved in dimension 2 in [5].

Note also that Lipschitz regularity of the state function u has been proved in [5] in dimension 2 without any positivity assumption and even for solutions of the Euler-Lagrange equation (8).

2. Study of $\Delta u + f\chi_{\Omega_u}$

In this section, u is a solution of $(\mathcal{P})$.

We show that $\Delta u + f\chi_{\Omega_u}$ is the difference of two Radon measure. We start with a technical proposition.

Proposition 2.1. *Let $p \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ with $p(0) = 0$. Then we have:*

$$p'(u)|\nabla u|^2 - \operatorname{div}(p(u)\nabla u) - fp(u) = 0 \text{ in } \mathcal{D}'(D).$$

Proof. Let $\Psi \in C_0^\infty(D)$ and $p \in W^{1,\infty}(\mathbb{R},\mathbb{R})$ with $p(0) = 0$. Let:

$$v_t(x) = u(x) + t\Psi(x)p(u(x)).$$

We have $v_t \in \mathrm{H}_0^1(D)$. Indeed, $|p(u(x))| \leq \|p'\|_\infty \||u(x)|$, so that v_t is in $L^2(D)$, and:

$$|\nabla p(u(x))| = |p'(u(x))\nabla u(x)| \leq \|p'\|_\infty \||\nabla u(x)|,$$

so $\nabla v_t \in L^2(D)$. Because $u(x) = 0$ imply $v_t(x) = 0$, we get $|\Omega_v| \leq |\Omega_u|$. By minimality of u we deduce:

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_D \left(|\nabla u + tp(u)\nabla\Psi + t\Psi p'(u)\nabla u|^2 - |\nabla u|^2\right) - \int_D f(u + t\Psi p(u) - u) \\ &= \int_D t\,\langle p(u)\nabla\Psi + \Psi p'(u)\nabla u \cdot \nabla u\rangle + t^2|p(u)\nabla\Psi + \Psi p'(u)\nabla u|^2 - tf\Psi p(u). \end{aligned}$$

Dividing by $t > 0$ and by $t < 0$ and letting t go to 0, we deduce:

$$0 = \int_D \nabla\Psi.p(u)\nabla u + \Psi p'(u)|\nabla u|^2 - f\Psi p(u),$$

which is Proposition 2.1. □

Theorem 2.2. *There exist two positive Radon measures μ_1 and μ_2 such that:*

$$\mu_1 = \Delta(u^+) + f\chi_{\{u>0\}}, \mu_2 = \Delta(u^-) - f\chi_{\{u<0\}}.$$

Moreover there exists $C = C(f,a)$ such that

$$\|u\|_\infty \leq C.$$

Proof. Let p_n be defined by:

$$p_n(r) = \begin{cases} 0 \text{ if } r & \leq & 0 \\ nr \text{ if } r & \in & [0,1/n] \\ 1 \text{ if } r & \geq & 1/n \end{cases}$$

and q_n by $q_n(r) = \int_0^r p_n(s)\mathrm{d}s$ for $r \geq 0$ and $q_n(r) = 0$ for $r \leq 0$. Applying Proposition 2.1 and using $p_n(u)\nabla u = \nabla q_n(u)$ we get:

$$n|\nabla u|^2\chi_{\{0<u<1/n\}} - \Delta(q_n(u)) - fp_n(u) = 0. \tag{17}$$

When n goes to infinity $p_n(u)$ converge in $L^1_{\mathrm{loc}}(D)$ to $\chi_{\{u>0\}}$ and $q_n(u)$ converge to u^+ in $L^2(D)$. So we have in $\mathcal{D}'(D)$:

$$\lim_{n\to\infty} \Delta q_n(u) + fp_n(u) = \Delta(u^+) + f\chi_{\{u>0\}}.$$

Let $\mu_1^n = n|\nabla u|^2\chi_{\{0<u<1/n\}}$ and $\mu_1 = \Delta(u^+) + f\chi_{\{u>0\}}$. Let $\Psi \in \mathrm{C}_0^\infty(D)$; by (17) we have

$$\begin{aligned} \int \Psi\mu_1^n &= \int (q_n(u)\Delta\Psi + fp_n(u)\Psi) \\ &\leq \|\Delta\Psi\|_\infty \int_{\mathrm{supp}\Psi} u^+ + \|f\|_\infty \|\|\Psi\|_\infty, \end{aligned}$$

because $p_n(u) \leq 1$ and $q_n(u) \leq u^+$. We deduce that the measures μ_n are uniformly bounded on compact sets and so the limit μ_1 of μ_1^n in $\mathcal{D}'(D)$ is a measure. Moreover, using the uniform bound on compact sets, the limit may be understood weakly in the sense of Radon's measures. We proceed in the same way to get a measure μ_2 such that:

$$\Delta(u^-) - f\chi_{\{u<0\}} = \mu_2.$$

Let us show the L^∞ estimate. We have:

$$-\Delta(u^+) = f\chi_{\{u>0\}} - \mu_1 \leq f\chi_{\{u>0\}}.$$

Because $|\Omega_u \cap D| < |D|$, there exists an open subset ω such that $\Omega_u \subset \omega \subset D$ and $|\omega| < 2|\Omega_u \cap D|$. We can use classical L^∞ elliptic estimates (see for example [8] Th. 8.16) to show that:

$$\|u^+\|_{L^\infty} \leq C\|f\chi_{u>0}\|_{L^q(\omega)} \leq C\|f\|_\infty a^{1/q},$$

with $C = C(d, |\omega|, q)$ and $q > d/2$. We have the same for u^-. □

We now show that, if u is Lipschitz on an open set D_1 included in D, then μ_1 and μ_2 are absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{d-1}$ in D_1. More precisely we have the following:

Proposition 2.3. *Let u be a solution of $(\mathcal{P})$ and D_1 be an open subset of D, such that u is Lipschitz continuous on D_1. Then there exists $C > 0$ such that, for every ball $B(x,r) \subset D_1$ with $r \leq 1$, we have:*

$$\mu_1(B(x,r)) \leq Cr^{d-1},$$

$$\mu_2(B(x,r)) \leq Cr^{d-1}.$$

Proof. We use the same notations as in the proof of Theorem 2.2. Let $B(x,r) \subset D_1$, then:

$$\begin{aligned}\int_{B(x,r)} (\Delta q_n(u) + f p_n(u)) &= \int_{\partial B(x,r)} p_n(u)\langle \nabla u.n\rangle + \int_{B(x,r)} f p_n(u) \\ &\leq \|\nabla u\|_{\infty, D_1} \mathrm{d}\omega_d r^{d-1} + \|f\|_\infty \omega_d r^d \\ &\leq Cr^{d-1},\end{aligned}$$

for $r \leq 1$. Because $\Delta q_n(u) + f p_n(u)$ converges weakly in the sense of Radon measures to $\Delta u + f\chi_{\{u>0\}}$, we deduce:

$$\mu_1(B(x,r)) \leq \liminf_{n\to\infty} \int_{B(x,r)} (\Delta q_n(u) + f p_n(u)) \leq Cr^{d-1}.$$

We can do the same for μ_2. □

We will now see that, under some extra hypotheses, $\Omega_u \cap D_1$ has a finite perimeter on bounded subsets D_1 of D where u is Lipschitz continuous. We have the following theorem:

Theorem 2.4. *Let u be a solution of $(\mathcal{P})$ with $\lambda > 0$ and let D_1 be a bounded open subset of D, with $\overline{D_1} \subset D$ and where u is Lipschitz continuous. Then $\Omega_u = \{|u| \neq 0\}$ has finite perimeter in D_1. Moreover, there exist constants C, C^1, r_0 depending on the data such that for every $B(x,r) \subset D_1$ with $r \leq r_0$:*

$$P(\Omega_u, B(x,r)) \leq C\left(\mu_1(B(x,r)) + \mu_2(B(x,r))\right) \leq C^1 r^{d-1},$$

(where μ_1 and μ_2 are the two measure defined in Th. 2.2).

The proof of Theorem 2.4 will require the following lemma which says in a very weak sense that "$|\nabla u|^2 = 2\lambda$" on $\partial\{u \neq 0\}$.

Proposition 2.5. *Let u be a solution of $(\mathcal{P})$ and let D_1 be an open subset of D where u is locally Lipschitz continuous. Then, for every $\varphi \in \mathrm{C}_0^\infty(D_1, \mathbb{R}^d)$ we have:*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{0<|u|<\varepsilon\}} \langle \varphi \cdot \nabla |u|\rangle \left(\lambda - \frac{1}{2}|\nabla u|^2\right) = 0.$$

Proof. Let $\varphi \in \mathrm{C}_0^\infty(D_1, \mathbb{R}^d)$ and $\Psi_\varepsilon(x) = \max(0, 1 - \frac{|x|}{\varepsilon})$. We write Euler-Lagrange's equation (8) with $\Phi = \varphi\Psi_\varepsilon(u) \in W^{1,\infty}(D)$, which has compact support. For this, because $\Phi \in W_0^{1,\infty}(D)$ with compact support in D, We can approximate Φ by $\Phi_n \in \mathrm{C}_0^\infty(D_1)$ in $W^{1,1}(D_1)$ with $\nabla\Phi_n$ uniformly bounded. The Euler-Lagrange's equation (8) is true also with Φ. We study each term:

$$\begin{aligned}\int_{\{u\neq 0\}} \mathrm{div}\ \Phi &= \int_{\{u\neq 0\}} \Psi_\varepsilon(u) \mathrm{div}\ \varphi + \int_{0<|u|<\varepsilon} \langle \varphi \cdot \nabla u\rangle \Psi'_\varepsilon(u) \\ &= \int_{\{u\neq 0\}} \Psi_\varepsilon(u) \mathrm{div}\ \varphi - \int_{\{u\neq 0\}} \frac{1}{\varepsilon}\langle \varphi \cdot \nabla |u|\rangle.\end{aligned}$$

Since $\Psi_\varepsilon(u)\chi_{\{u\neq 0\}}$ converge to 0 a.e when ε goes to 0, by dominated convergence, the first term goes to 0. For the same reason we get:

$$\lim_{\varepsilon\to 0} \int_D f\langle \nabla u \cdot \Phi\rangle = \lim_{\varepsilon\to 0} \int_{\{u\neq 0\}} f\Psi_\varepsilon(u)\langle \nabla u \cdot \varphi\rangle = 0.$$

$$\begin{aligned}\int_{\{u\neq 0\}} \text{div } \Phi|\nabla u|^2 &= \int_{\{u\neq 0\}} \Psi_\varepsilon(u)\text{div } \varphi|\nabla u|^2 + \langle \varphi\cdot\nabla u\rangle \Psi'_\varepsilon(u)|\nabla u|^2 \\ &= \int_{\{u\neq 0\}} \Psi_\varepsilon(u)\text{div } \varphi|\nabla u|^2 - \frac{1}{\varepsilon}\int_{0<|u|<\varepsilon} \langle \varphi\cdot\nabla|u|\rangle|\nabla u|^2.\end{aligned}$$

Using $\nabla u \in L^2$, the first term goes to 0. Finally we also have,

$$\int_D \langle D\Phi\nabla u\cdot\nabla u\rangle = \int_{\{u\neq 0\}} \Psi_\varepsilon(u)\langle D\phi\nabla u\cdot\nabla u\rangle - \frac{1}{\varepsilon}\int_{\{0<|u|<\varepsilon\}} |\nabla u|^2\langle \varphi\cdot\nabla|u|\rangle,$$

and the first the first term goes to 0. By writing Euler-Lagrange's (8) equation and letting ε goes to 0, we get the proposition. □

Proof of Theorem 2.4. Let $B(x,r)\subset D_1$ and $\varphi\in \mathrm{C}_0^\infty(B(x,r))$. For almost every $s>0$ the boundary of $\{|u|>s\}$ is regular (C^1), since on the open set $\{u\neq 0\}$ we have $-\Delta u = f$ so that u is C^1 and we can use Sard's lemma, which implies that $|\nabla u|>0$ on $\{|u|=s\}$ for almost every s. We can now write, using co-area formula (see 3.4.3 in [6]), and Gauss formula:

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\{0<|u|<\varepsilon\}}\langle\varphi\cdot\nabla|u|\rangle &= \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}s\int_{\{|u|=s\}}\left\langle \varphi\cdot\frac{\nabla|u|}{|\nabla u|}\right\rangle \\ = -\frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}s\int_{\{|u|=s\}}\langle\varphi\cdot\nu_s\rangle &= -\frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}s\int_{\{|u|>s\}}\text{div}\varphi,\end{aligned}$$

(here ν_s is the outward normal to $\{|u|>s\}$). We deduce,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\{0<|u|<\varepsilon\}}\langle\varphi\cdot\nabla|u|\rangle = \int_{\{|u|>0\}}\text{div}\varphi. \tag{18}$$

There exist $s_0<r$ such that $\varphi\in\mathrm{C}_0^\infty(B(x,s_0))$. If we suppose that $\|\varphi\|_\infty\le 1$, then:

$$\frac{1}{\varepsilon}\int_{\{0<|u|<\varepsilon\}}\langle\varphi\cdot\nabla|u|\rangle\frac{1}{2}|\nabla u|^2 \le \frac{1}{2\varepsilon}\|\nabla u\|_\infty\int_{\{|u|<\varepsilon\}\cap B(x,s)}|\nabla u|^2, \tag{19}$$

for every $s_0<s<r$. But we saw in the proof of Theorem 2.2 (see (17)) that

$$\lim_{n\to\infty} n|\nabla u|^2\chi_{\{0<u<1/n\}} = \mu_1,$$

weakly in the sense of Radon's measure, and it is the same on $\{-1/n<u<0\}$ with μ_2. For almost every $s<r$,

$$\mu_1(\partial B(x,s)) = \mu_2(\partial B(x,s)) = 0.$$

Let such a $s>s_0$, we get:

$$\lim_{n\to\infty} n\int_{\{|u|<1/n\}\cap B(x,s)}|\nabla u|^2 = \mu_1(B(x,s)) + \mu_2(B(x,s)).$$

From Proposition 2.5, (18) and (19) with $\varepsilon = 1/n$ and $n\to\infty$

$$\begin{aligned}\lambda\int_{\Omega_u\cap B(x,r)}\text{div}\varphi &\le \frac{\|\nabla u\|_{\infty,D_1}}{2}(\mu_1(B(x,s))+\mu_2(B(x,s))) \\ &\le \frac{\|\nabla u\|_{\infty,D_1}}{2}(\mu_1(B(x,r))+\mu_2(B(x,r))).\end{aligned}$$

That is, by taking the supremum over φ:

$$P(\Omega_u, B(x,r)) \leq \frac{\|\nabla u\|_{\infty,D_1}}{2\lambda}(\mu_1(B(x,r)) + \mu_2(B(x,r))).$$

To prove that Ω_u has finite perimeter, we use that μ_1 and μ_2 are finite on the bounded set D_1. Using Proposition 2.3, we get for $B(x,r) \subset D_1$ and r small enough:

$$P(\Omega_u, B(x,r)) \leq Cr^{d-1}.$$

□

3. Blow-up

In this section we study the blow up of a solution around a point x_0 of the boundary of $\Omega_u = \{u \neq 0\}$. We will throughout suppose that u is Lipschitz on an open set around x_0. In particular Ω_u is open. In the last proposition (see Prop. 3.5), we will also assume that u is nonnegative around x_0.

Notations. Let u be a solution of $(\mathcal{P})$ and let D_1 be an open subset of D such that u is Lipschitz continuous in D_1. Let $x_m \in \partial\Omega_u \cap D_1$ go to $x_0 \in \partial\Omega_u \cap D_1$ and r_m go to 0 with $B(x_m, r_m) \subset D_1$. We define:

$$u_m(x) = \frac{u(x_m + r_m x)}{r_m},$$

and

$$\Omega_m = \{x \in \mathbb{R}^d, x_m + r_m x \in \Omega_u\}.$$

We will refer to this as the blow-up of u relatively to $B(x_m, r_m)$.

Proposition 3.1. *There exists u_0 Lipschitz-continuous and a measurable set H^- included in $\mathbb{R}^d$ with locally finite perimeter such that, up to a subsequence, u_m converges to u_0 uniformly on every compact set, ∇u_m converges to ∇u_0 *-weakly in $L^\infty(\mathbb{R}^d)$ and χ_{Ω_m} converges to χ_{H^-} in $L^1_{\text{loc}}(\mathbb{R}^d)$ and almost everywhere. Moreover for almost every $x \notin H^-$, we have $u_0(x) = 0$.*

Proof. Let $R > 0$, for m large enough we have $B(x_m, r_m R) \subset D_1$ and $u(x_m) = 0$ so that for $x \in B(0,R)$:

$$|u_m(x)| \leq \|\nabla u\|_{\infty,D_1}|x|, |\nabla u_m(x)| \leq \|\nabla u\|_{\infty,D_1}.$$

So, up to a sub-sequence, u_m converges uniformly on $B(0,R)$ to Lipschitz continuous function u_0 and ∇u_m *-weakly converges in $L^\infty(\mathbb{R}^d)$ to ∇u_0.

$$P(\Omega_m, B(0,R)) = \frac{1}{r_m^{d-1}} P(\Omega_u, B(x_m, r_m R)) \leq CR^{d-1}.$$

This implies that that χ_{Ω_m} is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^d)$. So there exists H^- with locally finite perimeter such that, up to a sub-sequence, χ_{Ω_m} converge to χ_{H^-} in $L^1_{\text{loc}}(\mathbb{R}^d)$ and almost-everywhere. For almost every $x \notin H^-$, we have:

$$0 = \chi_{H^-}(x) = \lim_{m\to\infty} \chi_{\Omega_m}(x),$$

and for m large enough $x_m + r_m x \notin \Omega_u$, so that $u_m(x) = 0$. □

Proposition 3.2. *Let u_m be as above. Then, up to a subsequence, ∇u_m converges in $L^p_{\text{loc}}(\mathbb{R}^d)$ to ∇u_0 for all $1 \leq p < \infty$ and almost everywhere.*

Proof. Let $R > 0$. For m large enough so that $B(x_m, r_m R) \subset D_1$, we have in the sense of distributions:

$$\Delta u_m(x) = r_m \Delta u(x_m + r_m x).$$

According to Theorem 2.2, $\Delta u = -f\chi_{\Omega_u} + \mu_1 - \mu_2$ with μ_1 and μ_2 positive measures. We get

$$\begin{aligned}\int_{B(0,R)} |\Delta u_m| &= r_m^{1-d} \int_{B(0,r_m R)} |\Delta u| \\ &\leq \omega_d r_m R^d \|f\|_\infty + r_m^{1-d} \mu_1(B(0, r_m R)) + r_m^{1-d} \mu_2(B(0, r_m R)) \\ &\leq CR^d + CR^{d-1},\end{aligned}$$

according to Proposition 2.3. The measures $|\Delta u_m|$ are locally bounded uniformly and, up to a sub-sequence, we deduce the convergence of ∇u_m to ∇u_0 in $L^1_{\text{loc}}(\mathbb{R}^d)$ and so (up to a sub-sequence) the convergence almost-everywhere. Finally, using the convergence a.e and the uniformly bound on $\|\nabla u_m\|_\infty$ we deduce that ∇u_m converges to ∇u_0 in $L^p_{\text{loc}}(\mathbb{R}^d)$ for all $1 \leq p < +\infty$. □

Theorem 3.3 (Euler's equation of u_0)**.** *Let u_0 and H^- be as in Proposition 3.1. Then for every* $\Phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \langle D\Phi \nabla u_0 \cdot \nabla u_0 \rangle - \frac{1}{2} \int_{\mathbb{R}^d} \text{div}\Phi |\nabla u_0|^2 = \lambda \int_{H^-} \text{div}\Phi. \tag{20}$$

Proof. Let $\Phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $Phi_m(x) = \Phi\left((x - x_m)/r_m\right)$. For m large enough, $\Phi_m \in C_0^\infty(D_1, \mathbb{R}^d)$. Writing Euler-Lagrange's equation of u with Φ_m, we get

$$\int_{\mathbb{R}^d} \langle D\Phi_m(x) \nabla u(x) \cdot \nabla u(x) \mathrm{d}x \rangle - \frac{1}{2} \int_{\mathbb{R}^d} \text{div}\Phi_m(x) |\nabla u(x)|^2 \mathrm{d}x$$

$$= \int_{\mathbb{R}^d} f(x) \langle \nabla u(x) \cdot \Phi_m(x) \rangle \mathrm{d}x + \lambda \int_{\Omega_u} \text{div}\Phi_m(x) \mathrm{d}x.$$

Setting $x = x_m + r_m y$ in these integrals, we obtain

$$\int_{\mathbb{R}^d} \langle D\Phi(y) \nabla u_m \cdot \nabla u_m \rangle - \frac{1}{2} \int_{\mathbb{R}^d} \text{div}\Phi |\nabla u_m|^2 \mathrm{d}y$$

$$= r_m \int_{\mathbb{R}^d} f(x_m + r_m y) \langle \nabla u_m(y) \cdot \Phi(y) \rangle \mathrm{d}y + \lambda \int_{\{y,\ x_m + r_m y \in \Omega_u\}} \text{div}\Phi(y) \mathrm{d}y.$$

The last term converges to $\lambda \int_{H^-} \text{div}\Phi$. The third term converges to 0 because f and Φ are bounded and $\|\nabla u_m\|_\infty$ is uniformally bounded. Finally for the two first terms we use the convergence of ∇u_m to ∇u_0. The result follows. □

Proposition 3.4. *Let u_0 and H^- be as in Proposition 3.1 and let $B(x_1, R) \subset H^-$ almost-everywhere, then u_0 is harmonic on $B(x_1, R)$.*

Proof. Let v_m be defined by:

$$v_m = u_m \text{ on } \partial B(x_1, R), \Delta v_m = 0 \text{ in } B(x_1, R).$$

Then, we have:

$$\int_{B(x_1,R)} |\nabla v_m|^2 \leq \int_{B(x_1,R)} |\nabla u_m|^2 \leq C, \|v_m\|_\infty \leq \|u_m\|_\infty \leq C.$$

It follows that, up to a subsequence, v_m converges weakly in $\mathrm{H}^1(B(x_1,R))$ to some v_0 which is harmonic and satisfies. $v_0 = u_0$ on $\partial B(x_1,R)$. Since $v_m - v_0$ is harmonic in $B(x_1,R)$ and is equal to $u_m - u_0$ on $\partial B(x_1,R)$, we get:

$$\int_{B(x_1,R)} |\nabla v_m - \nabla v_0|^2 \leq \int_{B(x_1,R)} |\nabla u_m - \nabla u_0|^2,$$

which goes to 0, thanks to the convergence in L^2 of ∇u_m (see Prop. 3.2).

Let, for $x \in \overline{B(x_m + r_m x_1, r_m R)}$, $w_m(x) = r_m v_m((x - x_m)/r_m)$, for $x \in \partial B(x_m + r_m x_1, r_m R)$ we have, $w_m(x) = u(x)$. Therefore we can extend w_m by u outside $B(x_m + r_m x_1, r_m R)$. By minimality of u, we get,

$$\int_{B(x_m+r_mx_1,r_mR)} \left(\frac{1}{2}|\nabla u|^2 - fu\right) \leq \int_{B(x_m+r_mx_1,r_mR)} \left(\frac{1}{2}|\nabla w_m|^2 - fw_m\right) + \lambda^*|B(x_m + r_m x_1, r_m R) \cap \{u = 0\}|.$$

By change of variables $x = x_m + r_m(x + y)$ in the integrals, we obtain:

$$\int_{B(x_1,R)} \left(\frac{1}{2}r_m^d|\nabla u_m|^2 - r_m^{d+1}fu_m\right) \leq \int_{B(x_1,R)} \left(\frac{1}{2}r_m^d|\nabla v_m|^2 - r_m^{d+1}fv_m\right) + \lambda^* r_m^d|B(x_1,R) \setminus \Omega_m|.$$

We divide by r_m^d and let m go to infinity. Since $B(x_1,R) \subset H^-$, and $\chi_{H^-} = \lim \chi_{\Omega_m}$ we deduce:

$$\int_{B(x_1,R)} |\nabla u_0|^2 \leq \int_{B(x_1,R)} |\nabla v_0|^2,$$

and we get that u_0 is harmonic in $B(x_1,R)$. □

For the next proposition, we will suppose that $u \geq 0$ around the point where we study the blow-up.

Proposition 3.5. *Let u be a solution of $(\mathcal{P})$ and D_1 an open subset of D such that u* ***is nonnegative in*** *D_1. Let $x_0 \in \partial^*\Omega_u \cap D_1$ and ν the outward unit normal to $\partial\Omega_u$ at x_0. Let u_0 the limit of a subsequence of u_m for the blow-up relatively to $B(x_0,r_m)$. Then we have:*

$$u_0(x) = \begin{cases} -\sqrt{2\lambda}(x,\nu), & x \in H^- = \{x \in \mathbb{R}^d, (x,\nu) < 0\} \\ 0, & x \notin H^-. \end{cases} \tag{21}$$

Proof. By Theorem 1.8 u is locally Lipschitz continuous in D_1.

Since $x_0 \in \partial^*\Omega_u$, due to the properties on the reduced boundary, we have that, $\Omega_m = \{x, x_0 + r_m x \in \Omega_u\}$ converges to $H^- = \{x; (x.\nu) < 0\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and almost everywhere (up to a sub-sequence). Up to a rotation of coordinates we can suppose that $\nu = -e_1 = (-1, 0, .., 0)$ and so we have $H^- = \{x = (x_1, ..., x_d), x_1 > 0\}$. For almost every $x \notin H^-$, $u_m(x) = 0$ when m is large enough and so $u_0(x) = 0$, by continuity we have $u_0(x) = 0$ for all $x \notin H^-$.

Let $L = \{x \in \mathbb{R}^d, x_1 = 0\}$. At this point, u_0 is harmonic and nonnegative on $[x_1 > 0]$, vanishes on $\{x \in \mathbb{R}^d, x_1 \leq 0\}$ and is globally Lipschitz. By a classical reflexion argument and Liouville's Theorem, we get that there exists $a_1 \geq 0$ such that $u_0(x) = a_1 x_1$. Let $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^d, R)$ and $\Phi = -\varphi e_1$. We write the Euler's equation for u_0 (20) with this Φ:

$$\int_{H^-} (D\Phi\nabla u_0 \cdot \nabla u_0) = \int_L a_1^2\varphi,$$

$$\int_{H^-} \mathrm{div}\Phi|\nabla u_0|^2 = \int_L a_1^2\varphi, \lambda \int_{H^-} \mathrm{div}\Phi = \lambda \int_L \varphi.$$

We deduce that $a_1^2 = 2\lambda$ and because $u_0 \geq 0$ on D_1 we get (21). □

4. Pseudo-penalized problems

We will now show that a solution of $(\mathcal{P})$ is also a solution of a "pseudo-penalized" problem. In this section D_1 will be such that:

$$\left.\begin{array}{c} D_1\subset D \text{ open and bounded with } \partial D_1 \text{ regular}\\ u \text{ not identically equal to } 0 \text{ on } \partial D_1\\ \exists h_0 \text{ such that } 0<h_0<|D_1\cap\Omega_u|<|D_1|-h_0\\ 0<h_0<|(D\setminus D_1)\cap\Omega_u|<|D\setminus D_1|-h_0\end{array}\right\}. \tag{22}$$

Remark 4.1. These technical conditions just mean that there is a non-negligible part of D_1 intersecting Ω_u and a non negligible part intersecting the set $\{u=0\}$. It is, in particular, satisfied for all balls of small radius centered on $\partial^*\Omega_u$.

First, we give a technical lemma that we will use repeatedly in the next proofs. It is related to the Euler-Lagrange's equation (8), except that here we only assumed it to be satisfied in D_1. The proof is the same as the one of (6) and we do not reproduce it here.

Lemma 4.2. *Let $v\in \mathrm{H}^1_0(D)$ be such that there exists a $\lambda_v\geq 0$ such that, for all $\Phi\in \mathrm{C}^\infty_0(D_1,\mathbb{R}^d)$,*

$$\int_{D_1}(D\Phi\nabla v\cdot\nabla v)-\frac{1}{2}\int_{D_1}|\nabla v|^2\mathrm{div}\Phi=\int_{D_1}f\nabla v\cdot\Phi+\lambda_v\int_{\Omega_v}\mathrm{div}\Phi. \tag{23}$$

Let $v_t(x)=v(x+t\Phi(x))$ for $\Phi\in \mathrm{C}^\infty_0(D_1,\mathbb{R}^d)$. Then

$$\begin{aligned} J(v_t) &= \int_{D_1}(D\Phi\nabla v\cdot\nabla v)-\frac{1}{2}\int_{D_1}|\nabla v|^2\mathrm{div}\Phi-\int_{D_1}f\nabla v\cdot\Phi+o(t)\\ &= J(v)+t\lambda_v\int_{\Omega_v}\mathrm{div}\Phi+o(t),\\ |\Omega_{v_t}| &= |\Omega|-t\int_{\Omega_v}\mathrm{div}\Phi+o(t).\end{aligned}$$

We will now see that if u is a solution of $(\mathcal{P})$, it is also a solution of a pseudo-penalized problem. More precisely, we know that $J(u)-J(v)\leq 0$ if $|\Omega_v|<a$. We will see that we can control $J(u)-J(v)$ in term of $|\Omega_v|-|\Omega_u|$.

We will suppose from now and to the end of this section, except for the last Remark 4.6, that u is a solution of $(\mathcal{P})$ with $\lambda>0$ and D_1 verifies condition (22). We define:

$$\mathcal{F}_0=\left\{v\in \mathrm{H}^1_0(D); u-v\in \mathrm{H}^1_0(D_1)\right\},$$

and, for $h<h_0$,

$$\mu(h)=\sup\left\{\mu\geq 0; J(u)+\mu|\Omega_u|\leq J(v)+\mu|\Omega_v|, \forall v\in\mathcal{F}_0, a-h\leq|\Omega_v|\leq a\right\},$$

$$\lambda^*(h)=\inf\left\{\lambda^*\geq 0; J(u)\leq J(v)+\lambda^*(|\Omega_v|-a)^+, \forall v\in\mathcal{F}_0, |\Omega_v|\leq a+h\right\}.$$

So by definition we have: for every $v\in\mathcal{F}_0$

$$a-h\leq|\Omega_v|\leq a\Longrightarrow J(u)+\mu(h)|\Omega_u|\leq J(v)+\mu(h)|\Omega_v|, \tag{24}$$

$$a\leq|\Omega_v|\leq|\Omega_u|+a\Longrightarrow J(u)+\lambda^*(h)|\Omega_u|\leq J(v)+\lambda^*(h)|\Omega_v|. \tag{25}$$

The main result on $\mu(h)$ and $\lambda(h)$ is the following theorem:

Proposition 4.3. *The function μ is nondecreasing function, λ^* is non-increasing and for some h_0 and $h \in]0, h_0[$,*

$$0 < \mu(h) \leq \lambda \leq \lambda^*(h) < +\infty.$$

Moreover $\lim_{h\to 0} \mu(h) = \lim_{h\to 0} \lambda^*(h) = \lambda$.

We start by proving that the sets of λ^* appearing in the definition of $\lambda^*(h)$ is not empty. This is done also in [13] for $D_1 = D$. Here the proof is slightly different and is more local. Most of the arguments will be used later for the study of $\lambda^*(h)$.

Proposition 4.4. *Let u be a solution of $(\mathcal{P})$. There exists $\lambda^* > 0$ such that*

$$J(u) + \lambda^*|\Omega_u| \leq J(v) + \lambda^*|\Omega_v|, \tag{26}$$

for every $v \in \mathcal{F}_0$ with $|\Omega_v| \geq a$. Moreover we have $\lambda \leq \lambda^*$.

Proof. Let λ_n^* be an increasing sequence and $\lim_{n\to\infty} \lambda_n^* = +\infty$. Let $v_n \in \mathcal{F}_0$ such that

$$J(v_n) + \lambda_n^*(|\Omega_{v_n}| - a)^+ \leq J(v) + \lambda_n^*(|\Omega_v| - a)^+,$$

for all $v \in \mathcal{F}_0$ (existence of v_n is straightforward since $\mathcal{F}_0$ is closed). If there exists $n \geq 0$ such that $|\Omega_{v_n}| \leq a$, we have

$$J(u) \leq J(v_n) \leq J(v) + \lambda_n^*(|\Omega_v| - a)^+,$$

for all $v \in \mathcal{F}_0$ and we get (26) with $\lambda^* = \lambda_n^*$. So we argue by contradiction and we suppose that, for all $n \geq 0$, $|\Omega_{v_n}| > a$. Let $w \in \mathcal{F}_0$ such that $-\Delta w = f$ in D_1. We have:

$$0 \leq \lambda_n^*(|\Omega_{v_n}| - a) \leq J(u) - J(v_n) \leq J(u) - J(w),$$

so we can deduce that $\lim |\Omega_{v_n}| = a$. Since $J(v_n) \leq J(u)$ we have, up to a sub-sequence, that v_n converges weakly in $\mathrm{H}^1(D_1)$ and a.e to $v \in \mathcal{F}_0$. Then $|\Omega_v| \leq a$ and,

$$J(v) \leq \liminf_{n\to\infty} J(v_n) \leq J(u),$$

so v is also a solution of $(\mathcal{P})$ and $|\Omega_v| = a$. According to Proposition 1.2 there exists a λ_v such that v satisfies Euler-Lagrange's equation in D. Because of the condition on $|(D \setminus D_1) \cap \Omega_u| = |(D \setminus D_1) \cap \Omega_v| > 0$ there exists $\varphi \in \mathrm{C}_0^\infty(D \setminus \overline{D_1}, \mathbb{R}^d)$ such that $\int_{\Omega_u} \mathrm{div}\varphi \neq 0$. Because $u = v$ outside D_1, we get, writing (8) with φ for u and v, that $\lambda_v = \lambda$. Moreover, since $J(v_n) \leq J(w)$ for $w \in \mathcal{F}_0$ with $|\Omega_w| \leq |\Omega_{v_n}|$, as in Proposition 1.2, one can prove that v_n satisfies Euler-Lagrange's equation (23) with some $\lambda_{v_n} = \lambda_n$. Let us show that $\lambda_n \geq \lambda_n^*$. Let $\varphi \in \mathrm{C}_0^\infty(D_1)$ such that $\int_{\Omega_{v_n}} \mathrm{div}\varphi > 0$ (φ exists for n large enough, thanks to $|D_1 \cap \Omega_u| < |D_1|$ and $\lim |\Omega_{v_n}| = |\Omega_u|$). Set $v_n^t = v_n(x + t\varphi(x))$. According to Lemma 4.2 and because $|\Omega_{v_n}| > a$, we have for $t > 0$ small enough

$$|\Omega_{v_n^t}| = |\Omega_{v_n}| - t \int_{\Omega_{v_n}} \mathrm{div}\varphi + o(t) > a.$$

Using the definition of v_n with $v = v_n^t$ and Lemma 4.2, we get:

$$J(v_n) \leq J(v_n) + t\lambda_n \int_{\Omega_{v_n}} \mathrm{div}\varphi - \lambda_n^* t \int_{\Omega_{v_n}} \mathrm{div}\varphi + o(t),$$

and so $\lambda_n \geq \lambda_n^*$.

We now want to show that $\lim \lambda_n = \lambda$. Then using $\lambda_n \geq \lambda_n^*$ and $\lim \lambda_n^* = +\infty$, we will get a contradiction. For this we just have to show that v_n converge in the norm of $\mathrm{H}_0^1(D)$. Indeed let $\Phi \in \mathrm{C}_0^\infty(D_1, \mathbb{R}^d)$ such that $\int_{\Omega_v} \mathrm{div}\Phi > 0$. Writing Euler's equation (23) for v_n, we get:

$$\int_{D_1} (D\Phi\nabla v_n \cdot \nabla v_n) - \frac{1}{2}\int_{D_1} |\nabla v_n|^2 \mathrm{div}\Phi - \int_{D_1} f\nabla v_n \cdot \Phi = \lambda_n \int_{\Omega_{v_n}} \mathrm{div}\Phi,$$

and, letting n goes to infinity we get:

$$\Big(\lim_{n\to\infty} \lambda_n\Big) \int_{D_1} \mathrm{div}\Phi = \int_{D_1} (D\Phi\nabla v \cdot \nabla v) - \frac{1}{2}\int_{D_1} |\nabla v|^2 \mathrm{div}\Phi - \int_{D_1} f\nabla v \cdot \Phi,$$

and so, using Euler's equation for v in D, $\lim_{n\to\infty} \lambda_n = \lambda$.

To show the strong convergence of v_n, because of the weak convergence, we just have to show the convergence of the norm of ∇v_n. For this we just write: $J(v_n) \leq J(v)$, so

$$\frac{1}{2}\int_D |\nabla v_n|^2 \leq \frac{1}{2}\int_D |\nabla v|^2 + \int_D f(v - v_n),$$

and the last term goes to 0 (by weak convergence of v_n).

We now prove that $\lambda^* \geq \lambda$. For this let, $\varphi \in \mathrm{C}_0^\infty(D_1, \mathbb{R}^d)$ such that $\int_{\Omega_u} \mathrm{div}\varphi < 0$ and $u_t(x) = u(x + t\varphi(x))$. Using Lemma 4.2, for t small enough,

$$|\Omega_{u_t}| = |\Omega_u| - t\int_{\Omega_u} \mathrm{div}\varphi + o(t) \geq |\Omega_u|.$$

Writing

$$J(u) \leq J(u_t) + \lambda^*(|\Omega_{u_t}| - a),$$

and using Lemma 4.2 we get:

$$0 \leq t(\lambda - \lambda^*)\int_{\Omega_u} \mathrm{div}\varphi + o(t),$$

and so $\lambda \leq \lambda^*$. □

Proof of Proposition 4.3. We showed, in the previous proposition, that, for $h > 0, \lambda \leq \lambda^*(h) < +\infty$. The fact that λ^* is non-increasing and μ is nondecreasing comes directly from the definition. We will now show that $\mu(h) \leq \lambda$. Let $\varphi \in \mathrm{C}_0^\infty(D_1)$ such that $\int_{\Omega_u} \mathrm{div}\varphi > 0$ and $u_t(x) = u(x + t\varphi(x))$. From Lemma 4.2 we have for t small enough:

$$|\Omega_u| - h \leq |\Omega_{u_t}| = |\Omega_u| - t\int_{\Omega_u} \mathrm{div}\varphi + o(t) \leq |\Omega_u|.$$

Writing

$$J(u) + \mu(h)|\Omega_u| \leq J(u_t) + \mu(h)|\Omega_{u_t}|,$$

and using Lemma 4.2, we get:

$$0 \leq t(\lambda - \mu(h))\int_{\Omega_u} \varphi + o(t),$$

and so $\mu(h) \leq \lambda$. We will study the limit of $\lambda^*(h)$ and $\mu(h)$ as h tends to 0, and this will give us directly $\mu(h) > 0$ for h small enough. We will begin with the limit of $\lambda^*(h)$. The proof is very close to the one of Proposition 4.4. Let h_n decrease to 0. Since λ^* is non-increasing and $\lambda \leq \lambda^*(h)$ we just have to show that $\lim \lambda^*(h_n) \leq \lambda$ for a sub-sequence of h_n.

Let $\varepsilon \in]0, \lambda[$. By minimization, one proves existence of $v_n \in \mathcal{F}_0$, with $|\Omega_{v_n}| \leq a + h_n$ such that:

$$J(v_n) + (\lambda^*(h_n) - \varepsilon)(|\Omega_{v_n}| - a)^+ \leq J(v) + (\lambda^*(h_n) - \varepsilon)(|\Omega_v| - a)^+,$$

for all $v \in \mathcal{F}_0$ such that $|\Omega_v| \leq a + h_n$.

First we have that $|\Omega_{v_n}| > a$. If $|\Omega_{v_n}| \leq a$, we would have, for all $v \in \mathcal{F}_0$ with $|\Omega_v| \leq a + h_n$,

$$J(u) \leq J(v_n) = J(v_n) + (\lambda^*(h_n) - \varepsilon)(|\Omega_v| - a)^+ \leq J(v) + (\lambda^*(h_n) - \varepsilon)(|\Omega_{v_n}| - a)^+,$$

which contradicts the definition of $\lambda^*(h_n)$.

We have:

$$J(v_n) + (\lambda^*(h_n) - \varepsilon)(|\Omega_{v_n}| - a) \leq J(u). \tag{27}$$

Up to a sub-sequence v_n converge weakly in $\mathrm{H}^1(D_1)$ to a v, moreover we have that $v \in \mathcal{F}_0, |\Omega_v| \leq \lim |\Omega_{v_n}| = a$, $J(v) \leq \liminf J(v_n)$ and, passing to the limit in (27), we obtain

$$J(v) \leq J(u) = \min\{J(w), w \in \mathrm{H}_0^1(D), |\Omega_w| \leq a\} \leq J(v),$$

and we have $|\Omega_v| = a$, since from $|\Omega_v| < a$, we would easily prove that $-\Delta v = f$ in D and contradict assumption (7). Like in Proposition 4.4 we write Euler's equation in D for v with a λ_v and we get that $\lambda = \lambda_v$.

We can write an Euler's equation in D_1 for v_n with some λ_n: if $w \in \mathcal{F}_0$ such that $|\Omega_w| \leq |\Omega_v|$ we have $J(v_n) \leq J(w)$. Also v_n converges strongly in $\mathrm{H}_0^1(D)$ since, as in Proposition 4.4, $J(v_n) \leq J(v)$. So, writing the Euler's equation and letting n go to infinity, we get that $\lim \lambda_n = \lambda$. To conclude, we just have to see that $\lambda_n \geq (\lambda^*(h_n) - \varepsilon)$. Let $\varphi \in \mathrm{C}_0^\infty(D_1)$ such that $\int_{\Omega_{v_n}} \mathrm{div}\varphi > 0$ (φ exists thanks to the hypothesis on $|D_1 \cap \Omega_u|$) and $v_n^t = v_n(x + t\varphi(x))$. According to Lemma 4.2 and because $|\Omega_{v_n}| > a$, we have for $t > 0$ small enough

$$|\Omega_{v_n^t}| = |\Omega_{v_n}| - t\int_{\Omega_{v_n}} \mathrm{div}\varphi + o(t) > a.$$

Using the minimality of v_n with respect to v_n^t Lemma 4.2, we get:

$$J(v_n) \leq J(v_n) + t\lambda_n \int_{\Omega_{v_n}} \mathrm{div}\varphi + o(t) - (\lambda^*(h_n) - \varepsilon)t \int_{\Omega_{v_n}} \mathrm{div}\varphi + o(t)),$$

and so $\lambda_n \geq (\lambda(h_n) - \varepsilon)$.

Now we study the limit of $\mu(h)$. Many arguments are similar to those used in the proof of Proposition 4.4 and in the study of $\lambda^*(h)$. Let h_n decrease to 0 and $\varepsilon > 0$. Let $v_n \in \mathcal{F}_0$, with $|\Omega_{v_n}| \leq a$, solution of the following minimization problem:

$$J(v_n) + (\mu(h_n) + \varepsilon)(|\Omega_{v_n}| - (a - h_n))^+ \leq J(v) + (\mu(h_n) + \varepsilon)(|\Omega_v| - (a - h_n))^+, \tag{28}$$

for all $v \in \mathcal{F}_0, |\Omega_v| \leq a$. First we have $|\Omega_{v_n}| < a$: indeed, if $|\Omega_{v_n}| = a$, we get, taking $v \in \mathcal{F}_0$ with $a - h_n \leq |\Omega_v| \leq a$,

$$\begin{aligned} J(u) &\leq J(v_n) \\ &\leq J(v) + (\mu(h_n) + \varepsilon)(|\Omega_v| - (a - h_n))^+ - (\mu(h_n) + \varepsilon)(|\Omega_{v_n}| - (a - h_n))^+ \\ &= J(v) + (\mu(h_n) + \varepsilon)(|\Omega_v| - |\Omega_u|), \end{aligned}$$

which contradicts the definition of $\mu(h_n)$.

We now see that: $|\Omega_{v_n}| \geq a - h_n$. By contradiction if $|\Omega_{v_n}| < a - h_n$, we would have for all $\varphi \in \mathrm{C}_0^\infty(D_1)$, with $0 < |\{\varphi \neq 0\}| \leq (a - h_n) - |\Omega_{v_n}|$, $v_n + t\varphi \in \mathcal{F}_0, |\Omega_{v_n+t\varphi}| \leq a - h_n$ and, by minimality of v_n, $J(v_n) \leq J(v_n + t\varphi), t \in \mathbb{R}$. So we deduce that

$$-\Delta v_n = f \text{ in } D_1,$$

and because $J(u) \leq J(v_n)$, by uniqueness of the solution of the minimum of J in $\mathcal{F}_0$ we get also that $u = v$ and

$$-\Delta u = f \text{ in } D_1.$$

From this we can deduce that u is locally Lipschitz in D_1 and, by Theorem 2.4, Ω_u has finite perimeter in D_1. In particular, $\partial^*\Omega_u \cap D_1$ is not empty. Let $x_0 \in \partial^*\Omega_u \cap D_1$ and let r_m go to 0. We study the blow up of u relatively to $B(x_0, r_m)$ and denote by u_0 a limit (up to a sub-sequence). Since

$$-\Delta u_m(x) = r_m f(x_0 + r_m x), \text{ in } B(0, R),$$

as soon as $B(x_0, r_m R) \subset D_1$, we get that u_0 is harmonic in $\mathbb{R}^d$. If we write the Euler's equation for u_0 we get for $\Phi \in \mathrm{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} (D\Phi\nabla u_0, \nabla u_0) - \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{div}\Phi|\nabla u_0|^2 = \lambda \int_{H^-} \mathrm{div}\Phi.$$

Using that u_0 is harmonic (and so u_0 is C^∞) we get that the left hand side is equal to 0, which is a contradiction because $\lambda > 0$ and H^- is a half-space. (Note that if we know that $u \geq 0$ we can use Prop. 3.5.)

Now we known that $a - h_n \leq |\Omega_{v_n}| < a$ and we may use it in (28). For instance, we have $J(v_n) \leq J(v)$ for all $v \in \mathcal{F}_0$ with $|\Omega_v| \leq |\Omega_{v_n}|$. Therefore there exists λ_n such that v_n satisfies the Euler's equation in D_1. We take $\varphi \in \mathrm{C}_0^\infty(D_1, \mathbb{R}^d)$ with $\int_{\Omega_{v_n}} \mathrm{div}\varphi < 0$ and $v_n^t(x) = v_n(x + t\varphi(x)), x \in D_1$. Since $\Omega_{v_n} < a$, by Lemma 4.2, we have $|\Omega_{v_n^t}| \leq a$ for t small enough and, (as in the study of λ^*)

$$\lambda_n \leq \mu(h_n) + \varepsilon. \tag{29}$$

As before v_n weakly converges (up to a sub-sequence) to some $v \in \mathcal{F}_0$ with $|\Omega_v| = a$ and

$$J(v) \leq \liminf_{n\to\infty} J(v_n) \leq J(u),$$

(because $\lim |\Omega_{v_n}| = a$). Again, we get that v satisfies Euler's equation with the same λ as u. We have, as before, convergence in the norm of $\mathrm{H}_0^1(D)$ by using

$$J(v_n) \leq J(v) + (\mu(h_n) - (a - h_n))(|\Omega_{v_n}| - a),$$

and we conclude in the same way $\lim \lambda_n = \lambda$. This, together with (29) and $\lambda > 0$ implies that $\mu(h) > 0$ for h small enough. □

Remark 4.5. We can show, using the same methods, that in fact we have $\mu(h) > 0$ for every $h > 0$.

Remark 4.6. In the proof of Proposition 4.3, we use $\lambda > 0$ only for $\mu(h)$. So, if u is a solution of $(\mathcal{P})$ with $\lambda = 0$, we get the existence of $\lambda^*(h)$ such that (25) holds and $\lim_{h\to 0} \lambda^*(h) = 0$.

5. Regularity of the boundary

In this section we study the regularity of the boundary of Ω_u with u a solution of $(\mathcal{P})$ in regions where u does not change its sign. The main result will be the following theorem:

Theorem 5.1. *Let u be a solution of $(\mathcal{P})$ with $\lambda > 0$ and let $D_1 \subset D$ satisfy conditions (22) and:*

$$D_1 \subset \{u \geq 0\}.$$

Then: Ω_u has locally finite perimeter in D_1 and

$$\mathcal{H}^{d-1}((\partial\Omega_u \setminus \partial^*\Omega_u) \cap D_1) = 0,$$

$$\Delta u + f\chi_{\Omega_u} = \sqrt{2\lambda}\mathcal{H}^{d-1}\lfloor\partial\Omega_u, \text{ in } \mathcal{D}'(D_1).$$

The reduced boundary $\partial^*\Omega_u$ is a $C^{1,\alpha}$ hypersurface ($\alpha > 0$). If moreover $d = 2$, we have $\partial^*\Omega_u = \partial\Omega_u$ and so $\partial\Omega_u$ is regular in D_1.

Remark 5.2. If we know that $u \geq 0$ on D, Theorem 5.1 is true with D instead of D_1 (we apply the theorem with D_1 equal to balls with center on $\partial\Omega_u$).

The proofs of the following propositions and lemmas are very close to the ones in [2,10]. The main differences is that we have "pseudo-penalization" conditions (24) and (25). Here $\mu(h) \neq \lambda^*(h)$ in general, while equality occurs in [2, 10] (in fact with a term like $\int_{\Omega_u} g^2$ with $g^2 > c > 0$). So we have to verify that the approach in [2, 10] works even if $\lambda^*(h) \neq \mu(h)$. A main point here though is that $\mu(h) > 0$. We begin with a technical proposition (as in [2]):

Proposition 5.3. *Let u and D_1 as in Theorem 5.1. Then for all $0 < \tau < 1$ there exist $C > 0, r_0 > 0$ such that for all balls with $B(x_1, r) \subset D_1$ and $r \leq r_0$, we have:*

$$\frac{1}{r}\fint_{\partial B(x_1,r)} u \leq C,$$

implies $u = 0$ on $B(x_1, \tau r)$.

Remark 5.4. This result essentially means that the averages are bounded from below. This is one main step in proving that ∇u does not degenerate near the boundary.

Proof. The proof for $\tau = 1/4$ is the same as in Lemma 2.8 of [10] with $g = \sqrt{2\mu(h)}$, where $h = \omega_d r_0^d$, (by Prop. 4.3 $\mu(h) > 0$ if r_0 is small enough), and the same f. In this article the authors study ($B_r = B(x_1, r)$) minimizers of:

$$J_r(v) = \int_{B_{r/2}} \left(|\nabla v|^2 - 2fv + g^2\chi_{\Omega_v}\right), v \in \mathrm{H}^1(B_{r/2}).$$

So we have, using Proposition 4.3, in our case that, for $r \leq r_0$:

$$J_r(u) \leq J_r(w),$$

for all $w \in \mathcal{F}_0$ if $a - \omega_d r_0^d \leq |\Omega_v| \leq |\Omega_u|$. In [10] the authors use only the following perturbation: $w = \min\{u, v\}$ on $B_{r/2}$ and $w = u$ outside $B_{r/2}$ for a $v \geq 0$ such that $u < v$ on $\partial B_{r/2}$. Using the same w, we have $w \in \mathcal{F}_0$ and $a - \omega_d r_0^d \leq |\Omega_w| \leq |\Omega_u|$ so, using (24) and the same proof we obtain the proposition. $\square$

Proposition 5.5. *There exist C_1, C_2, r_0 such that, for every $B(x_0,r) \subset D_1$ and $r \leq r_0$, we have*

$$0 < C_1 \leq \frac{|B(x_0,r) \cap \Omega_u|}{|B(x_0,r)|} \leq C_2 < 1.$$

Moreover we have:

$$\mathcal{H}^{d-1}((\partial\Omega_u \setminus \partial^*\Omega_u) \cap D_1) = 0.$$

Proof. We know that Ω_u has finite perimeter (see Th. 2.4). The proof of the first part is the same as Lemma 3.7 in [2] (see also [10], Lem. 2.10). The second part comes from the theory of sets with finite perimeter (see 5.8 in [6]). □

Theorem 5.6. *We have in the sense of distribution in D_1 that:*

$$\Delta u + f\chi_{\Omega_u} = \sqrt{2\lambda}\mathcal{H}^{d-1}.\lfloor\partial\Omega_u.$$

Proof. The proof is the same as in Theorem 2.13 in [10] see also [2] (4.7, 5.5). The steps are as follows: we show that the measure $\Delta u + \chi_{\Omega_u}$ is absolutely continuous with respect to $\mathcal{H}^{d-1}.\lfloor\partial\Omega_u$ (both are radon measures). And, using Proposition 3.5, we get that the derivative of $\Delta u + \chi_{\Omega_u}$ is $\sqrt{2\lambda}$ on $\partial^*\Omega_u$ and therefore almost everywhere. □

At this stage, we showed that u is what is called in [10] and in [2] **a weak solution** (see Def. 3.1 in [10] and 5.1 in [2]). We directly get the following regularity for ∂D_1.

Theorem 5.7. *Let u as in Theorem 5.1, then*
1) $\partial^*\Omega_u \cap D_1$ is a $C^{1,\alpha}$ hypersurface for some $\alpha > 0$ and $\mathcal{H}^{d-1}((\partial\Omega_u \setminus \partial^*\Omega_u) \cap D_1) = 0$;
2) If $d = 2$ then $\partial\Omega_u = \partial^*\Omega_u$ and so $\partial\Omega_u$ is regular in D_1.

Remark 5.8. If f is more regular around Ω_u, then so is $\partial^*\Omega_u$ (or $\partial\Omega_u$ when $d = 2$).

Proof. The first point directly comes from [2], 6−8, generalized in [10], Section 5. One important thing is that the regularity of $\partial^*\Omega_u$ is shown for "weak solution" in [2], and we have proved that we do have such weak solutions here.

For the second point, we have to generalize Theorem 6.6 in [2] and his Corollary 6.7. The corollary is deduced from the theorem exactly in the same way as in [2].

So, when $d = 2$, we have to show that:

$$\lim_{r\to 0} \fint_{B_r \cap \Omega_u} \max\{\lambda - |\nabla u|^2, 0\} = 0.$$

For $B_r = B(x_0, r)$ with $x_0 \in \partial\Omega_u$. We take $\zeta \in C_0^1(D_1)$ as in [2] and the same $v = \max\{u - \varepsilon\zeta, 0\}$. Then we have $|\Omega_u| - |\Omega_v| = |\{0 < u \leq \varepsilon\zeta\}| \leq |\{\zeta \neq 0\}|$, so using the definition of $\mu(h)$ with $h = |\{\zeta \neq 0\}|$ we get, as in [2]:

$$\int_{\{0<u\leq\varepsilon\zeta\}} \left(\mu(h) - \frac{|\nabla u|^2}{2}\right) \leq \frac{\varepsilon^2}{2}\int_{\{u>\varepsilon\zeta\}} |\nabla\zeta|^2.$$

Now, using $\lambda \geq \mu(h)$:

$$\int_{\{0<u\leq\varepsilon\zeta\}} \left(\lambda - \frac{|\nabla u|^2}{2}\right) \leq \frac{\varepsilon^2}{2}\int_{\{u>\varepsilon\zeta\}} |\nabla\zeta|^2 + (\lambda - \mu(h))h.$$

Exactly as in [2], we take for $r \leq \rho \leq R$,

$$\zeta(x) = \begin{cases} \frac{\log(\rho/|x-x_0|)}{\log(\rho/r)} & \text{for} \quad r < |x - x_0| \leq \rho \\ 1 & \text{for} \quad |x - x_0| \leq r. \end{cases} \tag{30}$$

We have $u(x) \leq Cr$ in B_r ($x_0 \in \partial\Omega_u$ and u is Lipschitz) and we choose $\varepsilon = Cr$. We get, with this choice of ζ and ε:

$$\begin{aligned}\int_{B_r\cap\Omega_u} \max\left\{\lambda - \frac{|\nabla u|^2}{2}, 0\right\} &\leq \int_{B_\rho\cap\Omega_u} \max\left\{\frac{|\nabla u|^2}{2} - \lambda, 0\right\} \\ &\quad + (\lambda - \mu'(\rho))\pi\rho^2 + \frac{Cr^2}{\log(\rho/r)}.\end{aligned}$$

As in [2], we use the (modified) Theorem 6.3 with $Q \equiv \lambda/2$.

$$⨍_{B_r\cap\Omega_u} \max\left\{\lambda - \frac{|\nabla u|^2}{2}, 0\right\} \leq C_1\left(\frac{\rho}{r}\right)^2\left(\frac{\rho}{R}\right)^\alpha + (\lambda - \mu'(\rho))\pi\left(\frac{\rho}{r}\right)^2 + \frac{C}{\log(\rho/r)}.$$

For the choice of r, ρ and R, here we can take $R = R_0$ constant, but we have an other term in $\lambda - \mu'(\rho)$. We can choose $r = \rho\left(\pi(\lambda - \mu'(\rho)) + C_1(\rho/R)^\alpha\right)^{1/4} < \rho$ for ρ small enough (Prop. 4.3). With this choice we have:

$$\begin{aligned}⨍_{B_r\cap\Omega_u} \max\left\{\lambda - \frac{|\nabla u|^2}{2}, 0\right\} &\leq \frac{\pi((\lambda - \mu'(\rho)) + C_1(\rho/R)^\alpha)}{\pi((\lambda - \mu'(\rho)) + C_1(\rho/R)^\alpha)^{1/2}} \\ &\quad + \frac{C}{\log(\pi(\lambda - \mu'(\rho)) + C_1(\rho/R)^\alpha)^{1/4}},\end{aligned}$$

which goes to 0 when ρ goes to 0, using Proposition 4.3. □

6. The hypothesis $\lambda > 0$

In this section, we discuss the hypothesis $\lambda > 0$. A main result is the following.

Proposition 6.1. *Let u be a solution of $(\mathcal{P})$ with $f \geq 0, f \not\equiv 0$. Then we have $\lambda > 0$.*

Proof. This proposition comes from the more general Proposition 6.2 (where we do not make assumption on the sign of f). If $f \geq 0$, we take D_1 such that conditions (22) in Section 4 are true. By Proposition 6.2, $-\Delta u \geq 0$ in D_1, so that $u > 0$ in D_1. This contradicts the conditions on D_1. □

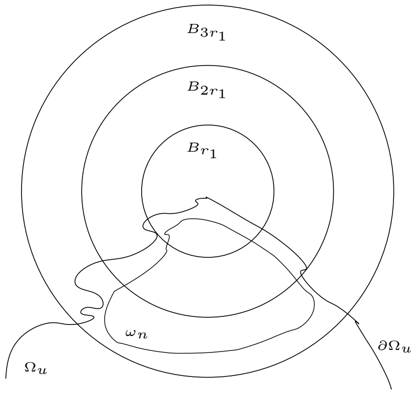
Proposition 6.2. *Let u be a solution of $(\mathcal{P})$ such that $\lambda = 0$. Let D_1 be an open subset satisfying conditions* (22) *and such that $u \geq 0$ on D_1. Then we have that:*

$$-\Delta u = f\chi_{\Omega u} \text{ in } D_1. \tag{31}$$

Proof. We will show that $-\Delta u = f\chi_{\Omega_u}$ in every regular open subset D_2 such that $\overline{D_2} \subset D_1$. To make the proof more clear, we will take $D_1 = B_3 = B(0, 3r_1)$ (r_1 small enough) and $D_2 = B_1 = B(0, r_1)$ but there is no changes for arbitrary open subset. Let (ω_n) be an increasing sequence of regular open subsets of $B_3 \cap \Omega_u$ such that ($B_2 = B(0, 2r_1)$)

$$\{x \in \overline{B_2 \cap \Omega_u}, d(x, \partial\Omega_u) \geq 1/n\} \subset \omega_n \subset \overline{\omega_n} \subset B_3 \cap \Omega_u, \tag{32}$$

(see the picture below). In particular, $B_2 \cap \omega_n$ increases to $B_2 \cap \Omega_u$.



For every $\varphi \in C_0^\infty(B_1)$ ($-\Delta u = f$ in ω_n and Δu is a measure),

$$\langle -\Delta u, \varphi\rangle = \int_{B_1} \varphi f \chi_{\omega_n} - \int_{B_1\setminus\omega_n} \varphi \mathrm{d}(\Delta u). \tag{33}$$

The first term goes to $\int_{B_1} \varphi f \chi_{\Omega_u}$ so we need to show that the second one converges to 0 to get $-\Delta u = f\chi_{\Omega_u}$ in B_1.

Let $\theta \in C_0^\infty(B_2)$ be such that $0 \le \theta \le 1$ and $\theta = 1$ in $\overline{B_1}$. Let H_n be the vector valued function defined by:

$$H_n = \begin{cases} \theta\nabla u \text{ on } B_3 \setminus \omega_n \\ \text{harmonic on } \omega_n, \end{cases}$$

(by harmonic, we mean that each component is harmonic). This function is continuous on the boundary of ω_n (∇u is continuous in Ω_u), so $\mathrm{div} H_n$ does not charge $\partial\omega_n$ and we get, using $\theta \equiv 1$ in B_1, for every $\varphi \in C_0^\infty(B_1)$,

$$\langle \mathrm{div}(H_n) \cdot \varphi\rangle = \int_{B_1\cap\omega_n} \mathrm{div}(H_n)\varphi + \int_{B_1\setminus\omega_n} \varphi \mathrm{d}(\Delta u). \tag{34}$$

We want to show that the last term of this equation goes to 0: for this we will show that the two first ones go to 0.

More precisely, we will prove that

$$H_n \text{ converges uniformly to 0 on } B_2 \tag{35}$$

$$\|\mathrm{div} H_n\|_{L^2, B_1\cap\omega_n} \le C. \tag{36}$$

From (35), we deduce that the first term in (34) tends to 0. We also deduce that, since $\mathrm{div} H_n$ is harmonic in ω_n, it converges to 0 on any compact set of $B_2 \cap \Omega_u$ (recall that $B_2 \cap \omega_n$ increases to $B_2 \cap \Omega_u$). Coupled with (36) this implies that $\mathrm{div} H_n \chi_{B_1\cap\omega_n}$ tends to 0 in $\mathcal{D}'(B_1)$ and the proof of Proposition 6.2 will be complete.

Using the maximum principle, we have that:

$$\|H_n\|_{\infty,\omega_n} \le \|\theta\nabla u\|_{\infty,\partial\omega_n} \le \|\nabla u\|_{\infty, B_2\cap\partial\omega_n}.$$

By the following Lemma 6.3 and since$\nabla u = 0$ outside Ω_u, we deduce (35).

We now prove (36). Let $i = 1...d$, we have $H_n^i - \theta u_{x_i} \in \mathrm{H}_0^1(\omega_n)$ (H_n^i is the i-th component of H_n), taking the Laplacian, we get

$$\Delta\left(H_n^i - \theta u_{x_i}\right) = \Delta\left((u\theta)_{x_i} - u\theta_{x_i}\right) = (\Delta(u\theta))_{x_i} - \Delta\left(u\theta_{x_i}\right).$$

Since in $\omega_n, \Delta(u\theta) = -f\theta + u\Delta\theta + 2\langle\nabla u \cdot \nabla\theta\rangle$ and $\nabla(u\theta_{x_i}) = \theta_{x_i}\nabla u + u\nabla\theta_{x_i}$ are bounded functions (u and ∇u are bounded), we can multiply by $H_n^i - \theta u_{x_i}$ and integrate by parts to have,

$$-\|\nabla\left(H_n^i - \theta u_{x_i}\right)\|^2_{L^2,\omega_n} = -\int_{\omega_n}\Delta(u\theta)\left(H_n^i - \theta u_{x_i}\right)_{x_i} + \int_{\omega_n}\left\langle\nabla\left(u\theta_{x_i}\right)\cdot\nabla\left(H_n^i - \theta u_{x_i}\right)\right\rangle,$$

and so,

$$\|\nabla\left(H_n^i - \theta u_{x_i}\right)\|^2_{L^2,\omega_n} \leq \|\Delta(u\theta)\|_{L^2,\omega_n}\|\nabla\left(H_n^i - \theta u_{x_i}\right)\|_{L^2,\omega_n} + \|\nabla\left(u\theta_{x_i}\right)\|_{L^2,\omega_n}\|\nabla\left(H_n^i - \theta u_{x_i}\right)\|_{L^2,\omega_n}.$$

This implies that $\|\nabla(H_n^i - \theta u_{x_i})\|_{L^2,\omega_n}$ is bounded. Since

$$\mathrm{div} H_n = \sum_{i=1}^d\left(\left(H_n^i - \theta u_{x_i}\right)_{x_i} + \theta_{x_i}u_{x_i} + \theta u_{x_ix_i}\right),$$

using that $-\Delta u = f$ in ω_n, we get that $\mathrm{div} H_n$ is bounded in $L^2(\omega_n)$. □

It remains to prove the following lemma.

Lemma 6.3. *Using the same notations as in the previous proposition, we have that:*

$$\lim_{n\to\infty}\|\nabla u\|_{\infty,B_2\cap(\Omega_u\setminus\omega_n)} = 0.$$

Proof. We will first show that there exists a decreasing function η with $\lim_{r\to 0}\eta(r) = 0$, such that if $x_0 \in B_2$, $0 \leq r \leq 1$ and $|\{u = 0\}\cap B(x_0, r)| > 0$ then,

$$\frac{1}{r}\fint_{\partial B(x_0,r)} u \leq \eta(r). \tag{37}$$

Let $B_r = B(x_0, r)$ be such a ball. Using exactly the same computation as in Proposition 2.5 in [10], we define v by $-\Delta v = f$ in B_r and $v = u$ outside B_r and we have ($B_r \subset D_1 = B(0, 3)$):

$$\int_{B_r}|\nabla(u - v)|^2 \leq \lambda^*\left(\omega_d r^d\right)|\{u = 0\}\cap B_r\}|, \tag{38}$$

(see Prop. 4.3 for the definition of λ^* and Rem. 4.6). We also get as in [2, 10],

$$|\{u = 0\}\cap B_r\}|\left(\frac{1}{r}\fint_{\partial B_r} u\right)^2 \leq C\int_{B_r}|\nabla(u - v)|^2. \tag{39}$$

Actually, this is true only if $\frac{1}{r}\fint_{\partial B_r} u - Cr \geq 0$ where C depends only on d and f, but otherwise $\eta(r) \geq Cr$ works. With (38) and (39), we get (37) with $\eta(r) = C\sqrt{\lambda^*(\omega_d r^d)} + Cr$. (Using Rem. 4.6, λ^* is decreasing and $\lim_{h\to 0}\lambda^*(h) = 0$.)

Let $x_0 \in (\Omega_u \setminus \omega_n)\cap B_2$ and $B_{r_0} = B(x_0, r)$ be the largest ball included in Ω_u. By definition of ω_n, we have $r_0 \leq 1/n$ (see (32)).

For $\delta > 0$ we have that $\{u = 0\} \cap B_{r_0+\delta} \neq \emptyset$. So there are two cases.

1) (Good case.) If $|\{u = 0\} \cap B_{r+\delta_0} = 0|$ for some δ_0 then we see from (38) that $-\Delta u = f$ in $B_{r_0+\delta_0}$. Let $0 < \delta < \delta_0$ and $x \in B_{r_0+\delta}$ be such that $u(x) = 0$, now we use Lemma 2.4 in [10] on $B_{r_0+\delta}$ in x and we get (C depends only on d and f):

$$\frac{1}{r_0+\delta} \fint_{\partial B_{r_0+\delta}} u \leq C(r_0+\delta) \leq \eta(2r_0).$$

And when δ goes to 0:

$$\frac{1}{r_0} \fint_{\partial B_{r_0}} u \leq \eta(2r_0). \tag{40}$$

2) If for every $0 < \delta < r_0$ $|\{u = 0\} \cap B_{r+\delta}| > 0$ we can use (37),

$$\frac{1}{r_0+\delta} \fint_{\partial B_{r_0+\delta}} u \leq \eta(r_0+\delta) \leq \eta(2r_0),$$

and, when δ goes to 0,

$$\frac{1}{r_0} \fint_{\partial B_{r_0}} u \leq \eta(2r_0). \tag{41}$$

Now we use that

$$|\nabla u(x_0)| \leq C \frac{1}{r_0} \fint_{\partial B_{r_0}} u + Cr_0$$

(see Prop. 2.4 in [10], in B_{r_0} we have $-\Delta u = f$). Using this in (41) or in (40) we have ($r_0 \leq \frac{1}{n}$)

$$|\nabla u(x_0)| \leq \eta(2/n) + C/n,$$

which goes to 0 when n goes to infinity. □

Remark 6.4. By moving D_1 in all D in Proposition 6.2, we could see that, if $\lambda = 0$ then

$$-\Delta u = f\chi_{\Omega u} \text{ in } D.$$

We can compare this with the hypothesis (7) taken at the beginning of the paper which excludes existence of w such that $-\Delta w = f$ in D and $|\Omega_w| \leq a$.

Remark 6.5. It is likely that a result similar to Proposition 6.2 is true when u changes its sign, but the proof of Lemma 6.3 strongly uses that $u \geq 0$. Let us mention that, if (7) is true, we can prove that the set of b for which $\lambda_{u_b} > 0$ is a dense open set in $(0, a)$ where u_b is any solution of

$$J(u_b) = \min \left\{ J(u), u \in \mathrm{H}^1(D), |\Omega_u| = b \right\}.$$

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