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ENTROPY AND COMPLEXITY OF A PATH IN SUB-RIEMANNIAN GEOMETRY

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Abstract. We characterize the geometry of a path in a sub-Riemannian manifold using two metric invariants, the entropy and the complexity. The entropy of a subset A of a metric space is the minimum number of balls of a given radius ε needed to cover A. It allows one to compute the Hausdorff dimension in some cases and to bound it from above in general. We define the complexity of a path in a sub-Riemannian manifold as the infimum of the lengths of all trajectories contained in an ε-neighborhood of the path, having the same extremities as the path. The concept of complexity for paths was first developed to model the algorithmic complexity of the nonholonomic motion planning problem in robotics. In this paper, our aim is to estimate the entropy, Hausdorff dimension and complexity for a path in a general sub-Riemannian manifold. We construct first a norm $\|\cdot\|_\varepsilon$ on the tangent space that depends on a parameter $\varepsilon > 0$. Our main result states then that the entropy of a path is equivalent to the integral of this ε-norm along the path. As a corollary we obtain upper and lower bounds for the Hausdorff dimension of a path. Our second main result is that complexity and entropy are equivalent for generic paths. We give also a computable sufficient condition on the path for this equivalence to happen.

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1. Introduction

In a sub-Riemannian geometry, submanifolds may have a Hausdorff dimension greater than their topological dimension. What values can the Hausdorff dimension have? More generally, what are the metric properties of sub-Riemannian submanifolds? In this paper we will answer these questions for one-dimensional submanifolds, or paths, using two geometric invariants, the entropy and the complexity.

The entropy of a subset A of a metric space is the minimum number of balls of a given radius ε needed to cover A. This value indicates how well one can approximate A by a finite set. The use of entropy in sub-Riemannian geometry was suggested by Gromov [7] (p. 277). It allows one to compute the Hausdorff dimension in some cases and to bound it from above in general.

We define the complexity of a path in a sub-Riemannian manifold as the infimum of the lengths of all trajectories contained in an ε-neighborhood of the path, having the same extremities as the path. Trajectories here refer to the control system associated with the sub-Riemannian manifold. The concept of complexity for paths was first developed for the nonholonomic motion planning problem in robotics (see Bellaïche *et al.* [2]), to

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model the algorithmic complexity of the problem. This application is described in [10], which uses the results of this paper. Gromov also introduced a concept equivalent to the complexity in [7] (p. 278). As with the entropy, this concept is based on an approximation by finite sets, there called ε-chains.

The Hausdorff dimension of paths, and more generally of submanifolds, is known for the case where the sub-Riemannian manifold is equiregular (Gromov [7]). However neither the entropy nore the Hausdorff dimension have been estimated for the general case.

In this paper, our aim is to estimate the entropy, Hausdorff dimension and complexity for the general case. On a sub-Riemannian manifold, we construct a norm $\|\cdot\|_\varepsilon$ on the tangent space that depends on a parameter $\varepsilon > 0$ (see Sect. 3.1). Our first main result, Theorem 3.10, is an estimate of the entropy in terms of this ε-norm.

Theorem. *The entropy $e\big(q(\cdot),\varepsilon\big)$ of a path $q(t)$, $t \in [0,T]$, satisfies*

$$e\big(q(\cdot),\varepsilon\big) \asymp \int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t,$$

where $\asymp$ denotes the equivalence up to multiplicative constants (uniform with respect to ε, for ε small enough).

As a corollary we obtain upper and lower bounds for the Hausdorff dimension of a path. Our second main result links complexity to entropy.

Theorem. *For a generic path $q(t)$, $t \in [0,T]$, the complexity $\sigma\big(q(\cdot),\varepsilon\big)$ satisfies*

$$\sigma\big(q(\cdot),\varepsilon\big) \asymp e\big(q(\cdot),\varepsilon\big) \asymp \int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t.$$

We give also a computable sufficient condition on the path for this equivalence to happen. This condition precises the meaning of generic here.

The organization of this paper is the following. In Section 2, we give some recalls of sub-Riemannian geometry and we define the entropy and the complexity of a path. We show there how the complexity of paths allows to compute the algorithmic complexity of nonholonomic motion planning. Section 3 contains the definition of the ε-norm and our main theorems which are illustrated by several examples. The proofs are postponed in Section 4 for a particular case and in Section 5 for the general case.

2. Paths in sub-Riemannian geometry

2.1. Sub-Riemannian manifolds

We recall here some definitions and basic results of sub-Riemannian geometry. More general presentations can be found in Bellaïche [1] (the main reference for this section) or in Kupka [13].

Let M be a real analytic manifold and $X_1, \ldots, X_m$ analytic vector fields on M. We denote by (Σ) the associated control system

$$\dot{q} = \sum_{i=1}^{m} u_i X_i(q), \quad q \in M, \tag{Σ}$$

where the control function $u(t) = (u_1(t), \ldots, u_m(t))$ takes values in $\mathbb{R}^m$. The choice of a measurable function $u(t)$, $t \in [0,\tau]$, and of an initial point $p \in M$ defines a trajectory γ of (Σ). The *length* of γ is

$$\operatorname{length}(\gamma) = \int_0^\tau \|\dot{\gamma}(t)\|_{\gamma(t)} \mathrm{d}t,$$

where for $q \in M$ and $v \in T_qM$, we set

$$\|v\|_q = \inf \left\{ \sqrt{u_1^2 + \cdots + u_m^2} \mid \sum_{i=1}^{m} u_i X_i(q) = v \right\}. \tag{1}$$

The *sub-Riemannian distance* is defined by $d(p,q) = \inf \operatorname{length}(\gamma)$, where the infimum is taken on all the trajectories of (Σ) joining p to q. The manifold M endowed with the distance d, denoted (M,d), is called the *sub-Riemannian manifold* attached to $X_1, \ldots, X_m$.

Let $\mathcal{L}^1(X_1, \ldots, X_m)$ be the set of linear combinations, with real coefficients, of the vector fields $X_1, \ldots, X_m$. We define recursively $\mathcal{L}^s = \mathcal{L}^s(X_1, \ldots, X_m)$ by setting $\mathcal{L}^s = \mathcal{L}^{s-1} + [\mathcal{L}^{s-1}, \mathcal{L}^1]$, for $s = 2, 3, \ldots$ Due to Jacobi identity $\mathcal{L}^s$ is the set of linear combinations of all commutators of $X_1, \ldots, X_m$ with a length $\leq s$. The union $\mathcal{L}$ of all $\mathcal{L}^s$ is a Lie sub-algebra of the Lie algebra of vector fields on M. It is generated by the commutators $[[X_{i_1}, X_{i_2}], \ldots, X_{i_k}]$. Such a commutator is denoted $[X_I]$, where I is the multi-index $I = (i_1, \ldots, i_k)$ and its length is $|I| = k$.

For $p \in M$, let $L^s(p)$ be the subspace of T_pM which consists of the values $X(p)$ taken, at the point p, by the vector fields X belonging to $\mathcal{L}^s$. The following condition on $X_1, \ldots, X_m$:

$$\bigcup_{s \geq 1} L^s(p) = T_pM \quad \text{at every point } p \in M,$$

is called the *rank condition*. When this condition is satisfied, at each point $p \in M$ there is a smallest integer $r = r(p)$ such that $L^{r(p)}(p) = T_pM$. This integer is called the *degree of nonholonomy* at p. For each point $p \in M$, there is an increasing sequence of dimensions, called the *growth vector*,

$$1 \leq \dim L^1(p) \leq \cdots \leq \dim L^s(p) \leq \cdots \leq \dim L^{r(p)}(p) = n.$$

We say that p is a *regular point for* (M,d) if the growth vector remains constant in some neighborhood of p. Otherwise we say that p is a *singular point for* (M,d).

Finally, one says that the system (Σ) is *controllable* if the sub-Riemannian distance between any two points of M is finite.

Chow's theorem (Chow [5])**.** *If M is connected and if $X_1, \ldots, X_m$ satisfy the rank condition, then the system (Σ) is controllable.*

Notice that the conditions are also necessary here, since $X_1, \ldots, X_m$ are analytic vector fields (Nagano [17], Sussmann [19]).

In the sequel we always assume that the rank condition is satisfied. The sub-Riemannian distance d is then continuous and the topology defined by d is the original topology of M.

2.2. Entropy and Hausdorff dimension

Let (X,d) be a metric space and $A \subset X$ a bounded subset.

Definition 2.1. The *metric entropy* of A is the function $\varepsilon \mapsto e(A,\varepsilon)$, defined for $\varepsilon > 0$, where $e(A,\varepsilon)$ is the minimal number of closed balls of radius ε in X needed to cover A.

Notice that originally, as conceived by Kolmogorov [12], the ε-entropy is defined as the logarithm $\log_2 e(A,\varepsilon)$. It represents the amount of information we need to describe a point in A with the accuracy ε or to digitally memorize A with this accuracy.

Definition 2.2. The *entropy dimension* of A, denoted $\dim_e A$, is the greatest lower bound of β for which $e(A,\varepsilon) \leq (1/\varepsilon)^\beta$ for ε small enough, that is

$$\dim_e A = \overline{\lim_{\varepsilon\to 0}} \frac{\log e(A,\varepsilon)}{\log(\frac{1}{\varepsilon})}\cdot$$

The entropy dimension characterizes the asymptotic behavior of $e(A,\varepsilon)$ as ε tends to 0 and reflects the geometry of A in X.

A maybe more usual characterization of the geometry of a space uses the Hausdorff dimension (and measure), introduced by Hurewicz and Wallman [8].

For $\beta \geq 0$, the *β-dimensional Hausdorff measure*, denoted $m_\beta(A)$, is defined as $\lim_{\varepsilon\to 0} m_\beta(A,\varepsilon)$, where

$$m_\beta(A,\varepsilon) = \inf\left\{\sum_{i=1}^{\infty} r_i^\beta,\ r_i \leq \varepsilon \text{ radius of balls } B_i, \text{ and } A \subset \bigcup_{i=1}^{\infty} B_i\right\}\cdot$$

For a given A, $m_\beta(A)$ is a decreasing function of β, infinite when β is less than a certain value, and zero when β is greater than this value.

Definition 2.3. The *Hausdorff dimension* of A is the real number:

$$\dim_{\mathrm{H}} A = \sup\{\beta \mid m_\beta(A) = \infty\} = \inf\{\beta \mid m_\beta(A) = 0\}\cdot$$

Entropy and Hausdorff dimension are related by the following properties (Comte and Yomdin [6]).

Proposition 2.4. *(i) For all $\beta \geq 0$, $m_\beta(A) \leq \lim_{\varepsilon\to 0} \varepsilon^\beta e(A,\varepsilon)$, which implies*

$$\dim_{\mathrm{H}} A \leq \dim_e A.$$

(ii) If $e(A,\varepsilon) \sim \varepsilon^{-\beta_0}$, then $\dim_{\mathrm{H}} A = \dim_e A = \beta_0$.

In the case where the metric space is a sub-Riemannian manifold (M,d), some results already exist for the Hausdorff dimension and the entropy of a submanifold of M.

When (M,d) is equiregular (every point in M is regular), the Hausdorff dimension of the manifold is given by

$$\dim_{\mathrm{H}} M = \sum_{s\geq 1} s\big(\dim L^s(p) - \dim L^{s-1}(p)\big)$$

for any point $p \in M$ (see Mitchell [16] and Vershik and Gershkovich [20]). Gromov [7] (p. 104) extends this result to Hausdorff dimension of a submanifold N, again in an equiregular manifold:

$$\dim_{\mathrm{H}} N = \max_{q\in N}\left\{\sum_{s\geq 1} s\big(\dim L_N^s(q) - \dim L_N^{s-1}(q)\big)\right\}, \tag{2}$$

where $L_N^s(q)$ is the linear subspace $L^s(q) \cap T_q N$.

On the other hand, Bellaïche [1] computes the entropy and the Hausdorff dimension of a non equiregular sub-Riemannian manifold, the Grušin plane. This last example is interesting because it does not fit in the case (ii) of Proposition 2.4. It follows that the asymptotic behavior of entropy cannot be deduced only from the entropy (or Hausdorff) dimension.

2.3. Complexity

Let (M,d) be the sub-Riemannian manifold attached to a system (Σ). We call *path* a one-dimensional analytic submanifold of M which is diffeomorphic to a closed interval in $\mathbb{R}$. A *parameterization* of a path $\mathcal{C}$ is an analytic diffeomorphism $q:[0,T]\to\mathcal{C}$, $t\mapsto q(t)$.

Submanifold means here embedded submanifold. This implies in particular the following property:

Proposition 2.5. *Let $\mathcal{C}$ be a path and $q(t)$ a parameterization of $\mathcal{C}$. There exist constants ρ and $\nu>0$ such that, if $d(q(t),q(t'))<\rho$, then*

$$d(q(t),q(t'))\geq\nu|t-t'|.$$

Proof. The assertion is clear when we replace d by a Riemannian distance on M. And there exists a Riemannian distance on M which is everywhere smaller than the sub-Riemannian distance (Gromov [7], p. 98). □

Given a path $\mathcal{C}$ in M and $\varepsilon>0$, we introduce the tubular neighborhood of $\mathcal{C}$

$$\mathrm{Tube}(\mathcal{C},\varepsilon)=\bigcup_{q\in\mathcal{C}}B(q,\varepsilon),$$

where $B(q,\varepsilon)$ is the sub-Riemannian ball centered at q of radius ε. We denote also by a and b the extremities of $\mathcal{C}$.

Definition 2.6. We call *complexity* of $\mathcal{C}$ the function $\varepsilon\mapsto\sigma(\mathcal{C},\varepsilon)$, defined for $\varepsilon>0$, with

$$\sigma(\mathcal{C},\varepsilon)=\inf\left\{\frac{\mathrm{length}(\gamma)}{\varepsilon}\;\middle|\;\begin{array}{l}\gamma\text{ trajectory of }(\Sigma)\\ \gamma\text{ joins }a\text{ to }b\\ \gamma\subset\mathrm{Tube}(\mathcal{C},\varepsilon)\end{array}\right\}. \tag{3}$$

We are interested here in the asymptotic behavior of the complexity. In particular we want to compare this behavior to the one of the entropy. With this aim in view, we introduce some notations. We write $f(\varepsilon)\preceq g(\varepsilon)$ if there exists κ and $\nu>0$ such that, for ε small enough, $f(\varepsilon)\leq\kappa g(\nu\varepsilon)$. We say that $f(\varepsilon)$ and $g(\varepsilon)$ are *equivalent*, and we write $f(\varepsilon)\asymp g(\varepsilon)$, when $f(\varepsilon)\preceq g(\varepsilon)$ and $g(\varepsilon)\preceq f(\varepsilon)$.

Remark 2.7.

- In [9], following ideas of Gromov [7] (p. 278), we propose a different definition of complexity: it is the minimum number of points in an ε-chain in $\mathcal{C}$, where an ε-chain in $\mathcal{C}$ is a sequence of points $v_0=a$, v_1, $\dots$, $v_k=b$ in $\mathcal{C}$ with $d(v_i,v_{i+1})\leq\varepsilon$ for $i=0,\dots,k-1$. Both definitions are however equivalent: the minimum number of points in an ε-chain is greater than $\sigma(\mathcal{C},\varepsilon)$ but smaller than $\sigma(\mathcal{C},3\varepsilon)$.
- As suggested by Gromov [7] (p. 278), the entropy and the complexity allow to define ε-lengths of a path, generalizing the notion of length for the trajectories:

  $$\mathrm{length}^e_\varepsilon(\mathcal{C})=\varepsilon\times e(\mathcal{C},\varepsilon)\qquad\text{and}\qquad\mathrm{length}^\sigma_\varepsilon(\mathcal{C})=\varepsilon\times\sigma(\mathcal{C},\varepsilon).$$

  A natural question is to ask if these ε-lengths are equivalent. The answer will follow directly from the comparison between entropy and complexity.

Proposition 2.8. *Let $\mathcal{C}$ be a path in the sub-Riemannian manifold (M,d). Then*

$$\sigma(\mathcal{C},\varepsilon)\preceq e(\mathcal{C},\varepsilon).$$

Proof. Let $\mathcal{C}\subset M$ be a path with extremities a and b, and $B_1,\dots,B_N$ be balls of radius ε covering $\mathcal{C}$.

Any two points in the same ball can be linked by a trajectory of length $\leq 2\varepsilon$ included in the ball. Consider now two balls B_i, B_j with a non empty intersection. Any couple of points in $B_i\cup B_j$ can be linked by a trajectory of length $\leq 4\varepsilon$ included in $B_i\cup B_j$.

By induction we show then that a and b can be linked by a trajectory of length $\leq 2N\varepsilon$ included in $\bigcup B_i$. The result follows from the definitions of σ and M. $\square$

Unfortunately the reverse inequality $e(\mathcal{C}, \varepsilon) \preceq \sigma(\mathcal{C}, \varepsilon)$ does not always hold. The obstruction to this inequality is the presence in the path of points with particular metric properties.

Definition 2.9. A point q in a path $\mathcal{C}$ is called a *metric cusp* if, for every positive constants k and ρ, there exist two points q_1 and q_2 in $\mathcal{C}$ different from q and such that:

- q lies between q_1 and q_2;
- $d(q_1, q_2) \leq \rho$;
- $d(q_1, q) \geq kd(q_1, q_2)$.

The sentence "$q \in \mathcal{C}$ between q_1 and q_2" is well defined since $\mathcal{C}$ is diffeomorphic to an interval of $\mathbb{R}$. Notice also that the extremities of the path are never metric cusps.

In Euclidean geometry (a special case of sub-Riemannian geometry), metric cusps coincide with the usual algebraic cusps (take $y = \sqrt{|x|}$ in $\mathbb{R}^2$ for instance). Of course a curve containing an algebraic cusp can not be a path (it is not a submanifold). In sub-Riemannian geometry however, paths may have metric cusps, as we will see in Section 3.6.

2.4. Application to nonholonomic motion planning

The complexity of paths is related to the algorithmic complexity of a Control Theory problem: nonholonomic motion planning in the presence of obstacles.

We consider a nonholonomic control system, that is a system (Σ). We assume it is controllable. Obstacles are closed subsets F of the configuration space M. The open set $M - F$ is called the *free space*. The motion planning problem is: *given a and b in $M - F$, find a trajectory of (Σ) contained in the free space, and joining a to b.*

This problem has a solution if and only if a and b are in the same connected component of $M - F$ (Chow's theorem). Since $M - F$ is an open set, connectivity is equivalent to arc connectivity. Then a and b belong to the same connected component if there is an arc in $M - F$ linking a to b.

This argument suggests a general method to solve the complete problem. This method, called "Approximation of a collision-free holonomic path" (see Laumond *et al.* [15]), has two steps:

- find a curve $\mathcal{C}$ in the free space linking a to b ($\mathcal{C}$ is the "collision-free holonomic path");
- approximate $\mathcal{C}$ by a trajectory of (Σ), close enough to be contained in the free space.

What is the algorithmic complexity of this method? The complexity of the first step (motion planning problem for holonomic systems) is well modeled and understood. It depends on the geometric complexity of the obstacles and the robot in the real world (see Canny [4] or Schwartz and Sharir [18]).

We are interested here in the second step which is due to the nonholonomy. Let us precise first what is a trajectory approximating a given curve. Let $\mathcal{C}$ be a curve and $\mathrm{Tube}(\mathcal{C}, \varepsilon)$ the tube of radius ε centered at $\mathcal{C}$. We denote by ρ the biggest radius ε for which $\mathrm{Tube}(\mathcal{C}, \varepsilon)$ is contained in the free space. We say that a trajectory of (Σ) approximates $\mathcal{C}$ if it is contained in $\mathrm{Tube}(\mathcal{C}, \rho)$ and if it has the same extremities as $\mathcal{C}$.

Now, recall that the complexity of an algorithm is the minimal number of elementary steps needed to get the result. Since the parameter ρ represents the size of the free space around $\mathcal{C}$, we consider that the elementary step of our method is "to build a trajectory of length ρ". The number of elementary steps in a trajectory is

$$\sigma(\gamma) = \frac{\mathrm{length}(\gamma)}{\rho}\cdot$$

We define the algorithmic complexity $\sigma_m(\mathcal{C})$ of the second step as the greatest lower bound of $\sigma(\gamma)$ for all trajectories γ approximating $\mathcal{C}$ (other kind of complexities can be defined, see Bellaïche *et al.* [3] and Jean [10]).

When $\mathcal{C}$ is a path, this algorithmic complexity can be deduced from the values of the complexity of the path $\mathcal{C}$. Indeed, $\sigma_m(\mathcal{C})$ is equal to $\sigma(\mathcal{C},\varepsilon)$ taken at $\varepsilon=\rho$.

3. Entropy and complexity of a path

Let (M,d) be the sub-Riemannian manifold attached to a system $X_1,\dots,X_m$. We assume that M is an orientable manifold and we denote by det the determinant n-form.

3.1. **Shape of sub-Riemannian balls**

We fix a path $\mathcal{C}\subset M$ and we denote by r the maximum of the degree of nonholonomy on $\mathcal{C}$.

We consider the families of vector fields $([X_{I_1}],\dots,[X_{I_n}])$ such that each bracket $[X_{I_j}]$ is of length $|I_j|\leq r$ and we denote by $\mathcal{I}_r$ the (finite) set of these families.

Definition 3.1. Let $p\in\mathcal{C}$ and $\varepsilon>0$. We say that the family $\underline{I}=([X_{I_1}],\dots,[X_{I_n}])\in\mathcal{I}_r$ is *associated with* (p,ε) *on* $\mathcal{C}$ if it achieves the maximum on $\mathcal{I}_r$ of the function

$$\Big|\det\Big([X_{I_1}]\varepsilon^{|I_1|},\dots,[X_{I_n}]\varepsilon^{|I_n|}\Big)(p)\Big|.$$

In particular the value at p of a family associated with (p,ε) forms a basis of T_pM.

Example 3.2 (A nilpotent system in $\mathbb{R}^3$)**.** Fix an integer $r\geq 2$ and consider the system defined in $\mathbb{R}^3$ by the vector fields

$$X_1=\partial_x,\quad X_2=\partial_y+\frac{x^{r-1}}{(r-1)!}\partial_z.$$

The Lie algebra $\mathcal{L}(X_1,X_2)$ is nilpotent of order r (for $r=2$, it is the Lie algebra associated with the Heisenberg group and for $r=3$, X_1,X_2 generate the Martinet distribution). The only non zero commutators are

$$[X_{I_i}]=(-1)^{i-2}\frac{x^{r-i}}{(r-i)!}\partial_z,\ i=2,\dots,r,\quad\text{where } I_i=\Bigg(1,2,\underbrace{1,\dots,1}_{(i-2)}\Bigg).$$

The set of singular points is the plane $\{x=0\}$. The degree of nonholonomy is 2 at regular points and r at singular ones. The families of vector fields with non identically zero determinant are $\underline{I}_i=(X_1,X_2,[X_{I_i}])$ and

$$\big|\det\big(\varepsilon X_1,\varepsilon X_2,\varepsilon^i[X_{I_i}]\big)(x,y,z)\big|=\frac{|x|^{r-i}}{(r-i)!}\varepsilon^{2+i}.$$

Choose any path $\mathcal{C}$ in $\mathbb{R}^3$ crossing the singular locus. The families associated with $\big((x,y,z),\varepsilon\big)$ on $\mathcal{C}$ are:

$$\underline{I}_2\ \text{ if }\ |x|\geq(r-2)\varepsilon,\qquad \underline{I}_i\ \text{ if }\ |x|\in\big[(r-i)\varepsilon,(r-i+1)\varepsilon\big],\quad i=3,\dots,r.$$

Notice that, when $|x|=(r-i)\varepsilon$, both families $\underline{I}_i$ and $\underline{I}_{i+1}$ are associated on $\mathcal{C}$.

The notion of associated basis allows to describe the shape of the sub-Riemannian balls.

Theorem 3.3. *There exist a constant $\delta_0>0$ and functions $\kappa(\delta)$, $K(\delta)$, $0<\kappa(\delta)<K(\delta)$, with* $\lim_{\delta\to 0}K(\delta)=0$, *such that:*

for every $p\in\mathcal{C}$, $\varepsilon<1$, $\delta<\delta_0$ and every family $\underline{I}$ associated with (p,ε) on $\mathcal{C}$,

$$B_{\underline{I}}\big(p,\kappa(\delta)\varepsilon\big)\subset B(p,\delta\varepsilon)\subset B_{\underline{I}}\big(p,K(\delta)\varepsilon\big),\tag{4}$$

where $B_{\underline{I}}(p,\varepsilon)=\big\{p\exp(u_n[X_{I_n}])\cdots\exp(u_1[X_{I_1}]),\ |u_i|<\varepsilon^{|I_i|},\ 1\leq i\leq n\big\}$.

This theorem is proved in [11].

Remark 3.4.

- We note on the right the action of diffeomorphisms: $p\exp(tX)$ results of the action of $\exp(tX)$ on point p. This notation is consistent with the notation for Lie group: diffeomorphisms which come from flows of left-invariant vector fields are defined by right multiplication.
- Theorem 3.3 gives a uniform estimate of sub-Riemannian balls which extends the classical Ball–Box theorem (Bellaïche [1], Gromov [7]). Indeed, p being fixed (and $\delta = \delta_0$), for ε smaller than some $\varepsilon(p)$, the estimate (4) is equivalent to the one of Ball–Box theorem. However $\varepsilon(p)$ can be infinitely small for p close to a singular point, though (4) holds for $\varepsilon < 1$.

Let $p \in M$ and $v \in T_pM$. For a family $\underline{I}$ which values at p form a basis of the tangent space T_pM, we denote by $v_1^{\underline{I}}, \ldots, v_n^{\underline{I}}$ the coordinates of v in this basis, that is

$$v = \sum_{i=1}^{n} v_i^{\underline{I}} [X_{I_i}](p).$$

Definition 3.5. The *ε-norm* of $v \in T_pM$ is defined as

$$\|v\|_{p,\varepsilon} = \max\left\{ \left| v_i^{\underline{I}} \right| \varepsilon^{-|I_i|},\ 1 \le i \le n,\ \underline{I} \text{ associated with } (p,\varepsilon) \text{ on } \mathcal{C} \right\}\cdot$$

When v is the tangent $\dot{q}(t)$ to the path, the dependence with respect to $q(t)$ is implicit and we write the ε-norm as $\|\dot{q}(t)\|_\varepsilon$. It becomes then a function of t, which is piecewise continuous and so integrable on $[0,T]$.

Example 3.6. Let us carry on the previous example. We choose for $\mathcal{C}$ the path $\{x = z^p, y = 0,\ 0 \le z \le 1\}$, $p \ge 1$ being an integer. The tangent is $\dot{q}(z) = pz^{p-1}\partial_x + \partial_z$, that is, for $i = 2, \ldots, r$,

$$\dot{q}(z) = pz^{p-1}X_1 + \frac{(r-i)!}{z^{p(r-i)}}[X_{I_i}].$$

When the family associated with $\big(q(z),\varepsilon\big)$ is $\underline{I}_i$, *i.e.* when $z \in \Big[\big((r-i)\varepsilon\big)^{1/p}, \big((r-i+1)\varepsilon\big)^{1/p}\Big]$ for $i = 3, \ldots, r$ and $z \ge \big((r-2)\varepsilon\big)^{1/p}$ for $i = 2$, the ε-norm is

$$\|\dot{q}(z)\|_\varepsilon = \max\left(\frac{pz^{p-1}}{\varepsilon},\ \frac{(r-i)!}{z^{p(r-i)}\varepsilon^i} \right) = \frac{(r-i)!}{z^{p(r-i)}\varepsilon^i}$$

for ε small enough.

In the same way, if we consider now the path $\{y = 0, z = x^p, 0 \le x \le 1\}$, the ε-norm is

$$\|\dot{q}(x)\|_\varepsilon = \frac{p(r-i)!}{x^{r-i-p+1}\varepsilon^i} \qquad \text{if } x \in \big[(r-i)\varepsilon, (r-i+1)\varepsilon\big],\ i \ge 3,$$
$$\|\dot{q}(x)\|_\varepsilon = \frac{p(r-2)!}{x^{r-1-p}\varepsilon^2} \qquad \text{if } x \ge (r-2)\varepsilon.$$

3.2. (H)-generic and regular points for a path

Let $\mathcal{C} \subset M$ be a path.

Definition 3.7. A point $q \in \mathcal{C}$ is said to be *regular for $\mathcal{C}$* if the growth vector is constant on $\mathcal{C}$ near q. Otherwise q is said to be *singular for $\mathcal{C}$*.

Regular points for $\mathcal{C}$ form an open dense set in $\mathcal{C}$ and singular points are isolated in $\mathcal{C}$. Notice that a regular point for $\mathcal{C}$ can be singular for M.

Let us consider the following property for a point q in the interior of $\mathcal{C}$:

(H) if $T_q\mathcal{C}$ belongs to $L^s(q)$, then there exist commutators $[X_{I_1}], \dots, [X_{I_l}]$ in $\mathcal{L}^s$ whose values taken at q form a basis of $L^s(q)$ and such that, for all $q' \in \mathcal{C}$ near q, $T_{q'}\mathcal{C}$ belongs to $\mathrm{span}\{[X_{I_1}](q'), \dots, [X_{I_l}](q')\}$.

For q a regular point for $\mathcal{C}$, this property simply reads as: if $T_q\mathcal{C}$ belongs to $L^s(q)$, then $T_{q'}\mathcal{C}$ belongs to $L^s(q')$ for all $q' \in \mathcal{C}$ near q.

Definition 3.8. We say that a point is *(*H*)-generic* either if it satisfies (H) or if it is an extremity of the path.

The (H)-generic points form an open dense set in $\mathcal{C}$. A consequence of this property is the following lemma, proved in Section 5.

Lemma 3.9. *The metric cusps are not (*H*)-generic and so are isolated points in $\mathcal{C}$.*

Finally we have two finite sets of points in $\mathcal{C}$: singular and non (H)-generic points. We say that $\mathcal{C}$ is *generic at singular points* when these two sets have an empty intersection, that is if each singular point for $\mathcal{C}$ is (H)-generic. It is a generic property on the paths space.

3.3. Estimate of entropy and complexity

Recall from Section 2.3 that we use the notations $\preceq$, $\asymp$ to denote the corresponding inequalities up to multiplicative constants (uniform with respect to ε, for ε small enough). Our first main result gives an equivalent for the entropy.

Theorem 3.10. *For a path $\mathcal{C}$ with a parameterization $q(t)$, $t \in [0, T]$,*

$$e(\mathcal{C}, \varepsilon) \asymp \int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t.$$

The proof is given in Section 5.1.

Remark 3.11. When ε tends to zero the ε-norm $\|v\|_{p,\varepsilon}$ is equivalent to $\|v\|_p/\varepsilon$ ($\|v\|_p$ is defined in (1)). If $\mathcal{C}$ is a trajectory of the system, the integral of the ε-norm – and so the entropy – is then equivalent to $\mathrm{length}(\mathcal{C})/\varepsilon$. Thus the ε-length $\mathrm{length}_\varepsilon^e(\mathcal{C})$ defined in Section 2.3 seems to be a good generalization of the notion of length.

This estimate allows to compute the entropy dimension. It gives also upper and lower bounds for the Hausdorff dimension.

Corollary 3.12. *The Hausdorff dimension of a path $\mathcal{C}$ belongs to the interval $[\beta_{\mathrm{reg}}, r]$, where*

- *β_{reg} is the smallest integer β such that $T_q\mathcal{C} \in L^\beta(q)$ for all point q regular for $\mathcal{C}$;*
- *r is the maximum of the degree of nonholonomy on $\mathcal{C}$.*

Moreover, for a path containing no singular points, the Hausdorff dimension is the smallest integer β such that $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$.

The proof is given in Section 5.1.

Notice that the Hausdorff dimension can take not only the integer values belonging to the interval $[\beta_{\mathrm{reg}}, r]$, but also rational ones.

Remark 3.13. In an equiregular manifold, the Hausdorff dimension of a path results from formula (2) of Gromov: $\dim_{\mathrm{H}} \mathcal{C}$ equals the smallest integer β such that $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$. Corollary 3.12 appears then as a generalization of Gromov's formula. However, when the path contains singular points, we may have both $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$ and $\dim_{\mathrm{H}} \mathcal{C} > \beta$ (see example (d) in Sect. 3.5 below). Thus, in general, the Hausdorff dimension of $\mathcal{C}$ is not characterized only by the dimension of the spaces $L^s_{\mathcal{C}}(q) = L^s(q) \cap T_q\mathcal{C}$.

Our second main result links the complexity to the integral of the ε-norm.

Theorem 3.14. *Let $\mathcal{C}$ be a path, with a parameterization $q(t)$, $t \in [0,T]$, and $t_1 < \cdots < t_s$ be the parameters of the points which are both metric cusps and singular for $\mathcal{C}$ ($0 < t_1$ and $t_s < T$). The complexity of $\mathcal{C}$ satisfies*

$$\int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t - \sum_{i=1}^{s} \int_{t_i-\varepsilon}^{t_i+\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \preceq \sigma(\mathcal{C},\varepsilon) \preceq \int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t.$$

The proof is given in Section 5.2.

This result does not always allow to estimate the complexity and to compare it with the entropy. Anyway complexity and entropy may be non equivalent, as shown by the example in Section 3.6. Moreover it is in general difficult to decide if a point is a metric cusp or not.

However Theorem 3.14 provides sufficient conditions for complexity and entropy to be equivalent. These conditions, expressed in the corollary below, are interesting for applications since they are generically satisfied and computable.

Corollary 3.15. *Let $\mathcal{C}$ be a path, with a parameterization $q(t)$, $t \in [0,T]$. If one of the following conditions holds:*

- *$\mathcal{C}$ is generic at singular points;*
- *denoting $t'_1 < \cdots < t'_N$ the parameters of the points which are both non (H)-generic and singular for $\mathcal{C}$, $t'_0 = 0$ and $t'_{N+1} = T$, we have, for $i = 1, \ldots, N+1$,*

$$\int_{t'_{i-1}}^{t'_i} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \asymp \int_{t'_{i-1}+\varepsilon}^{t'_i-\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t;$$

then $\sigma(\mathcal{C},\varepsilon) \asymp e(\mathcal{C},\varepsilon)$.

3.4. Example 1: The car control system

A classical example of nonholonomic control system is the car (see Laumond [14]). It is represented as two driving wheels connected by an axle. A state of the system is parameterized by the coordinates (x, y) of the center of the axle and by the orientation angle θ of the car. In the manifold $M = \mathbb{R}^2 \times \mathcal{S}^1$, the control system is

$$\dot{q} = u_1 X_1 + u_2 X_2, \text{ with } X_1 = \cos\theta\partial_x + \sin\theta\partial_y,\ X_2 = \partial_\theta.$$

The associated sub-Riemannian manifold has no singular point: the values of the family $\underline{I} = \{X_1, X_2, [X_1, X_2]\}$ form a basis everywhere. It is the only possible associated family.

Let $\mathcal{C}$ be a path in M and $q(t)$ a parameterization of $\mathcal{C}$. We define $\varphi(t)$ as the angle between $\dot{q}(t)$ and the plane generated by $X_1(q(t))$ and $X_2(q(t))$. The coordinate of $\dot{q}(t)$ on $[X_1, X_2](q(t))$ is $\sin\varphi(t)$. According to Theorem 3.14 we have

$$\sigma(\mathcal{C},\varepsilon) \asymp \frac{1}{\varepsilon^2} \int_0^T \max(\varepsilon, |\sin\varphi(t)|)\mathrm{d}t.$$

If the path is a trajectory, then $\varphi \equiv 0$, and the complexity equals length$(\mathcal{C})/\varepsilon$. On the other hand if the path is always perpendicular to the direction of the car, the complexity is equivalent to $1/\varepsilon^2$. So we show that to reverse into a parking place needs more maneuvers than going in a straight line!

This application of the complexity is developed in a more complete way in [10], including the case of the car pulling trailers.

3.5. Example 2: A nilpotent system in $\mathbb{R}^3$

As in the examples of Section 3.1, we consider the system of vector fields in $\mathbb{R}^3$

$$X_1 = \partial_x, \quad X_2 = \partial_y + \frac{x^{r-1}}{(r-1)!}\partial_z.$$

We are interested in paths with singular points. We consider for instance the paths $\{x = z^p, y = 0, 0 \leq x, z \leq 1\}$ where either p or $1/p$ is an integer and

– if $p \geq 1$, then p is the multiplicity of the intersection between $\mathcal{C}$ and the singular locus $\{x = 0\}$; $p = \infty$ means that $\mathcal{C}$ is included in the singular locus;
– if $1/p \geq 1$, then $1/p$ is the multiplicity of the intersection between $\mathcal{C}$ and the plane $\{z = 0\}$ (that is the plane $\{X_1(0), X_2(0)\}$); $1/p = \infty$ (or $p = 0$) means that $\mathcal{C}$ is a trajectory of the system.

For these paths, the only singular point is an extremity, the origin. They are then generic at singular points and so the complexity is always equivalent to the entropy. The ε-norm has already been computed in the examples of Section 3.1. Integrating it along the path yields the following entropy estimates.

(a) If $p = 0$ ($\mathcal{C}$ is a trajectory of the system):

$$e(\mathcal{C}, \varepsilon) \asymp \frac{1}{\varepsilon} \qquad \text{and} \qquad \dim_{\mathrm{H}} \mathcal{C} = 1.$$

It is the only case where the Hausdorff dimension equals the topological one.

(b) If $0 < p < 1/(r-2)$ ($\mathcal{C}$ is tangent to the distribution at 0):

$$e(\mathcal{C}, \varepsilon) \asymp \frac{1}{\varepsilon^2} \qquad \text{and} \qquad \dim_{\mathrm{H}} \mathcal{C} = 2.$$

The path is tangent to the distribution $\{X_1, X_2\}$ at the singular point 0, but not at regular points. The leading term in $e(\mathcal{C}, \varepsilon)$ is the entropy of a path contained in the regular locus (but which is not a trajectory). Notice that when there is no singular points, that is when $r = 2$, only this case or the previous one can occur.

(c) If $p = 1/(r-2)$ ($\mathcal{C}$ is transverse to the singular locus at 0 and, if $r \geq 4$, tangent to the distribution at 0):

$$e(\mathcal{C}, \varepsilon) \asymp \frac{1}{\varepsilon^2} \log(\frac{1}{\varepsilon}) \qquad \text{and} \qquad \dim_{\mathrm{H}} \mathcal{C} = 2.$$

In this case, the entropy dimension is $\dim_e \mathcal{C} = 2$ although $e(\mathcal{C}, \varepsilon)\varepsilon^2$ tends to infinity. For the Hausdorff dimension we have $\dim_{\mathrm{H}} \mathcal{C} \leq \dim_e \mathcal{C} = 2$ and the equality $\dim_{\mathrm{H}} \mathcal{C} = 2$ results from Corollary 3.12.

(d) If $1/(r-2) < p < \infty$ ($\mathcal{C}$ may be either transverse or tangent to the singular locus):

$$e(\mathcal{C}, \varepsilon) \asymp \frac{1}{\varepsilon^{r-1/p}} \qquad \text{and} \qquad \dim_{\mathrm{H}} \mathcal{C} = r - 1/p.$$

The Hausdorff dimension is greater than 2, the Hausdorff dimension of a path included in the regular set, but less than r, the one of a path included in the singular locus (see next case). It may not be an integer when p is greater than one (in this case $r - 1 \leq \dim_{\mathrm{H}} \mathcal{C} < r$).
Moreover we may have both $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$ and $\dim_{\mathrm{H}} \mathcal{C} > \beta$. Take for instance $p = 1/2$ and $r \geq 5$: the path is tangent everywhere to $L^2(q)$ and $\dim_{\mathrm{H}} \mathcal{C} = r - 2 \geq 3$.

(e) If $p = \infty$ (the path is included in the singular locus):

$$e(\mathcal{C}, \varepsilon) \asymp \frac{1}{\varepsilon^r} \qquad \text{and} \qquad \dim_{\mathrm{H}} \mathcal{C} = r.$$

In this case every point in the path is regular for $\mathcal{C}$.

3.6. Complexity and entropy may be non equivalent

Let us consider the system of vector fields in $\mathbb{R}^3$

$$X_1 = \partial_x, \quad X_2 = \partial_y + (x^9 - xz^2)\partial_z.$$

The singular locus is the set $\{9x^8 = z^2\}$. The degree of nonholonomy equals 10 on the subset $\{x = z = 0\}$, 3 on the remainder of the singular locus, and 2 elsewhere.

We choose a path tangent to $\{x = z = 0\}$ at 0 but which does not go through the singular locus, say $\mathcal{C} = \{(0, y, y^2), y \in [-1, 1]\}$. The origin is a singular point for $\mathcal{C}$ and is not (H)-generic. Thus $\mathcal{C}$ is not generic at singular points.

A short calculation gives the ε-norm

$$\|\dot{q}(y)\|_\varepsilon = \begin{cases} \max\left(\varepsilon^{-1}, \dfrac{2}{9!}y\varepsilon^{-10}\right) & \text{if } |y| \leq (9!)^{1/4}\varepsilon^2, \\ 2y^{-3}\varepsilon^{-2} & \text{if } (9!)^{1/4}\varepsilon^2 \leq |y| \leq 1. \end{cases}$$

On the other hand we have $d(q(\varepsilon), q(-\varepsilon)) = 2\varepsilon$. It is then easy to show that $\sigma(\mathcal{C}, \varepsilon) \asymp \sigma(q([-1, -\varepsilon]), \varepsilon) + \sigma(q([\varepsilon, 1]), \varepsilon)$. By applying Theorems 3.10 and 3.14, we obtain

$$\sigma(\mathcal{C}, \varepsilon) \asymp \int_\varepsilon^1 \|\dot{q}(y)\|_\varepsilon \mathrm{d}y \qquad \text{and} \qquad e(\mathcal{C}, \varepsilon) \asymp \int_0^1 \|\dot{q}(y)\|_\varepsilon \mathrm{d}y.$$

Using the expression of the ε-norm, this yields to

$$e(\mathcal{C}, \varepsilon) \asymp \varepsilon^{-6} \qquad \text{and} \qquad \sigma(\mathcal{C}, \varepsilon) \asymp \varepsilon^{-4}.$$

Thus entropy and complexity are not equivalent. It results also from Theorem 3.14 that 0 is a metric cusp.

4. Paths with only (H)-generic points

In this section we study paths containing only (H)-generic points (see Sect. 3.2 for the definition). In this case Theorems 3.10 and 3.14 take the following form, proved in Section 4.2 below.

Lemma 4.1. *Let $\mathcal{C}$ be a path containing only (*H*)-generic points and $q(t)$ a parameterization of $\mathcal{C}$. Then*

$$\sigma(\mathcal{C}, \varepsilon) \asymp e(\mathcal{C}, \varepsilon) \asymp \int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t.$$

4.1. The ε-norm along a path

Lemma 4.2. *Let $\mathcal{C}$ be a path containing only (*H*)-generic points and $q(t)$ a parameterization of $\mathcal{C}$. Then*

(a) *the following metric property (MP) is verified:*
there exist k and $\rho_0 > 0$ such that, for $q(t_0)$ and $q(t_1) \in \mathcal{C}$,

$$d(q(t_0), q(t_1)) < \rho_0 \quad \Rightarrow \quad d(q(t_0), q(t)) \leq k d(q(t_0), q(t_1)) \quad \forall t \in [t_0, t_1];$$

(b) *there exist constants ν, k_1 and $k_2 > 0$ such that, if $t_0 \in [0, T]$ and $\varepsilon \leq \rho_0$, then $d(q(t_0), q(t_1)) = \varepsilon$ implies*

$$k_1 \leq \int_{t_0}^{t_1} \|\dot{q}(t)\|_{\nu\varepsilon} \mathrm{d}t \leq k_2. \tag{5}$$

Remark 4.3. The property *(MP)* implies obviously that no points of $\mathcal{C}$ is a metric cusp.

Lemma 4.2 results from Propositions 4.6, 4.7 and 4.8 below.

4.1.1. *Notations*

For a family $\underline{I}$ of n brackets $([X_{I_1}],\dots,[X_{I_n}])$, we denote by $D(\underline{I})=|I_1|+\dots+|I_n|$ the total length and by $\det_{\underline{I}}(q)=\det([X_{I_1}],\dots,[X_{I_n}])(q)$ the determinant.

Given $p\in M$, the application $\phi_{\underline{I}}$ from an open neighborhood of $0\in\mathbb{R}^n$ to M is defined by

$$\phi_{\underline{I}}(u)=p\exp(u_n[X_{I_n}])\cdots\exp(u_1[X_{I_1}]).$$

For $\varepsilon>0$, $\mathrm{Box}(\varepsilon)$ denotes the set $\{|u_i|<\varepsilon^{|I_i|},\ 1\le i\le n\}$ in $\mathbb{R}^n$ and its image is $B_{\underline{I}}(p,\varepsilon)=\phi_{\underline{I}}\big(\mathrm{Box}(\varepsilon)\big)$.

Consider a path $\mathcal{C}$ with a parameterization $q(t)$, $t\in[0,T]$. Given a family $\underline{I}$ of vector fields and $i=1,\dots,n$, we set

$$\begin{aligned}\det{}_i^{\underline{I}}(t)&=\det\Big([X_{I_1}]\big(q(t)\big),\dots,\underbrace{\dot q(t)}_{i-th},\dots,[X_{I_n}]\big(q(t)\big)\Big),\\ \dot q_i^{\underline{I}}(t)&=\frac{\det_i^{\underline{I}}(t)}{\det_{\underline{I}}\big(q(t)\big)}\quad\text{if } \det{}_{\underline{I}}\big(q(t)\big)\neq 0.\end{aligned}$$

This is consistent with notations of Section 3.1: when defined, the $\dot q_i^{\underline{I}}(t)$'s are the coordinates of $\dot q(t)$ in the basis $\big([X_{I_1}](q(t)),\dots,[X_{I_n}](q(t))\big)$, that is

$$\dot q(t)=\sum_{i=1}^n \dot q_i^{\underline{I}}(t)\,[X_{I_i}]\big(q(t)\big). \tag{6}$$

Let $\mathcal{F}$ be the set of analytic functions $\det_{\underline{I}}\big(q(t)\big)$ and $\det_i^{\underline{I}}(t)$, for $i\in\{1,\dots,n\}$ and $\underline{I}$ such that each $|I_k|\le r$, where r is the maximum of the degree of nonholonomy on $\mathcal{C}$. Zeroes of functions in this set define on $[0,T]$ a finite number of real numbers $0=T_0<T_1<\dots<T_N=T$ such that, on $]T_j,T_{j+1}[$, a function in $\mathcal{F}$ is either identically zero or everywhere non zero. In particular a function non identically zero on $]T_j,T_{j+1}[$ has a constant sign on this interval.

There is also a constant Δt_0, $0<\Delta t_0<\frac{1}{2}\max_j(T_{j+1}-T_j)$, such that:
for all $j\in\{0,\dots,N\}$, $t\in[T_j-\Delta t_0,T_j+\Delta t_0]$, and f and $g\in\mathcal{F}$,

$$\text{if } f(T_j)\neq 0 \text{ and } g(T_j)=0, \text{ then } |f(t)|>|g(t)|. \tag{7}$$

Set

$$\begin{aligned}d_{\max}&=\max\Big\{\Big|\det{}_i^{\underline{I}}(t)\Big|\quad\text{s.t.}\quad t\in[0,T],\ i=1,\dots,n,\ \text{each } |I_k|\le r\Big\}\\ d_{\min}&=\min\Big\{\Big|\det{}_i^{\underline{I}}(T_j)\Big|\neq 0\quad\text{s.t.}\quad j=1,\dots,N,\ i=1,\dots,n,\ \text{each } |I_k|\le r\Big\}\cdot\end{aligned}$$

Both $d_{\max}$ and $d_{\min}$ are positive and finite. It follows from (7) that, if $\det_i^{\underline{I}}(T_j)\neq 0$ for some i, j and $\underline{I}$, then $|\det_i^{\underline{I}}(t)|>d_{\min}/2$ for every $t\in[T_j-\Delta t_0,T_j+\Delta t_0]$.

Now, set $\eta_0=d_{\min}/(2d_{\max})>0$. For any $i,l=1,\dots,n$, any $j\in\{0,\dots,N\}$ and any families $\underline{I}$, $\underline{J}$ of brackets of length $\le r$, if $\det_i^{\underline{I}}(T_j)\neq 0$, then for every $t\in[T_j-\Delta t_0,T_j+\Delta t_0]$

$$\frac{|\det_i^{\underline{I}}(t)|}{|\det_l^{\underline{J}}(t)|}>\eta_0. \tag{8}$$

4.1.2. *Preliminary results*

Fix a path $\mathcal{C}$ containing only (H)-generic points and a parameterization $q(t)$ of $\mathcal{C}$. Let δ_0 and $K(\cdot)$ be respectively the constant and the function given by Theorem 3.3. We will use the following lemma, which results from [11] (Lem. 7).

Lemma 4.4. *There exist a constant $C > 0$ and a function $\delta :]0,1[\to]0,\delta_0]$ such that, if $p \in \mathcal{C}$, $\varepsilon < 1$ and if $\underline{I}$ is a family associated with (p,ε) on $\mathcal{C}$, then the following properties are satisfied.*

(i) For every $q \in B_{\underline{I}}\big(p, K(\delta_0)\varepsilon\big)$,

$$\frac{1}{2}\big|\mathrm{det}_{\underline{I}}(p)\big| \le \big|\mathrm{det}_{\underline{I}}(q)\big| \le 2\big|\mathrm{det}_{\underline{I}}(p)\big|$$

$$\big|\mathrm{det}_{\underline{I}}(q)\big|\varepsilon^{D(\underline{I})} \ge \frac{1}{C}\max\big\{\big|\mathrm{det}_{\underline{K}}(q)\big|\varepsilon^{D(\underline{K})},\ \underline{K} \text{ s.t. each } |K_i| \le r\big\}\cdot$$

(ii) $\phi_{\underline{I}}$ is a local diffeomorphism in a neighborhood of every point of $\mathrm{Box}\big(K(\delta_0)\varepsilon\big)$.

(iii) Denoting by $(\psi_1,\ldots,\psi_n)$ the local inverse application of $\phi_{\underline{I}}$, we have, for every $q \in B_{\underline{I}}\big(p, K(\delta_0)\varepsilon\big)$,

$$\frac{1}{2} \le \big|[X_{I_i}]\cdot\psi_i(q)\big| \le 2.$$

Moreover, given $\tau > 0$, if $q \in B_{\underline{I}}\big(p, K\big(\delta(\tau)\big)\varepsilon\big)$, then,

$$\big|[X_{I_j}]\cdot\psi_i(q)\big| \le \tau\varepsilon^{|I_i|-|I_j|} \qquad \text{for } j \ne i.$$

Let $[t_0,t_1] \subset [0,T]$. We assume that $q([t_0,t_1])$ is included in $B_{\underline{I}}\big(q(t_0), K(\delta_0)\eta\big)$ for some $\eta < 1$ and a family $\underline{I}$ associated with $(q(t_0),\eta)$ on $\mathcal{C}$.

The application $\phi_{\underline{I}}$ is a local diffeomorphism on $\mathrm{Box}\big(K(\delta_0)\eta\big)$. So there is a unique absolutely continuous application $\theta : [t_0,t_1] \to \mathbb{R}^n$ such that $\theta(t) \in \mathrm{Box}\big(K(\delta_0)\eta\big)$ and $\phi_{\underline{I}}(\theta(t)) = q(t)$, for all $t \in [t_0,t_1]$.

Now, $q(t)$ and θ are one-to-one on $[t_0,t_1]$ and $\phi_{\underline{I}}$ is a local diffeomorphism on $\mathrm{Box}\big(K(\delta_0)\eta\big)$. Therefore $\phi_{\underline{I}}$ is a global diffeomorphism on a neighborhood of $\theta([t_0,t_1])$ and its inverse application $\phi_{\underline{I}}^{-1} = (\psi_1,\ldots,\psi_n)$ is well defined on this neighborhood.

To shorten we denote by $\big(\psi_1(t),\ldots,\psi_n(t)\big)$ the application $\phi_{\underline{I}}^{-1}\big(q(t)\big) = \theta(t)$ for $t \in [t_0,t_1]$.

Proposition 4.5. *Consider $t_0 \in [0,T[$, $t_1 \in [t_0, t_0+\Delta t_0]$, $\eta < \eta_0$ and a family $\underline{I}$ associated with $(q(t_0),\eta)$ on $\mathcal{C}$. Let p be an integer such that*

$$\int_{t_0}^{t_1}\big|\dot{q}_p^{\underline{I}}(s)\big|\,\eta^{-|I_p|}\mathrm{d}s = \max_{1\le i\le n}\int_{t_0}^{t_1}\Big|\dot{q}_i^{\underline{I}}(s)\Big|\,\eta^{-|I_i|}\mathrm{d}s.$$

If $q([t_0,t_1])$ is included in $B_{\underline{I}}\big(q(t_0), K(\delta_0)\eta\big)$, then

$$|\psi_p(t_1)| \ge \frac{1}{4}\int_{t_0}^{t_1}\big|\dot{q}_p^{\underline{I}}(s)\big|\,\mathrm{d}s, \tag{9}$$

$$|\psi_i(t)| \le 3\eta^{|I_i|}\int_{t_0}^{t_1}\big|\dot{q}_p^{\underline{I}}(s)\big|\eta^{-|I_p|}\mathrm{d}s, \quad \forall t \in [t_0,t_1],\ i = 1,\ldots,n. \tag{10}$$

As a consequence we have

$$|\psi_i(t)| \le 12\eta^{|I_i|-|I_p|}|\psi_p(t_1)|.$$

Proof. In the proof of this proposition – and below in those of Propositions 4.6 and 4.7 – the family $\underline{I}$ is fixed. We then write $\dot{q}_i$ and det_i instead of $\dot{q}_i^{\underline{I}}$ and $\mathrm{det}_i^{\underline{I}}$.

Let $t \in [t_0,t_1]$ and $i \in \{1,\ldots,n\}$. The application $\phi_{\underline{I}}^{-1}$ is defined at every point $q(s)$ with $s \in [t_0,t]$. We can write

$$\psi_i(t) = \int_{t_0}^{t}\frac{\mathrm{d}\psi_i}{\mathrm{d}t}(s)\mathrm{d}s.$$

Due to (6) the derivative of $\psi_i(s)$ is

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t}(s) = \sum_{l=1}^{n} \dot{q}_l(s)[X_{I_l}] \cdot \psi_i(s).$$

We integrate from t_0 to t and obtain

$$\psi_i(t) = \sum_{l=1}^{n} \int_{t_0}^{t} \dot{q}_l(s)[X_{I_l}] \cdot \psi_i(s)\mathrm{d}s.$$

According to Lemma 4.4(iii), reducing eventually δ_0 (in such a way that $\delta_0 \leq \delta(1/4n)$), we have,

$$\begin{aligned} \Big| \int_{t_0}^{t} \dot{q}_i(s)[X_{I_i}] \cdot \psi_i(s)\mathrm{d}s \Big| &\leq 2 \int_{t_0}^{t} \big|\dot{q}_i(s)\big|\mathrm{d}s, \\ \Big| \int_{t_0}^{t} \dot{q}_l(s)[X_{I_l}] \cdot \psi_i(s)\mathrm{d}s \Big| &\leq \frac{1}{4n}\eta^{|I_i|-|I_l|} \int_{t_0}^{t} \big|\dot{q}_l(s)\big|\mathrm{d}s, \quad \text{for } l \neq i. \end{aligned} \tag{11}$$

This implies that

$$\begin{aligned} \big|\psi_i(t)\big| &\leq 2 \int_{t_0}^{t} \big|\dot{q}_i(s)\big|\mathrm{d}s + \sum_{l\neq i} \frac{1}{4n}\eta^{|I_i|-|I_l|} \int_{t_0}^{t} \big|\dot{q}_l(s)\big|\mathrm{d}s \\ &\leq 2\eta^{|I_i|} \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big|\eta^{-|I_p|}\mathrm{d}s + \frac{1}{4n} \sum_{l\neq i} \eta^{|I_i|} \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big|\eta^{-|I_p|}\mathrm{d}s, \end{aligned}$$

by definition of p. It proves (10).

On the other hand we have

$$|\psi_p(t_1)| \geq \left| \int_{t_0}^{t_1} \dot{q}_p(s)[X_{I_p}] \cdot \psi_p(s)\mathrm{d}s \right| - \sum_{l\neq p} \left| \int_{t_0}^{t_1} \dot{q}_l(s)[X_{I_l}] \cdot \psi_p(s)\mathrm{d}s \right|.$$

Claim. *The function $\dot{q}_p(s)$ has a constant sign on $[t_0, t_1]$.*

The function $[X_{I_p}] \cdot \psi_p(s)$ has also a constant sign since $|[X_{I_p}] \cdot \psi_p(s)| \geq 1/2$ (Lem. 4.4(iii)). This observation and (11) applied to $i = p$ yield to

$$\big|\psi_p(t_1)\big| \geq \frac{1}{2} \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big|\mathrm{d}s - \sum_{l\neq p} \frac{1}{4n}\eta^{|I_p|} \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big|\eta^{-|I_p|}\mathrm{d}s,$$

which implies (9).

It remains to prove the claim. Observe first that the parameter t_0 belongs to some interval $[T_i, T_{i+1}[$, say $[T_0, T_1[$. As $\Delta t_0 < T_2 - T_1$, we have $[t_0, t_1] \subset [T_0, T_2]$. We distinguish two cases according to the position of t_1 with respect to T_1.

- If $t_1 \leq T_1$: each $\dot{q}_i$ – and then $\dot{q}_p$ – has a constant sign on $[t_0, t_1]$.
- If $t_1 > T_1$, then $q(T_1)$ belongs to $B_{\underline{I}}\big(q(t_0), K(\delta_0)\eta\big)$. It follows from Lemma 4.4(i) that $\det_{\underline{I}}\big(q(T_1)\big) \neq 0$, and so that the sign of $\det_{\underline{I}}(q(t))$ is constant on $[t_0, t_1]$. Since $\dot{q}_p(t) = \det_p(t)/\det_{\underline{I}}(q(t))$, its sign is constant if the one of $\det_p(t)$ does.

On the other hand, since $q(T_1)$ is (H)-generic, there exists l such that

$$\det_l(T_1) \neq 0 \ \text{ and, if } |I_i| > |I_l|, \ \det_i \equiv 0 \text{ on } [T_0, T_1].$$

Now, we can make two observations.

– If $|I_i| < |I_l|$, for $t \in [t_0, t_1]$, (8) is satisfied, that is

$$\eta_0 < \frac{|\det_l(t)|}{|\det_i(t)|} = \frac{|\dot{q}_l(t)|}{|\dot{q}_i(t)|}.$$

(Notice that both fractions might have an infinite value.) Thus, for $\eta < \eta_0$, we have $|\dot{q}_i(t)|\eta^{-|I_i|} < |\dot{q}_l(t)|\eta^{-|I_l|}$.

– If $|I_i| = |I_l|$ and $\det_i(T_1) = 0$, from definition (7) of Δt_0, we have $|\dot{q}_i(t)| < |\dot{q}_l(t)|$ for $t \in [t_0, t_1]$.

The only possibility for p is then $|I_p| = |I_l|$ and $\det_p(T_1) \neq 0$. Thus $\det_p(t)$, and so $\dot{q}_p(t)$, has a constant sign on $[t_0, t_1]$. This ends the proof of the claim. □

4.1.3. *First property of Lemma 4.2*

Proposition 4.6. *Let $\mathcal{C}$ be a path containing only (*H*)-generic points and $q(t)$ a parameterization of $\mathcal{C}$. Then the following property holds:*

(MP) there exist k and $\rho_0 > 0$ such that, for $q(t_0)$ and $q(t_1)$ in $\mathcal{C}$,

$$d\big(q(t_0), q(t_1)\big) < \rho_0 \ \Rightarrow\ d\big(q(t_0), q(t)\big) \leq k d\big(q(t_0), q(t_1)\big) \quad \forall t \in [t_0, t_1].$$

Proof. For $t \in [0, T]$ and $\Delta t \leq \Delta t_0$, we set

$$\eta(t, \Delta t) = \sup\big\{d\big(q(t), q(t')\big),\ t' \in [t, t + \Delta t]\big\}.$$

Since the parameterization is analytic, we can choose Δt such that $\eta(t, \Delta t)$ is not greater than a given constant for all t. Here we fix Δt in such a way that $\eta(t, \Delta t) < \delta_0\eta_0$ (recall that η_0 and Δt_0 are defined at the end of Sect. 4.1.1 and that δ_0 is given in Th. 3.3).

Let us consider now $t_0 \in [0, T]$ and $t_1 \geq t_0$ such that $t_1 - t_0 \leq \Delta t$. We set $\eta_1 = \sup\big\{d\big(q(t_0), q(t)\big),\ t \in [t_0, t_1]\big\}$. In particular we have $\eta_1 \leq \eta(t_0, \Delta t)$.

For each $t \in [t_0, t_1]$, $q(t)$ belongs to $B\big(q(t_0), \eta_1\big)$ and there exists $t_2 \in [t_0, t_1]$ such that $q(t_2)$ does not belong to $B\big(q(t_0), \frac{1}{2}\eta_1\big)$. With our choice of Δt, η_1 is smaller than $\delta_0\eta_0$ (which is of course smaller than δ_0). We can then use Theorem 3.3:

$$\forall t \in [t_0, t_1],\quad q(t) \in B_{\underline{I}}(q(t_0), \delta_+\eta) \qquad \text{and} \qquad q(t_2) \notin B_{\underline{I}}(q(t_0), \delta_-\eta),$$

with $\eta = \eta_1/\delta_0$, $\delta_+ = K(\delta_0)$, $\delta_- = \kappa(\delta_0/2)$ and $\underline{I}$ is a family associated with $(q(t_0), \eta)$ on $\mathcal{C}$.

Since $\eta < \eta_0$, we can apply Proposition 4.5 to t_2: for $i = 1, \ldots, n$,

$$|\psi_i(t_2)| \leq 12\eta^{|I_i| - |I_p|}|\psi_p(t_1)|.$$

Moreover there exists one specific i such that $|\psi_i(t_2)| \geq (\delta_-\eta)^{|I_i|}$. We obtain then an inequality for the p-th coordinate of $q(t_1)$:

$$|\psi_p(t_1)| \geq \frac{1}{12}(\delta_-)^{|I_i|}\eta^{|I_p|} \geq \left(\frac{1}{12}(\delta_-)^r\eta\right)^{|I_p|}.$$

Thus $q(t_1)$ does not belong to $B_{\underline{I}}\big(q(t_0), \delta_1\eta\big)$, where $\delta_1 = (\delta_-)^r/12$. Notice that $\delta_1 < \delta_0$ and that there exists $\delta_2 < \delta_0$ such that $K(\delta_2) = \delta_1$ (or at least $K(\delta_2) \leq \delta_1$).

Applying again Theorem 3.3, we obtain that $q(t_1)$ does not belong to $B\big(q(t_0), \delta_2\eta\big)$, that is $d\big(q(t_0), q(t_1)\big) \geq \eta_1/k$ where $k = \delta_0/\delta_2$ (this constant k depends only on δ_0, $\kappa(\cdot)$, $K(\cdot)$ and r and not on t_0).

From the definition of η_1 we have $d\big(q(t_0), q(t)\big) \leq \eta_1$ for $t \in [t_0, t_1]$. So we have shown $d\big(q(t_0), q(t)\big) \leq kd\big(q(t_0), q(t_1)\big)$.

We just proved the required property when $t_1 - t_0 \leq \Delta t$. On the other hand, we can apply Proposition 2.5: there exists a positive constant $\rho_0 (= \Delta t/\nu)$ such that $d\big(q(t_0), q(t_1)\big) < \rho_0$ implies $t_1 - t_0 \leq \Delta t$. This completes the proof. □

4.1.4. *Estimate of* $\|\dot{q}(t)\|_\varepsilon$

Proposition 4.7. *Let $\mathcal{C}$ be a path containing only (H)-generic points and $q(t)$, $t \in [0, T]$, a parameterization of $\mathcal{C}$. There exists positive constants ν, k_1 and k_2 such that, if $t_0 \in [0, T]$ and $\varepsilon \leq \rho_0$, then $d(q(t_0), q(t_1)) = \varepsilon$ implies*

$$k_1 \leq \int_{t_0}^{t_1} \left|\dot{q}_p^{\underline{I}}(s)\right| \eta^{-|I_p|} \mathrm{d}s \leq k_2 \tag{12}$$

where $\eta = \nu\varepsilon$, $\underline{I}$ is a family associated with $(q(t_0), \eta)$ on $\mathcal{C}$, and p is an integer such that

$$\int_{t_0}^{t_1} \left|\dot{q}_p^{\underline{I}}(s)\right| \eta^{-|I_p|} \mathrm{d}s = \max_{1 \leq i \leq n} \int_{t_0}^{t_1} \left|\dot{q}_i^{\underline{I}}(s)\right| \eta^{-|I_i|} \mathrm{d}s.$$

Proof. For $d\big(q(t_0), q(t_1)\big) = \varepsilon \leq \rho_0$, it results from Proposition 4.6 that

$$\forall t \in [t_0, t_1], \quad q(t) \in B\big(q(t_0), k\varepsilon\big) \ \text{ and } \ q(t_1) \notin B\left(q(t_0), \frac{1}{2}\varepsilon\right).$$

We use the same reasoning (and the same notations) as in the proof of Proposition 4.6.

We apply first Theorem 3.3 and obtain positive constants ν and δ_- (depending only on k, $\kappa(\cdot)$, $K(\cdot)$ and δ_0) such that, setting $\eta = \nu\varepsilon$,

$$\forall t \in [t_0, t_1], \quad q(t) \in B_{\underline{I}}\big(q(t_0), \delta_+\eta\big) \ \text{ and } \ q(t_1) \notin B_{\underline{I}}\big(q(t_0), \delta_-\eta\big). \tag{13}$$

Recall that the choice of ρ_0 allows to apply Proposition 4.5. Furthermore, we have $|\psi_p(t_1)| \leq (\delta_+\eta)^{|I_p|}$ and, for some i, $|\psi_i(t_1)| \geq (\delta_-\eta)^{|I_i|}$. This yields to

$$\int_{t_0}^{t_1} \big|\dot{q}_p(s)\big| \mathrm{d}s \leq 4(\delta_+\eta)^{|I_p|} \ \text{ and } \ (\delta_-\eta)^{|I_i|} \leq 3\eta^{|I_i|} \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big| \eta^{-|I_p|} \mathrm{d}s$$

and then to the required inequality

$$\frac{\delta_-}{3} \leq \int_{t_0}^{t_1} \big|\dot{q}_p(s)\big| \eta^{-|I_p|} \mathrm{d}s \leq 4(\delta_+)^{|I_p|}.$$ □

In order to complete the proof of Lemma 4.2, it remains to show that the inequality of Proposition 4.7 implies (5).

Proposition 4.8. *There exists constants c_1, $c_2 > 0$ such that, with the conditions of Proposition 4.7, we have*

$$c_1 \int_{t_0}^{t_1} \|\dot{q}(s)\|_\eta \mathrm{d}s \leq \max_{1 \leq i \leq n} \int_{t_0}^{t_1} \left|\dot{q}_i^{\underline{I}}(s)\right| \eta^{-|I_i|} \mathrm{d}s \leq c_2 \int_{t_0}^{t_1} \|\dot{q}(s)\|_\eta \mathrm{d}s.$$

Proof. Recall the definition of $\|\dot q(t)\|_\eta$ (Sect. 3.1):

$$\|\dot q(t)\|_\eta = \max\left\{\left|\dot q_i^{\underline{J}(t)}(t)\right|\eta^{-|J_i(t)|},\ 1\le i\le n,\ \underline{J}(t) \text{ associated with } (q(t),\eta) \text{ on } \mathcal{C}\right\}\cdot$$

The proof is based on the following relation. Fix $t\in[t_0,t_1]$ and let $\underline{J}$, $\underline{K}$ be two families of brackets of length $\le r$ and such that $\det_{\underline{J}}\big(q(t)\big)$ and $\det_{\underline{K}}\big(q(t)\big)\neq 0$. If there is a constant C such that

$$C\left|\det_{\underline{J}}\big(q(t)\big)\right|\eta^{D(\underline{J})} \ge \max\left\{\left|\det_{\underline{K}}\big(q(t)\big)\right|\eta^{D(\underline{K}')},\ \underline{K}' \text{ s.t. each } |K'_j|\le r\right\}, \tag{14}$$

then, for all $j\in\{1,\dots,n\}$,

$$\left|\dot q_j^{\underline{J}}(t)\right|\eta^{-|J_j|} \le C\sum_{k=1}^n \left|\dot q_k^{\underline{K}}(t)\right|\eta^{-|K_k|}. \tag{15}$$

This inequality arises from the basis change formulas

$$\dot q_j^{\underline{J}}(t) = \sum_{k=1}^n \dot q_k^{\underline{K}}(t)\frac{\det\big([X_{J_1}],\dots,[X_{J_{j-1}}],[X_{K_k}],\dots,[X_{J_n}]\big)\big(q(t)\big)}{\det_{\underline{J}}\big(q(t)\big)}$$

and from bounds on determinants provided by (14).

Fix $t\in[t_0,t_1]$. According to (13), $q(t)$ belongs to $B_{\underline{I}}\big(q(t_0),K(\delta_0)\eta\big)$. We consider a family $\underline{I}$ associated with $(q(t_0),\eta)$ and a family $\underline{J}(t)$ associated with $(q(t),\eta)$ on $\mathcal{C}$. Condition (14) is fulfilled by $\underline{J}(t)$ with $C=1$ (owing to the definition of an associated basis) and by $\underline{I}$ (owing to Lem. 4.4(i)).

We apply (15) both to the pair $\underline{I}$, $\underline{J}(t)$ and to the pair $\underline{J}(t)$, $\underline{I}$:

$$\left|\dot q_i^{\underline{I}}(t)\right|\eta^{-|I_i|} \le C\sum_{j=1}^n \left|\dot q_j^{\underline{J}(t)}(t)\right|\eta^{-|J_j(t)|}, \qquad 1\le i\le n,$$

$$\left|\dot q_j^{\underline{J}(t)}(t)\right|\eta^{-|J_j(t)|} \le \sum_{i=1}^n \left|\dot q_i^{\underline{I}}(t)\right|\eta^{-|I_i|}, \qquad 1\le j\le n.$$

We take the maximum on i, j and $\underline{J}(t)$ in each inequality, then integrate it between t_0 and t_1, and we obtain Proposition 4.8. □

4.2. **Proof of Lemma 4.1**

Let us first carry on the discussion of Section 2.3.

Proposition 4.9. *Let $\mathcal{C}$ be a path satisfying (MP), that is,*

there exist k and $\rho>0$ such that, for q_1 and $q_2\in\mathcal{C}$,

$$d(q_1,q_2)<\rho \quad\Rightarrow\quad d(q_1,q)\le kd(q_1,q_2) \quad \textit{for all } q\in\mathcal{C} \textit{ between } q_1 \textit{ and } q_2.$$

Then

$$e(\mathcal{C},\varepsilon)\preceq\sigma(\mathcal{C},\varepsilon).$$

Proof. Let γ be a trajectory of (Σ), contained in $\mathrm{Tube}(\mathcal{C},\varepsilon)$ and connecting the extremities of $\mathcal{C}$. We consider a piece of γ of length ε connecting some balls $B(q_1,\varepsilon)$ and $B(q_2,\varepsilon)$.

The distance between q_1 and q_2 is smaller than 3ε. If $3\varepsilon<\rho$, it follows from property *(MP)* that every $q\in\mathcal{C}$ between q_1 and q_2 belongs to $B(q_1,3k\varepsilon)$. Iterating this argument from $q_1=a$ until $q_2=b$, we cover $\mathcal{C}$ with N balls of radius $3k\varepsilon$, where N is not greater than $\mathrm{length}(\gamma)/\varepsilon$. It implies that $e(\mathcal{C},3k\varepsilon)\le\sigma(\mathcal{C},\varepsilon)$, which proves the proposition. □

We are now in a position to show Lemma 4.1.

Proof of Lemma 4.1. Let $\mathcal{C}$ be a path containing only (H)-generic points and $q(t)$ a parameterization of $\mathcal{C}$. According to Lemma 4.2–*1*, property *(MP)* holds. It follows from Propositions 2.8 and 4.9 that in this case $\sigma(\mathcal{C},\varepsilon) \asymp e(\mathcal{C},\varepsilon)$. It remains to prove that $e(\mathcal{C},\varepsilon)$ is equivalent to the integral of the ε-norm.

Since $q(t)$ is continuous, there is an integer M and parameters $t_0 = 0 < \cdots < t_M < t_{M+1} = T$ such that $d\big(q(t_j), q(t_{j+1})\big) = \varepsilon/k$ for $j = 0, \ldots, M-1$ and $d\big(q(t_M), q(t_{M+1})\big) < \varepsilon/k$ (where k is the constant given by property *(MP)*).

From Lemma 4.2–*2*, we have, for $j = 0, \ldots, M-1$,

$$k_1 \leq \int_{t_j}^{t_{j+1}} \|\dot{q}(t)\|_{\eta_1} \mathrm{d}t$$

where $\eta_1 = \nu\varepsilon/k$. Summing up on j, we obtain

$$M \leq \frac{1}{k_1} \int_0^{t_M} \|\dot{q}(t)\|_{\eta_1} \mathrm{d}t. \tag{16}$$

On the other hand, Lemma 4.2–*1* implies that, for every $t \in [t_j, t_{j+1}]$, $d\big(q(t_j), q(t)\big) \leq \varepsilon$. Thus the $M+1$ balls $B\big(q(t_j), \varepsilon\big)$ covers the whole path $\mathcal{C}$, *i.e.* $e(\mathcal{C},\varepsilon) \leq M+1$. Using (16), we obtain

$$e(\mathcal{C},\varepsilon) \leq 1 + \frac{1}{k_1} \int_0^{t_M} \|\dot{q}(t)\|_{\eta_1} \mathrm{d}t \preceq \int_0^T \|\dot{q}(t)\|_{\varepsilon} \mathrm{d}t.$$

Conversely, consider an integer M' and parameters $t_0 = 0 < \cdots < t_{M'} < t_{M'+1} = T$ such that, for $j = 0, \ldots, M'-1$, $d\big(q(t_j), q(t_{j+1})\big) = 3k\varepsilon$ and $d\big(q(t_{M'}), q(t_{M'+1})\big) < 3k\varepsilon$. Lemma 4.2–*2* implies that, setting $\eta_2 = 3k\nu\varepsilon$,

$$\frac{1}{k_2} \int_0^{t_{M'}} \|\dot{q}(t)\|_{\eta_2} \mathrm{d}t \leq M'. \tag{17}$$

Now, there exists a covering $\cup_l B(q_l, \varepsilon)$ of $\mathcal{C}$ with less than $2e(\mathcal{C},\varepsilon)$ balls. Notice that two points $q(t_i)$ and $q(t_j)$, $0 \leq i < j \leq M'$, can not belong to the same ball of the covering. Indeed, if it is the case, then $d\big(q(t_i), q(t_j)\big) \leq 2\varepsilon$. Since $t_{i+1} \in [t_i, t_j]$, Lemma 4.2–*1* implies $d\big(q(t_i), q(t_{i+1})\big) \leq 2k\varepsilon$. This contradicts the definition of the parameter t_i.

Hence the number of balls in the covering is greater than M'. Inequality (17) implies then

$$\frac{1}{k_2} \int_0^{t_{M'}} \|\dot{q}(t)\|_{\eta_2} \mathrm{d}t \leq 2e(\mathcal{C},\varepsilon).$$

It results from Lemma 4.2–*2* that the integral of $\|\dot{q}(t)\|_{\eta_2}$ between $t_{M'}$ and T is not greater than k_2. Hence we have

$$\int_0^T \|\dot{q}(t)\|_{\varepsilon} \mathrm{d}t \preceq e(\mathcal{C},\varepsilon).$$

Finally we have the equivalence between the entropy and the integral of the ε-norm, which concludes the proof. $\square$

5. General case

Consider a path $\mathcal{C}$ with a parameterization $q(t)$, $t \in [0,T]$. Let $\bar{t}_1 < \cdots < \bar{t}_{\bar{N}}$ be the parameters of the non (H)-generic points and set $\bar{t}_0 = 0$ and $\bar{t}_{\bar{N}+1} = T$. Since the extremities are (H)-generic, we have $\bar{t}_0 < \bar{t}_1$ and $\bar{t}_{\bar{N}} < \bar{t}_{\bar{N}+1}$.

Each pieces $\mathcal{C}_i = q\big([\bar{t}_i, \bar{t}_{i+1}]\big)$, $i = 0, \ldots, \bar{N}$, is a path and contains only (H)-generic points. It results from Lemma 4.2–*1* that $\mathcal{C}_i$ satisfies the property *(MP)* and so that it has no metric cusp. Thus $\mathcal{C}$ has no metric cusp between $q(\bar{t}_i)$ and $q(\bar{t}_{i+1})$. This shows Lemma 3.9 of Section 3.2:

Lemma 3.9. *The metric cusps are non (*H*)-generic points.*

5.1. **Proof of Theorem 3.10**

Proposition 5.1. *Let $\mathcal{C}$ be a path and q_0 a point in the interior of $\mathcal{C}$. Denote by $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the part of $\mathcal{C}$ lying between a and q_0 (resp. q_0 and b). Then*

$$e(\mathcal{C}, \varepsilon) \asymp e(\mathcal{C}_1, \varepsilon) + e(\mathcal{C}_2, \varepsilon).$$

Proof. It is clear that $e(\mathcal{C}, \varepsilon) \leq e(\mathcal{C}_1, \varepsilon) + e(\mathcal{C}_2, \varepsilon)$. On the other hand, a covering of $\mathcal{C}$ by ε-balls must contain a covering of $\mathcal{C}_1$ and one of $\mathcal{C}_2$. Thus $e(\mathcal{C}, \varepsilon)$ is greater than $e(\mathcal{C}_1, \varepsilon)$ and $e(\mathcal{C}_2, \varepsilon)$, which implies

$$e(\mathcal{C}, \varepsilon) \geq \frac{1}{2}(e(\mathcal{C}_1, \varepsilon) + e(\mathcal{C}_2, \varepsilon)). \qquad \square$$

With this result, we are now in a position to establish the estimate of the entropy.

Proof of Theorem 3.10. Proposition 5.1 implies that $e(\mathcal{C}, \varepsilon) \asymp \sum_{i=0}^{\bar{N}} e(\mathcal{C}_i, \varepsilon)$. Since each $\mathcal{C}_i$ contains only (H)-generic points, it follows from Lemma 4.1 that

$$e(\mathcal{C}_i, \varepsilon) \asymp \int_{\bar{t}_i}^{\bar{t}_{i+1}} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t,$$

which concludes the proof. $\square$

It remains to prove the results on the Hausdorff dimension.

Proof of Corollary 3.12. We are first going to prove the second assertion of the corollary: for a path $\mathcal{C}$ containing no singular points, $\dim_{\mathrm{H}} \mathcal{C}$ is the smallest integer β such that $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$.

Let $\mathcal{C}$ be a path containing no singular points and $q(t)$, $t \in [0, T]$ a parameterization of $\mathcal{C}$. To begin with, let us describe the set of associated families near a point of $\mathcal{C}$.

Fix a point $q \in \mathcal{C}$. For ε small enough, any family $\underline{I}$ associated with (q, ε) on $\mathcal{C}$ satisfies

$$\begin{aligned} D(\underline{I}) &= \min\big\{D(\underline{K}),\ \text{each } |K_j| \leq r,\ \det{}^{\underline{K}}(q) \neq 0\big\}, \\ \big|\det{}^{\underline{I}}(q)\big| &= \max\big\{\big|\det{}^{\underline{K}}(q)\big|,\ \text{each } |K_j| \leq r,\ D(\underline{K}) = D(\underline{I})\big\}\cdot \end{aligned}$$

We say that such a family is *minimal at q* and we denote by $\mathcal{M}_q$ the set of minimal families at q.

Consider now a point $q(t_1)$ in $\mathcal{C}$, $t_1 > 0$. It is a regular point for $\mathcal{C}$, so the function $\min\big\{D(\underline{K}),\ \det{}^{\underline{K}}\big(q(t)\big) \neq 0\big\}$ is constant for t near t_1. It implies two properties:

- there exists $t_0 \in [0, t_1[$ such that $\mathcal{M}_{q(t)}$ is independent of t on the interval $[t_0, t_1[$;
- there exists ε^1 such that, for $t \in [t_0, t_1]$ and $\varepsilon \leq \varepsilon^1$, a family is associated with $(q(t), \varepsilon)$ on $\mathcal{C}$ if and only if it is minimal at $q(t)$.

Thus, for all $t \in [t_0, t_1[$ and $\varepsilon \leq \varepsilon^1$, the set of the families associated with $(q(t), \varepsilon)$ on $\mathcal{C}$ is equal to $\mathcal{M}_{q(t_0)}$.

Now, let β be the smallest integer such that $T_q\mathcal{C} \in L^\beta(q)$ for all q in $\mathcal{C}$. There exists an interval $[t_0, t_1] \subset [0, T]$ such that $\dot{q}(t)$ belongs to $L^\beta(q(t))/L^{\beta-1}(q(t))$ for all $t \in [t_0, t_1]$. Reducing if needed the size of the interval, we can assume that, for all $t \in [t_0, t_1[$ and $\varepsilon \leq \varepsilon^1$, the set of the families associated with $(q(t), \varepsilon)$ is equal to $\mathcal{M}_{q(t_0)}$. The ε-norm is then equal to:

$$\|\dot{q}(t)\|_\varepsilon = \varepsilon^{-\beta} \max \left\{ \left| \dot{q}_k^{\underline{I}}(t) \right|, \underline{I} \in \mathcal{M}_{q(t_0)},\ |I_k| = \beta \right\}.$$

Due to Theorem 3.10, the entropy of the path $\mathcal{C}_0 = q([t_0, t_1])$ is equivalent to $\varepsilon^{-\beta}$, which implies $\dim_{\mathrm{H}} \mathcal{C}_0 = \beta$ by Proposition 2.4(ii). Since $\mathcal{C}_0$ is included in $\mathcal{C}$, we have $\dim_{\mathrm{H}} \mathcal{C} \geq \beta$.

The same kind of reasoning and Proposition 2.4(i) allows to show the reverse inequality. Thus we obtain $\dim_{\mathrm{H}} \mathcal{C} = \beta$.

The other assertion of the corollary is that the Hausdorff dimension of a path $\mathcal{C}$ belongs to $[\beta_{\mathrm{reg}}, r]$, where r is the maximum of the degree of nonholonomy on $\mathcal{C}$ and β_{reg} is the smallest integer β such that $T_q\mathcal{C} \in L^\beta(q)$ for all point q regular for $\mathcal{C}$.

The lower bound is a consequence of the previous result: $\mathcal{C}$ contains a path $\mathcal{C}_0$ with no singular point and such that $T_q\mathcal{C}_0$ belongs to $L^{\beta_{\mathrm{reg}}}(q)/L^{\beta_{\mathrm{reg}}-1}(q)$ everywhere. The Hausdorff dimension of $\mathcal{C}_0$ is β_{reg} and $\dim_{\mathrm{H}} \mathcal{C} \geq \dim_{\mathrm{H}} \mathcal{C}_0$.

The upper bound results directly from the estimate of the entropy. Indeed, the ε-norm has a uniform upper bound $\|\dot{q}(t)\|_\varepsilon \leq \text{const} \times \varepsilon^{-r}$ on $\mathcal{C}$. By Theorem 3.10, we have $e(\mathcal{C}, \varepsilon) \preceq \varepsilon^{-r}$ and $\dim_e(\mathcal{C}) \leq r$. The conclusion follows from Proposition 2.4(i). □

5.2. Proof of Theorem 3.14

Proposition 5.2. *Consider a path $\mathcal{C}$, a point q_0 in the interior of $\mathcal{C}$ and define $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) as the part of $\mathcal{C}$ lying between a and q_0 (resp. q_0 and b). Let q_1^ε be the first $q \in \mathcal{C}_1$ such that $d(q, \mathcal{C}_2) \leq 3\varepsilon$ and q_2^ε be the last $q \in \mathcal{C}_2$ such that $d(q, \mathcal{C}_1) \leq 3\varepsilon$. Denote by $\mathcal{C}_1^\varepsilon$ (resp. $\mathcal{C}_2^\varepsilon$) the part of $\mathcal{C}$ lying between a and q_1^ε (resp. q_2^ε and b). Assume $\mathcal{C}_1$ and $\mathcal{C}_2$ contain only (H)-generic points.*

If q_0 is not a metric cusp, then

$$\sigma(\mathcal{C}, \varepsilon) \asymp \sigma(\mathcal{C}_1, \varepsilon) + \sigma(\mathcal{C}_2, \varepsilon),$$

otherwise

$$\sigma(\mathcal{C}_1^\varepsilon, \varepsilon) + \sigma(\mathcal{C}_2^\varepsilon, \varepsilon) \preceq \sigma(\mathcal{C}, \varepsilon) \leq \sigma(\mathcal{C}_1, \varepsilon) + \sigma(\mathcal{C}_2, \varepsilon).$$

Proof. Recall from Lemma 4.2–*1* that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy property *(MP)*: there exist k and $\rho > 0$ such that, for q_1 and $q_2 \in \mathcal{C}_i$,

$$d(q_1, q_2) < \rho \quad \Rightarrow \quad d(q_1, q) \leq k d(q_1, q_2) \quad \text{for all } q \in \mathcal{C}_i \text{ between } q_1 \text{ and } q_2.$$

If q_0 is not a metric cusp, then the whole path $\mathcal{C}$ satisfies *(MP)*. It follows then from Propositions 2.8 and 4.9 that in this case $\sigma(\mathcal{C}, \varepsilon) \asymp e(\mathcal{C}, \varepsilon)$. Applying also this result to $\mathcal{C}_1$ and $\mathcal{C}_2$ and using Proposition 5.1, we obtain the conclusion.

Assume now that q_0 is a metric cusp. The inequality $\sigma(\mathcal{C}, \varepsilon) \leq \sigma(\mathcal{C}_1, \varepsilon) + \sigma(\mathcal{C}_2, \varepsilon)$ is straightforward. Let us prove the other inequality.

We consider a trajectory γ of (Σ), contained in $\mathrm{Tube}(\mathcal{C}, \varepsilon)$ and connecting the extremities a and b of $\mathcal{C}$. The length of γ is greater than $\varepsilon\sigma(\mathcal{C}, \varepsilon)$.

The intersection of γ with the closure of $\mathrm{Tube}(\mathcal{C}_1^\varepsilon, \varepsilon)$ contains a trajectory γ_1 starting from a. The end-point p_1 of γ_1 is such that $d(p_1, q_-) = \varepsilon$ for one point $q_- \in \mathcal{C}$ between a and q_1^ε and $d(p_1, q_+) \leq 2\varepsilon$ for one point $q_+ \in \mathcal{C}$ between q_1^ε and b.

What is the distance from p_1 to q_1^ε? Notice first that $d(q_-, q_+) \leq 3\varepsilon$. Now, if q_+ lies in $\mathcal{C}_2$, the definition of q_1^ε implies $q_- = q_1^\varepsilon$. Otherwise, if q_+ lies between q_1^ε and q_0, it follows from property *(MP)* that $d(q_-, q_1^\varepsilon) \leq 3k\varepsilon$. In both cases we have then $d(p_1, q_1^\varepsilon) \leq 3k\varepsilon$.

Finally we can construct a trajectory going from a to q_1^ε, staying in $\mathrm{Tube}(\mathcal{C}_1^\varepsilon, 3k\varepsilon)$ and of length smaller than $\mathrm{length}(\gamma_1) + 3k\varepsilon$. This last quantity is then greater than $3k\varepsilon\sigma(\mathcal{C}_1^\varepsilon, 3k\varepsilon)$.

The same reasoning applies to a part γ_2 of the trajectory in $\mathrm{Tube}(\mathcal{C}_2^\varepsilon, \varepsilon)$. Since the length of γ is greater than the sum of the lengths of γ_1 and γ_2, we obtain finally

$$\sigma(\mathcal{C}, \varepsilon) + 6k \geq 3k\sigma(\mathcal{C}_1^\varepsilon, 3k\varepsilon) + 3k\sigma(\mathcal{C}_2^\varepsilon, 3k\varepsilon),$$

and so the conclusion. □

Corollary 5.3. *Let $\mathcal{C}$ be a path with a parameterization $q(t)$, $t \in [0,T]$, and $t_1 < \cdots < t_N$ be the parameters of the metric cusps. Set $t_0 = t_{0,+}^\varepsilon = 0$, $t_{N+1} = t_{N+1,-}^\varepsilon = T$ and, for $i = 1, \ldots, N$, define $t_{i,-}^\varepsilon = \min\{t \text{ s.t. } d\big(q(t), q([t_i, T])\big) \leq 3\varepsilon\}$ and $t_{i,+}^\varepsilon = \max\{t \text{ s.t. } d\big(q(t), q([0, t_i])\big) \leq 3\varepsilon\}$. Then*

$$\sum_{i=0}^{N} \int_{t_{i,+}^\varepsilon}^{t_{i+1,-}^\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \preceq \sigma(\mathcal{C}, \varepsilon).$$

Proof. Remark first that, as a consequence of Lemma 3.9, $t_1, \ldots, t_N$ is a sub-sequence of $\bar{t}_1, \ldots, \bar{t}_{\bar{N}}$ (the parameters of the non (H)-generic points).

Reasoning as in the proof of Proposition 5.2, we obtain

$$\sum_{i=0}^{N} \sigma\big(q([t_{i,+}^\varepsilon, t_{i+1,-}^\varepsilon]), \varepsilon\big) \preceq \sigma(\mathcal{C}, \varepsilon).$$

Moreover each path $q([t_{i,+}^\varepsilon, t_{i+1,-}^\varepsilon])$ satisfies *(MP)*: it results then from Propositions 2.8 and 4.9 that the complexity of that path is equivalent to its entropy, that is, to the integral of the ε-norm by Theorem 3.10. □

Let us estimate now the parameters $t_{i,-}^\varepsilon$ and $t_{i,+}^\varepsilon$.

Proposition 5.4. *Fix $i \in \{1, \ldots, N\}$ and let T_1 and T_2 be parameters in $[0,T]$ such that $T_1 < t_{i,-}^\varepsilon$ and $T_2 > t_{i,+}^\varepsilon$ (with the notations of Cor. 5.3). Then,*

$$\int_{T_1}^{t_i - \varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \preceq \int_{T_1}^{t_{i,-}^\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \tag{18}$$

$$\int_{t_i + \varepsilon}^{T_2} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \preceq \int_{t_{i,+}^\varepsilon}^{T_2} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t. \tag{19}$$

Moreover, if $q(t_i)$ is a regular point for $\mathcal{C}$, then

$$\int_{T_1}^{t_{i,-}^\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \asymp \int_{T_1}^{t_i} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \qquad \text{and} \qquad \int_{t_{i,+}^\varepsilon}^{T_2} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \asymp \int_{t_i}^{T_2} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t.$$

Proof. Let us use Proposition 2.5: there exists a constant $\nu > 0$ such that, if $d\big(q(t), q(t')\big)$ is small enough, then

$$d\big(q(t), q(t')\big) \geq \nu |t - t'|. \tag{20}$$

Since $d\big(q(t_{i,-}^\varepsilon), q(t)\big) = 3\varepsilon$ for one $t \geq t_i$, we have $t_i - t_{i,-}^\varepsilon \leq 3\nu\varepsilon$. This implies inequality (18). The other inequality (19) is obtained in the same way.

Assume now that $q(t_i)$ is a regular point for $\mathcal{C}$.

Claim. *If $q(t_i)$ is regular for $\mathcal{C}$, then for ε small enough,*

$$\begin{cases} \displaystyle\int_{t_i-\varepsilon}^{t_i} \|\dot q(t)\|_\varepsilon \mathrm{d}t \le 2\int_{t_i-2\varepsilon}^{t_i-\varepsilon} \|\dot q(t)\|_\varepsilon \mathrm{d}t, \\ \displaystyle\int_{t_i}^{t_i+\varepsilon} \|\dot q(t)\|_\varepsilon \mathrm{d}t \le 2\int_{t_i+\varepsilon}^{t_i+2\varepsilon} \|\dot q(t)\|_\varepsilon \mathrm{d}t. \end{cases}$$

This result implies that the integral of the ε-norm between T_1 and $t_i-\varepsilon$ is equivalent to the one between T_1 and t_i and that the integral between $t_i+\varepsilon$ and T_2 is equivalent to the one between t_i and T_2. This ends the proof of Proposition 5.4.

It remains to prove the claim. We use a result shown in the proof of Corollary 3.12 in the preceding section. If $q(t_i)$ is a regular point for $\mathcal{C}$, then there exists δ and $\varepsilon^1>0$ such that: for all $t\in\ [t_i-\delta,t_i[$ and $\varepsilon\le\varepsilon^1$, the set of the families associated with $(q(t),\varepsilon)$ is equal to $\mathcal{M}_{q(t_0)}$, with $t_0=t_i-\delta$.

Recall also the definition of the ε-norm (Sect. 3.1):

$$\|\dot q(t)\|_\varepsilon = \max\left\{\left|\dot q_k^{\underline{I}}(t)\right|\varepsilon^{-|I_k|},\ 1\le k\le n,\ \underline{I}\text{ associated with }\big(q(t),\varepsilon\big)\text{ on }\mathcal{C}\right\}\cdot \tag{21}$$

Now, reducing eventually δ, each function $\left|\dot q_k^{\underline{I}}(t)\right|$ ($k\in\{1,\dots,n\}$, $\underline{I}\in\mathcal{M}_{q(t_0)}$) satisfies: if $\varepsilon\le\delta/2$ and $t\in\,]t_i-\varepsilon,t_i[$, then $\left|\dot q_k^{\underline{I}}(t)\right|\le 2\left|\dot q_k^{\underline{I}}(t-\varepsilon)\right|$ (it results from the analyticity of $|\dot q_k^{\underline{I}}(t)|$ on $]t_0,t_i[$).

For $\varepsilon\le\min\{\delta/2,\varepsilon^1\}$ and $t\in\,]t_i-\varepsilon,t_i[$, any family associated with $(q(t),\varepsilon)$ on $\mathcal{C}$ belongs to $\mathcal{M}_{q(t_0)}$. So there exist $\underline{J}\in\mathcal{M}_{q(t_0)}$ and $j\in\{1,\dots,n\}$ such that $\|\dot q(t)\|_\varepsilon=|\dot q_j^{\underline{J}}(t)|\varepsilon^{-|J_j|}$. The ε-norm at t satisfies then

$$\|\dot q(t)\|_\varepsilon\le 2\left|\dot q_j^{\underline{J}}(t-\varepsilon)\right|\varepsilon^{-|J_j|}\le 2\|\dot q(t-\varepsilon)\|_\varepsilon.$$

The second inequality above holds because $\underline{J}\in\mathcal{M}_{q(t_0)}$ is also a family associated with $\big(q(t-\varepsilon),\varepsilon\big)$ on $\mathcal{C}$.

This proves the first inequality of the claim (remind that $\|\dot q(t)\|_\varepsilon$ is piecewise continuous). The second inequality is proved in the same way. □

Remark 5.5. When $q(t_i)$ is a singular point for $\mathcal{C}$, it is not possible to find a set of associated families independent of t and ε. In this case, indeed, a family associated with $(q(t),\varepsilon)$ on $\mathcal{C}$ is minimal at $q(t)$ if $\varepsilon\le O(|t-t_i|)$ (take for instance the nilpotent system in $\mathbb{R}^3$ of Sect. 3.1 with $\mathcal{C}$ equals to the x-axis).

We are now in a position to prove the estimate for complexity.

Proof of Theorem 3.14. Consider a path $\mathcal{C}$ with a parameterization $q(t)$, $t\in[0,T]$. By Proposition 2.8 and Theorem 3.10, we obtain a first inequality:

$$\sigma(\mathcal{C},\varepsilon)\preceq\int_0^T\|\dot q(t)\|_\varepsilon\mathrm{d}t.$$

Let $t_1<\dots<t_N$ be the parameters of the metric cusps. Corollary 5.3 and the first part of Proposition 5.4 imply

$$\int_0^T\|\dot q(t)\|_\varepsilon\mathrm{d}t-\sum_{i=1}^N\int_{t_i-\varepsilon}^{t_i+\varepsilon}\|\dot q(t)\|_\varepsilon\mathrm{d}t\preceq\sigma(\mathcal{C},\varepsilon).$$

It follows from the second part of Proposition 5.4 that, if $q(t_i)$ is regular for $\mathcal{C}$, then the integral of the ε-norm between $t_i-\varepsilon$ and $t_i+\varepsilon$ can be neglected. Hence, denoting $t'_1,\dots,t'_s$ the parameters (among $t_1,\dots,t_N$) of the points which are both metric cusps and singular for $\mathcal{C}$, we obtain:

$$\int_0^T \|\dot{q}(t)\|_\varepsilon \mathrm{d}t - \sum_{i=1}^{s} \int_{t'_i-\varepsilon}^{t'_i+\varepsilon} \|\dot{q}(t)\|_\varepsilon \mathrm{d}t \preceq \sigma(\mathcal{C}, \varepsilon),$$

which ends the proof. $\square$

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