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ABSTRACT VARIATIONAL PROBLEMS WITH VOLUME CONSTRAINTS

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Abstract. Existence results for a class of one-dimensional abstract variational problems with volume constraints are established. The main assumptions on their energy are additivity, translation invariance and solvability of a transition problem. These general results yield existence results for nonconvex problems. A counterexample shows that a naive extension to higher dimensional situations in general fails.

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1. Introduction

Recently, constrained variational problems of the type

$$\begin{cases} \text{Minimize } E(u) := \displaystyle\int_\Omega f(u(x), \nabla u(x))\,\mathrm{d}x, \\ u \in W^{1,p}(\Omega, \mathbb{R}),\ \Omega \subset \mathbb{R}^n, \\ |\{u = 0\}| = \alpha,\ |\{u = 1\}| = \beta \end{cases} \tag{1}$$

have been studied under different assumptions on the energy density f [1,3,5]. This was initially suggested by Gurtin in 1992, see also [2]. The aim of the present article is to find general conditions on the energy $E(u)$ which entail the existence of solutions to volume constrained problems of type (1) in the one-dimensional case, and which generalize the results of Morini and Rieger [3] to energy functionals which are not neccessarily of integral form.

In Section 2 two existence results for a very general class of admissible energies are presented. Their proofs turn out to be surprisingly simple. In Section 3 these results are applied to minimization problems of integral type which are nonconvex in u'. In Section 4 a higher dimensional example illustrates how non-smoothness of the boundary may lead to non-existence of solutions.

Keywords and phrases. Level set constraints, nonconvex problems, minimization.

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2. Abstract variational problems

We consider the following one-dimensional variational problem on an open interval $I = (a, b)$:

$$\begin{cases} \text{Minimize } E(u, I), \text{ for } u \in W^{1,p}(I, [0, 1]), \\ |\{u = 0\}| = \alpha, \ |\{u = 1\}| = \beta, \end{cases} \tag{2}$$

where $\alpha + \beta < b - a = |I|$ and $1 < p \leq +\infty$.

The aim of this section is to prove two existence results under certain assumptions on the energy E. We collect the main assumptions in the following definition:

Definition 2.1 (Admissible energies)**.** Let $I = (a, b)$ be an open interval in $\mathbb{R}$, let $1 < p \leq +\infty$, and let $\mathcal{I}$ denote the set of all open intervals in I. Consider a function

$$E \colon W^{1,p}(I, [0, 1]) \times \mathcal{I} \to \mathbb{R},$$

bounded from below, with $E(u, T) = E(v, T)$ for $T \in \mathcal{I}$ whenever $u|_T = v|_T$. (Thus we sometimes write $E(u, T)$ even if u is only defined on T.)

Then E is called an *admissible energy* if it satisfies the following three conditions:

(i) let $T \in \mathcal{I}$ be an arbitrary open interval $(x_0, x_1) \subset I$. Consider the problem

$$\begin{cases} \text{Minimize } E(u, T), \text{ for } u \in W^{1,p}(T, [0, 1]), \\ \text{such that there exist } \xi_0, \xi_1 \in [x_0, x_1] \\ \text{with } u(\xi_0) = 0, \text{ and } u(\xi_1) = 1. \end{cases} \tag{3}$$

This problem admits a solution (u, ξ_0, ξ_1);

(ii) (Additivity) for $x_0, x_1, x_2 \in I$, $x_0 < x_1 < x_2$ and $u \in W^{1,p}(I, [0, 1])$ we have:

$$E(u, (x_0, x_1)) + E(u, (x_1, x_2)) = E(u, (x_0, x_2));$$

(iii) (Translation invariance) for all x_0, $x_1 \in \bar{I}$, $u \in W^{1,p}((x_0, x_1), [0, 1])$ and $\tau \in \mathbb{R}$ such that $x_0+\tau$, $x_1+\tau \in \bar{I}$ we have

$$E(u, (x_0, x_1)) = E(u(\cdot - \tau), (x_0 + \tau, x_1 + \tau)).$$

It turns out to be useful to extend the notion of admissible energies to functions which are only piecewise in $W^{1,p}$:

Definition 2.2. Let E be an admissible energy on $W^{1,p}(I, [0, 1]) \times \mathcal{I}$. Let $(x_i)_{i=1,\dots,n} \in I$ with $x_{i+1} > x_i$, and $u|_{(x_i, x_{i+1})} \in W^{1,p}((x_i, x_{i+1}), [0, 1])$, then define $E(u) := \sum_{i=1}^n E(u, (x_i, x_{i+1}))$.

A special type of admissible energies is given in the following definition:

Definition 2.3. We call an energy E *symmetric* if for all $(x_0, x_1) \in \mathcal{I}$ and all $u \in W^{1,p}((x_0, x_1), [0, 1])$ we have

$$E(u(x_0 + \cdot), (0, x_1 - x_0)) = E(u(x_1 - \cdot), (0, x_1 - x_0)).$$

The properties (ii) and (iii) of Definition 2.1 can be easily verified for a given energy E. However, condition (i) is a little involved. The following remark gives an easy characterization of (i) for symmetric energies:

Remark 2.4. If E is symmetric, and satisfies (ii) and (iii), then E is an admissible energy if and only if for every $T := (x_0, x_1) \subset I$ the Dirichlet boundary value problem

$$\begin{cases} \text{Minimize } E(u, T), \text{ for } u \in W^{1,p}(T, [0, 1]), \\ u(x_0) = 0, \text{ and } u(x_1) = 1, \end{cases} \tag{4}$$

admits a solution.

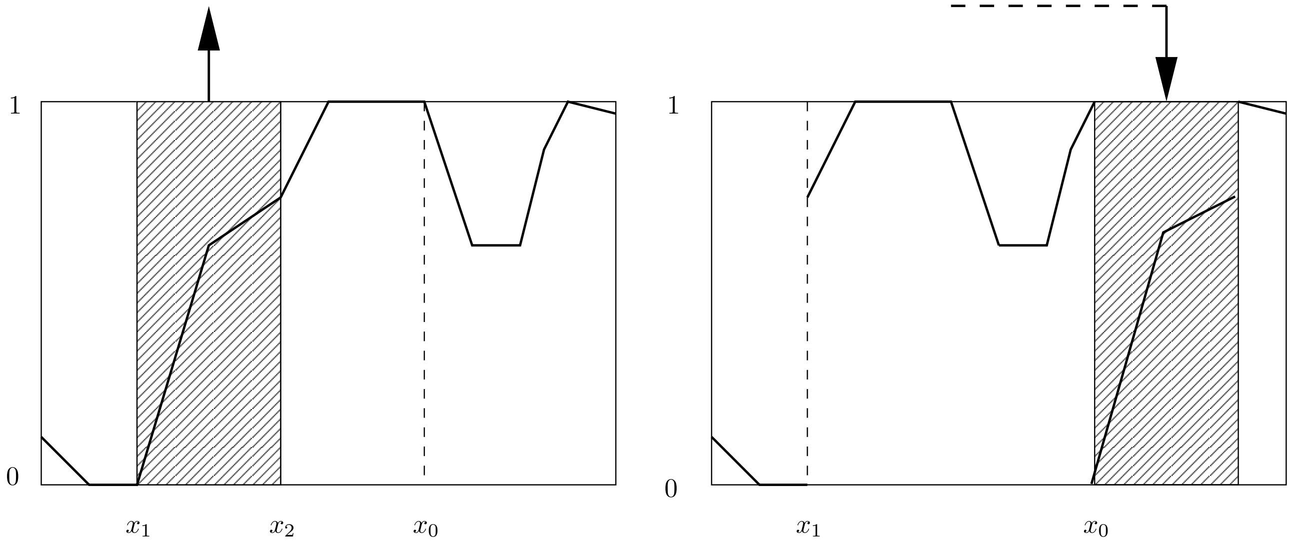

FIGURE 1. The shift operator S cuts out the piece of the function between x_1 and x_2 (left picture), glues the remaining parts of the function together and inserts the cut out piece at x_0 (right).

Proof. Any solution of (4) obviously solves (3). Hence the condition is sufficient. Necessity can be proved using the same argument as in Lemma 2.7, see below. $\square$

Before we state existence results for problem (2) we give some useful lemmata:

Lemma 2.5 (Energy conserving shifts). *Let E be an admissible energy. Let $x_0, x_1, x_2 \in I = (a, b)$ with $x_2 > x_1$ and $x_2 - x_1 \leq b - x_0$. We define an operation $S((x_1, x_2), x_0)$ on the functions which are piecewise $W^{1,p}$ on I in the following way (compare Fig. 1): If $x_1 \geq x_0$ we define*

$$S((x_1,x_2),x_0)u(x) := \begin{cases} u(x), & x < x_0, \\ u(x + x_1 - x_0), & x_0 \leq x < x_0 + x_2 - x_1, \\ u(x - (x_2 - x_1)), & x_0 + x_2 - x_1 \leq x < x_2, \\ u(x), & x \geq x_2, \end{cases}$$

else we define

$$S((x_1,x_2),x_0)u(x) := \begin{cases} u(x), & x < x_1, \\ u(x + x_2 - x_1), & x_1 \leq x < x_0, \\ u(x + x_1 - x_0), & x_0 \leq x < x_0 + x_2 - x_1, \\ u(x), & x \geq x_0 + x_2 - x_1. \end{cases}$$

The so defined operator S is energy conserving, i.e. $E(S((x_1, x_2), x_0)u, I) = E(u, I)$.

The shift operator $S((x_1, x_2), x_0)$ can be described as "cutting out" the values of a function on the interval (x_1, x_2) and inserting them at the position x_0.

Proof. The proof follows immediately from (ii) and (iii) of the definition of admissible energies. $\square$

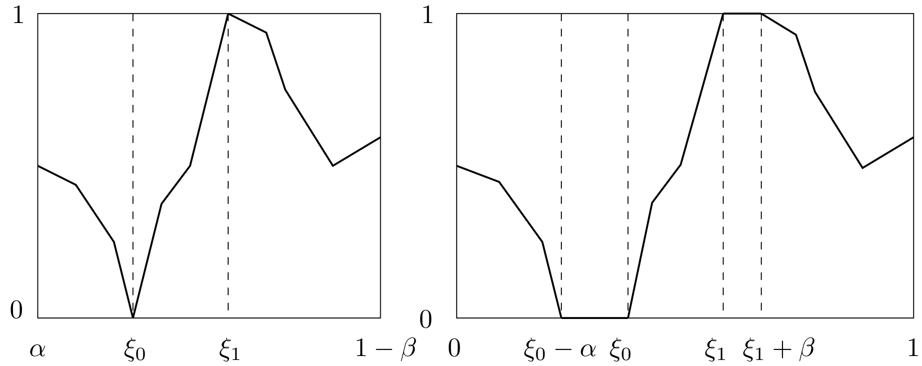

FIGURE 2. Left: the function $\tilde{u}$ solves problem (3) on the interval $(\alpha, 1-\beta)$. Right: construction of the function u.

The operation defined in this lemma is the basic step in the constructions of the following results.

Lemma 2.6 (Existence for the relaxed problem)**.** *Let E be an admissible energy for the open interval I, continuous with respect to the strong $W^{1,p}$-topology on all open sets in I. Then for α, $\beta > 0$ with $\alpha + \beta < |I|$ the relaxed problem*

$$\begin{cases} \text{Minimize } E(u, I), \text{ where } u \in W^{1,p}(I, [0,1]), \\ |\{u = 0\}| \geq \alpha, \ |\{u = 1\}| \geq \beta, \end{cases} \tag{5}$$

admits a solution u.

Moreover we can construct a solution with the following properties:

$$\begin{aligned} \{x \in I, \ u(x) = 0\} &= A_0 \cup R_0, \\ \{x \in I, \ u(x) = 1\} &= A_1 \cup R_1, \end{aligned} \tag{6}$$

where A_0, A_1 are intervals closed in I and R_0, R_1 are countable sets.

Proof. For simplicity take $I = (0,1)$. We will first prove that the following function u (compare Fig. 2) is a minimizer to the relaxed problem. Let $(\tilde{u}, \xi_0, \xi_1)$ be a minimizer of (3) on $(\alpha, 1-\beta)$. Without loss of generality assume $\xi_0 < \xi_1$. Define

$$u(x) := \begin{cases} \tilde{u}(x+\alpha), & x < \xi_0 - \alpha, \\ 0, & \xi_0 - \alpha \leq x < \xi_0, \\ \tilde{u}(x), & \xi_0 \leq x \leq \xi_1, \\ 1, & \xi_1 < x \leq \xi_1 + \beta, \\ \tilde{u}(x-\beta), & x > \xi_1 + \beta. \end{cases}$$

Take an arbitrary function $w \in W^{1,p}(I, [0,1])$ with $E(w, I) \leq E(u, I)$, satisfying $|\{w = 0\}| \geq \alpha$ and $|\{w = 1\}| \geq \beta$. The following construction will exclude the case $E(w, I) < E(u, I)$. Moreover it will show that u can be chosen such that the additional properties as stated in the lemma are satisfied.

We denote $T := \{x \in I, \ w(x) \in (0,1)\}$. We refer to T as the "transition set".

T is an open set, hence there exists a countable set of disjoint open intervals T_j with $T = \cup_j T_j$. Consider one of these intervals $T_j = (l, r)$. Since w is continuous, there are only three possibilities for $w(l)$:

(i) $w(l) = 0$;
(ii) $w(l) = 1$;
(iii) $w(l) \in (0,1)$ and $l = 0$, *i.e.* the set T_j lies on the left boundary of $(0,1)$.

Case (iii) can only occur once, call the corresponding set $T^L = (0, \hat{a})$. (In the case that $w(0) \in \{0,1\}$, you may set $T^L = \emptyset$ and $\hat{a} = 0$.) Using the same reasoning for $w(r)$ we define a set $T^R = (\hat{b}, 1)$ where $w(r) \in (0,1)$ and

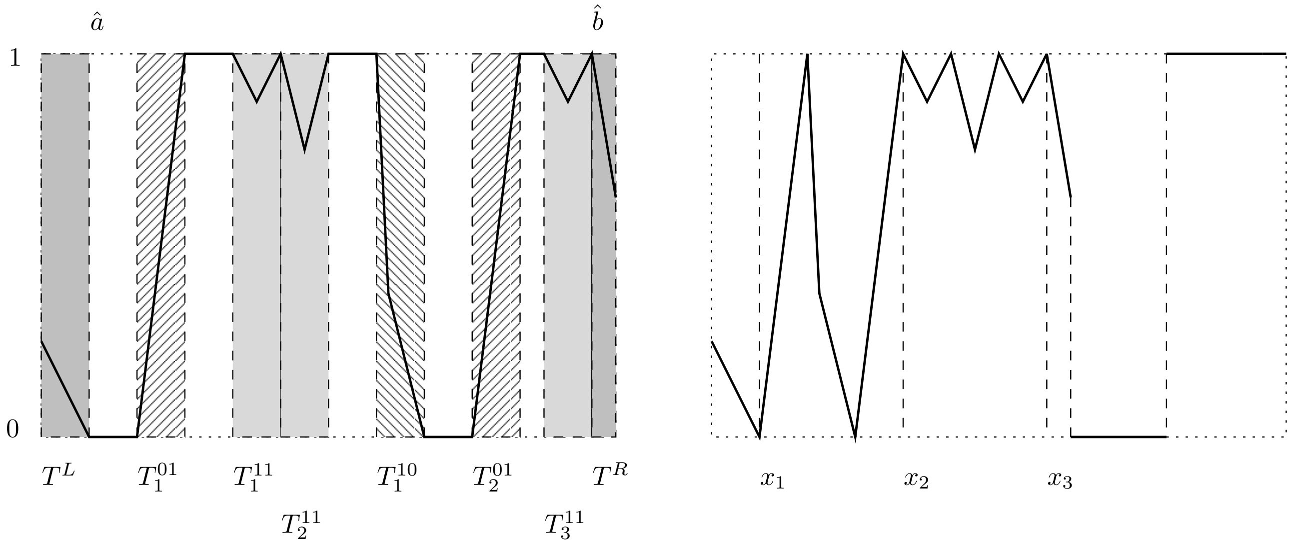

FIGURE 3. Left: illustration of the decomposition of the transition set T into open intervals T^L (left boundary), T^R (right boundary), T_j^{00} and T_j^{11} (possibly infinitely many), T_j^{01} and T_j^{10} (finitely many). Right: the construction in the proof of Lemma 2.6 collects iteratively the sets T^L, then T_j^{00}, then T_j^{01} and T_j^{10}, then T_j^{11} and finally T^R using energy preserving shifts. The construction shows that an optimal minimizer of the relaxed problem (5) is given by u (see text).

get the following decomposition of T:

$$T = \left(\bigcup_i T_i^{00}\right) \cup \left(\bigcup_{i=1}^{i^{01}} T_i^{01}\right) \cup \left(\bigcup_{i=1}^{i^{10}} T_i^{10}\right) \cup \left(\bigcup_i T_i^{11}\right) \cup T^L \cup T^R,$$

such that for $T_i^{yz} = (l, r)$ we have $u(l) = y$, $u(r) = z$.

Since $w \in W^{1,p}(I, [0,1])$, we see that i^{01} and i^{10} are finite. (Otherwise $\int |w'|^p = +\infty$.) By continuity of w, there are now three possibilities: either $i^{01} = i^{10} + 1$, $i^{01} = i^{10} - 1$, or $i^{01} = i^{10}$. We consider here the first case and assume without loss of generality that $w(\hat{a}) = 0$ and $w(\hat{b}) = 1$. The other cases can be handled in a similar way. We define a function $\hat{u}$ with $E(\hat{u}, I) = E(w, I)$ by an iterative energy conserving construction using Lemma 2.5 in several steps (see Fig. 3).

Step 1: First define $T_0^{00} := T^L$ and $v_0 := w$. For $T_j^{00} =: (l_j, r_j)$ and $s := \sum_{k=0}^{j-1}(r_k - l_k)$ define iteratively $v_j := S(T_j^{00}, s)v_{j-1}$ for $j \geq 1$. The limit function $\lim_j v_j$ denote by $\hat{u_1}$. (Since $\sum |T_j^{00}|$ is bounded, it is easy to see that v_j converges in $W^{1,p}$.) By continuity of E we have $E(\hat{u_1}, I) = \lim_j E(v_j, I) = E(w, I)$, and for the limit point $x_1 := \sum_k (r_k - l_k)$ we have $\lim_{x \to x_1} \hat{u_1}(x) = 0$, since $w \in W^{1,p}$. The result of step 1 is a function $\hat{u_1}$ with transition set $\hat{T} := \{\hat{u_1}(x) \in (0,1)\}$ represented by the disjoint sum

$$\hat{T} = \hat{T}_0 \cup \left(\bigcup_{i=1}^{i^{01}} \hat{T}_i^{01}\right) \cup \left(\bigcup_{i=1}^{i^{10}} \hat{T}_i^{10}\right) \cup \left(\bigcup_i \hat{T}_i^{11}\right) \cup \hat{T}^R,$$

such that $\hat{T}_0 = (0, x_1) \setminus R$, R countable set, and for all i and $y, z \in \{0,1\}$ and for $\hat{T}_i^{yz} = (l, r)$ we have again $\hat{u_1}(l) = y$, $\hat{u_1}(r) = z$ and $\hat{T}^R = (\hat{b}, 1)$.

Step 2: Now continue in the same way by shifting the values of $\hat{u_1}$ on $\hat{T}_i^{01}$ and $\hat{T}_i^{10}$ for all (finitely many) i in the order $\hat{T}_1^{01}, \hat{T}_1^{10}, \hat{T}_2^{01}, \hat{T}_2^{10} \dots$ Call the result $\hat{u_2}$ and define $x_2 := x_1 + \cup_i^{i^{01}} |\hat{T}_i^{01}| + \cup_i^{i^{10}} |\hat{T}_i^{10}|$. We have now for $\check{T} := \{\hat{u_2}(x) \in (0,1)\}$ the representation

$$\check{T} = \check{T}_0 \cup \left(\bigcup_{i=1} \check{T}_i^{11} \right) \cup \check{T}^R,$$

where $\check{T}_0 = (0, x_2) \setminus R$, R countable set, and $\check{T}_i^{11}$ and $\check{T}^R$ satisfy the same conditions as $\hat{T}_i^{11}$ and $\hat{T}^R$ above. Since we assumed $i^{01} = i^{10} + 1$, we get $\lim_{x \to x_2} \hat{u_2}(x) = 1$.

Step 3: Finally shift the values of $\hat{u_2}$ on $\check{T}_i^{11}$ following the idea of Step 1. The resulting function $\hat{u_3}$ has only finitely many non-trivial intervals on which $\hat{u_3} \in \{0, 1\}$. By applying apropriate shifts, we can assume that $\tilde{u}_3$ has only two such intervals. Call the closure of these intervals $\tilde{A}_0 = [a_0, b_0]$ and $\tilde{A}_1 = [a_1, b_1]$.

Step 4: Now define $\delta := |\tilde{A}_0| - \alpha$ and $\varepsilon := |\tilde{A}_1| - \beta$. By construction be have $\delta, \varepsilon \geq 0$. If $\delta > 0$ then choose a $xi_0 \in [0, 1 - \beta + \alpha]$ such that $\tilde{u_3}(\xi_0) = 0$ and define $\hat{u_4} := S((a_0, a_0 + \delta), xi_0$. Applying the same idea for the case $\varepsilon > 0$ we obtain a function $\hat{u_5}$, and we immediately see that the triple $(\hat{u_5}|_{(0,1-(\beta+\alpha))}, x_1, x_2)$ solves (3). Hence $E(\hat{u_5}, (0, 1 - (\beta + \alpha))) = E(\tilde{u}, (\alpha, 1 - \beta))$, and $E(\hat{u_5}, I) = E(u, I)$. Finally shift $\tilde{A}_0$ and $\tilde{A}_1$ in order to get a continuous function with only two non-trivial intervals on which the function is zero resp. one. Call the resulting function $\hat{u}$ and these intervals A_0 and A_1. Taking everything together we deduce that $E(w, I) = E(\hat{u}, I) = E(u, I)$. Thus u and $\hat{u}$ are both minimizers of (5), and by construction $\hat{u}$ satisfies the additional properties (6). □

Lemma 2.7 (Boundary conditions)**.** *Let E be a symmetric admissible energy on $I = (a, b)$ and there exists a solution to the relaxed problem (5). Then there exists a solution u to (5) with $u(a) = 0$ and $u(b) = 1$.*

Proof. For simplicity take $I = (0, 1)$. Let w be a solution to (5) as given by Lemma 2.6. First, we show that we can construct a solution $\tilde{w}$ to (5) with $\tilde{w}(0) = 0$ or $\tilde{w}(0) = 1$.

If $w(0) = 0$ or $w(0) = 1$ we are done. Hence let us assume that $L := \min\{x \in I,\ w(x) \in \{0, 1\}\}$ is positive. In the following we assume that $w(L) = 0$ and construct a function $\tilde{w}$ with $\tilde{w}(0) = 0$. In the case where $w(L) = 1$ the same construction would give us $\tilde{w}(0) = 1$.

We distinguish two cases (see Fig. 4):

Case 1: $E(w, (0, L/2)) \geq E(w, (L/2, L))$.

In this case, we define

$$\tilde{w}(x) := \begin{cases} w(L - x), & x < L/2, \\ w(x), & x \geq L/2. \end{cases}$$

By the symmetry of E, we have $E(\tilde{w}, I) \leq E(w, I)$, moreover $\tilde{w}(0) = w(L) = 0$.

Case 2: $E(w, (0, L/2)) \leq E(w, (L/2, L))$.

In this case, the construction of $\tilde{w}$ is nearly as easy: we choose $x_0 > L$ such that $w(x_0) = w(L/2)$. (By the intermediate value theorem such an x_0 exists.) Then we define

$$\tilde{w}(x) := \begin{cases} w(x + L), & x \leq x_0 - L, \\ w(x_0 - x - L/2), & x_0 - L < x \leq x_0 - L/2, \\ w(x - x_0 + L/2), & x_0 - L/2x \leq x_0, \\ w(x), & x > x_0. \end{cases}$$

Again by the symmetry of E we have $E(\tilde{w}, I) \leq E(w, I)$. Moreover $\tilde{w}(0) = w(L) = 0$.

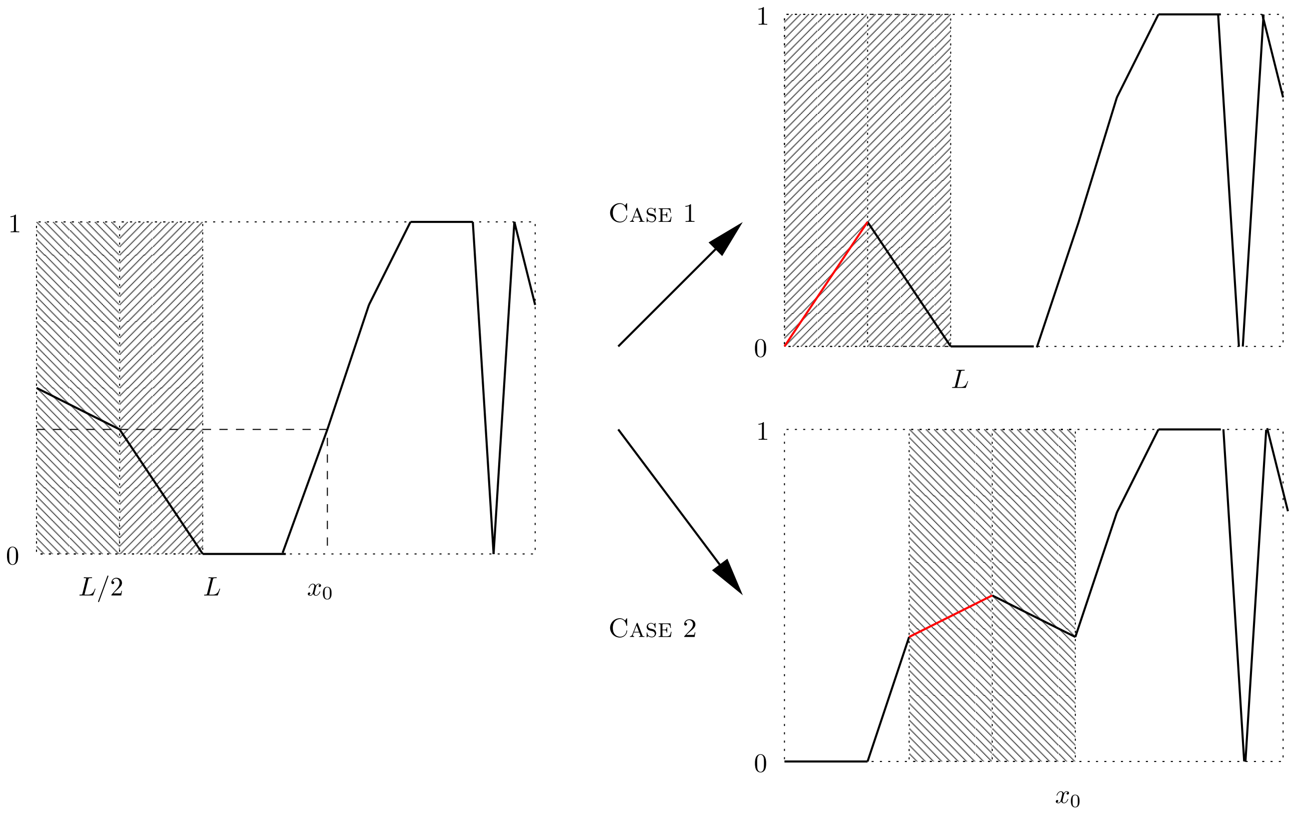

FIGURE 4. Starting from a solution w of the relaxed problem, we construct another solution $\tilde{w}$ with $\tilde{w}(0) = 0$. To this aim, we compare the energy of w on the shaded regions (left). Depending on which is larger, we perform one of the constructions illustrated on the right.

Applying the same construction at $x = 1$, we get a function v with $v(0), v(1) \in \{0, 1\}$. If $v(0) = 0$ and $v(1) = 1$ we just set $u := v$. If $v(0) = 1$ and $v(1) = 0$ we set $u := v(1 - \cdot)$. For the remaining cases we use the following construction (performed without loss of generality for the case $v(0) = v(1) = 1$).

Let $L_1 := \min\{x \in I,\ v(x) = 0\}$ and $L_2 := \max\{x < L_1,\ v(x) = 1\}$. Define $\tilde{v} := S((0, L_2), 1 - L_2)v$, then $\tilde{v}$ is continuous since $v(0) = v(L_2) = v(1) = 1$ and $E(\tilde{v}, I) = E(v, I)$. By a final application of the construction above (see Fig. 4) we can now get a function u with $u(0) = 0$, $u(1) = 1$ and $E(u, I) = E(w, I)$. □

Using the lemmata proved above we can prove the following existence result:

Theorem 2.8 (Existence for small γ)**.** *Let $I = (a, b)$ be an open interval, $1 < p \le +\infty$, and α, $\beta > 0$ with $\alpha + \beta < |I|$. Let E be a symmetric admissible energy, continuous with respect to the $W^{1,p}$-topology on all open sets in I. Then there exists a constant $\gamma_0 > 0$, such that the abstract variational problem (2) admits a solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.*

Proof. First, we see that by (ii) and (iii) the solvability of the problem depends only on γ, but not on α and β. Hence we can speak of a solution for a specific transition width γ instead of α and β. An illustration of our construction is given in Figure 5.

Now we consider the relaxed problem (5) for $|\{u = 0\}| \ge \alpha_0$ and $|\{u = 1\}| \ge \beta_0$, where we assume without loss of generality that $I = (0, 1)$. By Lemma 2.6 and Lemma 2.7 there exists a solution u to this problem with $(0, \alpha_0) \subset \{u = 0\}$, $(1 - \beta_0, 1) \subset \{u = 1\}$ and $T := u^{-1}((0, 1))$ satisfying $\bar{T} \subset [\alpha_0, 1 - \beta_0]$.

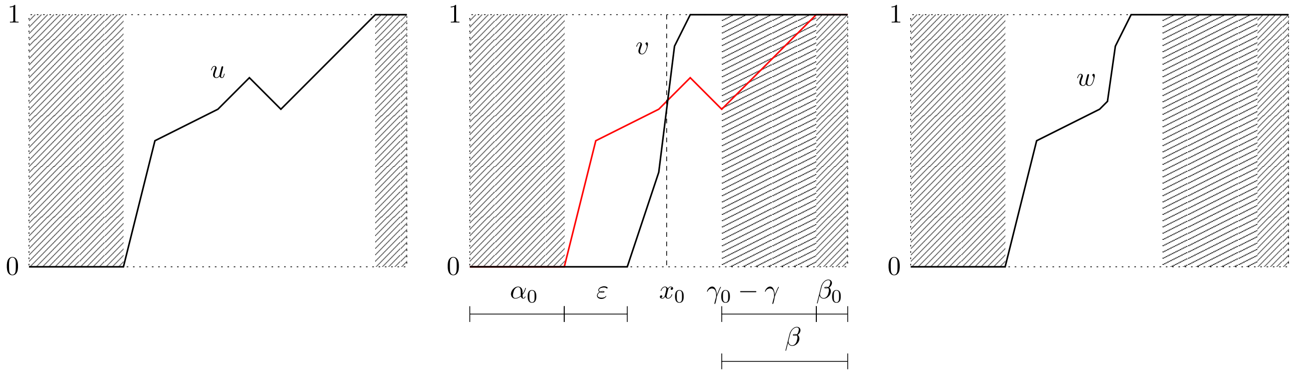

FIGURE 5. Left: a solution u for the relaxed problem (5) also solves the problem (2) for certain α_0 and β_0. Middle: the relaxed solution v for $\alpha := \alpha_0$ and $\beta = \beta_0 + (\gamma_0 - \gamma) > \beta_0$ intersects with u at some point x_0. Right: a solution w of the relaxed problem (5) satisfying the first constraint of (2) can be constructed by combining u and v. An analogous construction for the second constraint yields a solution for (2).

Now define $\gamma_0 := |T|$. Since u is continuous we have $\gamma_0 > 0$. Moreover, u is a solution of the non-relaxed problem (2) for the transition width γ_0. Now take $\gamma \in (0, \gamma_0)$, and let v be a solution of (5) with the constraints $|\{v = 0\}| = \alpha_0 =: \alpha$ and $|\{v = 1\}| = \beta_0 + (\gamma_0 - \gamma) =: \beta$. Our goal is to modify v to a function w such that $|\{x \in (\alpha, 1-\beta) : w(x) = 0\}| = 0$ and $|\{x \in (\alpha, 1-\beta) : w(x) = 1\}| = 0$. We prove the first statement, the latter one can be asserted in a similar way. By Lemma 2.6 and Lemma 2.7 we can assume that $v_{|(\alpha,\alpha+\varepsilon]} = 0$, $v|_{[1-\beta,1]} = 1$ and $v_{|(\alpha+\varepsilon,1-\beta)} > 0$ a.e. for some $\varepsilon \geq 0$. If $\varepsilon = 0$, the function v already satisfies $|\{x \in (\alpha, 1-\beta),\ v(x) = 0\}| = 0$, hence we have only to consider the case where $\varepsilon > 0$. Since $u(\alpha+\varepsilon) \geq v(\alpha+\varepsilon) = 0$ and $u(1-\beta) \leq v(1-\beta) = 1$, by the intermediate value theorem for continuous functions there exists a point $x_0 \in [\alpha+\varepsilon, 1-\beta]$ such that $u(x_0) = v(x_0)$. Now define the function

$$w(x) := \begin{cases} u(x), & x < x_0, \\ v(x), & x \geq x_0. \end{cases}$$

Since u is a minimizer of (5) with transition width γ_0 we have in particular $E(u, (\alpha, 1-\beta_0)) \leq E(w, (\alpha, 1-\beta_0))$ and $E(u, (\alpha, x_0)) + E(u, (x_0, 1-\beta_0)) \leq E(v, (\alpha, x_0)) + E(u, (x_0, 1-\beta_0))$. Since $u|_{(\alpha,x_0)} = w|_{(\alpha,x_0)}$, we conclude that $E(w, (\alpha, x_0)) = E(u, (\alpha, x_0)) \leq E(v, (\alpha, x_0))$. Hence $E(w, (\alpha, 1-\beta_0)) \leq E(v, (\alpha, 1-\beta_0))$ and thus w is a solution of (5) with $|\{x \in (\alpha, 1-\beta) : w(x) = 0\}| = 0$. □

The continuity condition of Lemma 2.6 and Theorem 2.8 is only used if a minimizer u of the relaxed problem may have infinitely many transition layers $T_j = (l, r)$ with $u(l) = u(r) \in \{0, 1\}$. The following corollary catches a situation where this can be excluded *a priori*:

Corollary 2.9. *Let $I = (a, b)$ be an open interval, $1 < p \leq +\infty$, and $\alpha, \beta > 0$ with $\alpha + \beta < |I|$. Let E be a symmetric admissible energy such that for all $T \in \mathcal{I}$*

$$E(0, T) = \min\{E(u, T),\ u|_{\partial T} = 0\},$$
$$E(1, T) = \min\{E(u, T),\ u|_{\partial T} = 1\}.$$

Then there exists a constant $\gamma_0 > 0$, such that the abstract variational problem (2) admits a solution whenever $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

Proof. We follow the proof of Lemma 2.6 and Theorem 2.8. The transition layers of the form T_j^{00} and T_j^{11} can be omitted, by the following argument: if *e.g.* $u|_{(x_0,x_1)} \in (0, 1)$ and $u(x_0) = u(x_1) = 0$, we have

$E(u,(x_0,x_1)) \geq E(0,(x_0,x_1))$ and we can replace u by 0 on (x_0,x_1). Hence the proofs work without using the continuity of E. $\square$

The following theorem gives existence for arbitrarily large transition layers, but only for a special class of energies:

Theorem 2.10 (Existence for special energies)**.** *Let $I=(a,b)$ be an open interval, $1<p\leq+\infty$, and let α, $\beta>0$ be such that $\alpha+\beta<|I|$. Let E be an admissible energy continuous with respect to the strong $W^{1,p}$-topology on all open set in I. Suppose that there exists a constant function $\lambda\in(0,1)$ such that*

$$E(\lambda,T)=\min_{\mu\in[0,1]}E(\mu,T) \tag{7}$$

for all nontrivial $T\in\mathcal{I}$. Then there exists a solution to the abstract variational problem (2).

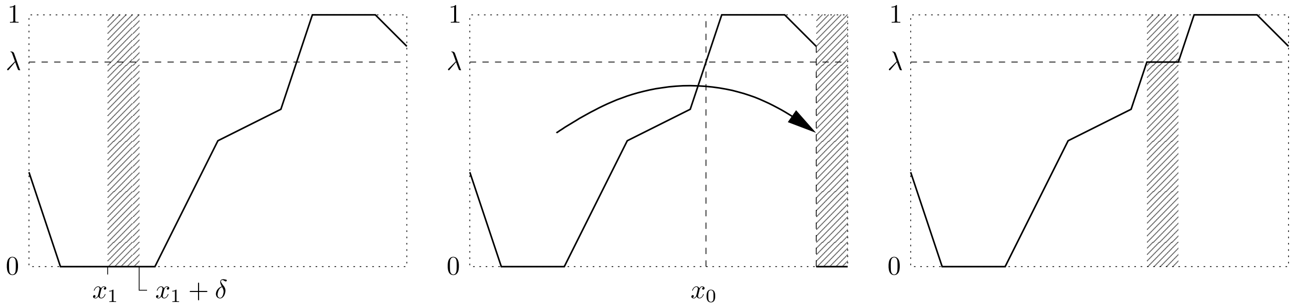

FIGURE 6. Left: assume the solution of the relaxed problem is zero on the interval $[x_1,x_1+\delta]$. Middle: shift this interval to the right. Right: insert a constant piece with the optimal value λ such that the resulting function is continuous.

Proof. The following construction is illustrated in Figure 6. Let v be a solution of the relaxed problem (5) for I, α and β as given above satisfying for some $\delta,\varepsilon\geq 0$ the constraints $|\{v=0\}|=\alpha+\delta$, $|\{v=1\}|=\alpha+\varepsilon$. By Lemma 2.6 such a v exists and we can assume that there exists an $x_1\in I$ such that $v|_{[x_1,x_1+\delta]}=0$. Let $\tilde v:=S((x_1,x_1+\delta),1-\delta)v$. Let x_0 be such that $\tilde v(x_0)=\lambda$. Then define

$$u(x):=\begin{cases}\tilde v(x), & x<x_0,\\ \lambda, & x_0\leq x\leq x_0+\delta,\\ \tilde v(x-\delta), & x>x_0+\delta.\end{cases}$$

By applying Lemma 2.5 and Condition (7) we have $E(u,I)\leq E(v,I)$, moreover $|\{u=0\}|=|\{v=0\}|-\delta=\alpha$. In the same way we can take care of the second constraint, and the so defined u solves the original problem (2). $\square$

From Theorem 2.8 we have immediately the following corollary which was proved in Theorem 2.1 (H2) of [3] using an ODE method:

Corollary 2.11. *Let θ be a continuous function with* $\operatorname{argmin}\theta\subset[0,1]$ *and* $\min\theta=0$, *let I be an open interval, and α, $\beta>0$ with $\alpha+\beta<|I|$. Then there exists a constant $\gamma_0>0$ such that the problem*

$$\begin{cases}\textit{Minimize } E(u):=\displaystyle\int_I \frac{1}{2}|u'(x)|^2+\theta(u(x))\,\mathrm{d}x,\\ |\{u=0\}|=\alpha,\ |\{u=1\}|=\beta,\end{cases}$$

admits a minimizer in $W^{1,2}(I,\mathbb{R})$, provided that $\gamma:=|I|-(\alpha+\beta)<\gamma_0$.

Proof. For $T \in \mathcal{I}$ take

$$E(u,T) := \int_T \frac{1}{2}|u'(x)|^2 + \theta(u(x))\ \mathrm{d}x.$$

Let u be a minimizer for $E(\cdot,\cdot)$ obtained by Theorem 2.8. The only condition that has to be checked is that no function $v \in W^{1,2}(I,\mathbb{R}) \setminus W^{1,2}(I,[0,1])$ has a lower energy then u, but this can be excluded by a simple construction, cutting out the set I_0 where $v(x) \notin [0,1]$ and inserting an interval of length $|I_0|$ where we define the function as a constant v_0 such that $\theta(v_0) = 0$. This gives a new function w with values in $[0,1]$ and using the assumption $\operatorname{argmin}\theta \subset [0,1]$ one easily checks that

$$\begin{aligned}
E(v) &= \int_I \frac{1}{2}|v'(x)|^2 + \theta(v(x))\,\mathrm{d}x \\
&= \int_{I\setminus I_0} \frac{1}{2}|v'(x)|^2 + \theta(v(x))\,\mathrm{d}x + \int_{I_0} \frac{1}{2}|v'(x)|^2 + \theta(v(x))\,\mathrm{d}x \\
&> \int_{I\setminus I_0} \frac{1}{2}|v'(x)|^2 + \theta(v(x))\,\mathrm{d}x \ = \ \int_I \frac{1}{2}|w'(x)|^2 + \theta(w(x))\,\mathrm{d}x \\
&= E(w).
\end{aligned}$$

This gives a contradiction to our assumption, and hence such a v cannot exist. □

Similarly, the following result is an immediate consequence of Theorem 2.10 (compare Th. 2.1 (H1) in [3]):

Corollary 2.12. *Let $1 < p < +\infty$, and let $f \in \mathcal{C}^1([0,1]\times\mathbb{R},\mathbb{R})$ satisfy $f(a,b) \geq C|b|^p$ for some constant $C > 0$. Furthermore assume that f is convex in the second variable, and that there exists a constant $\lambda \in (0,1)$ such that $f(\lambda,0) = 0$. Then the problem*

$$\begin{cases} \textit{Minimize } E(u) := \displaystyle\int_I f(u(x),u'(x))\,\mathrm{d}x, \\ |\{u=0\}| = \alpha,\ |\{u=1\}| = \beta, \end{cases}$$

admits a minimizer in $W^{1,p}(I)$.

Admissible energies are not neccessarily of (Lebesgue) integral form. This is illustrated by the following example (suggested to me by Massimiliano Morini):

Remark 2.13. Let $F\colon [0,1] \to \mathbb{R}$ be a continuous function, then

$$E(u,(x_0,x_1)) := F(u(x_0)) - F(u(x_1))$$

is an admissible energy on $W^{1,2}$ which is continuous with respect to the strong $W^{1,2}$-norm.

We can even have symmetric admissible energies which are not of integral form:

Remark 2.14. Let $F\colon [0,1] \to \mathbb{R}$ be a continuous function and define $v(x) := u'(x)$ whenever $u'(x)$ exists, and $v(x) := 0$ elsewhere, then

$$E(u,(x_0,x_1)) := F(v(x_0)) - F(v(x_1))$$

is a symmetric admissible energy on $W^{1,p}$ for $p > 2$.

In the next section we will apply Theorem 2.8 to nonconvex variational problems with volume constraints.

3. Applications to nonconvex problems

The main result of this section is:

Theorem 3.1 (Existence for nonconvex problems)**.** *Let $1 < p < +\infty$, and let $f \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R})$ be a nonnegative function with $f(a,b) \geq C|b|^p$ for a constant $C > 0$. Moreover, assume that*

(a) $f(0,0) = 0$*;*
(b) $\operatorname{sgn} \partial_a f(a, b_1) = \operatorname{sgn} \partial_a f(a, b_2)$ *for all* $a \in [0,1]$ *and all* $b_1, b_2 \in \mathbb{R}$*;*
(c) $f(a,b) = f(a,-b)$ *for all* $a \in [0,1]$ *and all* $b \in \mathbb{R}$*.*

Then there exists $\gamma_0 > 0$ such that the abstract variational problem (2) with the energy

$$E(u,T) := \int_T f(u(x), u'(x))\,\mathrm{d}x$$

admits a $W^{1,p}$-solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

The main purpose of this theorem is to illustrate a concrete application of the abstract theory, hence we are not attempting to give the most general result possible.

Proof. To apply Theorem 2.8 we have to prove that E is symmetric, continuous and satisfies the conditions (i–iii). E is obviously continuous and symmetric and satisfies (ii) and (iii), hence it remains to assert (i). Here we apply Remark 2.4, so we only have to check the existence of a solution to the transition problem. For this purpose we consider the convexified transition problem (4), where we replace f by f^{**} (the convexification of f with respect to the second variable). Due to the convexity of f^{**} and the coercivity condition that we have assumed, standard results from the calculus of variation guarantee the existence of a solution $u \in W^{1,p}(I)$. If $f(u(x), u'(x)) = f^{**}(u(x), u'(x))$ for a.e. $x \in I$, then the function u also solves the transition problem for f. If not, let $M \subset I$ denote the set of points x such that $f(u(x), u'(x)) \neq f^{**}(u(x), u'(x))$. M is open and hence a union of disjoint open intervals. We deduce from the coercivity bound $f(a,b) \geq C|b|^p$ that for all $y \in \mathbb{R}$ there exists a minimal $|y_1|$ with $y_1/y \geq 1$ such that $f(\cdot, y_1) = f^{**}(\cdot, y_1)$ and a maximal $|y_0|$ with $y_0/y \leq 1$ such that $f(\cdot, y_0) = f^{**}(\cdot, y_0)$. By this and Condition (b) we can decompose M (up to a set of measure zero) into open intervals $M_i = (m_i, n_i)$ such that on each M_i the function $\operatorname{sgn} \partial_u f(u, u')$ is constant and $u'(x) \in (y_0, y_1)$ for all $x \in M_i$ such that

$$f^{**}(u(x), u'(x)) = \frac{y_1 - u'(x)}{y_1 - y_0} f(u(x), y_0) + \frac{u'(x) - y_0}{y_1 - y_0} f(u(x), y_1)$$

and $f(\cdot, u'(y_i)) = f^{**}(\cdot, u'(y_i))$ for $i = 0, 1$.

Consider now the case that $\partial_u f(u(x), u'(x)) < 0$ on M_i. Then we define the affine interpolation

$$v_i(x) := \begin{cases} u(x) & \text{for } x \notin M_i, \\ u(m_i) + y_0(x - m_i) & \text{for } m_i < x \leq \xi, \\ u(n_i) - y_1(x - n_i) & \text{for } \xi < x < n_i, \end{cases}$$

where

$$\xi := \frac{u(n_i) - u(m_i) - (n_i - m_i)y_1}{y_0 - y_1}.$$

Now $\int_{M_i} f^{**}(v_i'(x), v_i(x))\,\mathrm{d}x \leq \int_{M_i} f^{**}(u'(x), u(x))\,\mathrm{d}x$, and for every $x \in M_i$ we have by construction that $f^{**}(v_i(x), v_i'(x)) = f(v_i(x), v_i'(x))$. An analogous argument applies to the cases where $\partial_u f(u(x), u'(x)) \geq 0$. The constraint $v_i \in [0,1]$ is ensured by the Condition (a).

Define $v|_{\bar{M}_i} := v_i$ to get a solution to the nonconvex problem (4). Thus Condition (i) is valid. It remains to apply Theorem 2.8 to get the existence result. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $1 < p < +\infty$, let $\theta \in \mathcal{C}^1([0,1], \mathbb{R}_+)$ and $\phi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}_+)$. Assume that ϕ is symmetric and $\phi(b) \geq C|b|^p$ for a constant $C > 0$, and that $\phi(0) = 0$. Then there exists a $\gamma_0 > 0$ such that*

$$\begin{cases} \textit{Minimize } E(u) := \int_I \phi(u') + \theta(u)\,\mathrm{d}x, \\ |\{u = 0\}| = \alpha,\ |\{u = 1\}| = \beta, \end{cases}$$

admits a solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

Without a coercivity condition, existence for the transition problem fails in general. A typical example is given by $f(u, u') := \exp(-|u'|^2)$. However, we have the following result, which can be derived from the proof of Theorem 3.1:

Corollary 3.3. *Let $1 < p \leq +\infty$, and let $f \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R})$ be a nonnegative function, symmetric in the second variable. Moreover assume that*

(a) $f(0,0) = 0$*;*
(b) $\operatorname{sgn} \partial_u f(a, b_1) = \operatorname{sgn} \partial_u f(a, b_2)$ *for all $a \in [0,1]$ and all $b_1, b_2 \in \mathbb{R}$;*
(c) *the transition problem (4) for f^{**}, the convexification of f with respect to the second variable, admits a $W^{1,p}$-solution;*
(d) *for all $y \in \mathbb{R}$ there exists b' with $b'/y \geq 1$ such that $\phi(b') = \phi^{**}(b')$.*

Then there exists a $\gamma_0 > 0$ such that the abstract variational problem (2) with the energy $E(u, I_0) := \int_{I_0} f(u(x), u'(x))\,\mathrm{d}x$ admits a $W^{1,p}$-solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

As a trivial example of an energy density which satisfies the conditions of Corollary 3.3, but violates the coercivity condition of Theorem 3.1, take $f(a, b) := \sin^2 b$.

4. Higher dimensional problems

It would seem natural to extend the one-dimensional existence results above, and in particular Theorem 2.10, to higher dimensional problems. However, without specific assumptions, *e.g.* on the smoothness of the boundary of the domain $\Omega \subset \mathbb{R}^n$, an extension of Theorem 2.10 to the higher dimensional case fails. We demonstrate this with the following counterexample:

Example 4.1. Let $p > 2$. There exist a bounded domain $\Omega \subset \mathbb{R}^2$ and a smooth, positive, convex function ψ with $C_1|Y|^p \leq \psi(Y) \leq C_2|Y|^p$ for some $C_1, C_2 > 0$ such that there exists **no** $\gamma_0 > 0$ with the property that the volume constraint problem

$$\begin{cases} \text{Minimize } E(u) := \int_\Omega \psi(|\nabla u(x)|) + |u(x)|\,\mathrm{d}x, \\ |\{u = 0\}| = \alpha,\ |\{u = 1\}| = \beta \end{cases} \tag{8}$$

admits a $W^{1,p}$-solution for all α, β with $\gamma := |\Omega| - (\alpha + \beta) \leq \gamma_0$.

Outline of the proof: first we define ψ. Let $\psi|_{(-1,1)} := \frac{1}{2}|\cdot|^2$, $\psi|_{\mathbb{R}\setminus(-2,2)} := |\cdot|^p$ and interpolate smoothly, such that ψ is convex. We consider now an auxiliary problem: We want to find sequences α_j, β_j with $\alpha_j + \beta_j < |\Omega|$ and $\gamma_j := |\Omega| - (\alpha_j + \beta_j) \to 0$, such that the volume constraint problems

$$\begin{cases} \text{Minimize } E(u) := \int_\Omega \frac{1}{2}|\nabla u(x)|^2 + |u(x)|\,\mathrm{d}x, \\ |\{u = 0\}| = \alpha_j,\ |\{u = 1\}| = \beta_j \end{cases} \tag{9}$$

do not admit a $W^{1,p}$-solution.

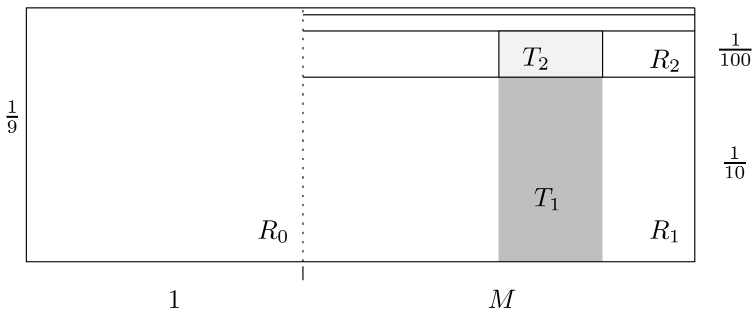

FIGURE 7. The set Ω and the transition layers T_j.

We use Proposition 2.6 of [3] which asserts that the one-dimensional volume constrained problem (2) with $E(u, I) := \int_I \frac{1}{2}|u'|^2 + |u| \, \mathrm{d}x$ does not admit a solution for $\gamma > \sqrt{2}$. Define $\Omega := R_0 \cup R \subset \mathbb{R}^2$ (see Fig. 7), where

$$R_0 := (-1, 0) \times (0, 1/9), \quad R := \bigcup_{i=1}^{\infty} R_i, \quad R_i := [0, M) \times (10^{1-i}, 10^{-i}),$$

and $M > 0$ will be chosen later. As sequences α_j and β_j we set

$$\alpha_j := \frac{1+M}{9} - \frac{1}{4} 10^{-j} M - 2 \times 10^{-j}, \quad \beta_j := \frac{1}{4} 10^{-j} M,$$

hence $\gamma_j = 2 \times 10^{-j}$. We denote the $W^{1,p}$-solution of the relaxed problem

$$\begin{cases} \text{Minimize } E(u) := \displaystyle\int_\Omega \frac{1}{2} |\nabla u(x)|^2 + |u(x)| \, \mathrm{d}x, \\ |\{u = 0\}| \geq \alpha_j, \ |\{u = 1\}| \geq \beta_j \end{cases}$$

by u_j.

We claim that the transition layer $T_j := \{x \in \Omega : u_j(x) \in (0, 1)\}$ of the minimizer u_j has the form $T_j = \{(x_1, x_2) \in R_j : 3M/4 - \sqrt{2} < x_1 < 3M/4\}$. Since $\gamma_j > 10^{-j}\sqrt{2} = |T_j|$, we conclude that u_j cannot be a solution of problem (9). Since u_j can be approximated by functions satisfying the volume constraints with energy arbitrarily close to the energy of u_j, this proves the non-solvability of the auxiliary problem (9).

Since $W^{1,p}(\Omega) \subset \mathcal{C}(\Omega)$, the transition layer is open. The main idea is now to observe that the energy of a transition layer T can be estimated from below by $Cl(T)$, where $l(T)$ is the "length" of the transition layer and $C > 0$ is a constant. More precisely we define for every line g in Ω

$$l(g) := |\{x \in g : n_x \cap \{u = 0\} \neq \emptyset, \ n_x \cap \{u = 1\} \neq \emptyset, \\ \text{where } n_x \text{ is the line through } x \text{ orthogonal to } g\}\cdot$$

We denote the energy of the solution of the one-dimensional transition problem on $(0, t)$ by $\phi(t)$. By Proposition 2.6 in [3] we know that $\phi(t) \geq \phi(\sqrt{2})$. Hence we conclude that

$$E(T_j) := \int_{T_j} \frac{1}{2} |\nabla u|^2 + |u| \, \mathrm{d}x \geq \int_0^{l(g)} \phi\left(\sqrt{2}\right) = l(g) \phi\left(\sqrt{2}\right).$$

Now define

$$l(T_j) := \sup_g l(g).$$

If T_j does not connect the upper and lower boundary in some R_i, $i \leq j$, then $l(T_j) \geq C_0\, 10^{-j} M$, and hence the energy is bounded from below by $E_1 := C_0 \phi(\sqrt{2})\, 10^{-j} M$. If T_j connects the upper and lower boundary in some R_i, $i < j$, then its energy can be estimated from below by $E_2 := 10^{-i}\phi(\sqrt{2})$, this can be shown by means of an argument similar to the one given below.

If T_j is as in our claim and $u_j(x_1, x_2) = v(x_1 - 3M/4 - 2)$ on T_j, where v is the solution of the one dimensional transition problem on $(0, \sqrt{2})$, then the energy on the transition layer is $10^{-j}\phi(\sqrt{2})$. This is smaller than E_1 and E_2 for M sufficiently large.

Now we use the estimate

$$\int_{T_j} \frac{1}{2}|\nabla u_j|^2 + |u_j|\,\mathrm{d}x \geq \int_l^h \int_0^M \frac{1}{2}|\partial_x u_j|^2 + |u_j|\,\mathrm{d}x\,\mathrm{d}y$$

with $l := \sum_{i=1}^{j-1} 10^{-i}$, $h := \sum_{i=1}^{j} 10^{-i}$. We can estimate further where we denote $x_1(y) := \inf\{x \in (0, M) : (x, y) \in T_j\}$, $x_2(y) := \sup\{x \in (0, M) : (x, y) \in T_j\}$ and get

$$\int_l^h \int_0^M \frac{1}{2}|\partial_x u_j|^2 + |u_j|\,\mathrm{d}x\,\mathrm{d}y \geq \int_l^h \int_{x_1(y)}^{x_2(y)} \phi(x_2(y) - x_1(y))\,\mathrm{d}x\,\mathrm{d}y.$$

Now, since ϕ is convex (see [3]), applying Jensen's inequality we deduce that a straight transition layer

$$\begin{aligned} T_j &= (3M/4 - \sqrt{2}, 3M/4) \times (l, h) \\ &= \{(x_1, x_2) \in R_j : 3M/4 - \sqrt{2} < x_1 < 3M/4\} \end{aligned}$$

is optimal. Since $|T_j| = 10^{-j}\sqrt{2} < \gamma_j$, we have non-existence for the auxiliary problem (9).

Now let w_j be a solution for the relaxed version of (8) for α_j, β_j, then since $\psi(w_j) \geq \frac{1}{2}|w|^2$ and $\psi(u_j) = \frac{1}{2}|u_j|^2$, we have $E(w_j) \geq \int \frac{1}{2}|\nabla w_j|^2 + |w_j| \geq \int \frac{1}{2}|\nabla u_j|^2 + |u_j|$. And since $|\nabla u_j| \leq 1$ (see [3]), we get $E(w_j) \geq E(u_j)$, where equality holds only if $u_j = w_j$. Hence we have proved non-existence for the original problem. ☐

In the last example we made crucial use of the non-smoothness of the boundary of Ω. In fact it seems natural to propose the following conjecture:

Conjecture 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. Suppose that $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is positive, continuous, strictly convex in the second variable and that there exists a constant $C > 0$ such that $f(a, b) \geq C|b|^p$ for all $a \in [0, 1]$. Then there exists a $\gamma_0 > 0$ with the property that the volume constraint problem*

$$\begin{cases} \text{Minimize } E(u) := \int_\Omega f(u(x), |\nabla u(x)|)\,\mathrm{d}x, \\ |\{u = 0\}| = \alpha,\ |\{u = 1\}| = \beta \end{cases}$$

admits a $W^{1,p}$-solution for all α, β with $\gamma := |\Omega| - (\alpha + \beta) \leq \gamma_0$.

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