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A STABILITY RESULT IN THE LOCALIZATION OF CAVITIES IN A THERMIC CONDUCTING MEDIUM

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Abstract. We prove a logarithmic stability estimate for a parabolic inverse problem concerning the localization of unknown cavities in a thermic conducting medium Ω in $\mathbb{R}^n$, $n \geq 2$, from a single pair of boundary measurements of temperature and thermal flux.

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1. Introduction and the main result

In the present paper we are concerned with the study of a problem in thermal imaging. This is a technique used to determine some physical and geometrical proprieties of a thermic conducting medium *via* boundary measurements of temperature and thermal flux. More precisely we denote by Ω a thermic conducting medium, *i.e.* a sufficiently smooth, bounded domain in $\mathbb{R}^n$, $n \geq 2$, and by D a *cavity* in Ω (*i.e.* D is a domain compactly contained in Ω), of which neither the form nor the position is known. On the other hand we can measure the *temperature* f and the *thermal flux* g on the boundary of the medium $\partial\Omega$. The goal is then to identify the cavity D *via* the boundary data f, g. This problem can occur in nondestructive tests of materials, for example in detecting the corrosion parts of an aircraft which are inaccessible to direct inspections (see Bryan and Caudill [5–7], and their references).

We denote by $u(t,x)$ the temperature at the time t and at the point $x \in \Omega\backslash\overline{D}$, u_0 the initial temperature in $\Omega\backslash\overline{D}$, f the temperature on $(0,T)\times\partial\Omega$, and $k(x)$ the anisotropic thermal diffusion coefficient, that is k is an $n \times n$ symmetric matrix-valued function in $\overline{\Omega}$ satisfying the following conditions:

(i) there exists a constant $\lambda \geq 1$, such that for all $x \in \overline{\Omega}$, and for all $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \leq k(x)\xi\cdot\xi \leq \lambda|\xi|^2 \quad \text{(ellipticity)}, \tag{1.1}$$

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(ii) there exists a constant $\Lambda \geq 0$, such that for all $x, y \in \overline{\Omega}$,

$$|k(x) - k(y)| \leq \Lambda \frac{|x-y|}{R_0} \quad \text{(Lipschitz continuity)}, \tag{1.2}$$

where R_0 is a positive constant related to the size of Ω (see Th. 1.1 and Sect. 2 below for a precise definition).

For Ω, D, k, u_0, f assigned, suppose that u solves the following parabolic problem, which we call the *direct problem*:

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 & \text{in } (0,T) \times \Omega \backslash \overline{D}, \\ u(0) = u_0 & \text{in } \Omega \backslash \overline{D}, \\ u = 0 & \text{on } (0,T) \times \partial D, \\ u(t,\sigma) = f(t,\sigma) & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{1.3}$$

It is well-known that, under reasonable assumptions on the data, problem (1.3) has a unique solution, and that the thermal flux

$$k(\sigma)\nabla u(t,\sigma) \cdot \mathbf{n}(\sigma)$$

is well-defined for $(t,\sigma) \in (0,T) \times \partial\Omega$. (Here and in the sequel $\mathbf{n}(\sigma)$ denotes the exterior unit normal at $\sigma \in \partial\Omega$.) In the present paper we are interested in the following two problems:

(a) *uniqueness* result: for *any* u_0, f assigned in (1.3), does the thermal flux $k\nabla u \cdot \mathbf{n}_{|(0,T)\times\Gamma}$ on $(0,T) \times \Gamma$ of the corresponding solution u determine *uniquely* the domain D in Ω?
(b) *stability* result: for *any* u_0 and f assigned in (1.3), does D depend *continuously* on the thermal flux $k\nabla u \cdot \mathbf{n}_{|(0,T)\times\Gamma}$?

Here and in the sequel Γ denotes a relatively open piece of $\partial\Omega$.

We begin by observing that, following a counterexample of Bryan and Caudill [6], uniqueness result (a) can fail without additional hypotheses on the data u_0, f. In fact let D_1, D_2 be the following two rectangles in $\mathbb{R}^2$: $D_1 := (0,\pi) \times (0,2\pi)$, $D_2 := (0,\pi) \times (0,\pi)$, and let Ω be a bounded domain in $\mathbb{R}^2$ containing D_1. For $u(t,x_1,x_2) := \mathrm{e}^{-2t} \sin x_1 \sin x_2$, let us define the functions u_1, u_2 as follows: $u_1 := u_{|(0,T)\times\Omega\backslash\overline{D_1}}$, $u_2 := u_{|(0,T)\times\Omega\backslash\overline{D_2}}$. It is clear that u_1, u_2 are solutions of (1.3), respectively when $D := D_i$, $i = 1,2$, $k(x) := I_2$ (I_2 is the 2×2 identity matrix). Moreover $\frac{\partial}{\partial\mathbf{n}} u_1 = \frac{\partial}{\partial\mathbf{n}} u_2$ on $(0,T) \times \partial\Omega$. So in this case uniqueness fails.

On the other hand if we assume that in (1.3) the initial temperature u_0 is *constant* (but *a priori unknown*), then it is not difficult to prove uniqueness result (a) for *any* datum $f \in C^1([0,T], H^{\frac{1}{2}}(\partial\Omega))$, $f \not\equiv 0$. In fact suppose that there exist two domains D_1, D_2 (here and in the sequel $\Omega\backslash\overline{D_i}$ is supposed connected) and two constants c_1, c_2 such that the corresponding solutions $u_i \in C((0,T), H^1(\Omega\backslash D_i)) \cap C^1([0,T], L^2(\Omega\backslash D_i))$ of (1.3), when $D := D_i$, and the initial temperature $u_{i0} \equiv c_i$, have the same thermal flux on $(0,T) \times \Gamma$, that is

$$k\nabla u_1(t) \cdot \mathbf{n}_{|\Gamma} = k\nabla u_2(t) \cdot \mathbf{n}_{|\Gamma} \quad \text{in } H^{-\frac{1}{2}}(\Gamma) \quad \text{for all } t \in (0,T).$$

We denote by G the connected component of $\Omega\backslash(\overline{D_1 \cup D_2})$ such that $\partial\Omega \subset \partial G$. Let us define

$$u := u_1 - u_2 \quad \text{in } (0,T) \times G.$$

Then u solves

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 \text{ in } (0,T)\times G, \\ u = 0 \text{ on } (0,T)\times\partial\Omega, \\ k\nabla u\cdot\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma. \end{cases}$$

By the unique continuation principle (see Lin [15]) it follows that $u\equiv 0$ in $[0,T)\times\overline{G}$, that is

$$u_1 = u_2 \quad \text{in } [0,T)\times\overline{G}. \tag{1.4}$$

This in particular implies that $c_1 = c_2$. Next let us denote by

$$v_i := u_{it} \quad \text{in } [0,T)\times\Omega\backslash\overline{D_i}.$$

Let assume, for instance, that $D_2\backslash D_1\neq\emptyset$. We have that $(\Omega\backslash D_1)\backslash G\neq\emptyset$, and v_1 solves

$$\begin{cases} v_{1t} - \operatorname{div}(k(x)\nabla v_1) = 0 \ \text{ in } (0,T)\times\Omega\backslash\overline{D_1}, \\ v_1(0) = 0 \ \text{ in } \Omega\backslash\overline{D_1}, \\ v_1 = 0 \ \text{ on } (0,T)\times\partial D_1, \\ v_1 = f_t \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{1.5}$$

Let $t_0\in(0,T]$ be fixed. Multiplying the equation in (1.5) by v_1, and integrating by parts over $(0,t_0)\times(\Omega\backslash D_1)\backslash G$, we obtain

$$\begin{aligned} \frac{1}{2}\int\limits_{(\Omega\backslash D_1)\backslash G} |v_1(t_0)|^2\,\mathrm{d}x &= -\int\limits_0^{t_0}\int\limits_{(\Omega\backslash D_1)\backslash G} k(x)\nabla v_1(t)\cdot\nabla v_1(t)\mathrm{d}x\mathrm{d}t + \int\limits_0^{t_0}\int\limits_{\partial((\Omega\backslash D_1)\backslash G)} k\nabla v_1(t)\cdot\mathbf{n}v_1(t)\mathrm{d}\sigma\mathrm{d}t \\ &\le \int\limits_0^{t_0}\int\limits_{\partial((\Omega\backslash D_1)\backslash G)} k\nabla v_1(t)\cdot\mathbf{n}v_1(t)\mathrm{d}\sigma\mathrm{d}t. \end{aligned} \tag{1.6}$$

Since $v_i\equiv 0$ on $(0,T)\times\partial D_i$, from (1.4) and (1.6) we derive

$$\int\limits_{(\Omega\backslash D_1)\backslash G} |v_1(t_0)|^2\,\mathrm{d}x = 0 \quad \text{for all } t_0\in[0,T].$$

Hence the unique continuation principle implies $v_1\equiv 0$ in $[0,T)\times\Omega\backslash D_1$, that is $u_1\equiv c$ in $[0,T)\times\Omega\backslash D_1$, where $c := c_1 = c_2$. Again, since $u_1\equiv 0$ on $(0,T)\times\partial D_1$, we derive that $c=0$, that is $u_1\equiv 0$ in $[0,T)\times\Omega\backslash D_1$. This implies that $f\equiv 0$ on $(0,T)\times\partial\Omega$, which yields a contradiction. The uniqueness result (a) is then proved.

Concerning the stability result (b), we recall that Vessella [18] proved a continuous dependence of logarithmic type of D from $\frac{\partial}{\partial\mathbf{n}}u_{|(t_0,t_1)\times\Gamma}$ (here the interval $(t_0,t_1)\subset[0,T]$), in the case where in (1.3) $n=3$, $k=I_3$ (I_3 is the 3×3 identity matrix), and the temperature f on $(0,T)\times\partial\Omega$ is *monotone* with respect to the time variable t. In [8] Canuto *et al.* have considered the analogous of problem (1.3), but for Neumann boundary conditions (that is the Dirichlet boundary conditions $u=0$ on $(0,T)\times\partial D$, $u=f$ on $(0,T)\times\partial\Omega$ appearing in (1.3) are replaced by $k\nabla u\cdot\mathbf{n}=0$ on $(0,T)\times\partial D$, $k\nabla u\cdot\mathbf{n}=0$ on $(0,T)\times\partial D$, $k\nabla u\cdot\mathbf{n}=g$ on $(0,T)\times\partial\Omega$ respectively). They proved a continuous dependence of logarithmic type of D from $u_{|(0,T)\times\Gamma}$.

The corresponding problem for the elliptic case has been studied too, in a previous paper by Alessandrini *et al.* [4] who proved a logarithmic stability estimate. Let us point out that, to fix ideas, we have considered in the present paper the problem of determination of cavities. More generally, we can prove logarithmic stability estimates also when unknown portions of $\partial\Omega$ are to be determined (see [4,8] for analogous results). Finally we stress that, in the elliptic case, counterexamples by Alessandrini and Rondi [3] show that logarithmic stability is best possible. This suggests that also in the parabolic case stability estimates better than logarithmic cannot be expected.

We give now a list of our *a priori* assumptions on the domains Ω, D, and on the boundary datum f in (1.3), under which we shall prove Theorem 1.1.

We assume that Ω is a bounded domain in $\mathbb{R}^n$ of class

$$C^{1,1} \quad \text{with constants } R_0, E, \tag{1.7}$$

and that D is a bounded domain in $\mathbb{R}^n$ of class

$$C^{1,\alpha},\ 0<\alpha\leq 1, \quad \text{with constants } R_0, E, \tag{1.8}$$

such that $D\subset\Omega$, $\mathrm{dist}(\partial D,\partial\Omega)\geq R_0$, and $\Omega\backslash D$ is connected. For a precise definition of (1.7, 1.8) see Definition 2.1 below. Given $M>0$, we assume:

$$|\Omega|\leq MR_0^n. \tag{1.9}$$

Here and in the sequel $|\Omega|$ denotes the Lebesgue measure of Ω. We observe that (1.7) and (1.8) imply a lower bound on the diameter of Ω and D respectively. Moreover, by combining (1.7) with (1.9), an upper bound on the diameter of Ω can also be obtained.

We shall assume the following on the Dirichlet datum f:

$$f\in H^{3/4}((0,T),H^{1/2}(\partial\Omega)), \quad f\not\equiv 0,$$

and, for a given constant $F>0$,

$$\frac{\|f\|_{1/4,1/2}}{\|f\|_{L^2((0,T)\times\partial\Omega)}}\leq F, \tag{1.10}$$

where in order to simplify the notations, here and below $\|f\|_{1/4,1/2}$ denotes the norm $\|f\|_{H^{1/4}((0,T),H^{1/2}(\partial\Omega))}$.

We now state the main result of the present paper.

Theorem 1.1. *Let Ω be a bounded and connected domain in $\mathbb{R}^n$ of class $C^{1,1}$, with constants R_0, E, and let Γ be a relatively open piece of $\partial\Omega$. Let $k(x)$ be a $n\times n$ symmetric matrix-valued function in $\overline{\Omega}$ satisfying assumptions (1.1, 1.2). Let D_i, $i=\{1,2\}$, be two domains of class $C^{1,\alpha}$, $0<\alpha\leq 1$, with constants R_0, E, such that $D_i\subset\Omega$, $\mathrm{dist}(\partial D_i,\partial\Omega)\geq R_0$, and $\Omega\backslash D_i$ is connected. Let $f\in H^{3/4}((0,T),H^{1/2}(\partial\Omega))$ satisfy (1.10) such that $u_i\in H^1((0,T),H^1(\Omega\backslash D_i))$ is solution of (1.3) when $D:=D_i$, and the initial temperature $u_{i0}=0$ in $\Omega\backslash\overline{D_i}$. If*

$$R_0\,\|k\nabla u_1\cdot\mathbf{n}-k\nabla u_2\cdot\mathbf{n}\|_{L^2((0,T)\times\Gamma)}\leq T^{\frac{1}{2}}R_0^{(n-1)/2}\epsilon, \tag{1.11}$$

then

$$d_{\mathcal{H}}(D_1,D_2)\leq CR_0\left|\ln\left(\frac{T^{\frac{1}{2}}R_0^{(n-1)/2}}{\|f\|_{1/4,1/2}}\epsilon\right)\right|^{-\frac{\kappa}{n}}, \tag{1.12}$$

where the constants C, κ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only.

We recall that the Hausdorff distance $d_{\mathcal{H}}(D_1, D_2)$ between bounded sets D_1 and D_2 of $\mathbb{R}^n$ is the number

$$d_{\mathcal{H}}(D_1, D_2) := \max\left\{\sup_{x\in D_1} \operatorname{dist}(x, D_2), \sup_{x\in D_2} \operatorname{dist}(D_1, x)\right\}\cdot$$

The proof of Theorem 1.1 has the same structure of that in [4] (Ths. 2.1, 2.2) and in [8] (Th. 4.1). As a first step we prove a ln ln-type estimate of the Hausdorff distance between the domains Ω_1, Ω_2 (where $\Omega_i := \Omega\backslash D_i$), by using as main tools the so-called three spheres and three cylinders inequality for solutions of parabolic equations given in Section 4 (see Ths. 4.1, 4.3, and Cor. 4.2). As a second step, employing in a more refined way the above mentioned inequalities and a geometric lemma (Prop. 5.5), which has been proved in [4], we obtain a logarithmic stability estimate of the Hausdorff distance between Ω_1, Ω_2, which implies, by a simple reasoning, the desired result, *i.e.* estimate (1.12). The main difference between the stability result established in [8] (Th. 4.1), and our result, *i.e.* Theorem 1.1, lies in the hypothesis of *regularity* of the unknown (a part of the boundary I in [8], and a cavity D in our result), which is of class $C^{1,1}$ in [8], and is of class $C^{1,\alpha}$, $0 < \alpha \le 1$, in our result. This difference on the regularity is a consequence of the strong unique continuation principle at the boundary for elliptic operators established by Adolfsson and Escauriaza [1], which need, for the Neumann case, that the boundary of the domain is of class $C^{1,1}$, while, for the Dirichlet case, it is sufficient that the boundary is of class $C^{1,\alpha}$, $0 < \alpha \le 1$.

The remainder of the paper is organized as follows: in Section 2 we give some notations and definitions; in Section 3 we introduce the so-called technique of elliptic continuation for solutions of parabolic equations which allow us to define, starting from a solution of a parabolic problem, a solution for a related corresponding elliptic problem. In Section 3 we establish also a Cauchy estimate for the solution of such an elliptic problem. This estimate will be crucial in Section 4 to prove a three cylinders inequality at the boundary for a parabolic equation. In Section 5 we prove some auxiliary propositions which we shall use in Section 6 to prove Theorem 1.1. Finally, the appendix (Sect. 7) contains the proof of Lemma 3.3 and some interpolation and traces inequalities, which we use throughout the paper.

2. Notations and definitions

We shall fix the space dimension $n \ge 2$ throughout the paper. Therefore we shall omit the dependence of the various quantities on n.

We shall use the letter c to denote absolute constants, and the letters C, $\tilde{C}$ to denote constants depending on some *a priori* data. The value of the constants may change from line to line, but we have specified their dependence everywhere they appear.

We shall identify $\mathbb{R}^2$ and $\mathbb{C}$.

As usual we shall denote by $x = (x_1, \cdots, x_n)$ a point in $\mathbb{R}^n$ and by $x' = (x_1, \cdots, x_{n-1})$ the first $(n-1)$-components of x. $X = (y, x)$ is a point in $\mathbb{R}^{n+1}$, for $x \in \mathbb{R}^n$, whereas $X' = (y, x')$ are the first n-components of X.

By $B_r(a)$ ($\Delta_r(a)$, $\Delta'_r(a)$, $D_r(a)$ respectively) we shall denote the open ball in $\mathbb{R}^{n+1}$ ($\mathbb{R}^n$, $\mathbb{R}^{n-1}$, $\mathbb{C}$ respectively) centered at a, of radius r. Sometimes we shall write for brevity B_r, Δ_r, Δ'_r, D_r instead of $B_r(0)$, $\Delta_r(0)$, $\Delta'_r(0)$, $D_r(0)$, respectively. We shall denote by $B_r^+ = \{X \in B_r \text{ s.t. } y > 0\}$, $\Delta_r^+ = \{x \in \Delta_r \text{ s.t. } x_n > 0\}$.

When dealing with $n+1$ variables (y, x), we shall denote $\nabla = \nabla_x$, $\operatorname{div} = \operatorname{div}_x$, $D^2 = D_x^2$. Sometimes we shall write $\partial_y^k w$ instead of $\frac{\partial^k w}{\partial y^k}$, w_y instead of $\frac{\partial w}{\partial y}$ and w_{yy} instead of $\frac{\partial^2 w}{\partial y^2}$. Similarly, for brevity, we shall write, for example, $\|w(y)\|_{L^2(\Omega)}$ instead of $\|w(y,\cdot)\|_{L^2(\Omega)}$, and $\int\limits_\Omega |w(y)|^2\,\mathrm{d}x$ instead of $\int\limits_\Omega |w(y,x)|^2\,\mathrm{d}x$.

When representing locally a boundary as a graph, it will be convenient to use the following notation:

Definition 2.1. Let Ω be a bounded open set in $\mathbb{R}^n$. We shall say that a portion Γ of $\partial\Omega$ is of Lipshitz class (resp. of class $C^{1,\alpha}$, $0 < \alpha \le 1$) with constants R_0, $E > 0$, if, for any $P \in \Gamma$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$\Omega \cap \Delta_{R_0} = \{x \in \Delta'_{R_0} \ s.t. \ x_n > \varphi(x')\},$$

where φ is a $C^{0,1}$ function (resp. φ is a $C^{1,\alpha}$ function) on $\Delta'_{R_0} \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = 0 \quad \text{(and resp. } \varphi(0) = |\nabla\varphi(0)| = 0)$$

and

$$\|\varphi\|_{C^{0,1}(\Delta'_{R_0})} \le ER_0 \quad (\text{resp. } \|\varphi\|_{C^{1,\alpha}(\Delta'_{R_0})} \le ER_0). \qquad \square$$

Remark 2.2. We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous, and coincide with the standard definition when $R_0 = 1$ and $T = 1$. For instance, the norm appearing above is meant as follows

$$\|\varphi\|_{C^{1,\alpha}(\Delta'_{R_0})} := \|\varphi\|_{L^\infty(\Delta'_{R_0})} + R_0 \|\nabla\varphi\|_{L^\infty(\Delta'_{R_0})} + R_0^{1+\alpha} [\nabla\varphi]_{\alpha,\Delta'_{R_0}},$$

where

$$[\nabla\varphi]_{\alpha,\Delta'_{R_0}} := \sup_{\substack{x,y\in\Delta'_{R_0} \\ x\neq y}} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x-y|^\alpha},$$

and $|\cdot|$ is the Euclidean norm. Similarly we shall set

$$\|u\|_{C^{0,1}((0,T)\times\Omega)} := \|u\|_{L^\infty((0,T)\times\Omega)} + R_0 [u]_{1,(0,T)\times\Omega},$$

where

$$[u]_{1,(0,T)\times\Omega} := \sup_{\substack{(t,x),(s,y)\in(0,T)\times\Omega \\ (t,x)\neq(s,y)}} \frac{|u(t,x) - u(s,y)|}{|(t,x) - (s,y)|},$$

$$\|u\|^2_{H^1((0,T),H^1(\Omega))} := \int_0^T \int_\Omega (|u|^2 + T^2 |u_t|^2 + R_0^2 |\nabla u|^2) \mathrm{d}x\mathrm{d}t,$$

and so on for boundary and trace norms such as $\|\cdot\|_{L^2((0,T)\times\Omega)}$, $\|\cdot\|_{H^{1/4}((0,T),H^{1/2}(\partial\Omega))}$.

3. Elliptic continuation for solutions of parabolic equations

In this section we introduce the so-called technique of elliptic continuation for solutions of parabolic equations (see Landis and Oleinik [14] or Lin [15]), which can be traced back to the pioneering work by Ito and Yamabe [12], who introduced this technique in 1959 to prove unique continuation properties for solutions of

$$\partial_t u - \operatorname{div}(k(x)\nabla u) = 0 \quad \text{in } (0,T)\times\Omega'. \tag{3.1}$$

Roughly speaking this technique consists in the following idea: fixing $t_0 \in (0,T)$, a solution of the parabolic equation (3.1) can be continued to a function $w(t_0; y, x)$ (for values of y in an appropriate interval) which satisfies an elliptic equation in y, x (see Prop. 3.1 below). In this way many properties of the solutions of elliptic equations can be transferred to solutions of parabolic equations.

Here and below we assume that Ω' is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, of class $C^{1,\alpha}$, $0 < \alpha \leq 1$, with constant R_0, E, $x_0 \in \partial\Omega'$, $R \in (0, R_0/2]$, and $t_0 \in (0,T)$. Moreover we suppose that k is a $n \times n$ symmetric matrix-valued function in Ω' satisfying assumptions (1.1, 1.2) (with Ω replaced by Ω'), and $u \in H^1((0,T), H^1(\Omega' \cap \Delta_{2R}(x_0)))$ is a nonidentically zero solution of

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 \text{ in } (0,T) \times (\Omega' \cap \Delta_{2R}(x_0)), \\ \qquad\qquad u = 0 \text{ on } (0,T) \times ((\partial\Omega') \cap \Delta_{2R}(x_0)). \end{cases} \tag{3.2}$$

The main result in this section is the following:

Proposition 3.1. *Let $A := \min\left\{\sqrt{2}\delta, t_0\sqrt{a_R}\right\}$, where $\delta := \frac{R}{8e\pi\lambda}$, $a_R := \frac{1}{\lambda c_P R^2}$, and c_P is the Poincaré constant. There exists a function $w \in C^\omega((-A,A),\ H^1(\Omega' \cap \Delta_{R/2}(x_0)))$ solution of the following problem*

$$\begin{cases} w_{yy} + \operatorname{div}(k(x)\nabla w) = 0 & \text{in} \quad (-A,A) \times (\Omega' \cap \Delta_{R/2}(x_0)), \\ w(0) = u(t_0) & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w_y(0) = 0 & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w = 0 & \text{on} \quad (-A,A) \times ((\partial\Omega') \cap \Delta_{R/2}(x_0)). \end{cases} \tag{3.3}$$

Moreover, for

$$r \leq \frac{3A}{32e\pi\lambda}, \tag{3.4}$$

and

$$\rho := \frac{8}{3}\sqrt{2}e\pi\lambda r, \quad \widetilde{\rho} := 2\sqrt{2}\rho,$$

the following inequality holds:

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_r(X_0)} |w|^2 \,\mathrm{d}X \leq Cr \left(\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \right)^{\overline{\beta}} \left(\frac{1}{r} \int\limits_{(\mathbb{R}\times\Omega')\cap B_{\widetilde{\rho}}(X_0)} |w|^2 \,\mathrm{d}X \right)^{1-\overline{\beta}}, \tag{3.5}$$

where $X_0 \in \mathbb{R}^{n+1}$ is the point $(0, x_0)$, the constant $C \geq 1$ depends on λ only, and $\overline{\beta} := \frac{\alpha\beta}{1+\alpha}$, $\beta \in (0,1)$ depending on λ only.

(We observe that the choice of r in (3.4) implies that $\widetilde{\rho} < A$.) We recall that $C^\omega(\mathbb{R}, Z)$ denotes the space of real analytic variable functions with values in a Banach space Z, and $\mathrm{d}X$ (resp. $\mathrm{d}x$) is the $(n+1)$-dimensional (resp. n-dimensional) volume Lebesgue measure.

We precede the proof of Proposition 3.1 by some preliminary lemmas.

Lemma 3.2. *Under the assumptions of Proposition 3.1, let $\eta \in C^2[0,+\infty)$ be a cut-off function satisfying:*

$$\eta(t) = \begin{cases} 1 & \text{for } t \in [0,t_0] \\ 0 & \text{for } t \in [T,+\infty) \end{cases}, \quad \text{and} \quad |\eta'| \leq \frac{c}{T-t_0}.$$

There exists a unique solution $u_1 \in C((0,T), H^1(\Omega' \cap \Delta_{2R}(x_0))) \cap C^1([0,T), L^2(\Omega' \cap \Delta_{2R}(x_0)))$ *of the problem:*

$$\begin{cases} u_{1t} - \operatorname{div}(k(x)\nabla u_1) = 0 & in \quad (0,+\infty) \times (\Omega' \cap \Delta_{2R}(x_0)), \\ u_1(0) = 0 & in \quad \Omega' \cap \Delta_{2R}(x_0), \\ u_1 = g & on \quad (0,+\infty) \times \partial(\Omega' \cap \Delta_{2R}(x_0)), \end{cases} \tag{3.6}$$

where $g := \eta(t)u$. *Moreover, for all* $t \geq 0$, *we have*

$$\|u_1(t)\|_{H^1(\Omega' \cap \Delta_{2R}(x_0))} \leq c\mathrm{e}^{-a_R(t-T)_+} C_1 H, \tag{3.7}$$

where $(t-T)_+ := \max(0,(t-T))$,

$$C_1 := \left(\lambda \frac{T}{T-t_0}\left(\mathrm{e}^{\frac{T}{T-t_0}} + \frac{R_0^2}{T-t_0}\right)\right)^{\frac{1}{2}}, \tag{3.8}$$

and

$$H := \max_{0 \leq t \leq T} \|u(t)\|_{H^1(\Omega' \cap \Delta_{2R}(x_0))}.$$

(η' denotes the derivative of η, and a_R is as in Prop. 3.1.)

Proof of Lemma 3.2. The proof follows step by step, up to the obvious changes, from the proof of Lemma 3.1.2 in [8]. □

Let us still denote by u_1 the extension by 0 of u_1 to $\mathbb{R} \times (\Omega' \cap \Delta_{2R}(x_0))$, and let $\widetilde{u_1}(\mu,x)$ be the Fourier transform of $u_1(t,x)$ with respect to the time variable t, that is

$$\widetilde{u_1}(\mu,x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-i\mu t} u_1(t,x)\mathrm{d}t. \tag{3.9}$$

The following result holds:

Lemma 3.3. *Under the assumptions of Proposition 3.1, let* $\widetilde{u_1}(\mu,x)$ *be as above. Then* $\widetilde{u_1} \in C^\omega(\mathbb{R}, H^1(\Omega' \cap \Delta_{2R}(x_0)))$ *solves*

$$\begin{cases} i\mu\widetilde{u_1} - \operatorname{div}(k(x)\nabla\widetilde{u_1}) = 0 & in \quad \mathbb{R} \times (\Omega' \cap \Delta_{2R}(x_0)), \\ \widetilde{u_1} = 0 & on \quad \mathbb{R} \times ((\partial\Omega') \cap \Delta_{2R}(x_0)). \end{cases}$$

Moreover

$$\|\widetilde{u_1}(\mu)\|_{H^1(\Omega' \cap \Delta_{R/2}(x_0))} \leq cC_1 H\mathrm{e}^{-\sqrt{|\mu|}\delta}\left(T + \frac{1}{a_{R/4}}\right). \tag{3.10}$$

(Here the constants c, C_1, H are as in Lem. 3.2, and $\delta := \frac{R}{8e\pi\lambda}$ is as in Prop. 3.1.)

Proof of Lemma 3.3. See the appendix, Section 7. □

Lemma 3.4. *Under the assumptions of Proposition 3.1, let $\ell > 0$, and let $\rho \in (0,R)$. For $f \in H^1(\Omega' \cap \Delta_{2\rho}(x_0))$, assume that $w \in C^\omega((-2\ell, 2\ell), H^1(\Omega' \cap \Delta_{2\rho}(x_0))$ solves*

$$\begin{cases} w_{yy} + \operatorname{div}(k(x)\nabla w) = 0 & in \quad (-2\ell, 2\ell) \times (\Omega' \cap \Delta_{2\rho}(x_0)), \\ w(0) = f & in \quad \Omega' \cap \Delta_{2\rho}(x_0), \\ w_y(0) = 0 & in \quad \Omega' \cap \Delta_{2\rho}(x_0), \\ w = 0 & on \quad (-2\ell, 2\ell) \times ((\partial\Omega') \cap \Delta_{2\rho}(x_0)). \end{cases}$$

Then, for

$$\rho_1 = \left(\pi e \lambda \left(\frac{1}{\rho^2} + \frac{1}{\ell^2}\right)^{1/2}\right)^{-1}, \tag{3.11}$$

$$\rho_2 = \min\left\{\rho_1, \frac{\rho}{4\sqrt{\lambda}}\right\}, \tag{3.12}$$

$$\rho_3 = \frac{1}{2}(\rho - \sqrt{\lambda}\rho_2), \tag{3.13}$$

and for every $y \in \left(-\frac{3}{8}\rho_1, \frac{3}{8}\rho_1\right)$, the following inequality holds:

$$\int\limits_{\Omega' \cap \Delta_{\rho_3}(x_0)} \left(|w_y(y)|^2 + |\nabla w(y)|^2\right) \mathrm{d}x \le C \left(\|\nabla f\|^2_{L^2(\Omega' \cap \Delta_{\rho/4}(x_0))}\right)^\beta$$
$$\times \left(\frac{1}{\ell\rho^2}\|w\|^2_{L^2((-2\ell,2\ell)\times(\Omega' \cap \Delta_{2\rho}(x_0)))} + \|\nabla f\|^{21-\beta}_{L^2(\Omega' \cap \Delta_{\rho/4}(x_0))}\right)^{1-\beta}, \tag{3.14}$$

where the constant C depends on λ and $\ell\rho^{-1}$ only, and β is as in Proposition 3.1.

Proof of Lemma 3.4. We divide the proof into three steps.

Step 1: In this step we prove that the power series

$$\sum_{j=0}^{+\infty} \partial_y^j w(0) \frac{z^j}{j!} \tag{3.15}$$

converges in $C^{1,\alpha}(\overline{\Omega' \cap \Delta_{\frac{1}{4}\rho}(x_0)}) \cap H^2_{\mathrm{loc}}(\Omega' \cap \Delta_{\frac{3}{4}\rho}(x_0))$ for every complex number z such that $|z| < \rho_1$. Let us denote $Q_0 := (-2\ell, 2\ell) \times (\Omega' \cap \Delta_{2\rho}(x_0))$, $Q_1 := (-\ell, \ell) \times (\Omega' \cap \Delta_\rho(x_0))$. By a slight modification of the arguments used to prove Lemma 3.3, we obtain

$$\|\partial_y^j w\|^2_{L^2(Q_1)} \le (C_2 j^2)^j \|w\|^2_{L^2(Q_0)} \quad \text{for every } j \ge 1, \tag{3.16}$$

where

$$C_2 = \pi^2 \lambda^2 \left(\frac{1}{\rho^2} + \frac{1}{\ell^2}\right) \cdot \tag{3.17}$$

Let us fix $j \ge 1$ and let us denote

$$U(y,x) = \partial_y^j w(y,x) \quad \text{in} \quad (-\ell, \ell) \times (\Omega' \cap \Delta_\rho(x_0)). \tag{3.18}$$

We have that $U \in C^\omega((-\ell,\ell), H^1(\Omega' \cap \Delta_\rho(x_0)))$ solves

$$\begin{cases} \partial^2_{yy} U + \operatorname{div}(k(x)\nabla U) = 0 & \text{in} \quad (-\ell,\ell) \times (\Omega' \cap \Delta_\rho(x_0)), \\ U = 0 & \text{on} \quad (-\ell,\ell) \times ((\partial\Omega') \cap \Delta_\rho(x_0)). \end{cases} \tag{3.19}$$

By standard $C^{1,\alpha}$ estimates (see Gilbarg and Trudinger [10]) we have

$$\|U\|_{C^{1,\alpha}((-\frac{\ell}{4},\frac{\ell}{4})\times(\Omega'\cap\Delta_{\frac{\rho}{4}}(x_0))} \leq \frac{C}{\rho^{\frac{n+1}{2}}} \|U\|_{L^2((-\ell,\ell)\times(\Omega'\cap\Delta_\rho(x_0))}, \tag{3.20}$$

where the constant C depends on E, α, λ, Λ, R_0/ρ. From (3.16, 3.18, 3.20) we obtain, for every $y \in (-\frac{\ell}{4}, \frac{\ell}{4})$, and for every $j \geq 1$,

$$\left\|\partial^j_y w(y)\right\|_{C^{1,\alpha}(\Omega'\cap\Delta_{\frac{\rho}{4}}(x_0))} \leq \frac{C}{\rho^{\frac{n+1}{2}}} C_2^{\frac{j}{2}} j^j \|w\|_{L^2(Q_0)}. \tag{3.21}$$

So (3.21) yields the convergence in $C^{1,\alpha}(\overline{\Omega' \cap \Delta_{\frac{1}{4}\rho}(x_0)})$ of the power series (3.15) in the disk D_{ρ_1}, where ρ_1 is given by (3.11).

For any $\varphi \in L^2(\Omega' \cap \Delta_\rho(x_0))$, let

$$F(y) := \int\limits_{\Omega'\cap\Delta_\rho(x_0)} w(y)\varphi \mathrm{d}x.$$

By (3.16) and by the interpolation inequality (7.10) (see the Appendix) we obtain, for every $j \geq 1$,

$$|F^{(j)}(y)|^2 \leq \frac{C}{\ell} C_2^j (j+1)^{2(j+1)} \|w\|^2_{L^2(Q_0)} \|\varphi\|^2_{L^2(\Omega'\cap\Delta_\rho(x_0))} \tag{3.22}$$

for every $y \in (-\ell,\ell)$, where C depends on λ and $\ell\rho^{-1}$ only. Therefore, for every $j \geq 1$,

$$\int\limits_{\Omega'\cap\Delta_\rho(x_0)} |\partial^j_y w(y)|^2 \mathrm{d}x \leq \frac{C}{\ell} C_2^j (j+1)^{2(j+1)} \|w\|^2_{L^2(Q_0)} \tag{3.23}$$

for every $y \in (-\ell,\ell)$. Let us fix $j \geq 1$ and $y \in (-\ell,\ell)$, and let us denote

$$g(y) = \partial^{j+2}_y w(y) \quad \text{in} \quad \Omega' \cap \Delta_\rho(x_0), \tag{3.24}$$

$$U(y) = \partial^j_y w(y) \quad \text{in} \quad \Omega' \cap \Delta_\rho(x_0). \tag{3.25}$$

We have that $U(y) \in H^1(\Omega' \cap \Delta_\rho(x_0))$ solves

$$\begin{cases} \operatorname{div}(k\nabla U(y)) = -g(y) & \text{in} \quad \Omega' \cap \Delta_\rho(x_0), \\ U(y) = 0 & \text{on} \quad (\partial\Omega') \cap \Delta_\rho(x_0). \end{cases} \tag{3.26}$$

From Caccioppoli inequality we have

$$\|\nabla U(y)\|^2_{L^2(\Omega'\cap\Delta_{\frac{3}{4}\rho}(x_0))} \leq C\left(\rho^2 \|g(y)\|^2_{L^2(\Omega'\cap\Delta_\rho(x_0))} + \frac{1}{\rho^2}\|U(y)\|^2_{L^2(\Omega'\cap\Delta_\rho(x_0))}\right), \tag{3.27}$$

where C depends on λ only. Now let $\overline{x} \in \Omega' \cap \Delta_{\frac{3}{4}\rho}(x_0)$ and let r be such that

$$\Delta_r(\overline{x}) \subset \Omega' \cap \Delta_{\frac{3}{4}\rho}(x_0).$$

Choosing as test functions $V(y) = (\eta^2 U_{x_i}(y))_{x_i}$, $i = 1, ..., n$, where η is a cut off function, we obtain, by standard H^2_{loc} estimates [10], and by (3.27)

$$\|D^2U(y)\|^2_{L^2(\Delta_{\frac{r}{2}}(\overline{x}))} \leq C\left(\frac{1}{r^2} + \frac{\Lambda^2}{R_0^2}\right)\left(\rho^2\|g(y)\|^2_{L^2(\Omega'\cap\Delta_\rho(x_0))} + \frac{1}{\rho^2}\|U(y)\|^2_{L^2(\Omega'\cap\Delta_\rho(x_0))}\right), \tag{3.28}$$

where C depends on λ only. By (3.23–3.25, 3.27, 3.28) we have, for every $j \geq 1$,

$$\int\limits_{\Delta_{\frac{r}{2}}(\overline{x})} |\nabla\partial_y^j w(y)|^2 \mathrm{d}x \leq \frac{C}{\ell\rho^2} C_2^j (j+3)^{2(j+3)} \|w\|^2_{L^2(Q_0)}, \tag{3.29}$$

$$\int\limits_{\Delta_{\frac{r}{2}}(\overline{x})} |D^2\partial_y^j w(y)|^2 \mathrm{d}x \leq \frac{C}{\ell\rho^2}\left(\frac{1}{r^2} + \frac{\Lambda^2}{R_0^2}\right) C_2^j (j+3)^{2(j+3)} \|w\|^2_{L^2(Q_0)}, \tag{3.30}$$

where the constant C in (3.29, 3.30) depends on λ and $\ell\rho^{-1}$ only. Finally (3.29, 3.30) yield the convergence in $H^2_{\text{loc}}(\Omega' \cap \Delta_{\frac{3}{4}\rho}(x_0))$ of the power series (3.15) in the disk D_{ρ_1}, where ρ_1 is given by (3.11).

Let us denote, for $x \in \Omega' \cap \Delta_{\frac{1}{4}\rho}(x_0)$,

$$\begin{aligned} W(z,x) : &= \sum_{j=0}^{+\infty} \partial_y^j w(0,x)\frac{z^j}{j!}, \quad \text{for } z \in D_{\rho_1}, \\ v(\xi,x) : &= W(i\xi,x), \quad \text{for } |\xi| < \rho_1. \end{aligned}$$

Step 2: In this step we prove that for every $\xi \in (-\rho_2, \rho_2)$ (ρ_2 as in (3.12)) we have

$$\int\limits_{\Omega'\cap\Delta_{\rho(\xi)}(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi)\cdot\nabla v(\xi))\mathrm{d}x \leq \int\limits_{\Omega'\cap\Delta_{\frac{\rho}{4}}(x_0)} |\nabla f|^2 \,\mathrm{d}x, \tag{3.31}$$

where

$$\rho(\xi) = \frac{\rho}{4} - \sqrt{\lambda}|\xi|. \tag{3.32}$$

First, let us observe that v is real and solves the following hyperbolic initial boundary value problem:

$$\begin{cases} v_{\xi\xi}(\xi) - \mathrm{div}(k\nabla v(\xi)) = 0 & \text{in} \quad (-\rho_1,\rho_1)\times(\Omega'\cap\Delta_{\frac{\rho}{4}}(x_0)), \\ v(0,x) = f(x) & \text{in} \quad \Omega'\cap\Delta_{\frac{\rho}{4}}(x_0), \\ v_\xi(0,x) = 0 & \text{in} \quad \Omega'\cap\Delta_{\frac{\rho}{4}}(x_0), \\ v = 0 & \text{on} \quad (-\rho_1,\rho_1)\times((\partial\Omega')\cap\Delta_{\frac{\rho}{4}}(x_0)). \end{cases} \tag{3.33}$$

We shall derive estimate (3.31) from an energy estimate for the problem (3.33). To this aim, let us denote

$$E(\xi) = \frac{1}{2}\int\limits_{\Omega'\cap\Delta_{\rho(\xi)}(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi)\cdot\nabla v(\xi))\mathrm{d}x. \tag{3.34}$$

Since $\xi \to v(\xi)$ is an analytic function from $(-\rho_1, \rho_1)$ to $C^{1,\alpha}(\overline{\Omega' \cap \Delta_{\frac{1}{4}\rho}(x_0)})$ we have that $\partial_\xi^j v(\xi) \in C^{1,\alpha}(\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0))$ for every $\xi \in (-\rho_2, \rho_2)$ and for every $j \geq 1$, where ρ_2 is given by (3.12). For every $\xi \in (-\rho_2, \rho_2)$, by the coarea formula we have the following equality

$$E(\xi) = \frac{1}{2} \int_0^{\rho(\xi)} \mathrm{d}\eta \int_{\Omega' \cap \partial\Delta_\eta(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi) \cdot \nabla v(\xi))\mathrm{d}\sigma, \tag{3.35}$$

where $\mathrm{d}\sigma$ is the $(n-1)$-dimensional surface Lebesgue measure. The derivative of $E(\xi)$ is equal to

$$\begin{aligned} E'(\xi) = & \int_0^{\rho(\xi)} \mathrm{d}\eta \int_{\Omega' \cap \partial\Delta_\eta(x_0)} (v_\xi(\xi)v_{\xi\xi}(\xi) + k\nabla v(\xi) \cdot \nabla v_\xi(\xi))\mathrm{d}\sigma \\ & -\frac{\sqrt{\lambda}}{2} \int_{\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi) \cdot \nabla v(\xi))\mathrm{d}\sigma \\ = & \int_{\Omega' \cap \Delta_{\rho(\xi)}(x_0)} (v_\xi(\xi)v_{\xi\xi}(\xi) + k\nabla v(\xi) \cdot \nabla v_\xi(\xi))\mathrm{d}x \\ & -\frac{\sqrt{\lambda}}{2} \int_{\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi) \cdot \nabla v(\xi))\mathrm{d}\sigma. \end{aligned} \tag{3.36}$$

Moreover, since $v(\xi) \in H^2_{\mathrm{loc}}(\Omega' \cap \Delta_{\frac{3}{4}\rho}(x_0))$, a simple calculation gives

$$k\nabla v(\xi) \cdot \nabla v_\xi(\xi) = -\mathrm{div}(k\nabla v(\xi))v_\xi + \mathrm{div}(kv_\xi \nabla v(\xi)) \quad \text{in } \Omega' \cap \Delta_{\rho(\xi)}(x_0). \tag{3.37}$$

So by Green's formula and the fact that $v = 0$ on $(-\rho_1, \rho_1) \times ((\partial\Omega') \cap \Delta_{\frac{\rho}{4}}(x_0))$, from (3.36, 3.37) we obtain

$$E'(\xi) = \int_{\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0)} k\nabla v(\xi) \cdot \mathbf{n} v_\xi(\xi)\mathrm{d}\sigma - \frac{\sqrt{\lambda}}{2} \int_{\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0)} (|v_\xi(\xi)|^2 + k\nabla v(\xi) \cdot \nabla v(\xi))\mathrm{d}\sigma,$$

where $\mathbf{n}$ denotes the outer unit normal to $\Omega' \cap \partial\Delta_{\rho(\xi)}(x_0)$. We have

$$\begin{aligned} |k\nabla v(\xi) \cdot \mathbf{n} v_\xi(\xi)| &\leq (k\nabla v(\xi) \cdot \nabla v(\xi))^{1/2}(k\mathbf{n} \cdot \mathbf{n})^{1/2}|v_\xi(\xi)| \\ &\leq \frac{\sqrt{\lambda}}{2}(|v_\xi(\xi)|^2 + k\nabla v(\xi) \cdot \nabla v(\xi)). \end{aligned}$$

Therefore $E'(\xi) \leq 0$, hence the function E is decreasing, so that $E(\xi) \leq E(0)$ and (3.31) follows.

Step 3: In this step we prove the assertion of Lemma 3.4. For every $z \in D_{\rho_1}$ let us set

$$G(z) := \int_{\Omega' \cap \Delta_{\rho_3}(x_0)} (W_z(z)^2 + k\nabla W(z) \cdot \nabla W(z))\mathrm{d}x \tag{3.38}$$

(where ρ_3 is defined in (3.13)), and let

$$\epsilon^2 = \int_{\Omega' \cap \Delta_{\frac{\rho}{4}}(x_0)} k \nabla f \cdot \nabla f \mathrm{d}x. \tag{3.39}$$

Let $\rho_1' \in (0, \rho_1)$. By (3.23) and (3.27) we obtain

$$|G(z)| \leq \frac{C}{\ell \rho^2 (1 - \rho_1' \rho_1^{-1})^8} \|w\|^2_{L^2(Q_0)}, \quad \text{for every } z \in D_{\rho_1'}, \tag{3.40}$$

where C depends on λ and $\ell \rho^{-1}$ only. On the other side (3.31) gives

$$|G(i\xi)| \leq \epsilon^2, \quad \text{for every } \xi \in (-\rho_2, \rho_2). \tag{3.41}$$

From (3.40, 3.41) and the analytic continuation estimate (see Isakov [11]) we obtain

$$G(y) = \int_{\Omega' \cap \Delta_{\rho_3}(x_0)} (w_y(y)^2 + k \nabla w(y) \cdot \nabla w(y)) \mathrm{d}x \leq \frac{1}{(1 - \rho_1' \rho_1^{-1})^8} \left(\int_{\Omega' \cap \Delta_{\frac{\rho}{4}}(x_0)} k \nabla f \cdot \nabla f \mathrm{d}x \right)^{\omega(0,y)}$$
$$\times \left(\frac{C}{\ell \rho^2} \|w\|^2_{L^2(Q_0)} + \int_{\Omega' \cap \Delta_{\frac{\rho}{4}}(x_0)} k \nabla f \cdot \nabla f \mathrm{d}x \right)^{1 - \omega(0,y)}, \tag{3.42}$$

where $\omega(\xi, y)$ is the harmonic measure of $\{i\xi \text{ s.t. } \xi \in [-\frac{\rho_2}{2}, \frac{\rho_2}{2}]\}$ with respect to $\{y + i\xi \in \mathbb{C} \text{ s.t. } y^2 + \xi^2 = (\rho_1')^2\}$ and C depends on λ and $\ell\rho^{-1}$ only. Now, let us choose $\rho_1' = \frac{3}{4}\rho_1$, so that $\frac{\rho_2}{2} < \rho_1' < \rho_1$. We have that $\omega(0, y) \geq \beta > 0$ for every $y \in (-\frac{3}{8}\rho_1, \frac{3}{8}\rho_1)$, where β depends on λ and Λ only. Therefore estimate (3.14) follows by (3.42).

The proof of Lemma 3.4 is complete. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Let us define

$$w_1(y, x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{it_0\mu} \widetilde{u_1}(\mu, x) \cosh(\sqrt{-i\mu}y) \mathrm{d}\mu,$$

where $\widetilde{u_1}$ has been introduced in (3.9). By (3.10) it follows that $w_1(y) \in H^1(\Omega' \cap \Delta_{R/2}(x_0))$, for $y \in (-\sqrt{2}\delta, \sqrt{2}\delta)$. Moreover $w_1 \in C^\omega((-\sqrt{2}\delta, \sqrt{2}\delta), H^1(\Omega' \cap \Delta_{R/2}(x_0)))$ and solves

$$\begin{cases} w_{1yy} + \mathrm{div}(k(x) \nabla w_1) = 0 & \text{in} \quad (-\sqrt{2}\delta, \sqrt{2}\delta) \times (\Omega' \cap \Delta_{R/2}(x_0)), \\ w_1(0) = u_1(t_0) & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w_{1y}(0) = 0 & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w_1 = 0 & \text{on} \quad (-\sqrt{2}\delta, \sqrt{2}\delta) \times ((\partial\Omega') \cap \Delta_{R/2}(x_0)). \end{cases}$$

By the classical theory of semigroups (see for example Pazy [17]) we know that there exists a unique $u_2 \in C((0,T), H^1(\Omega' \cap \Delta_{2R}(x_0))) \cap C^1([0,T), L^2(\Omega' \cap \Delta_{2R}(x_0)))$ solution of the problem

$$\begin{cases} u_{2t} - \operatorname{div}(k(x)\nabla u_2) = 0 & \text{in} \quad (0,+\infty) \times (\Omega' \cap \Delta_{2R}(x_0)), \\ u_2(0) = u(0) & \text{in} \quad \Omega' \cap \Delta_{2R}(x_0), \\ u_2 = 0 & \text{on} \quad (0,+\infty) \times \partial(\Omega' \cap \Delta_{2R}(x_0)). \end{cases}$$

We have that $u(t_0) = u_1(t_0) + u_2(t_0)$.

Next let $(\mu_j)_{j=1}^{+\infty}$, $(\varphi_j)_{j=1}^{+\infty}$ be respectively the (negatives) eigenvalues (in nonincreasing order) and the corresponding eigenfunctions of the problem

$$\begin{cases} \operatorname{div}(k\nabla\varphi_j) = \mu_j\varphi_j & \text{in} \quad \Omega' \cap \Delta_{2R}(x_0), \\ \varphi_j = 0 & \text{on} \quad \partial(\Omega' \cap \Delta_{2R}(x_0)), \\ \displaystyle\int\limits_{\Omega' \cap \Delta_{2R}(x_0)} |\varphi_j|^2 \,\mathrm{d}x = 1. \end{cases}$$

Since $(\varphi_j)_{j=1}^{+\infty}$ is an Hilbertian basis in $L^2(\Omega' \cap \Delta_{2R}(x_0))$, we have

$$u_2(t) = \sum_{j=1}^{+\infty} \alpha_j \mathrm{e}^{\mu_j t}\varphi_j \quad \text{in} \quad L^2(\Omega' \cap \Delta_{2R}(x_0)), \tag{3.43}$$

where $\alpha_j := \displaystyle\int\limits_{\Omega' \cap \Delta_{2R}(x_0)} u(0)\varphi_j \mathrm{d}x$. Let us define

$$w_2(y,x) := \sum_{j=1}^{+\infty} \alpha_j \mathrm{e}^{\mu_j t_0}\varphi_j(x)\cosh\left(\sqrt{|\mu_j|}y\right). \tag{3.44}$$

Since, for all $j \geq 1$,

$$c_1 |\mu_j| \leq \|\varphi_j\|^2_{H^1(\Omega' \cap \Delta_{2R}(x_0))} \leq c_2 |\mu_j|,$$

where the constants c_1, c_2 depend on λ only,

$$|\mu_j| \sim Cj^{\frac{2}{n}} \quad \text{as } j \to +\infty,$$

where the constant C depends on λ, Λ, $|\Omega'|$ (see for example Courant and Hilbert [9]), and $a_R \leq |\mu_j|$ for all $j \in \mathbb{N}$, we have that for $y \in (-t_0\sqrt{a_R}, t_0\sqrt{a_R})$ the series in (3.44) converges to $w_2(y)$ in $H^1(\Omega' \cap \Delta_{2R}(x_0))$. Therefore $w_2 \in C^\omega((-t_0\sqrt{a_R}, t_0\sqrt{a_R}), H^1(\Omega' \cap \Delta_{2R}(x_0)))$, and solves

$$\begin{cases} w_{2yy} + \operatorname{div}(k(x)\nabla w_2) = 0 & \text{in} \quad (-t_0\sqrt{a_R}, t_0\sqrt{a_R}) \times (\Omega' \cap \Delta_{2R}(x_0)), \\ w_2(0) = u_2(t_0) & \text{in} \quad \Omega' \cap \Delta_{2R}(x_0), \\ w_{2y}(0) = 0 & \text{in} \quad \Omega' \cap \Delta_{2R}(x_0), \\ w_2 = 0 & \text{on} \quad (-t_0\sqrt{a_R}, t_0\sqrt{a_R}) \times ((\partial\Omega') \cap \Delta_{2R}(x_0)). \end{cases}$$

Defining $w := w_1 + w_2$, we have that $w \in C^\omega((-A, A), H^1(\Omega' \cap \Delta_{R/2}(x_0)))$ (where $A := \min\{\sqrt{2}\delta, t_0\sqrt{a_R}\}$), and solves (3.3).

Let us choose

$$\rho = \ell = \frac{8}{3}\sqrt{2}e\pi\lambda r$$

in estimate (3.14). This choice gives $\rho_1 = \frac{8}{3}r$, ρ_1 as in (3.13). Moreover we have

$$(\mathbb{R} \times \Omega') \cap B_r(X_0) \subset \left(-\frac{3}{8}\rho_1, \frac{3}{8}\rho_1\right) \times (\Omega' \cap \Delta_{\rho_3}(x_0)), \tag{3.45}$$

and

$$(-2\ell, 2\ell) \times (\Omega' \cap \Delta_{2\rho}(x_0)) \subset (\mathbb{R} \times \Omega') \cap B_{\widetilde{\rho}}(X_0). \tag{3.46}$$

Integrating both the sides of inequality (3.14) on $(0, r)$ for $f := u(t_0)$, we obtain, by the inclusions (3.45, 3.46),

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_r(X_0)} |\nabla w|^2 \mathrm{d}X \le Cr\left(\frac{1}{r^3}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_{\tilde{\rho}}(X_0))}\right.$$
$$\left.+\|\nabla u(t_0)\|^2_{L^2(\Omega'\cap\Delta_{\rho/4}(x_0))}\right)^{1-\beta}\|\nabla u(t_0)\|^{2\beta}_{L^2(\Omega'\cap\Delta_{\rho/4}(x_0))}, \tag{3.47}$$

where C only depends on λ. By standard $C^{1,\alpha}$ elliptic estimates [10], we get

$$\rho^{1+\alpha}\left|\nabla w\right|_{\alpha,(\mathbb{R}\times\Omega')\cap B_{\rho/3}(X_0)} \le \frac{C}{\rho^{\frac{n+1}{2}}}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_\rho(X_0))}, \quad \text{for every } \alpha \in (0,1],$$

where C depends on α, λ, Λ only, and by (7.11) we obtain

$$\rho^2 \int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |\nabla u(t_0)|^2 \mathrm{d}x \le C\left(\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} \left|u(t_0)\right|^2 \mathrm{d}x\right)^{\frac{\alpha}{1+\alpha}}$$
$$\times\left(\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} \left|u(t_0)\right|^2 \mathrm{d}x + \frac{1}{\rho}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_\rho(X_0))}\right)^{\frac{1}{1+\alpha}}, \tag{3.48}$$

where C depends on α, λ, Λ only. By (7.12) and by Caccioppoli inequality we have

$$\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} \left|u(t_0)\right|^2 \mathrm{d}x \le \frac{c}{\rho}\left(\int\limits_{(\mathbb{R}\times\Omega')\cap B_{\frac{3}{4}\rho}(X_0)} \left(\left|w\right|^2 + \rho|\nabla w|^2\right)\mathrm{d}X\right) \le \frac{C}{\rho}\int\limits_{(\mathbb{R}\times\Omega')\cap B_\rho(X_0)} \left|w\right|^2 \mathrm{d}X, \tag{3.49}$$

where C depends on λ only. By (3.48, 3.49) we have

$$\rho^2 \int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |\nabla u(t_0)|^2 \mathrm{d}x \le C\left(\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} \left|u(t_0)\right|^2 \mathrm{d}x\right)^{\frac{\alpha}{1+\alpha}}\left(\frac{1}{\rho}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_\rho(X_0))}\right)^{\frac{1}{1+\alpha}}, \tag{3.50}$$

where C depends on α, λ, Λ only. By (3.49, 3.50) we have

$$\rho^2 \int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |\nabla u(t_0)|^2 \mathrm{d}x \leq \frac{C}{\rho}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_\rho(X_0))}, \tag{3.51}$$

where C depends on α, λ, Λ only. By (3.47, 3.50) and (3.51), we obtain

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_r(X_0)} |\nabla w|^2 \mathrm{d}X \leq Cr\left(\frac{1}{r^2}\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |u(t_0)|^2\,\mathrm{d}x\right)^{\bar\beta}\left(\frac{1}{r^3}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_{\bar\rho}(X_0))}\right)^{1-\bar\beta}, \tag{3.52}$$

where C depends on α, λ, Λ only, and $\bar\beta = \frac{\beta\alpha}{1+\alpha}$. By (7.13, 3.52, 3.49) we have

$$\begin{aligned}\int\limits_{(\mathbb{R}\times\Omega')\cap B_r(X_0)} |w|^2\,\mathrm{d}X &\leq c\left(r\int\limits_{\Omega'\cap\Delta_r(x_0)} |u(t_0)|^2\,\mathrm{d}x + r^2\int\limits_{(\mathbb{R}\times\Omega')\cap B_r(X_0)} |\nabla w|^2\mathrm{d}X\right)\\ &\leq Cr\left(\int\limits_{\Omega'\cap\Delta_{\rho/4}(x_0)} |u(t_0)|^2\,\mathrm{d}x\right)^{\bar\beta}\left(\frac{1}{r}\|w\|^2_{L^2((\mathbb{R}\times\Omega')\cap B_{\bar\rho}(X_0))}\right)^{1-\bar\beta},\end{aligned}$$

where C depends on α, λ, Λ only.

The proof of Proposition 3.1 is complete. □

4. A three cylinders inequality at the boundary for a parabolic equation

The main result in the present section is the following three spheres inequality and three cylinders inequality at the *boundary*:

Theorem 4.1 (Three spheres inequality and three cylinders inequality at the boundary)**.** *Let Ω' be a bounded domain in $\mathbb{R}^n$, of class $C^{1,\alpha}$, $0 < \alpha \leq 1$, with constants R_0, E, and let k be a $n\times n$ symmetric matrix-valued function in Ω' satisfying assumptions (1.1, 1.2) (with Ω replaced by Ω'). Let $x_0 \in \partial\Omega'$, and let $R \in (0, R_0/2]$. Assume that $u \in H^1((0,T), H^1(\Omega'))$ is a nonidentically zero solution of the following problem*

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 & \text{in} \quad (0,T)\times\Omega',\\ \qquad\qquad\qquad\quad u = 0 & \text{on} \quad (0,T)\times((\partial\Omega')\cap\Delta_{2R}(x_0)).\end{cases} \tag{4.1}$$

Let $t_0 \in (0,T)$, and let $A := \min\left\{\sqrt{2}\delta, t_0\sqrt{a_R}\right\}$, where δ, a_R are as in Proposition 3.1. There exist constants $\theta^* \in (0,1]$, $C_4 \geq \frac{1}{2}$, with θ^*/R_0, C_4 depending on E and α only, such that for any three numbers r_1, r_2, r_3 verifying

$$0 < r_1 < r_2 < \frac{r_3}{6C_4\lambda},$$

where $r_3 < \min\{\theta^* A, \delta\}$, the following three spheres inequality holds:

$$\int\limits_{\Omega' \cap \Delta_{r_2}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \le \widetilde{C} \left(\frac{r_3}{r_2}\right)^{n+2} \left(\int\limits_{\Omega' \cap \Delta_{r_1}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \right)^{\gamma} \left(\frac{12 C_4 \lambda r_2 r_3}{(6 C_4 \lambda r_2 - r_3)^2} \right)^{1-\tau} \left(\left(1 + \frac{T^2}{R^4}\right)^2 H^2 \right)^{1-\gamma}. \tag{4.2}$$

Here the constant $\widetilde{C} \ge 1$ depends on E, α, λ, Λ, $\frac{T}{T-t_0}$, $\frac{R_0^2}{T-t_0}$, only, and $H := \max_{0 \le t \le T} \|u(t)\|_{H^1(\Omega' \cap \Delta_{2R}(x_0))}$. Moreover $\gamma \in (0,1)$, $\gamma := \overline{\beta}\tau$, where $\overline{\beta} := \frac{\alpha\beta}{1+\alpha}$, $\beta \in (0,1)$ depending on λ only, and

$$\tau := \frac{\alpha_0}{\alpha_0 + \beta_0}, \quad \alpha_0 := \ln\left(\frac{1}{2} + \frac{r_3}{12 C_4 \lambda r_2}\right), \quad \beta_0 := \mathrm{e}^{C \frac{(\theta^* A)^\alpha}{R_0^\alpha}} \ln\left(12 C_4 \lambda \frac{r_2}{\widetilde{r_1}}\right), \tag{4.3}$$

where $\widetilde{r_1} := \frac{3 r_1}{64 e \pi \lambda}$, and C depends on E, α, λ, Λ only.

Let $t_0 \in (sT, (1-s)T)$, for some fixed $s \in (0, \frac{1}{2})$, and let $A_1 := \min\{\sqrt{2}\delta, sT\sqrt{a_R}\}$. There exist constants θ^*, C_4 such that for any three numbers r_1, r_2, r_3, $r_3 < \min\{\theta^* A_1, \delta\}$ (θ^*, C_4, r_1, r_2 as above), the following three cylinders inequality holds:

$$\begin{aligned} \int\limits_{sT}^{(1-s)T} \int\limits_{\Omega' \cap \Delta_{r_2}(x_0)} |u|^2 \,\mathrm{d}x\mathrm{d}t \le \widetilde{C} \left(\frac{r_3}{r_2}\right)^{n+2} & \left(\int\limits_{sT}^{(1-s)T} \int\limits_{\Omega' \cap \Delta_{r_1}(x_0)} |u|^2 \,\mathrm{d}x\mathrm{d}t \right)^{\gamma} \\ & \times \left(\frac{12 C_4 \lambda r_2 r_3}{(6 C_4 \lambda r_2 - r_3)^2} \right)^{1-\tau} \left(T \left(1 + \frac{T^2}{R^4}\right)^2 H^2 \right)^{1-\gamma}, \end{aligned} \tag{4.4}$$

where the constant $\widetilde{C} \ge 1$ depends on E, α, λ, Λ, $\frac{1}{s}$, $\frac{R_0^2}{sT}$, only, and C, H, γ are as above (with obviously A replaced by A_1 in (4.3)).

If we suppose moreover that in (4.1) $u(0) = 0$ in Ω', then the following result holds:

Corollary 4.2 (Three spheres inequality and three cylinders inequality at the boundary when $u(0) = 0$)**.** *Under the assumptions of Theorem 4.1 assume that $u \in H^1((0,T), H^1(\Omega'))$ is a nonidentically zero solution of the following problem*

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 & \text{in} \quad (0,T) \times \Omega', \\ u(0) = 0 & \text{in} \quad \Omega', \\ u = 0 & \text{on} \quad (0,T) \times ((\partial\Omega') \cap \Delta_{2R}(x_0)). \end{cases} \tag{4.5}$$

There exist constants θ^*, C_4 such that for any three numbers r_1, r_2, r_3, $r_3 < \theta^*\delta$ (θ^*, C_4, r_1, r_2 as in Th. 4.1), the following three spheres inequality holds:

$$\int\limits_{\Omega' \cap \Delta_{r_2}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \le \widetilde{C} \left(\frac{r_3}{r_2}\right)^{n+2} \left(\int\limits_{\Omega' \cap \Delta_{r_1}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \right)^{\gamma} \left(\frac{12 C_4 \lambda r_2 r_3}{(6 C_4 \lambda r_2 - r_3)^2} \right)^{1-\tau} \left(\left(1 + \frac{T^2}{R^4}\right)^2 H^2 \right)^{1-\gamma} \tag{4.6}$$

uniformly in $t_0 \in (0, \frac{T}{2})$, where $\widetilde{C} \ge 1$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$ only, and C, H, γ, are as in Theorem 4.1 (with A replaced by $\sqrt{2}\delta$ in (4.3)).

For any three numbers r_1, r_2, r_3 as above, the following three cylinders inequality holds:

$$\int_0^{T/2}\int_{\Omega'\cap\Delta_{r_2}(x_0)} |u|^2\,\mathrm{d}x\mathrm{d}t \le \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+2}\left(\int_0^{T/2}\int_{\Omega'\cap\Delta_{r_1}(x_0)} |u|^2\,\mathrm{d}x\mathrm{d}t\right)^{\gamma}$$

$$\times\left(\frac{12C_4\lambda r_2 r_3}{(6C_4\lambda r_2 - r_3)^2}\right)^{1-\tau}\left(T\left(1+\frac{T^2}{R^4}\right)^2 H^2\right)^{1-\gamma}, \tag{4.7}$$

where the constants $\widetilde{C}$, C, H, γ, are as above.

We recall also the following three spheres and three cylinders inequality at the *interior* when $u(0) = 0$ established in [8].

Theorem 4.3 (Three spheres inequality and three cylinders inequality at the interior)**.** *Let Ω' be a bounded and connected domain in $\mathbb{R}^n$, and let k be a $n\times n$ symmetric matrix-valued function satisfying assumptions (1.1, 1.2) (with Ω replaced by Ω'). Let $x_0 \in \Omega'$, and let $R > 0$ be such that $\Delta_{2R}(x_0) \subset \Omega'$. Assume that $u \in H^1((0,T), H^1_{\mathrm{loc}}(\Omega'))$ is a nonidentically zero solution of the problem:*

$$\begin{cases} u_t - \mathrm{div}(k(x)\nabla u) = 0 & \text{in} \quad (0,T)\times\Omega', \\ u(0) = 0 & \text{in} \quad \Omega'. \end{cases} \tag{4.8}$$

There exists $\theta^* \in (0,1]$ depending on λ and Λ, such that for any three numbers r_1, r_2, r_3 verifying

$$0 < r_1 < r_2 < \frac{r_3}{6\lambda},$$

$r_3 < \theta^*\delta$, the following three spheres inequality holds:

$$\int_{\Delta_{r_2}(x_0)} |u(t_0)|^2\,\mathrm{d}x \le \widetilde{C}\frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^{C}\left(\int_{\Delta_{r_1}(x_0)} |u(t_0)|^2\,\mathrm{d}x\right)^{\gamma}\left(\left(1+\frac{T^2}{R^4}\right)H^2\right)^{1-\gamma} \tag{4.9}$$

uniformly in $t_0 \in (0,\frac{T}{2})$, where $\widetilde{C} \ge 1$ depends on λ, Λ, $\frac{R_0^2}{T}$ only, $H := \max_{0\le t\le T}\|u(t)\|_{H^1(\Delta_{2R}(x_0))}$, $\gamma \in (0,1)$, $\gamma := \overline{\beta}\tau$, $\overline{\beta}$ as in Proposition 3.1, and

$$\tau := \frac{\alpha_0}{\alpha_0+\beta_0}, \quad \alpha_0 := \ln\left(\frac{1}{2}+\frac{r_3}{6\lambda r_2}\right), \quad \beta_0 := C\ln\left(6\lambda\frac{r_2}{\widetilde{r_1}}\right),$$

where $\widetilde{r_1} := \frac{3r_1}{64e\pi\lambda}$, and $C > 0$ depends on λ and Λ only.

For any three numbers r_1, r_2, r_3 as above, the following three cylinders inequality holds:

$$\int_0^{T/2}\int_{\Delta_{r_2}(x_0)} |u|^2\,\mathrm{d}x\mathrm{d}t \le \widetilde{C}\frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^{C}\left(\int_0^{T/2}\int_{\Delta_{r_1}(x_0)} |u|^2\,\mathrm{d}x\mathrm{d}t\right)^{\gamma}\left(T\left(1+\frac{T^2}{R^4}\right)^2 H^2\right)^{1-\gamma}, \tag{4.10}$$

where the constants $\widetilde{C}$, H, γ, are as above.

In order to prove Theorem 4.1 we proceed in the following way. We begin by establishing a three spheres inequality at the boundary for the function w defined in Proposition 3.1, solution of the following elliptic equation

$$w_{yy} + \operatorname{div}(k(x)\nabla w) = 0 \quad \text{in} \quad (-A, A) \times (\Omega' \cap \Delta_{R/2}(x_0)), \tag{4.11}$$

and satisfying the following Cauchy and boundary conditions:

$$\begin{cases} w(0) = u(t_0) & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w_y(0) = 0 & \text{in} \quad \Omega' \cap \Delta_{R/2}(x_0), \\ w = 0 & \text{on} \quad (-A, A) \times ((\partial\Omega') \cap \Delta_{R/2}(x_0)), \end{cases} \tag{4.12}$$

where u is solution of (4.1), and $t_0 \in (0, T)$ is a fixed time. Once a three spheres inequality at the boundary for w is at hand (see Prop. 4.4 below), we derive inequality (4.2) by using Cauchy estimate (3.5), and a suitable trace inequality for w.

We begin by establishing a three spheres inequality at the boundary for w. More precisely we prove the following:

Proposition 4.4. *Under the assumptions of Theorem 4.1, let w be solution of (4.11, 4.12). For any three numbers r_1, r_2, r_3 verifying*

$$0 < r_1 < r_2 < \frac{r_3}{2C_4\lambda}, \quad r_3 < \theta^* A$$

(θ^*, C_4, A as in Th. 4.1), the following inequality holds:

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_{r_2}(X_0)} |w|^2 \,\mathrm{d}X \le \widetilde{C} \left(\frac{r_3}{r_2}\right)^{n+1} \left(\int\limits_{(\mathbb{R}\times\Omega')\cap B_{r_1}(X_0)} |w|^2 \,\mathrm{d}X \right)^{\tau'} \times \left(\frac{4C_4\lambda r_2 r_3}{(2C_4\lambda r_2 - r_3)^2} \int\limits_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)} |w|^2 \,\mathrm{d}X \right)^{1-\tau'}, \tag{4.13}$$

where

$$\tau' := \frac{\alpha_0'}{\alpha_0' + \beta_0'}, \quad \alpha_0' := \ln\left(\frac{1}{2} + \frac{r_3}{4C_4\lambda r_2}\right), \quad \beta_0' := \mathrm{e}^{C\frac{(\theta^* A)^\alpha}{R_0^\alpha}} \ln\left(4\lambda C_4 \frac{r_2}{r_1}\right),$$

and the constants $\widetilde{C}$, C depend on E, α, λ, Λ only.

We recall that $B_r(X_0)$ is the ball in $\mathbb{R}^{n+1}$ of center X_0 and radius r, $X_0 \in \mathbb{R}^{n+1}$ is the point $(0, x_0)$, and $\mathrm{d}X$ is the $(n+1)$-dimensional volume Lebesgue measure.

In order to prove Proposition 4.4 we need some auxiliary results. First of all let us introduce the following notations. We denote by Θ a domain in $\mathbb{R}^{n+1}$ such that $\partial\Theta$ is of *Lipschitz* class with constants r_0, L. Assume that $0 \in \partial\Theta$. For some $\rho > 0$, let $\widetilde{w}$ be a nonidentically zero solution of the problem

$$\begin{cases} \operatorname{div}(\widetilde{K}(X)\nabla\widetilde{w}) = 0 & \text{in} \quad \Theta \cap B_\rho, \\ \widetilde{w} = 0 & \text{on} \quad (\partial\Theta) \cap B_\rho, \end{cases} \tag{4.14}$$

where $\widetilde{K}(X) := (\widetilde{K}_{ij}(X))_{1\le i,j\le n+1}$ is an $(n+1)\times(n+1)$ symmetric matrix-valued function in Θ, satisfying the following assumptions:

(i) there exists a constant $\lambda_0 \geq 1$ such that for all $X \in \Theta$, and all $\xi \in \mathbb{R}^{n+1}$

$$\lambda_0^{-1} |\xi|^2 \leq \widetilde{K}(X)\xi \cdot \xi \leq \lambda_0 |\xi|^2 ; \tag{4.15}$$

(ii)

$$\widetilde{K}(0) = I_{n+1}; \tag{4.16}$$

(iii) for all $X \in (\partial\Theta) \cap B_\rho$

$$\widetilde{K}(X)X \cdot \mathbf{n} \geq 0; \tag{4.17}$$

(iv) for $0 < \alpha \leq 1$, there exists a constant $c > 0$ such that for all $X \in \Theta$

$$\left|\widetilde{K}(X) - \widetilde{K}(0)\right| \leq c\frac{\alpha\, |X|^{\alpha}}{r_0^\alpha}, \quad \left|\nabla \widetilde{K}(X)\right| \leq c\frac{\alpha\, |X|^{\alpha-1}}{r_0^\alpha} \cdot \tag{4.18}$$

(Here I_{n+1} denotes the $(n+1)\times(n+1)$ identity matrix, and $\mathbf{n}$ is the outer unit normal at $(\partial\Theta)\cap B_\rho$.) Under assumptions (4.15–4.18), we prove a three spheres inequality at the boundary for a nonidentically zero solution $\widetilde{w}$ of (4.14). More precisely the following result holds:

Lemma 4.5. *Let Θ be a domain in $\mathbb{R}^{n+1}$ such that $\partial\Theta$ is of Lipschitz class with constants r_0, L. Assume that $0 \in \partial\Theta$, and, under assumptions (4.15–4.18), let $\widetilde{w}$ be a nonidentically zero solution of (4.14). There exists a positive constant $\overline{r}$, $\overline{r} \in (0,\rho]$, with $\overline{r}/\rho$ depending on r_0, L, λ_0 only, such that for any three numbers r_1, r_2, r_3 verifying*

$$0 < r_1 < r_2 < r_3 < \overline{r},$$

the following inequality holds:

$$\int\limits_{\Theta\cap B_{r_2}} |\widetilde{w}|^2 \,\mathrm{d}X \leq \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+1} \left(\int\limits_{\Theta\cap B_{r_1}} |\widetilde{w}|^2 \,\mathrm{d}X\right)^{s} \left(\frac{2r_2r_3}{(r_2-r_3)^2}\int\limits_{\Theta\cap B_{r_3}} |\widetilde{w}|^2 \,\mathrm{d}X\right)^{1-s}, \tag{4.19}$$

where the constants $\widetilde{C}$, C depend on λ_0, α, c, $\frac{\rho}{r_0}$ only, and

$$s := \frac{\alpha_1}{\alpha_1+\beta_1}, \quad \alpha_1 := \ln\left(\frac{1}{2} + \frac{r_3}{2r_2}\right), \quad \beta_1 := \mathrm{e}^{C\frac{\overline{r}^\alpha}{r_0^\alpha}} \ln\frac{2r_2}{r_1} \cdot$$

The proof of Lemma 4.5 is based on the following result due to Adolfsson and Escauriaza [1].

Lemma 4.6. *Under the assumptions of Lemma 4.5, let $\widetilde{w}$ be a nonidentically zero solution of (4.14). For $r \in (0, \rho)$, let us define the following functions:*

$$H(r) := \frac{1}{r^n} \int\limits_{\Theta \cap \partial B_r} \mu \, |\widetilde{w}|^2 \, \mathrm{d}\sigma,$$

$$D(r) := \frac{1}{r^{n-1}} \int\limits_{\Theta \cap B_r} \widetilde{K}(X) \nabla \widetilde{w} \cdot \nabla \widetilde{w} \mathrm{d}X,$$

$$N(r) := \frac{D(r)}{H(r)},$$

where $\mu(X) := \widetilde{K}(X) X \cdot X \frac{1}{|X|^2}$. There exist positive constants $\overline{r}$, C ($\overline{r}$, C as in Lem. 4.5), such that

(i)

$$r^n H(r) \textit{ is a nondecreasing function of } r \in (0, \overline{r}); \tag{4.20}$$

(ii)

$$\left| H'(r) - \frac{2}{r} D(r) \right| \leq C \frac{\alpha}{r_0^\alpha} r^{\alpha - 1} H(r) \quad \textit{for } r \in (0, \overline{r}); \tag{4.21}$$

(iii)

$$N'(r) \geq -C \frac{\alpha}{r_0^\alpha} r^{\alpha - 1} N(r) \quad \textit{for } r \in (0, \overline{r}). \tag{4.22}$$

(Here $\mathrm{d}\sigma$ denotes the n-dimensional surface Lebesgue measure, and H' (resp. N') the derivative of H (resp. of N).)

Proof of Lemma 4.5. First of all we observe that multiplying (4.22) by $\mathrm{e}^{C \frac{r^\alpha}{r_0^\alpha}}$ we have

$$0 \leq (N'(r) + C\alpha \frac{r^{\alpha-1}}{r_0^\alpha} N(r)) \mathrm{e}^{C \frac{r^\alpha}{r_0^\alpha}} = \frac{\mathrm{d}}{\mathrm{d}r} \left(\mathrm{e}^{C \frac{r^\alpha}{r_0^\alpha}} N(r) \right).$$

This implies that $\mathrm{e}^{C \frac{r^\alpha}{r_0^\alpha}} N(r)$ is a nondecreasing function of $r \in (0, \overline{r})$. Hence

$$N(r) \leq \mathrm{e}^{C \frac{\overline{r}^\alpha}{r_0^\alpha}} N(s) \quad \text{for all } r, s, \ 0 < r \leq s < \overline{r}. \tag{4.23}$$

Dividing (4.21) by $H(r)$, integrating over (r_1, r_2), and using (4.23) we have

$$\ln \frac{H(r_2)}{H(r_1)} \leq C \frac{1}{r_0^\alpha} (r_2^\alpha - r_1^\alpha) + 2\mathrm{e}^{C \frac{\overline{r}^\alpha}{r_0^\alpha}} N(r_2) \ln \frac{r_2}{r_1} \cdot \tag{4.24}$$

Similarly, dividing (4.21) by $H(r)$, integrating over (r_2, r_3), and using (4.23) we obtain

$$\ln \frac{H(r_3)}{H(r_2)} \geq -C \frac{1}{r_0^\alpha} (r_3^\alpha - r_2^\alpha) + 2\mathrm{e}^{-C \frac{\overline{r}^\alpha}{r_0^\alpha}} N(r_2) \ln \frac{r_3}{r_2} \cdot \tag{4.25}$$

Hence (4.24, 4.25) imply

$$\frac{\ln \frac{H(r_2)}{H(r_1)}}{\ln \frac{r_2}{r_1}} \leq \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}} \left(\frac{C\frac{1}{r_0^\alpha}(r_2^\alpha - r_1^\alpha)}{\ln \frac{r_2}{r_1}} + \frac{C\frac{1}{r_0^\alpha}(r_3^\alpha - r_2^\alpha)}{\ln \frac{r_3}{r_2}} + \frac{\ln \frac{H(r_3)}{H(r_2)}}{\ln \frac{r_3}{r_2}} \right) \cdot \tag{4.26}$$

Multiplying both terms of (4.26) by $\ln\left(\frac{r_2}{r_1}\right)\ln\left(\frac{r_3}{r_2}\right)$, and by the inequality:

$$\left(\frac{r_3}{r_2}\right)^{r_2^\alpha - r_1^\alpha} \left(\frac{r_3}{r_1}\right)^{r_3^\alpha - r_2^\alpha} \leq \left(\frac{r_3}{r_1}\right)^{r_3^\alpha},$$

we obtain

$$\left(\frac{H(r_2)}{H(r_1)}\right)^{\ln \frac{r_3}{r_2}} \leq \left(\frac{r_3}{r_1}\right)^{\frac{C}{r_0^\alpha} r_3^\alpha \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}} \left(\frac{H(r_3)}{H(r_2)}\right)^{\mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}} \ln \frac{r_2}{r_1}}. \tag{4.27}$$

Therefore (4.27) yields

$$H(r_2) \leq \left(\frac{r_3}{r_1}\right)^{a'} H(r_1)^{s'} H(r_3)^{1-s'}, \tag{4.28}$$

where

$$s' := \frac{\alpha_1'}{\alpha_1' + \beta_1'}, \quad \alpha_1' := \ln \frac{r_3}{r_2}, \quad \beta_1' := \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}} \ln \frac{r_2}{r_1},$$

and

$$a' := \frac{\frac{C}{r_0^\alpha} r_3^\alpha \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}}{\alpha_1' + \beta_1'}\cdot$$

Now by (4.28) with r_1, r_2, r_3 replaced by $\frac{r_1}{2}$, r_2, $\frac{r_2+r_3}{2}$ respectively, it follows that

$$H(r_2) \leq \left(\frac{r_2+r_3}{r_1}\right)^a \left(H\left(\frac{r_1}{2}\right)\right)^s \left(H\left(\frac{r_2+r_3}{2}\right)\right)^{1-s}, \tag{4.29}$$

where

$$s := \frac{\alpha_1}{\alpha_1 + \beta_1}, \quad \alpha_1 := \ln \frac{r_2 + r_3}{2r_2}, \quad \beta_1 := \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}} \ln \frac{2r_2}{r_1},$$

and

$$a := \frac{\frac{C}{r_0^\alpha} r_3^\alpha \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}}{\alpha_1 + \beta_1}\cdot$$

We recall that the classical trace inequality yields, for $r \leq \rho$,

$$\int\limits_{\Theta\cap\partial B_{\frac{r}{2}}} |\widetilde{w}|^2 \,\mathrm{d}\sigma \leq c \left(\frac{2}{r} \int\limits_{\Theta\cap B_{\frac{r}{2}}} |\widetilde{w}|^2 \,\mathrm{d}X + \frac{r}{2} \int\limits_{\Theta\cap B_{\frac{r}{2}}} |\nabla\widetilde{w}|^2 \,\mathrm{d}X \right), \tag{4.30}$$

and that the Caccioppoli inequality gives

$$\int_{\Theta\cap B_{r'}} |\nabla\widetilde{w}|^2\,\mathrm{d}X \le C\frac{1}{(r-r')^2}\int_{\Theta\cap B_r} |\widetilde{w}|^2\,\mathrm{d}X \quad \text{for } r'<r\le\rho, \tag{4.31}$$

where the constant C depends on λ_0 only. So by (4.30) for $r=r_1$, $r=\frac{r_2+r_3}{2}$ and by (4.31) we get

$$\int_{\Theta\cap\partial B_{\frac{r_1}{2}}} |\widetilde{w}|^2\,\mathrm{d}\sigma \le C\frac{1}{r_1}\int_{\Theta\cap B_{r_1}} |\widetilde{w}|^2\,\mathrm{d}X, \tag{4.32}$$

and

$$\int_{\Theta\cap\partial B_{\frac{r_2+r_3}{2}}} |\widetilde{w}|^2\,\mathrm{d}\sigma \le C\frac{r_2+r_3}{(r_2-r_3)^2}\int_{\Theta\cap B_{r_3}} |\widetilde{w}|^2\,\mathrm{d}X, \tag{4.33}$$

respectively, where the constant C depends on λ_0 only.

Next we rewrite (4.29) as follows:

$$H(r_2)\le\left(\frac{r_2+r_3}{r_1}\right)^a\left(\frac{2}{r_1}\right)^{ns}\left(\frac{2}{r_2+r_3}\right)^{n(1-s)}\left(\left(\frac{r_1}{2}\right)^n H\left(\frac{r_1}{2}\right)\right)^s\left(\left(\frac{r_2+r_3}{2}\right)^n H\left(\frac{r_2+r_3}{2}\right)\right)^{1-s}.$$

So, by the definition of H, we have

$$H(r_2)\le\left(\frac{r_2+r_3}{r_1}\right)^a\left(\frac{2}{r_1}\right)^{ns}\left(\frac{2}{r_2+r_3}\right)^{n(1-s)}\left(\int_{\Theta\cap\partial B_{\frac{r_1}{2}}}|\widetilde{w}|^2\,\mathrm{d}\sigma\right)^s\left(\int_{\Theta\cap\partial B_{\frac{r_2+r_3}{2}}}|\widetilde{w}|^2\,\mathrm{d}\sigma\right)^{1-s}.$$

Therefore using (4.32, 4.33) we obtain

$$H(r_2)\le C\left(\frac{r_2+r_3}{r_1}\right)^a\left(\frac{2}{r_1}\right)^{ns}\left(\frac{2}{r_2+r_3}\right)^{n(1-s)}\left(\frac{1}{r_1}\int_{\Theta\cap B_{r_1}}|\widetilde{w}|^2\,\mathrm{d}X\right)^s\left(\frac{r_2+r_3}{(r_3-r_2)^2}\int_{\Theta\cap B_{r_3}}|\widetilde{w}|^2\,\mathrm{d}X\right)^{1-s} \tag{4.34}$$

where the constant C depends on λ_0 only. Now since from (4.20) we know that $r^nH(r)$ is a nondecreasing function of $r\in(0,\overline{r})$, equation (4.34) yields

$$\eta^nH(\eta)\le Cr_2^n\left(\frac{r_2+r_3}{r_1}\right)^a\left(\frac{2}{r_1}\right)^{ns}\left(\frac{2}{r_2+r_3}\right)^{n(1-s)}\left(\frac{1}{r_1}\int_{\Theta\cap B_{r_1}}|\widetilde{w}|^2\,\mathrm{d}X\right)^s\left(\frac{r_2+r_3}{(r_3-r_2)^2}\int_{\Theta\cap B_{r_3}}|\widetilde{w}|^2\,\mathrm{d}X\right)^{1-s}$$

for $\eta\le r_2$. Finally, integrating over $(0,r_2)$, a simple calculation gives:

$$\int_{\Theta\cap B_{r_2}}|\widetilde{w}|^2\,\mathrm{d}X \;\le\; \int_0^{r_2}\eta^nH(\eta)\mathrm{d}\eta \;\le\; C\left(\frac{r_3}{r_1}\right)^a\left(\frac{r_2}{r_1}\right)^{(n+1)s}\left(\int_{\Theta\cap B_{r_1}}|\widetilde{w}|^2\,\mathrm{d}X\right)^s\left(\frac{r_2r_3}{(r_3-r_2)^2}\int_{\Theta\cap B_{r_3}}|\widetilde{w}|^2\,\mathrm{d}X\right)^{1-s}, \tag{4.35}$$

where the constant C depends on λ_0 only. We complete the proof of Lemma 4.5 by proving that the ratios $\left(\frac{r_3}{r_1}\right)^a$, $\left(\frac{r_2}{r_1}\right)^{(n+1)s}$ in (4.35) are bounded. In fact the term $\left(\frac{r_3}{r_1}\right)^a$ can be bounded in the following way:

$$\left(\frac{r_3}{r_1}\right)^a = \exp\left(\frac{\frac{C}{r_0^\alpha} r_3^\alpha \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}(\ln\frac{r_3}{r_2}+\ln\frac{r_2}{r_1})}{\ln\frac{r_2+r_3}{2r_2}+\mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}\ln\frac{2r_2}{r_1}}\right) \leq \exp\left(\frac{C}{r_0^\alpha}\overline{r}^\alpha \mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}\right).$$

Finally we increase the term $\left(\frac{r_2}{r_1}\right)^{(n+1)s}$ as follows:

$$\left(\frac{r_2}{r_1}\right)^{(n+1)s} = \exp\left((n+1)\frac{\ln\frac{r_2+r_3}{2r_2}}{\ln\frac{r_2+r_3}{2r_2}+\mathrm{e}^{2C\frac{\overline{r}^\alpha}{r_0^\alpha}}\ln\frac{2r_2}{r_1}}\ln\frac{r_2}{r_1}\right) \leq \left(\frac{r_3}{r_2}\right)^{n+1}.$$

The proof of Lemma 4.5 is complete. □

We are now in a position to prove Proposition 4.4.

Proof of Proposition 4.4. We shall follow the main lines of the proof of Theorem 0.4 in [1]. The idea is to construct a $C^{1,\alpha}$ diffeomorphism Φ from $\Theta\cap B_{\theta_2 A}$ to $(\mathbb{R}\times\Omega')\cap B_{\theta_1 A}(X_0)$, for a suitable domain $\Theta\subset\mathbb{R}^{n+1}$, and some constants $\theta_1,\theta_2\in(0,1]$, showing that $\widetilde{w}(Z):=w(\Phi(Z))$ satisfies the assumptions of Lemma 4.5 and hence inequality (4.19). From (4.19) one derives a similar inequality for w.

First of all, up to a rigid motion, we can suppose that $X_0=0$, and

$$(\mathbb{R}\times\Omega')\cap B_{2R} := \{X\in B_{2R} \text{ s.t. } x_n>\varphi(x')\}$$

(as usual $X\in\mathbb{R}^{n+1}$ is the point (y,x), and $x':=(x_1,\cdots x_{n-1})$ are the first $(n-1)$-components of $x\in\mathbb{R}^n$) where φ is a $C^{1,\alpha}$ function on $\Delta'_{2R}\subset\mathbb{R}^{n-1}$ satisfying

$$\varphi(0)=|\nabla\varphi(0)|=0,$$

and

$$\|\varphi\|_{C^{1,\alpha}(\Delta'_{2R})}\leq ER_0.$$

Next let us denote by

$$K(X):=\begin{pmatrix}1 & 0\\ 0 & k(x)\end{pmatrix}$$

the $(n+1)\times(n+1)$ matrix-valued function in $\mathbb{R}\times\overline{\Omega'}$, and by $w(X)$ the solution of:

$$\begin{cases} \mathrm{div}_X(K(X)\nabla_X w)=0 & \text{in} \quad (-A,A)\times\Omega'\cap\Delta_{R/2},\\ w(0,x)=u(t_0,x) & \text{in} \quad \Omega'\cap\Delta_{R/2},\\ w_y(0,x)=0 & \text{in} \quad \Omega'\cap\Delta_{R/2},\\ w=0 & \text{on} \quad (-A,A)\times((\partial\Omega')\cap\Delta_{R/2}), \end{cases} \tag{4.36}$$

where we recall that $A := \min\left\{\sqrt{2}\delta, t_0\sqrt{a_R}\right\}$, $\delta := \frac{R}{8e\pi\lambda}$, and $a_R := \frac{1}{\lambda c_P R^2}$, c_P being the Poincaré constant. Two cases can happen: either

(i) $K(0) = I_{n+1}$ (I_{n+1} is the $(n+1)\times(n+1)$ identity matrix), or
(ii) $K(0) \neq I_{n+1}$.

We begin by studying case (i), that is we suppose that $K(0) = I_{n+1}$. Let us denote by $Z = (z_0, \dots, z_n)$ the new variable in $\mathbb{R}^{n+1}$, and by $Z' = (z_0, \cdots, z_{n-1})$ the first n-components of Z. For $C_3 := \frac{3(2^\alpha-1)^2 E}{(\alpha \ln 2)^2}$, let us define

$$\Phi(Z) := \left(Z', z_n + \frac{C_3}{R_0^\alpha}|Z|^{\alpha+1}\right),$$

that is

$$\begin{aligned}
\Phi_0(Z) &:= z_0,\\
\Phi_1(Z) &:= z_1,\\
\vdots \quad &\ \vdots\ \vdots\\
\Phi_n(Z) &:= z_n + \frac{C_3}{R_0^\alpha}|Z|^{\alpha+1}.
\end{aligned}$$

Moreover we denote by

$$\Theta := \left\{Z \in B_A, \text{ s.t. } z_n > \varphi(z') - \frac{C_3}{R_0^\alpha}|Z|^{\alpha+1}\right\},$$

where $z' := (z_1, \cdots, z_{n-1})$ are the first $(n-1)$-components of $z = (z_1, \cdots, z_n)$. Following the computations in [1], we have that there exist $\theta_1, \theta_2 \in (0,1]$ with θ_1/R_0, θ_2/R_0 only depending on E and α, such that $\Phi \in C^{1,\alpha}(B_{\theta_2 A}, \mathbb{R}^n)$ satisfies

$$\Phi\left(Z', \varphi(z') - \frac{C_3}{R_0^\alpha}|Z|^{\alpha+1}\right) = (Z', \varphi(z')),$$

$$\Phi(\Theta \cap B_{\theta_2 A}) \subset (\mathbb{R}\times\Omega') \cap B_{\theta_1 A},$$

$$\frac{1}{2}|Z| \le |\Phi(Z)| \le C_4|Z| \quad \forall Z \in B_{\theta_2 A}, \tag{4.37}$$

$$\frac{1}{2^{n+1}} \le |\det D\Phi(Z)| \le 2 \quad \forall Z \in B_{\theta_2 A}, \tag{4.38}$$

$$(\partial\Theta) \cap B_{\theta_2 A} \text{ is of Lipschitz class with constants } \theta_2 A, L, \tag{4.39}$$

where the constants C_4 in (4.37) and L in (4.39) depend on E and α only.

Now, let us denote

$$\widetilde{w}(Z) := w(\Phi(Z)),$$

and

$$\widetilde{K}(Z) := |\det D\Phi(Z)|\,(D\Phi^{-1})(\Phi(Z))K(\Phi(Z))(D\Phi^{-1})^*(\Phi(Z)),$$

where $(D\Phi^{-1})^*$ denotes the transpose matrix of $D\Phi^{-1}$. One can verify that $\widetilde{w}$ solves

$$\begin{cases} \operatorname{div}(\widetilde{K}(Z)\nabla\widetilde{w}) = 0 & \text{in} \quad \Theta\cap B_{\theta_2 A}, \\ \widetilde{w} = 0 & \text{on} \quad (\partial\Theta)\cap B_{\theta_2 A}, \end{cases} \tag{4.40}$$

and that $\widetilde{K}$ satisfies the following properties:

(i) for all $Z\in\Theta$, and all $\xi\in\mathbb{R}^{n+1}$

$$\frac{1}{8}\lambda^{-1}|\xi|^2 \le \widetilde{K}(Z)\xi\cdot\xi \le 8\lambda|\xi|^2; \tag{4.40a}$$

(ii)

$$\widetilde{K}(0) = I_{n+1}; \tag{4.40b}$$

(iii) for all $Z\in(\partial\Theta)\cap B_{\theta_2 A}$

$$\widetilde{K}(Z)Z\cdot\mathbf{n}\ge 0; \tag{4.40c}$$

(iv) there exists a constant $C>0$ only depending on E, λ, Λ such that for all $Z\in\Theta$

$$\left|\nabla\widetilde{K}(Z)\right| \le C\frac{\alpha}{R_0^\alpha}|Z|^{\alpha-1}, \quad \left|\widetilde{K}(Z)-\widetilde{K}(0)\right| \le C\frac{\alpha}{R_0^\alpha}|Z|^\alpha. \tag{4.40d}$$

(As usual $\mathbf{n}$ denotes the unit outer normal to $(\partial\Theta)\cap B_{\theta_2 A}$.) Hence we can apply Lemma 4.5 to solution $\widetilde{w}$ of (4.40) with $r_0=\rho=\theta_2 A$, $\lambda_0=\lambda$, $c=C$. Then there exists $\theta^*\in(0,\theta_2]$, with θ^*/θ_2 only depending on E and α, such that for any three numbers r_1, r_2, r_3 satisfying

$$0<r_1<r_2<\frac{r_3}{2C_4}, \quad r_3<\theta^* A,$$

inequality (4.19) holds for $\widetilde{w}$ with radii respectively $\frac{r_1}{C_4}$, $2r_2$, $\frac{r_3}{C_4}$, that is

$$\int_{\Theta\cap B_{2r_2}} |\widetilde{w}|^2\,\mathrm{d}Z \le \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+1}\left(\int_{\Theta\cap B_{r_1/C_4}}|\widetilde{w}|^2\,\mathrm{d}Z\right)^{\tau''}\left(\frac{4C_4r_2r_3}{(2C_4r_2-r_3)^2}\int_{\Theta\cap B_{r_3/C_4}}|\widetilde{w}|^2\,\mathrm{d}Z\right)^{1-\tau''}, \tag{4.41}$$

where

$$\tau'' := \frac{\alpha_0''}{\alpha_0''+\beta_0''}, \quad \alpha_0'' := \ln\left(1+\frac{r_3}{4C_4r_2}\right), \quad \beta_0'' := \mathrm{e}^{C\frac{(\theta^*A)^\alpha}{R_0^\alpha}}\ln\frac{4C_4r_2}{r_1}, \tag{4.42}$$

and the constants $\widetilde{C}$, C depend on E, α, λ, Λ only.

Next we decrease $\int_{\Theta\cap B_{2r_2}}|\widetilde{w}|^2\,\mathrm{d}Z$ in the left hand side of (4.41) in terms of $\int_{(\mathbb{R}\times\Omega')\cap B_{r_2}}|w|^2\,\mathrm{d}X$, and we increase $\int_{\Theta\cap B_{r_j/C_4}}|\widetilde{w}|^2\,\mathrm{d}Z$, $j=1,3$, in the right hand side of (4.41) in terms of $\int_{(\mathbb{R}\times\Omega')\cap B_{r_j}}|w|^2\,\mathrm{d}X$. We begin by observing that from (4.37) we have

$$(\mathbb{R}\times\Omega')\cap B_{r/2}\subset\Phi(\Theta\cap B_r)\subset(\mathbb{R}\times\Omega')\cap B_{C_4 r} \quad \forall r\le\theta_2 A. \tag{4.43}$$

Hence from (4.38), and the left hand side of (4.43) we have

$$\int_{\Theta\cap B_{2r_2}} |\widetilde{w}|^2 \,\mathrm{d}Z = \int_{\Phi(\Theta\cap B_{2r_2})} |w|^2 \left|\det D\Phi^{-1}(X)\right| \mathrm{d}X \geq \frac{1}{2} \int_{(\mathbb{R}\times\Omega')\cap B_{r_2}} |w|^2 \,\mathrm{d}X. \tag{4.44}$$

Similarly from (4.38), and the right hand side of (4.43) we have, for $j = 1, 3$,

$$\int_{\Theta\cap B_{r_j/C_4}} |\widetilde{w}|^2 \,\mathrm{d}Z = \int_{\Phi(\Theta\cap B_{r_j/C_4})} |w|^2 \left|\det D\Phi^{-1}(X)\right| \mathrm{d}X \leq 2^{n+1} \int_{(\mathbb{R}\times\Omega')\cap B_{r_j}} |w|^2 \,\mathrm{d}X. \tag{4.45}$$

Finally (4.41, 4.44, 4.45) yield

$$\int_{(\mathbb{R}\times\Omega')\cap B_{r_2}} |w|^2 \,\mathrm{d}X \leq C\left(\frac{r_3}{r_2}\right)^{n+1} \left(\int_{(\mathbb{R}\times\Omega')\cap B_{r_1}} |w|^2 \,\mathrm{d}X\right)^{\tau''} \left(\frac{4C_4 r_2 r_3}{(2C_4 r_2 - r_3)^2} \int_{(\mathbb{R}\times\Omega')\cap B_{r_3}} |w|^2 \,\mathrm{d}X\right)^{1-\tau''},$$

where the constant $\widetilde{C}$ depends on E, α, λ, Λ only.

Now we treat case (ii), that is we assume that $K(0) \neq I_{n+1}$. We can consider a linear transformation $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that, setting $\overline{K}(SX) = \frac{SK(X)S^T}{|\det S|}$, we have $\overline{K}(0) = I_{n+1}$. We have that, under such a transformation, the modified coefficient $\overline{K}$, the transformed domain $S((-A, A) \times (\Omega' \cap \Delta_{R/2}))$ and the transformed boundary portion $S((-A, A) \times ((\partial\Omega') \cap \Delta_{R/2}))$ satisfy assumptions analogous to Proposition 4.4, with constants which are dominated by the *a priori* constants λ, Λ, R_0, E, up to multiplicative factors which only depend on λ. We also have that the ellipsoids $S((\mathbb{R} \times \Omega') \cap B_r)$ for $r < \frac{1}{\sqrt{\lambda}}A$, satisfy

$$(\mathbb{R} \times \Omega') \cap B_{\frac{1}{\sqrt{\lambda}}r} \subset S((\mathbb{R} \times \Omega') \cap B_r) \subset (\mathbb{R} \times \Omega') \cap B_{\sqrt{\lambda}r}. \tag{4.46}$$

Therefore, by a change of variables, using the result just proved when $K(0) = I_{n+1}$, we obtain

$$\int_{(\mathbb{R}\times\Omega')\cap B_{r_2}} |w|^2 \,\mathrm{d}X \leq \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+1} \left(\int_{(\mathbb{R}\times\Omega')\cap B_{r_1}} |w|^2 \,\mathrm{d}X\right)^{\tau'} \left(\frac{4C_4\lambda r_2 r_3}{\left(2C_4\lambda r_2 - r_3\right)^2} \int_{(\mathbb{R}\times\Omega')\cap B_{r_3}} |w|^2 \,\mathrm{d}X\right)^{1-\tau'},$$

where

$$\tau' := \frac{\alpha_0'}{\alpha_0' + \beta_0'}, \quad \alpha_0' := \ln\left(\frac{1}{2} + \frac{r_3}{4C_4\lambda r_2}\right), \quad \beta_0' := \mathrm{e}^{C\frac{(\theta^* A)^\alpha}{R_0^\alpha}} \ln\left(\frac{4C_4\lambda r_2}{r_1}\right),$$

and the constants $\widetilde{C}$, C depend on E, α, λ, Λ only.

The proof of Proposition 4.4 is complete. □

Now we prove Theorem 4.1.

Proof of Theorem 4.1. Let r_1, r_2, r_3 be three numbers satisfying

$$0 < r_1 < r_2 < \frac{r_3}{6C_4\lambda},$$

where $r_3 < \min\{\theta^* A, \delta\}$. Let us define $\widetilde{r_1} := \frac{3r_1}{64e\pi\lambda}$. Using the three spheres inequality at the boundary (4.13) for w with radii $\widetilde{r_1}$, $3r_2$, r_3, we have

$$\int_{(\mathbb{R}\times\Omega')\cap B_{3r_2}(X_0)} |w|^2 \,\mathrm{d}X \le \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+1}\left(\int_{(\mathbb{R}\times\Omega')\cap B_{\widetilde{r_1}}(X_0)} |w|^2 \,\mathrm{d}X\right)^{\tau} \times\left(\frac{12C_4\lambda r_2 r_3}{(6C_4\lambda r_2 - r_3)^2}\int_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)} |w|^2 \,\mathrm{d}X\right)^{1-\tau}, \tag{4.47}$$

where

$$\tau := \frac{\alpha_0}{\alpha_0+\beta_0}, \quad \alpha_0 := \ln\left(\frac{1}{2}+\frac{r_3}{12C_4\lambda r_2}\right), \quad \beta_0 := \mathrm{e}^{C\frac{(\theta^* A)^\alpha}{R_0^\alpha}}\ln\left(\frac{12C_4\lambda r_2}{\widetilde{r_1}}\right),$$

the constants $\widetilde{C}$, C depend on E, α, λ, Λ only, and the constant $C_4 \ge \frac{1}{2}$ depends on E and α only. Recalling the Cauchy estimate (3.5) for w established in Proposition 3.1, with $r = \widetilde{r_1}$, $\rho = \frac{8}{3}\sqrt{2}e\pi\lambda\widetilde{r_1}$, $\widetilde{\rho} = 2\sqrt{2}\rho$, we obtain

$$\int_{(\mathbb{R}\times\Omega')\cap B_{\widetilde{r_1}}(X_0)} |w|^2 \,\mathrm{d}X \le C\widetilde{r_1}\left(\int_{\Omega'\cap\Delta_{\rho/4}(x_0)} |u(t_0)|^2 \,\mathrm{d}x\right)^{\overline{\beta}}\left(\frac{1}{\widetilde{r_1}}\int_{(\mathbb{R}\times\Omega')\cap B_{\widetilde{\rho}}(X_0)} |w|^2 \,\mathrm{d}X\right)^{1-\overline{\beta}},$$

where the constant $C \ge 1$ depends on λ only, and $\overline{\beta} := \frac{\alpha\beta}{1+\alpha}$. Hence (4.47) becomes (since $\frac{\rho}{4} < r_1$, and $\widetilde{\rho} < r_3$)

$$\int_{(\mathbb{R}\times\Omega')\cap B_{3r_2}(X_0)} |w|^2 \,\mathrm{d}X \le \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+1}\widetilde{r_1}^{\overline{\beta}\tau}\left(\int_{\Omega'\cap\Delta_{r_1}(x_0)} |u(t_0)|^2 \,\mathrm{d}x\right)^{\overline{\beta}\tau} \times\left(\frac{12C_4\lambda r_2 r_3}{(6C_4\lambda r_2 - r_3)^2}\right)^{1-\tau}\left(\int_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)} |w|^2 \,\mathrm{d}X\right)^{1-\overline{\beta}\tau}, \tag{4.48}$$

where the constant $\widetilde{C}$ depends on E, α, λ, Λ only. Next we decrease $\int_{(\mathbb{R}\times\Omega')\cap B_{3r_2}(X_0)} |w|^2 \,\mathrm{d}X$ in the left hand side of (4.48) in terms of $\int_{\Omega'\cap\Delta_{r_2}(x_0)} |u(t_0)|^2 \,\mathrm{d}x$. By inequality (7.12) (see the Appendix) for $F(y,x) = w(y,x)$, $\rho = r_2$, $r = 2r_2$, and by Caccioppoli inequality we have

$$\int_{\Omega'\cap\Delta_{r_2}(x_0)} |u(t_0)|^2 \,\mathrm{d}x \le C\frac{1}{r_2}\int_{(\mathbb{R}\times\Omega')\cap B_{3r_2}(X_0)} |w|^2 \,\mathrm{d}X, \tag{4.49}$$

where the constant C depends on λ only. Finally we increase the integral $\int_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)} |w|^2 \,\mathrm{d}X$ in the right hand side of (4.48) in terms of the *a priori* data. We recall that $w(y,x) := w_1(y,x) + w_2(y,x)$, where $w_1(y,x) :=$

$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{it_0\mu}\widetilde{u_1}(\mu,x)\cosh(\sqrt{-i\mu}y)\mathrm{d}\mu$, and $w_2(y,x) := \sum\limits_{j=1}^{+\infty}\alpha_j e^{\mu_j t_0}\varphi_j(x)\cosh(\sqrt{|\mu_j|}y)$ (see Prop. 3.1). Since $|y| < \min\{\theta^* A, \delta\}$, using (3.10) we have, for $y \geq 0$,

$$\begin{aligned}\|w_1(y)\|_{L^2(\Omega'\cap\Delta_{r_3}(x_0))} &\leq \|w_1(y)\|_{H^1(\Omega'\cap\Delta_{R/2}(x_0))} \leq \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\|\widetilde{u_1}(\mu)\|_{H^1(\Omega'\cap\Delta_{R/2}(x_0))}\left|\cosh(\sqrt{-i\mu}y)\right|\mathrm{d}\mu \\ &\leq CC_1H\left(T+\frac{1}{a_{R/4}}\right)\frac{1}{\delta^2(1-1/\sqrt{2})^2},\end{aligned} \tag{4.50}$$

where the constant C depends on λ only, and C_1, defined in (3.8), depends on λ, $\frac{T}{T-t_0}$, $\frac{R_0^2}{T-t_0}$ only. Inequality (4.50) implies (recalling that $w_1(y,x)$ is even in y, $\delta := \frac{R}{8e\pi\lambda}$ and $a_R := \frac{1}{\lambda c_P R^2}$)

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)}|w_1|^2\,\mathrm{d}X \leq 2\int\limits_0^{r_3}\|w_1(y)\|^2_{L^2(\Omega'\cap\Delta_{r_3}(x_0)}\,\mathrm{d}y \leq CC_1^2 r_3 H^2\left(\frac{T^2}{R^4}+1\right)^2. \tag{4.51}$$

Similarly, for $y \geq 0$ we have (since $|y| < \min\{\theta^* A, \delta\}$)

$$\|w_2(y)\|^2_{L^2(\Omega'\cap\Delta_{r_3}(x_0))} \leq \|w_2(y)\|^2_{L^2(\Omega'\cap\Delta_{2R}(x_0))} \leq \int\limits_{\Omega'\cap\Delta_{2R}(x_0)}|u(0)|^2\,\mathrm{d}x \leq H^2. \tag{4.52}$$

Therefore (4.52) implies

$$\int\limits_{(\mathbb{R}\times\Omega')\cap B_{r_3}(X_0)}|w_2|^2\,\mathrm{d}X \leq 2\int\limits_0^{r_3}\|w_2(y)\|^2_{L^2(\Omega'\cap\Delta_{r_3}(x_0)}\,\mathrm{d}y \leq 2r_3H^2. \tag{4.53}$$

Hence (4.48, 4.49, 4.51, 4.53) yield

$$\int\limits_{\Omega'\cap\Delta_{r_2}(x_0)}|u(t_0)|^2\,\mathrm{d}x \leq \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+2}\left(\int\limits_{\Omega'\cap\Delta_{r_1}(x_0)}|u(t_0)|^2\,\mathrm{d}x\right)^{\overline{\beta}\tau}\left(\frac{12C_4\lambda r_2 r_3}{(6C_4\lambda r_2 - r_3)^2}\right)^{1-\tau}\left(\left(1+\frac{T^2}{R^4}\right)^2 H^2\right)^{1-\overline{\beta}\tau}, \tag{4.54}$$

that is (4.2), where the constant $\widetilde{C} \geq 1$ depends on E, α, λ, Λ, $\frac{T}{T-t_0}$, $\frac{R_0^2}{T-t_0}$ only.

Now if we suppose that $t_0 \in (sT, (1-s)T)$, for some fixed $s \in (0, \frac{1}{2})$, then (4.54) holds uniformly in $(sT, (1-s)T)$. So integrating (4.54) over the interval $(sT, (1-s)T)$, and using Hölder inequality, we obtain

$$\begin{aligned}\int\limits_{sT}^{(1-s)T}\int\limits_{\Omega'\cap\Delta_{r_2}(x_0)}|u|^2\,\mathrm{d}x\mathrm{d}t &\leq \widetilde{C}\left(\frac{r_3}{r_2}\right)^{n+2}\left(\int\limits_{sT}^{(1-s)T}\int\limits_{\Omega'\cap\Delta_{r_1}(x_0)}|u|^2\,\mathrm{d}x\mathrm{d}t\right)^{\overline{\beta}\tau} \\ &\times\left(\frac{12C_4\lambda r_2 r_3}{(6C_4\lambda r_2 - r_3)^2}\right)^{1-\tau}\left(T\left(1+\frac{T^2}{R_0^4}\right)^2 H^2\right)^{1-\overline{\beta}\tau},\end{aligned}$$

where the constant $\widetilde{C} \geq 1$ depends on E, α, λ, Λ, $\frac{1}{s}$, $\frac{R_0^2}{sT}$ only, which, putting $\gamma := \overline{\beta}\tau$, conclude the proof of Theorem 4.1. $\square$

We conclude this section by proving Corollary 4.2.

Proof of Corollary 4.2. The proof follows step by step from the proof of Theorem 4.1, by observing that $w_2(y,x) \equiv 0$. $\square$

5. Auxiliary propositions

In the present section we give a sequence of propositions which we shall use in the next section to prove Theorem 1.1. The proofs of these propositions are very similar to those of the corresponding Neumann case studied in [8]. Therefore the reader interested in more complete and detailed proofs can see [8], and also [4].

In what follows Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, satisfying assumptions (1.7, 1.9), and D_i, $i = 1, 2$, are two domains satisfying (1.8) such that $D_i \subset \Omega$, $\mathrm{dist}(\partial\Omega, \partial D_i) \geq R_0$, and $\Omega\backslash D_i$ is connected. Moreover we shall denote

$$\Omega_i := \Omega\backslash\overline{D_i},$$

and

$$G \quad \text{the connected component of } \Omega_1 \cap \Omega_2 \text{ s.t. } \partial\Omega \subset \partial G.$$

Proposition 5.1 (Stability estimates of continuation from Cauchy data on time-like surfaces)**.** *Under the assumptions of Theorem 1.1, let $u := u_1 - u_2$ in $(0,T) \times G$. There exists a constant $\overline{\gamma}$, $\overline{\gamma} \in (0,1)$, depending on λ and Λ only, such that we have*

$$\|u(t_0)\|_{L^2(\Delta_{\overline{R}}(P_2))} \leq C R_0^{\frac{n}{2}} \left(\frac{\|f\|_{1/4,1/2}}{T^{1/2} R_0^{(n-1)/2}} \right) \widetilde{\epsilon}^{\overline{\gamma}/3} \tag{5.1}$$

uniformly in $t_0 \in [0, T/2]$. Here $\overline{R} := \frac{R_0}{16(1+E^2)}$, $P_2 := P_1 - \frac{R_0}{16\sqrt{1+E^2}}\mathbf{n}$ ($\mathbf{n}$ denotes the outer unit normal at $P_1 \in \partial\Omega$), the constant C depends on E, λ, Λ $\frac{R_0^2}{T}$ only, and

$$\widetilde{\epsilon} := \left(\frac{T^{1/2} R_0^{(n-1)/2} \epsilon}{\|f\|_{1/4,1/2}} \right). \tag{5.2}$$

Proof of Proposition 5.1. By Theorem 3.3.1 in [8], we know that if

$$R_0 \left\| \frac{\partial}{\partial \mathbf{n}} u \right\|_{H^{1/4}((0,T), H^{1/2}(\Gamma))} \leq T^{1/2} R_0^{(n-1)/2} \overline{\epsilon},$$

then

$$\|u(t_0)\|_{L^2(\Delta_{\overline{R}}(P_2))} \leq C R_0^{\frac{n}{2}} \left(\frac{\|f\|_{1/4,1/2}}{T^{1/2} R_0^{(n-1)/2}} \right)^{1-\overline{\gamma}} \overline{\epsilon}^{\overline{\gamma}}, \tag{5.3}$$

uniformly in $t_0 \in [0, T/2]$, where the constant $C \geq 1$ depends on E, λ, Λ, $\frac{R_0^2}{T}$ only. The aim is then to estimate $\left\|\frac{\partial}{\partial \mathbf{n}} u\right\|_{H^{1/4}((0,T), H^{1/2}(\Gamma))}$ in terms of $\left\|\frac{\partial}{\partial \mathbf{n}} u\right\|_{L^2((0,T) \times \Gamma)}$ and the *a priori* data. We observe that the functions u,

u_t, u_{tt} satisfy

$$\begin{cases} u_t - \operatorname{div}(k(x)\nabla u) = 0 & \text{in} \quad (0,T)\times G, \\ u(0) = 0 & \text{in} \quad G, \\ u = 0 & \text{in} \quad (0,T)\times \partial\Omega. \end{cases}$$

Hence we may apply boundedness estimates (see for example Ladyzhenskaja *et al.* [13]) obtaining

$$\|u_t\|_{L^\infty((0,T)\times\overline{G})} \le CT^{-\frac{1}{2}} R_0^{-\frac{n-1}{2}} \|f\|_{1/4,1/2}, \tag{5.4}$$

$$\|u_{tt}\|_{L^\infty((0,T)\times\overline{G})} \le CT^{-\frac{3}{2}} R_0^{-\frac{n-1}{2}} \|f\|_{1/4,1/2}, \tag{5.5}$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$ only. We may think at $u(t)$ as solution of

$$\begin{cases} \operatorname{div}(k(x)\nabla u(t)) = u_t(t) & \text{in} \quad G, \\ u(t) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Similarly, we may think at $u_t(t)$ as solution of

$$\begin{cases} \operatorname{div}(k(x)\nabla u_t(t)) = u_{tt}(t) & \text{in} \quad G, \\ u_t(t) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By L^p regularity estimates (see [10]), by (5.4, 5.5), by trace inequalities and by the immersion of $W^{1-1/p,p}(\Gamma)$ in $H^{1-1/p}(\Gamma)$, for $p > 2$, we have

$$\sup_{t\in[0,T]} \left(R_0 \left\| \frac{\partial}{\partial \mathbf{n}} u(t) \right\|_{H^{1-1/p}(\Gamma)} + R_0 T \left\| \frac{\partial}{\partial \mathbf{n}} u_t(t) \right\|_{H^{1-1/p}(\Gamma)} \right) \le CT^{-1/2} \|f\|_{1/4,1/2},$$

where C depends on E, λ, Λ, $\frac{R_0^2}{T}$ only. Therefore

$$R_0 \left\| \frac{\partial}{\partial \mathbf{n}} u \right\|_{H^{\alpha/2}((0,T),H^\alpha(\Gamma))} \le C\|f\|_{1/4,1/2}, \tag{5.6}$$

with $\alpha := 1 - 1/p > 1/2$, where C depends on E, λ, Λ, $\frac{R_0^2}{T}$ only. By interpolation (see Lions and Magenes [16]), we have

$$\left\| \frac{\partial}{\partial \mathbf{n}} u \right\|_{H^{1/4}((0,T),H^{1/2}(\Gamma))} \le C \left\| \frac{\partial}{\partial \mathbf{n}} u \right\|^{1-\theta}_{H^{\alpha/2}((0,T),H^\alpha(\Gamma))} \cdot \left\| \frac{\partial}{\partial \mathbf{n}} u \right\|^{\theta}_{L^2((0,T)\times\Gamma)}, \tag{5.7}$$

where θ is given by $(1-\theta)\alpha = 1/2$. By (1.11, 5.6) and (5.7), choosing $p = 4$, we obtain

$$R_0 \left\| \frac{\partial}{\partial \mathbf{n}} u \right\|_{H^{1/4}((0,T),H^{1/2}(\Gamma))} \le C\|f\|_{1/4,1/2} \left(\frac{T^{1/2} R_0^{(n-1)/2} \epsilon}{\|f\|_{1/4,1/2}} \right)^{1/3},$$

where C depends on E, λ, Λ, $\frac{R_0^2}{T}$ only. Finally by (5.3) we derive

$$\|u(t_0)\|_{L^2(\Delta_{\overline{R}}(P_2))} \leq CR_0^{\frac{n}{2}} \left(\frac{\|f\|_{1/4,1/2}}{T^{1/2}R_0^{(n-1)/2}} \right) \widetilde{\epsilon}^{\widetilde{\gamma}/3}$$

uniformly in $t_0 \in [0, T/2]$, where C depends on E, λ, Λ $\frac{R_0^2}{T}$ only.

The proof of Proposition 5.1 is complete. □

Proposition 5.2 (Stability estimate of continuation from Cauchy data)**.** *(I) Under the assumptions of Theorem 1.1, let $f \in H^{3/4}((0,T), H^{1/2}(\partial\Omega))$ be such that $u_i \in H^1((0,T), H^1(\Omega\backslash D_i))$, $i = 1, 2$, is solution of (1.3) when $D := D_i$, and the initial temperature $u_{i0} = 0$ in $\Omega\backslash\overline{D_i}$. Then the following inequality holds*

$$\int_0^{T/2} \int_{\Omega_i\backslash G} |u_i|^2 \mathrm{d}x\mathrm{d}t \leq R_0 \|f\|_{1/4,1/2}^2 \omega \left(\frac{T^{1/2}R_0^{(n-1)/2}\epsilon}{\|f\|_{1/4,1/2}} \right), \tag{5.8}$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(t) \leq C(\ln|\ln t|)^{-\frac{1}{n}}, \quad \textit{for every } t < \mathrm{e}^{-1},$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only.

(II) Moreover if we suppose that there exist positive constants r_0, L, $r_0 \in (0, R_0]$, such that ∂G is of Lipschitz class with constants r_0, L, then (5.8) holds with ω given by

$$\omega(t) \leq C\,|\ln t|^{-\nu}, \quad \textit{for every } t < \mathrm{e}^{-1}, \tag{5.9}$$

where C, ν depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, L, R_0/r_0 only.

The proof of Proposition 5.2 will be given at the end of this section.

In the sequel, for $\rho > 0$ and A a bounded domain in $\mathbb{R}^n$, we shall denote

$$A_\rho := \{x \in A \text{ s.t. } \mathrm{dist}(x, A^c) > \rho\}, \tag{5.10}$$

where $A^c := \mathbb{R}^n\backslash A$, *i.e.* the complementary of A.

Proposition 5.3 (Stability estimate of continuation from the interior)**.** *Let $f \in H^{3/4}((0,T), H^{1/2}(\partial\Omega))$ satisfy (1.10) such that $u_i \in H^1((0,T), H^1(\Omega\backslash D_i))$ is solution of (1.3) when $D := D_i$, and the initial temperature $u_{i0} = 0$ in $\Omega\backslash\overline{D_i}$. Then, for every $\rho > 0$ and every $x_i \in (\Omega_i)_\rho$, we have*

$$\int_0^{T/2} \int_{\Delta_\rho(x_i)} |u_i|^2 \,\mathrm{d}x\mathrm{d}t \geq CR_0 \|f\|_{1/4,1/2}^2, \tag{5.11}$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F, R_0/ρ only.

Proof of Proposition 5.3. The proof follows from Proposition 4.3 in [4], and from Proposition 5.5 in [8], up to obvious changes. □

At this stage, we recall the notion of *modified distance* introduced in [4].

Definition 5.4. We call modified distance between bounded domains Ω_1 and Ω_2 in $\mathbb{R}^n$ the number

$$d_m(\Omega_1, \Omega_2) = \max\left\{ \sup_{x\in\partial\Omega_1} \operatorname{dist}(x, \Omega_2), \sup_{x\in\partial\Omega_2} \operatorname{dist}(\Omega_1, x) \right\}\cdot$$

□

Notice that obviously we have

$$d_m(\Omega_1, \Omega_2) \le d_{\mathcal{H}}(\Omega_1, \Omega_2),$$

but, in general, d_m does not dominate the Hausdorff distance, and indeed it does not satisfy the axioms of a distance function. This is made clear by the following example: $\Omega_1 := B_1(0)$, $\Omega_2 := B_1(0) \setminus \overline{B_{1/2}(0)}$. In this case $d_m(\Omega_1, \Omega_2) = 0$, whereas $d_{\mathcal{H}}(\Omega_1, \Omega_2) = 1/2$.

Proposition 5.5. *Let Ω_1, Ω_2 be bounded domains satisfying (3.7). There exist positive numbers d_0, r_0, $r_0 \in (0, R_0]$, for which the ratios d_0/R_0, r_0/R_0 only depend on E and α, such that if we have*

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) \le d_0,$$

then the following facts hold:

(i) *any connected component G of $\Omega_1 \cap \Omega_2$ has boundary of Lipschitz class with constants r_0, L, where r_0 is as above and $L > 0$ only depends on E and α;*
(ii) *there exists an absolute constant $c > 0$ such that*

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) \le c d_m(\Omega_1, \Omega_2).$$

Proof of Proposition 5.5. The proof is contained in [4]. □

In the proof of Proposition 5.2 we shall need to approximate the domains Ω_r with *regularized domains*, say $\widetilde{\Omega_r}$, $r > 0$. To this aim let us recall the following result, which was obtained in [4] (Lem. 5.3).

Lemma 5.6 (Regularized domains)**.** *Let Ω be a bounded domain such that $\partial\Omega$ is of Lipschitz class with constants R_0, E. There exists a family of regularized domains $\widetilde{\Omega_h} \subset \Omega$, for $0 < h \le aR_0$, having C^1 boundary such that*

$$\widetilde{\Omega_{h_2}} \subset \widetilde{\Omega_{h_1}}, \quad 0 < h_1 \le h_2,$$

$$\gamma_0 h \le \operatorname{dist}(x, \partial\Omega) \le \gamma_1 h, \quad \textit{for every } x \in \partial\widetilde{\Omega_h}, \tag{5.12}$$

$$|\Omega \setminus \widetilde{\Omega_h}| \le \gamma_2 M R_0^{n-1} h, \tag{5.13}$$

$$|\partial\widetilde{\Omega_h}|_{n-1} \le \gamma_3 M R_0^{n-1}, \tag{5.14}$$

for every $x \in \partial\widetilde{\Omega_h}$ there exits $y \in \partial\Omega$ s.t.

$$|y - x| = \operatorname{dist}(x, \partial\Omega), \tag{5.15}$$

where a, γ_i, $i = 0, 1, 2, 3$, are positive constants depending on E, α only.

(Here $|\cdot|_{n-1}$ denotes the surface measure.)

We are now in a position to prove Proposition 5.2.

Proof of Proposition 5.2. For $r < R_0$, let us denote

$$\mathcal{U}^r = \left\{ x \in \overline{\Omega} \backslash D \text{ s.t. } \operatorname{dist}(x, \partial\Omega) < r \right\}\cdot$$

From regularity estimates for solutions of parabolic equations [13], we have, for $i=1,2$,

$$\|u_i\|_{C^{1,\alpha}([0,T]\times\overline{(\Omega_i\setminus\mathcal{U}^{R_0/8})})}\leq CT^{-1/2}R_0^{-(n-1)/2}\|f\|_{1/4,1/2},\tag{5.16}$$

$$\|u_1-u_2\|_{C^{1,\alpha}([0,T]\times\overline{G})}\leq CT^{-1/2}R_0^{-(n-1)/2}\|f\|_{1/4,1/2},\tag{5.17}$$

where $C>0$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only.

We prove Proposition 5.2 for $i=1$, the case $i=2$ being analogous.

Proof of Part (I). With no loss of generality we can assume that $\widetilde{\epsilon}\leq\tilde{\mu}$, where $\widetilde{\epsilon}$ is defined in (5.2), and $\tilde{\mu}$, $0<\tilde{\mu}<\mathrm{e}^{-1}$, is a constant only depending on E, α, λ, Λ, M, which will be chosen later on, since, otherwise, equation (5.8) becomes trivial. Let $\widetilde{\theta}=\min\{a,\frac{1}{16(1+E^2)\gamma_1}\}$, where a, γ_1 have been introduced in Lemma 5.7. We have that $\widetilde{\theta}$ depends on E and α only. Let $\bar{r}:=\widetilde{\theta}R_0$ and let

$$\Sigma_{\gamma_1\bar{r}}=\{x\in G \text{ s.t. } \operatorname{dist}(x,\partial\Omega)=\gamma_1\bar{r}\}\cdot$$

For $r\leq\bar{r}$, let $\widetilde{V_r}$ be the connected component of $\widetilde{\Omega_{1,r}}\cap\widetilde{\Omega_{2,r}}$ whose closure contains $\Sigma_{\gamma_1\bar{r}}$. We have

$$\Omega_1\setminus G\subset[(\Omega_1\setminus\widetilde{\Omega_{1,r}})\setminus G]\cup[\widetilde{\Omega_{1,r}}\setminus\widetilde{V_r}],$$

$$\partial(\widetilde{\Omega_{1,r}}\setminus\widetilde{V_r})=\widetilde{\Gamma_{1,r}}\cup\widetilde{\Gamma_{2,r}},$$

where $\widetilde{\Gamma_{1,r}}$ is the part of boundary contained in $\partial\widetilde{\Omega_{1,r}}$, and $\widetilde{\Gamma_{2,r}}$ is the part contained in $\partial\widetilde{\Omega_{2,r}}\cap\partial\widetilde{V_r}$. Therefore we have

$$\int_0^{T/2}\int_{\Omega_1\setminus G}|u_1|^2\mathrm{d}x\mathrm{d}t\leq\int_0^{T/2}\int_{(\Omega_1\setminus\widetilde{\Omega_{1,r}})\setminus G}|u_1|^2\mathrm{d}x\mathrm{d}t+\int_0^{T/2}\int_{\widetilde{\Omega_{1,r}}\setminus\widetilde{V_r}}|u_1|^2\mathrm{d}x\mathrm{d}t.\tag{5.18}$$

By (5.16) and (5.13) we have

$$\int_0^{T/2}\int_{(\Omega_1\setminus\widetilde{\Omega_{1,r}})\setminus G}|u_1|^2\mathrm{d}x\mathrm{d}t\leq Cr\|f\|^2_{1/4,1/2},\tag{5.19}$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only. By the divergence theorem, we have, for $\tau\in(0,T/2)$,

$$\frac{1}{2}\int_{\widetilde{\Omega_{1,r}}\setminus\widetilde{V_r}}|u_1(\tau)|^2\mathrm{d}x\leq\int_0^{T/2}\int_{\widetilde{\Gamma_{1,r}}\cup\widetilde{\Gamma_{2,r}}}|k\nabla u_1\cdot\mathbf{n}u_1|\,\mathrm{d}\sigma\mathrm{d}t.$$

Hence, integrating over the interval $(0,T/2)$, we obtain

$$\int_0^{T/2}\int_{\widetilde{\Omega_{1,r}}\setminus\widetilde{V_r}}|u_1|^2\mathrm{d}x\mathrm{d}t\leq T\int_0^{T/2}\int_{\widetilde{\Gamma_{1,r}}\cup\widetilde{\Gamma_{2,r}}}|k\nabla u_1\cdot\mathbf{n}u_1|\,\mathrm{d}\sigma\mathrm{d}t.\tag{5.20}$$

Let $x \in \widetilde{\Gamma_{1,r}}$. By (5.15, 5.12) there exists $y \in \partial D_1$ such that $|x-y| = \mathrm{dist}(x, \partial D_1) \leq \gamma_1 r$. Since $u_1 \equiv 0$ on $(0,T) \times \partial D_1$, from (5.16) it follows that

$$|u_1(t,x)| = |u_1(t,x) - u_1(t,y)| \leq CT^{-1/2} R_0^{-(n-1)/2} \frac{r}{R_0} \|f\|_{1/4,1/2} \tag{5.21}$$

uniformly in $t \in [0, T/2]$, and $x \in \widetilde{\Gamma_{1,r}}$, where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only. Similarly, for $x \in \widetilde{\Gamma_{2,r}}$, there exists $y \in \partial D_2$ such that $|x-y| \leq \gamma_1 r$. Since $u_2 \equiv 0$ on $(0,T) \times \partial D_2$, by (5.16) it follows that

$$|u_1(t,x)| \leq |u(t,x)| + CT^{-1/2} R_0^{-(n-1)/2} \frac{r}{R_0} \|f\|_{1/4,1/2} \tag{5.22}$$

uniformly in $t \in [0, T/2]$, and $x \in \widetilde{\Gamma_{1,r}}$, where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only. (5.16, 5.13, 5.18–5.22) yield

$$\int_0^{T/2} \int_{\Omega_1 \setminus G} |u_1|^2 \mathrm{d}x\mathrm{d}t \leq CR_0 \|f\|_{1/4,1/2} \left(\frac{r}{R_0} \|f\|_{1/4,1/2} + R_0^{(n+1)/2} \|u\|_{L^\infty([0,T/2] \times \widetilde{\widetilde{V_r}})} \right), \tag{5.23}$$

where the constant C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only.

In order to estimate $\|u\|_{L^\infty([0,T/2] \times \widetilde{\widetilde{V_r}})}$, we shall make use of Proposition 5.1. So let $(\overline{t}, \overline{x})$ be such that $\left|u(\overline{t}, \overline{x})\right| = \|u\|_{L^\infty([0,T/2] \times \widetilde{\widetilde{V_r}})}$. Since $\min\{\mathrm{dist}(P_2, \partial\Omega_1),$ $\mathrm{dist}(P_2, \partial\Omega_2)\} \geq \frac{R_0}{16(1+E^2)} \geq \gamma_1 r$, we have that $P_2 \in \widetilde{\widetilde{V_r}}$, where P_2 has been introduced in Proposition 5.1. Let φ be an arc in $\widetilde{\widetilde{V_r}}$ joining $\overline{x}$ to P_2. Let us define $\{x_i\}$, $i = 1, ..., s$, as follows: $x_1 = P_2$, $x_{i+1} = \varphi(t_i)$, where $t_i = \max\{t \text{ s.t. } |\varphi(t) - x_i| = \frac{\theta^*}{2} \frac{\gamma_0 r}{12 \cdot 16 e\pi\lambda^2}\}$ if $|x_i - \overline{x}| > \frac{\theta^*}{2} \frac{\gamma_0 r}{12 \cdot 16 e\pi\lambda^2}$, otherwise let $i = s$ and stop the process (θ^* is the constant defined in Th. 4.3). By construction, the balls $\Delta_{\frac{\gamma_0 r}{4}}(x_i)$ are pairwise disjoint $|x_{i+1} - x_i| = \frac{\theta^*}{2} \frac{\gamma_0 r}{12 \cdot 16 e\pi\lambda^2}$, for $i = 1, ..., s-1$, $|x_s - \overline{x}| \leq \frac{\theta^*}{2} \frac{\gamma_0 r}{12 \cdot 16 e\pi\lambda^2}$. Hence we have $s \leq S \left(\frac{R_0}{r}\right)^n$, with S only depending on E, α, M.

At this stage, since $\Delta_{\gamma_0 r}(P_2) \subset G$, by iterated application of the three spheres inequality at the interior (4.9) for $t_0 = \overline{t}$, with radii

$$r_1 := \frac{\theta^*}{2} \frac{\gamma_0 r}{24 \cdot 16 e\pi\lambda^2}, \quad r_2 := \frac{\theta^*}{2} \frac{\gamma_0 r}{8 \cdot 16 e\pi\lambda^2}, \quad r_3 := \frac{\theta^*}{2} \frac{\gamma_0 r}{16 e\pi\lambda^2},$$

we have

$$\left\|u(\overline{t})\right\|^2_{L^2(\Delta_{r_1}(\overline{x}))} \leq C \left(\left\|u(\overline{t})\right\|^2_{L^2(\Delta_{r_1}(P_2))} \right)^{\gamma^s} \left(\frac{1}{R_0} \left(1 + \frac{T^2}{r^4}\right) \|f\|^2_{1/4,1/2} \right)^{1-\gamma^s}, \tag{5.24}$$

where $C \geq 1$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$ only, $\gamma \in (0,1)$ depends on λ and Λ only. (In (4.9) we have used the estimate $H^2 \leq C\frac{R_0}{T} \|f\|^2_{1/4,1/2}$, the constant C depending on E, α, λ only, see for example [13].) By (5.1) we obtain

$$\left\|u(\overline{t})\right\|^2_{L^2(\Delta_{r_1}(\overline{x}))} \leq C \frac{1}{R_0} \left(1 + \frac{T^2}{r^4}\right)^{1-\gamma^s} \|f\|^2_{1/4,1/2} \,\widetilde{\epsilon}^{\frac{2}{3}\overline{\gamma}\gamma^s}, \tag{5.25}$$

where the constant $C \geq 1$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$ only. Let us recall now the following interpolation inequality

$$\|v\|_{L^\infty(\Delta_\rho)} \leq c\left(\left(\int_{\Delta_\rho} |v|^2\,\mathrm{d}x\right)^{\frac{\alpha}{2\alpha+n}} [v]_{\alpha,\Delta_\rho}^{\frac{n}{2\alpha+n}} + \frac{1}{\rho^{n/2}}\left(\int_{\Delta_\rho} |v|^2\,\mathrm{d}x\right)^{1/2}\right), \tag{5.26}$$

which holds for any function $v \in C^{0,\alpha}(\Delta_\rho)$ defined in the ball $\Delta_\rho \subset \mathbb{R}^n$ and for any α, $0 < \alpha \leq 1$. By applying (5.26) to $u(\overline{t})$ in $\Delta_{r_1}(\overline{x})$, with $\alpha = 1$, by (5.25) and (5.17) we have

$$\left|u(\overline{t}, \overline{x})\right| \leq C\frac{1}{R_0^{(n+1)/2}}\left(\frac{R_0}{r}\right)^{\frac{n}{2}}\left(1 + \frac{T^2}{r^4}\right)^{(1-\gamma^s)/2} \|f\|_{1/4,1/2}\tilde{\epsilon}^{\gamma'\gamma^s}, \tag{5.27}$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$ only, and $\gamma' := \frac{2}{3(2+n)}\overline{\gamma}$, $\gamma' \in (0,1)$. From $r \leq \widetilde{\theta}R_0$, we have that $r \leq CT^{-\frac{1}{2}}$, with C depending on E, λ, Λ, $\frac{R_0^2}{T}$ only. Therefore we can estimate

$$1 + \frac{T^2}{r^4} \leq C\left(\frac{R_0}{r}\right)^4,$$

where C depends on E, λ, Λ, $\frac{R_0^2}{T}$ only. By substituting (5.27) in (5.23), and by the above inequality, we have

$$\int_0^{T/2}\int_{\Omega_1\setminus G} |u_1|^2 \mathrm{d}x\mathrm{d}t \leq CR_0\|f\|_{1/4,1/2}^2\left(\frac{r}{R_0} + \left(\frac{R_0}{r}\right)^{(n+4)/2}\tilde{\epsilon}^{\gamma'\gamma^s}\right), \tag{5.28}$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only.

Let us set $\bar{\mu} := \exp\left\{-\frac{1}{\gamma'}\exp\left(\frac{2S|\ln\gamma|}{\widetilde{\theta}^n}\right)\right\}$, $\tilde{\mu} := \min\{\bar{\mu}, \exp(-\frac{1}{\gamma'^2})\}$. We have that $\tilde{\mu} < \mathrm{e}^{-1}$ and it depends on E, α, λ, Λ, M only. Let $\tilde{\epsilon} \leq \tilde{\mu}$ and let

$$r(\tilde{\epsilon}) := R_0\left(\frac{2(S+1)\,|\ln\gamma|}{\ln|\ln\tilde{\epsilon}^{\gamma'}|}\right)^{1/n}.$$

Since $r(\tilde{\epsilon})$ is increasing in $(0, e^{-1})$ and since $r(\tilde{\mu}) \leq r(\bar{\mu}) = R_0\widetilde{\theta}$, inequality (5.28) is applicable when $r = r(\tilde{\epsilon})$ and we obtain

$$\int_0^{T/2}\int_{\Omega_1\setminus G} |u_1|^2 \mathrm{d}x\mathrm{d}t \leq CR_0\|f\|_{1/4,1/2}^2\left(\ln\left|\ln\tilde{\epsilon}^{\gamma'}\right|\right)^{-1/n},$$

where C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only. On the other hand, since $\tilde{\epsilon} \leq \exp(-\frac{1}{\gamma'^2})$, we have that $\ln\gamma' \geq -\frac{1}{2}\ln|\ln\tilde{\epsilon}|$, so that

$$\ln\left|\ln\tilde{\epsilon}^{\gamma'}\right| \geq \frac{1}{2}\ln|\ln\tilde{\epsilon}|\,.$$

The proof of Part (I) is complete.

Proof of Part (II). By the divergence theorem we have

$$\int_0^{T/2}\int_{\Omega_1\setminus G} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \le T \int_0^{T/2}\int_{\Gamma_1\cup\Gamma_2} |k\nabla u_1\cdot \mathbf{n} u_1| \,\mathrm{d}\sigma\mathrm{d}t, \tag{5.29}$$

where Γ_1 is a part of ∂D_1 and Γ_2 is a part of ∂D_2. Since $u_i = 0$ on $[0,T]\times\partial D_i$, $i = 1,2$, by (5.17), and the fact that $|\Gamma_i| \le C R_0^{n-1}$, C depending on E, α, M only (see [2], Lem. 2.8), we have

$$\int_0^{T/2}\int_{\Omega_1\setminus G} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \le C R_0^{(n+3)/2} \|f\|_{1/4,1/2} \max_{(0,T/2)\times\partial G} |u|,$$

where $u = u_1 - u_2$ in $(0,T)\times G$, and C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M only. Arguing as in the proof of Proposition 5.4 in [7], up to obvious changes, we obtain (5.9).

The proof of Proposition 5.2 is complete. □

6. Proof of Theorem 1.1

Our task in this section is to prove Theorem 1.1. Before doing so, we need to establish the following preliminary:

Proposition 6.1. *Under the assumptions of Theorem 1.1, suppose that*

$$\max_{i\in\{1,2\}} \int_0^{T/2}\int_{\Omega_i\setminus G} |u_i|^2 \,\mathrm{d}x\mathrm{d}t \le \eta,$$

Then

$$d_{\mathcal{H}}(\Omega_1,\Omega_2) \le C R_0 \left(\frac{\eta}{R_0 \|f\|^2_{1/4,1/2}}\right)^{\theta}, \tag{6.1}$$

where the constants C and θ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only.

Once this result is at hand we can prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two steps.

Step 1: By Proposition 5.2 Part (I), and Proposition 6.1 we have

$$d_{\mathcal{H}}(\Omega_1,\Omega_2) \le C R_0 \left(\ln\left|\ln\left(\frac{T^{1/2} R_0^{(n-1)/2}\epsilon}{\|f\|_{1/4,1/2}}\right)\right|\right)^{-\theta/n}, \tag{6.2}$$

where C, θ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. Let ϵ_0 be such that, for $\epsilon \le \epsilon_0$, the following inequality holds

$$C R_0 \left(\ln\left|\ln\left(\frac{T^{1/2} R_0^{(n-1)/2}\epsilon}{\|f\|_{1/4,1/2}}\right)\right|\right)^{-\theta/n} \le d_0, \tag{6.3}$$

where d_0 is the positive constant introduced in Proposition 5.5. Hence, equations (6.2) and (6.3) yield

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) \le d_0. \tag{6.4}$$

By Proposition 5.5 we know that (6.4) implies that G is of Lipschitz class (we recall that G is the connected component of $\Omega_1 \cap \Omega_2$ such that $\partial\Omega \subset \partial G$). Therefore, by Proposition 5.2 Part (II), we can improve (6.2) and we obtain

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) \le CR_0 \left| \ln \left(\frac{T^{1/2} R_0^{(n-1)/2} \epsilon}{\|f\|_{1/4,1/2}} \right) \right|^{-\kappa}. \tag{6.5}$$

Step 2: In this step we prove the assertion of Theorem 1.1. In order to simplify the notations, let us denote

$$d := d_{\mathcal{H}}(D_1, D_2), \quad \text{and} \quad \delta := d_{\mathcal{H}}(\Omega_1, \Omega_2).$$

First we observe that, by the definition of δ, we have

$$(D_1)_\delta \subset D_2,$$

where the set $(D_1)_\delta$ is defined in (5.10). So

$$D_1 \backslash D_2 \subset D_1 \backslash (D_1)_\delta,$$

and

$$|D_1 \backslash D_2| \le |D_1 \backslash (D_1)_\delta| \le CR_0^{n-1}\delta,$$

C depending on E, α, M only (see [2], Lem. 2.8). Similarly we have

$$|D_2 \backslash D_1| \le CR_0^{n-1}\delta,$$

and so

$$|D_1 \triangle D_2| \le CR_0^{n-1}\delta. \tag{6.6}$$

Without loss of generality, let $x_0 \in D_1$ be such that $d = \mathrm{dist}(x_0, D_2)$. Up to a rigid motion, we can suppose that $x_0 = (0, x_{0n})$. Then is not difficult to prove that

$$C(x_0, R_0/2) := \left\{ x \in \mathbb{R}^n \text{ s.t. } \frac{E}{R_0} |x' - x_0'| < x_n - x_{0n}, |x - x_0| < R_0/2 \right\},$$

that is the intersection of the ball $\Delta_{R_0/2}(x_0)$ with the cone having vertex x_0, and axis in the x_n-direction, is contained in D_1. Since

$$\Delta_d(x_0) \subset (\mathbb{R}^n \backslash D_2),$$

then

$$(\Delta_d(x_0) \cap C(x_0, R_0/2)) \subset D_1 \backslash D_2.$$

Hence (6.6) implies

$$|\Delta_d(x_0) \cap C(x_0, R_0/2)| \le CR_0^{n-1}\delta. \tag{6.7}$$

Let c be an absolute constant such that

$$|C(x_0, R_0)| = cR_0^n.$$

If we suppose that

$$\delta < \frac{c}{2^n C} R_0,$$

then (6.7) yields that $d < \frac{R_0}{2}$. In fact if $d \geq \frac{R_0}{2}$, then

$$c(R_0/2)^n = |\Delta_d(x_0) \cap C(x_0, R_0/2)| \leq CR_0^{n-1}\delta < c(R_0/2)^n,$$

which leads a contradiction. Then $\Delta_d(x_0) \cap C(x_0, R_0/2) = C(x_0, d)$, and

$$cd^n = |\Delta_d(x_0) \cap C(x_0, R_0/2)| \leq CR_0^{n-1}\delta.$$

Hence

$$d_{\mathcal{H}}(D_1, D_2) \leq CR_0 \left| \ln \left(\frac{T^{1/2} R_0^{(n-1)/2} \epsilon}{\|f\|_{1/4,1/2}} \right) \right|^{-\frac{\kappa}{n}},$$

where C, κ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. The proof of Theorem 1.1 is complete. $\square$

Now we prove Proposition 6.1.

Proof of Proposition 6.1. Let d_m be the modified distance between Ω_1 and Ω_2 introduced in Definition 5.4. In the sequel we shall denote

$$d_m := d_m(\Omega_1, \Omega_2).$$

(We recall that $\delta := d_{\mathcal{H}}(\Omega_1, \Omega_2)$.) We begin by establishing (6.1) for d_m. Without loss of generality, assume that

$$\int_0^{T/2} \int_{\Omega_1 \setminus G} |u_1|^2 \, \mathrm{d}x\mathrm{d}t \leq \eta. \tag{6.8}$$

Two cases can occur: either
(I) $d_m < \frac{\theta^* R_0}{2}$, or
(II) $d_m \geq \frac{\theta^* R_0}{2}$,
where θ^* is the constant introduced in Theorem 4.1.

We begin by studying case (I). Without loss of generality let $x_0 \in \partial\Omega_1$ be such that

$$d_m = \mathrm{dist}(x_0, \Omega_2).$$

By using the three cylinders inequality at the boundary (4.7) for $u = u_1$, with radii

$$r_1 := d_m \frac{1}{7 \cdot 16 C_4 e\pi\lambda^2}, \quad r_2 := \frac{\theta^*}{2} \frac{R_0}{7 \cdot 16 C_4 e\pi\lambda^2}, \quad r_3 := \frac{\theta^*}{2} \frac{R_0}{16 e\pi\lambda},$$

we have

$$\int_0^{T/2}\int_{\Omega_1\cap\Delta_{r_2}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \le C\left(\int_0^{T/2}\int_{\Omega_1\cap\Delta_{r_1}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t\right)^{\gamma} \left(R_0 \|f\|^2_{1/4,1/2}\right)^{1-\gamma}, \tag{6.9}$$

where the constant $C \ge 1$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, only, and $\gamma \in (0,1)$ depends on E, α, λ, Λ only. (In (4.5) $2R$ is replaced by R_0 and in (4.7) H^2 is replaced by $H^2 \le \frac{C}{T} R_0 \|f\|^2_{1/4,1/2}$, C depending on E, α, λ only.) Now let $\overline{d} := \frac{r_2}{1+\sqrt{1+E^2}}$, $\overline{x} := x_0 - \overline{d}\sqrt{1+E^2}\mathbf{n}$, where $\mathbf{n}$ denotes the outer normal at $x_0 \in \partial\Omega_1$. We have

$$\Delta_{\overline{d}}(\overline{x}) \subset (\Omega_1 \cap \Delta_{r_2}(x_0)). \tag{6.10}$$

By (6.10), and (5.11) we have

$$\int_0^{T/2}\int_{\Omega_1\cap\Delta_{r_2}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \ge C R_0 \|f\|^2_{1/4,1/2}, \tag{6.11}$$

where the constant C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. Now (6.8), and the fact that $r_1 < d_m$, yield

$$\eta \ge \int_0^{T/2}\int_{\Omega_1\cap\Delta_{r_1}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t. \tag{6.12}$$

So by (6.9, 6.11, 6.12) we have

$$1 \le C\left(\frac{\eta}{R_0 \|f\|^2_{1/4,1/2}}\right)^{\gamma}, \tag{6.13}$$

where the constant C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. If $\frac{\eta}{R_0\|f\|^2_{1/4,1/2}} < 1$ a simple calculation gives

$$d_m \le C R_0 \left(\frac{\eta}{R_0 \|f\|^2_{1/4,1/2}}\right)^{\theta}, \tag{6.14}$$

where the constants C and θ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. On the other hand if $\frac{\eta}{R_0\|f\|^2_{1/4,1/2}} \ge 1$, since

$$d_m \le C R_0, \tag{6.15}$$

where C depends on E and M only, equation (6.14) follows trivially.

Now we consider case (II), that is we assume that $\frac{\theta^* R_0}{2} \le d_m$. As above we have

$$(\Omega_1 \cap \Delta_{\overline{d}}(\overline{x})) \subset (\Omega_1 \cap \Delta_{r_2}(x_0)) \subset (\Omega_1 \cap \Delta_{d_m}(x_0)) \subset \Omega_1 \backslash G,$$

and

$$\eta \geq \int_0^{T/2} \int_{\Omega_1 \cap \Delta_{\overline{d}}(\overline{x})} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \geq CR_0 \|f\|^2_{1/4,1/2}, \tag{6.16}$$

where the constant C depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. Hence (6.14) follows trivially from (6.16) and (6.15).

Without loss of generality let $x_0 \in \Omega_1$, $x_0 \notin \partial\Omega_1$, be such that

$$\delta = \mathrm{dist}(x_0, \Omega_2).$$

Denoting by

$$h := \mathrm{dist}(x_0, \partial\Omega_1),$$

let us distinguish the following three cases:

(i) $h \leq \frac{\delta}{2}$;

(ii) $h > \frac{\delta}{2}$ and $h > \frac{d_0}{2}$;

(iii) $h > \frac{\delta}{2}$ and $h \leq \frac{d_0}{2}$,

where d_0 is the constant defined in Proposition 5.4.

If case (i) occurs, taking $z_0 \in \partial\Omega_1$ such that $h = |x_0 - z_0|$, and taking $y_0 \in \Omega_2$ such that $\delta = |x_0 - y_0|$, by the triangular inequality we have

$$\delta \leq h + |y_0 - z_0| \leq 2d_m$$

so that (6.1) follows from (6.14).

If case (ii) occurs let us set

$$d_1 := \min\left\{\frac{\delta}{2}, \frac{d_0}{2}\right\}.$$

By using the three cylinders inequality (4.10) for $u := u_1$, with radii

$$r_1 := \frac{\theta^*}{2}\frac{d_1}{7 \cdot 32e\pi\lambda^2}, \quad r_2 := \frac{\theta^*}{2}\frac{d_0}{7 \cdot 32e\pi\lambda^2}, \quad r_3 := \frac{\theta^*}{2}\frac{d_0}{32e\pi\lambda},$$

and by the bound $H^2 \leq C\frac{R_0}{T}\|f\|^2_{1/4,1/2}$, we have

$$\int_0^{\frac{T}{2}} \int_{\Delta_{r_2}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \leq C \left(\int_0^{\frac{T}{2}} \int_{\Delta_{r_1}(x_0)} |u_1|^2 \,\mathrm{d}x\mathrm{d}t \right)^{\gamma} \left(R_0 \|f\|^2_{1/4,1/2} \right)^{(1-\gamma)}, \tag{6.17}$$

where the constant $C \geq 1$ depends on E, α, λ, Λ, $\frac{R_0^2}{T}$, only, and $\gamma \in (0,1)$ depends on λ and Λ only. Repeating the same arguments in order to obtain (6.14), we derive

$$d_1 \leq CR_0 \left(\frac{\eta}{R_0 \|f\|^2_{1/4,1/2}} \right)^{\theta}, \tag{6.18}$$

where the constants C and θ depend on E, α, λ, Λ, $\frac{R_0^2}{T}$, M, F only. Now, if $d_1 = \frac{\delta}{2}$, the thesis follows. On the other hand if $d_1 = \frac{d_0}{2}$, then (6.1) follows trivially since

$$\frac{\delta}{d_0} \leq C,$$

where C depends on E, α, M only.

Finally, if case (iii) occurs, that is $\delta \leq d_0$, by Proposition 5.5 we know that there exists an absolute constant c such that

$$\delta \leq c d_m.$$

Hence (6.1) follows from (6.14).

The proof of Proposition 6.1 is complete. □

7. Appendix

Proof of Lemma 3.3. Let us denote, for every μ, $\xi \in \mathbb{R}$, $x \in \Omega' \cap \Delta_{2R}(x_0)$

$$v(\mu;\xi,x) := \mathrm{e}^{i\sqrt{|\mu|}\xi}\widetilde{u_1}(\mu,x).$$

For every $\mu \in \mathbb{R}\setminus\{0\}$, the function $v(\mu)$ solves the uniformly elliptic problem

$$\begin{cases} i\mathrm{sgn}(\mu)v_{\xi\xi}(\mu)+\mathrm{div}(k(x)\nabla v(\mu)) = 0 & \text{in} \quad \mathbb{R}\times(\Omega'\cap\Delta_{2R}(x_0)), \\ v(\mu) = 0 & \text{on} \quad \mathbb{R}\times((\partial\Omega')\cap\Delta_{2R}(x_0)). \end{cases} \tag{7.1}$$

Let us denote $a_j := 2 - \frac{j}{m}$, for every $j \in \{0, 1, ..., m\}$, and $m \in \mathbb{N}$. Moreover, let

$$h_j(s) := \begin{cases} 0 & \text{if } |s| > a_j, \\ \frac{1}{2}\left(1+\cos\left(\frac{\pi(a_{j+1}-s)}{a_{j+1}-a_j}\right)\right) & \text{if } a_{j+1} \leq |s| \leq a_j, \\ 1 & \text{if } |s| < a_{j+1}, \end{cases}$$

and

$$v_j(\mu) := \frac{\partial^j v(\mu)}{\partial \xi^j}.$$

We have that $v_j(\mu)$ solves

$$\begin{cases} i\mathrm{sgn}(\mu)v_{j\xi\xi}(\mu)+\mathrm{div}(k(x)\nabla v_j(\mu)) = 0 & \text{in} \quad \mathbb{R}\times(\Omega'\cap\Delta_{2R}(x_0)), \\ v_j(\mu) = 0 & \text{on} \quad \mathbb{R}\times((\partial\Omega')\times\Delta_{2R}(x_0)). \end{cases} \tag{7.2}$$

Multiplying equation in (7.2) by $\overline{v_j(\mu)}\eta_j^2$, where

$$\eta_j(\xi,x) := h_j\left(\frac{\xi}{R}\right)h_j\left(\frac{|x|}{R}\right),$$

and integrating over $D_j(x_0) := (-a_jR, a_jR) \times (\Omega' \cap \Delta_{a_jR}(x_0))$, we obtain

$$\left(\int\limits_{D_j(x_0)} k(x)\nabla v_j(\mu)\cdot\nabla\overline{v_j(\mu)}\eta_j^2 \mathrm{d}x\mathrm{d}\xi\right)^2 + \left(\int\limits_{D_j(x_0)} \left|\frac{\partial v_j(\mu)}{\partial \xi}\right|^2 \eta_j^2 \mathrm{d}x\mathrm{d}\xi\right)^2 \leq \frac{8\lambda^2\pi^4 m^4}{R^4}\left(\int\limits_{D_j(x_0)} |v_j(\mu)|^2 \mathrm{d}x\mathrm{d}\xi\right)^2.$$

Therefore, for every $j \in \{0, 1, ..., m\}$ we obtain

$$\int\limits_{D_{j+1}(x_0)} |v_{j+1}(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi \leq \frac{\sqrt{8}\lambda\pi^2 m^2}{R^2}\int\limits_{D_j(x_0)} |v_j(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi. \tag{7.3}$$

By iteration of (7.3) for $j = 0, ..., m-1$, we have

$$\int\limits_{-R}^{R}\int_{\Omega'\cap\Delta_R(x_0)} |v_m(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi \leq 4R\left(\frac{\sqrt{8}\lambda\pi^2 m^2}{R^2}\right)^m \int\limits_{\Omega'\cap\Delta_{2R}(x_0)} |\widetilde{u_1}(\mu)|^2 \,\mathrm{d}x. \tag{7.4}$$

Now, let us estimate the integral on the right hand side of (7.4). By (3.7) we obtain

$$\|\widetilde{u_1}(\mu)\|^2_{L^2(\Omega'\cap\Delta_{2R}(x_0))} \leq \frac{1}{4\pi}\left(\int\limits_{-\infty}^{+\infty} \|u_1(t)\|_{L^2(\Omega'\cap\Delta_{2R}(x_0))} \,\mathrm{d}t\right)^2 \leq cC_1^2H^2\left(T + \frac{1}{a_R}\right)^2.$$

Therefore, by (7.4), we have, for every $\mu \in \mathbb{R}\setminus\{0\}$, and for every $m \in \mathbb{N}$,

$$\int\limits_{-R}^{R}\int_{\Omega'\cap\Delta_R(x_0)} |v_m(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi \leq cRC_1^2H^2\left(T + \frac{1}{a_R}\right)^2\left(\frac{\sqrt{8}\lambda\pi^2 m^2}{R^2}\right)^m. \tag{7.5}$$

Moreover by Caccioppoli inequality we have

$$\int\limits_{-\frac{R}{2}}^{\frac{R}{2}}\int_{\Omega'\cap\Delta_{R/2}(x_0)} |\nabla_x v_m(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi \leq \frac{C}{R^2}\int\limits_{-R}^{R}\int_{\Omega'\cap\Delta_R(x_0)} |v_m(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi,$$

where the constant C depends on λ only. So from (7.5) it follows that

$$\int\limits_{-\frac{R}{2}}^{\frac{R}{2}}\int_{\Omega'\cap\Delta_{R/2}(x_0)} |\nabla_x v_m(\mu)|^2 \,\mathrm{d}x\mathrm{d}\xi \leq \frac{CC_1^2H^2}{R}\left(T + \frac{1}{a_R}\right)^2\left(\frac{\sqrt{8}\lambda\pi^2 m^2}{R^2}\right)^m, \tag{7.6}$$

where the constant C depends on λ only.

For fixed $\mu \in \mathbb{R}\setminus\{0\}$ and $\varphi \in L^2(\Omega'\cap\Delta_{R/2}(x_0), \mathbb{C})$, let us denote

$$F(\xi) := \int\limits_{\Omega'\cap\Delta_{R/2}(x_0)} \partial_{x_i} v(\mu;\xi,x)\overline{\varphi(x)}\mathrm{d}x \quad \text{for } i = 1, \cdots n.$$

By the interpolation inequality (7.10), and by inequality (7.6) we have

$$|F^{(m)}(\xi)| \leq \frac{cCC_1H}{R^{m+2}}\left(T+\frac{1}{a_R}\right)(2\lambda\pi(m+1))^{m+1}\,\|\varphi\|_{L^2(\Omega'\cap\Delta_{R/2}(x_0))}. \tag{7.7}$$

By using inequality (7.7) for every $m\in\mathbb{N}$ and the power series of F at any point ξ such that $\Re e\xi\in(-R/2,R/2)$, $\Im m\xi=0$, we have that the function F can be analytically extended to the rectangle $\{\xi\in\mathbb{C}$ s.t. $\Re e\xi\in(-R/2,R/2), \Im m\xi\in(-\bar\rho,\bar\rho)\}$, where $\bar\rho:=\frac{R}{4e\pi\lambda}$. We continue to denote by F the analytic extension of F. In particular, choosing $\xi=-i\delta$, where $\delta:=\frac{R}{8e\pi\lambda}$, by (3.7) we obtain the estimate

$$|F(-i\delta)| \leq cC_1H\left(T+\frac{1}{a_{R/4}}\right)\|\varphi\|_{L^2(\Omega'\cap\Delta_{R/2}(x_0))}. \tag{7.8}$$

On the other side, by the definition of v, we have

$$F(-i\delta)=\int\limits_{\Omega'\cap\Delta_{R/2}(x_0)} \mathrm{e}^{\sqrt{|\mu|}\delta}\partial_{x_i}\widetilde{u_1}(\mu,x)\overline{\varphi(x)}\mathrm{d}x \quad \text{for } i=1,\cdots,n, \tag{7.9}$$

so that by choosing $\varphi(x)=\partial_{x_i}\widetilde{u_1}(\mu,x)$ in (7.9) we obtain (3.10) from (7.8).

The proof of Lemma 3.3 is complete. □

Interpolation and trace inequalities

Given an interval I in $\mathbb{R}$, and $f\in H^1(I)$, we have

$$\|f\|_{L^\infty(I)} \leq c\left(|I|\|f'\|^2_{L^2(I)}+\frac{1}{|I|}\|f\|^2_{L^2(I)}\right)^{1/2}, \tag{7.10}$$

where $|I|$ denotes the length of the interval I.

$$\left(\frac{1}{\rho^{n-2}}\int\limits_{\Delta_\rho}|\nabla f(x)|^2\mathrm{d}x\right)^{1/2} \leq C\left(\left(\rho^{1+\alpha}\,|\nabla f|_{\alpha,\Delta_{\rho'}}\right)^{\frac{1}{1+\alpha}}\left(\frac{1}{\rho^n}\int\limits_{\Delta_\rho}|f(x)|^2\mathrm{d}x\right)^{\frac{\alpha}{2(1+\alpha)}}+\left(\frac{1}{\rho^n}\int\limits_{\Delta_\rho}|f(x)|^2\mathrm{d}x\right)^{1/2}\right), \tag{7.11}$$

where $\rho<\rho'<2\rho$, $0<\alpha<1$ and C depends on α only.

For every $\rho<r$, we have

$$\int\limits_{\Delta_\rho}|F(0,x)|^2\mathrm{d}x \leq c\left(\frac{r}{r^2-\rho^2}\int\limits_{B_r^+}|F(X)|^2\mathrm{d}X+r\int\limits_{B_r^+}|F_y(X)|^2\mathrm{d}X\right). \tag{7.12}$$

For every $h>0$, we have

$$\int\limits_{\Delta_r}|F(0,x)|^2\mathrm{d}x \leq c\left(\frac{1}{h}\int\limits_0^h\int\limits_{\Delta_r}|F(y,x)|^2\mathrm{d}x\mathrm{d}y+h\int\limits_0^h\int\limits_{\Delta_r}|F_y(y,x)|^2\mathrm{d}x\mathrm{d}y\right). \tag{7.13}$$

$$\int_{B_r^+} |F(X)|^2 \mathrm{d}X \leq c \left(r \int_{\Delta_r} |F(0,x)|^2 \mathrm{d}x + r^2 \int_{B_r^+} |F_y(X)|^2 \mathrm{d}X \right). \tag{7.14}$$

$$\int_0^h \int_{\Delta_r} |F(y,x)|^2 \mathrm{d}y \mathrm{d}x \leq c \left(h \int_{\Delta_r} |F(0,x)|^2 \mathrm{d}x + h^2 \int_0^h \int_{\Delta_r} |F_y(y,x)|^2 \mathrm{d}y \mathrm{d}x \right). \tag{7.15}$$

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