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INTEGRAL REPRESENTATION AND Γ-CONVERGENCE OF VARIATIONAL INTEGRALS WITH $P(X)$-GROWTH

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Abstract. We study the integral representation properties of limits of sequences of integral functionals like $\int f(x, Du)\,\mathrm{d}x$ under nonstandard growth conditions of (p, q)-type: namely, we assume that

$$|z|^{p(x)} \leq f(x, z) \leq L(1 + |z|^{p(x)}) .$$

Under weak assumptions on the continuous function $p(x)$, we prove Γ-convergence to integral functionals of the same type. We also analyse the case of integrands $f(x, u, Du)$ depending explicitly on u; finally we weaken the assumption allowing $p(x)$ to be discontinuous on nice sets.

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Introduction

The aim of this paper is the study of the Γ-convergence and integral representation properties for sequences of integral functionals of the type

$$F(u, \Omega) := \int_\Omega f(x, u(x), Du(x))\,\mathrm{d}x, \tag{0.1}$$

where Ω is an open subset of $\mathbb{R}^n$ and f is a non-negative Borel function defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$. Under the assumption of p-growth

$$|z|^p \leq f(x, u, z) \leq L(1 + |z|^p) \tag{0.2}$$

existence and integral representation of the Γ-limit with respect to the strong topology of L^p of a sequence of functionals as (0.1) was proved in the scalar case in [11, 15, 16], and in the vector-valued case in [21], under suitable assumptions on the dependence of f on u, see also [10, 14].

In the context of regularity theory for minimizers, ten years ago Marcellini [22] replaced (0.2) with the more flexible (p, q)-growth assumption

$$|z|^p \leq f(x, u, z) \leq L(1 + |z|^q) , \qquad q \geq p > 1 ; \tag{0.3}$$

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the theory of integrals with (p,q)-growth received contributions from various authors, see the references in [23].

Dealing with the passage to the limit for variational problems, *i.e.*, convergence of energies as $j \to +\infty$, and in the context of Γ-convergence, Zhikov [27] proved several results, under the (p,q)-growth assumption (0.3), when $N = 1$ and $f(x,u,\cdot)$ is convex. If (0.3) is satisfied, the functional F is coercive if regarded on the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$, whereas it is bounded (and, in case of convexity, continuous) if regarded on the smaller space $W^{1,q}(\Omega;\mathbb{R}^N)$. This fact is responsible for the presence of the so called Lavrentiev effect, due to the lack of $W^{1,q}$-density of smooth functions in $W^{1,p}(\Omega;\mathbb{R}^N)$, see [28]. On the other hand, for the same reason it is not clear how to obtain existence of the Γ-limit with respect to L^p-convergence in the whole space, see *e.g.* [10] (Ch. 21).

In the context of cavitation and related theories, dealing with integral functionals satisfying a q-growth condition from above and taken to be $+\infty$ outside $W^{1,q}(\Omega;\mathbb{R}^N)$, measure representation of the relaxation with respect to weak $W^{1,p}$ convergence is obtained in [6] and [1], assuming $z \mapsto f(x,z)$ is convex and $p > q - q/n$.

A borderline case lying between (0.2) and (0.3) is the one of $p(x)$-growth:

$$|z|^{p(x)} \le f(x,u,z) \le L(1 + |z|^{p(x)}), \qquad p(x) \ge 1. \tag{0.4}$$

This kind of growth was first considered by Zhikov in the context of homogenization, see [31], and in recent years the subject gained importance by providing variational models for many problems from Mathematical Physics. For instance, recently Rajagopal and Růžička elaborated a model for the electrorheological fluids; they are special non-Newtonian fluids which are characterized by their ability to change their mechanical properties when in presence of an electromagnetic field, see [25] and [26]. Other models of this type arise for fluids whose viscosity is influenced in a similar way by the temperature, see [30]; the mathematical model for Zhikov's thermistor problem includes equations like

$$-\mathrm{div}(p(x)|Du|^{p(x)-2}Du) = 0\,,$$

whose solutions correspond to minimizers of $\int_\Omega |Du|^{p(x)}\,\mathrm{d}x$.

For the regularity of minimizers of functionals with $p(x)$-growth, we refer to [2, 3, 5, 12, 13, 20, 22, 28]. In particular, Zhikov proved higher integrability of the gradient under the following condition about the modulus of continuity $\omega(R)$ of $p(x)$

$$\limsup_{R\to 0^+} \omega(R)\,\log(1/R) < +\infty\,, \tag{0.5}$$

a condition which is sharp since, in general, dropping it causes the loss of any type of regularity of minimizers, see [29].

Moreover, condition (0.5) seems to play a central role in the theory of functionals with $p(x)$-growth since Zhikov proved in [29] that such functionals exhibit the Lavrentiev phenomenon if (0.5) is violated. On the other side, in [1] it is proved that the singular part of the measure representation of relaxed functionals with growth (0.4) disappears if (0.5) holds true.

In this paper we show that if the growth exponent $p(x)$ satisfies a local continuity estimate of the type (0.5), then integral representation holds for non-negative local functionals satisfying a $p(x)$-growth condition from above in the set $W^{1,p(x)}(\Omega;\mathbb{R}^N)$ of functions $u \in W^{1,1}$ with $|Du|^{p(x)} \in L^1$.

Also, we prove Γ-compactness in $W^{1,p(x)}(\Omega;\mathbb{R}^N)$ for sequences of local functionals satisfying the $p(x)$-growth condition (0.4). Moreover, the Γ-limit turns out to be the integral of a quasi-convex function in the sense of Morrey [24], satisfying the same $p(x)$-growth condition from above. More precisely, we organize the paper as follows.

After giving the notation and some preliminary results in Section 1, we analyse in Section 2 the properties of functionals with $p(x)$-growth, in particular recalling a density result due to Zhikov in the set $W^{1,p(x)}(\Omega;\mathbb{R}^N)$, see Proposition 2.18. Here a central role is played by the local assumption (0.5) on the modulus of continuity of $p(x)$. In Section 3 we prove an integral representation result of the type $\int f(x,Du)\,\mathrm{d}x$ for non-negative local functionals satisfying a $p(x)$-growth condition from above, see Theorem 3.1. Then, in Section 4 we prove existence and integral representation of the $\Gamma(L^1)$-limit of sequences of local functionals with $p(x)$-growth, see Theorems 4.1 and 4.2. Section 5 is dedicated to the more general case of integrands $f(x,u,Du)$ with explicit dependence on u, obtaining the same conclusions under the assumption of a continuous dependence of f on u, see Theorems 5.1 and 5.2. Finally, in Section 6 we generalize the results allowing the growth exponent $p(x)$ to be discontinuous on a negligible set given by the interfaces between nice subsets of the domain Ω, see Theorems 6.1 and 6.2. We finally remark that, in the particular case of relaxation (*i.e.*, the Γ-limit of a constant sequence) we expect the relaxed functional to be the integral of the quasi-convex envelope of f: see Corollary 6.3 for the case of $p(x)$ piecewise constant.

1. Notation and preliminaries

In the sequel Ω is a fixed bounded open subset of $\mathbb{R}^n$ and $\mathcal{A}$ is the family of its open subsets; if $A, B \in \mathcal{A}$, by $A \subset\subset B$ we mean that the closure $\overline{A}$ of A is a compact set contained in B, and by $\mathcal{A}_0$ we denote the class of all $A \in \mathcal{A}$ such that $A \subset\subset \Omega$. Also, $B_\delta(x)$ denotes the ball of radius $\delta > 0$ centred at $x \in \mathbb{R}^n$.

As usual, $L^p(\Omega;\mathbb{R}^N)$ and $W^{1,p}(\Omega;\mathbb{R}^N)$ will denote the standard Lebesgue and Sobolev spaces of functions $u : \Omega \to \mathbb{R}^N$, for any $p \geq 1$. Finally, we will denote by p^* the Sobolev conjugate of p, *i.e.*, $p^* := np/(n-p)$ if $1 \leq p < n$, and $p^* = +\infty$ if $p \geq n$. We recall that if $u \in L^1(\Omega;\mathbb{R}^N)$ and $Du \in L^p(\Omega;\mathbb{R}^{nN})$, then $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ provided Ω has Lipschitz boundary (also weaker conditions are sufficient, see *e.g.* [4]). For such sets Ω, if a sequence $\{u_j\}$ is bounded in $L^1(\Omega;\mathbb{R}^N)$ and $\{Du_j\}$ is bounded in $L^p(\Omega;\mathbb{R}^{nN})$, then $\{u_j\}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^N)$.

We recall the main definitions and results from the theory of Γ-convergence. We refer to [14] or [10] for a more extensive introduction to the subject.

Definition 1.1. Let $F_j : X \to \overline{\mathbb{R}}$ be a sequence of functions defined on a metric space (X,d). We say that $\{F_j\}$ is $\Gamma(d)$-converging to $F : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have

1. (lim inf *inequality*) for every sequence $\{x_j\}$, if $d(x_j,x) \to 0$, then

$$F(x) \leq \liminf_{j\to+\infty} F_j(x_j)\,;$$

2. (*existence of a recovery sequence*) there exists a sequence $\{x_j\}$, with $d(x_j,x) \to 0$, such that

$$F(x) = \lim_{j\to+\infty} F_j(x_j)\,.$$

The function F is called the $\Gamma(d)$-limit of $\{F_j\}$ and we write $F = \Gamma(d)\text{-}\lim\limits_{j\to+\infty} F_j$.

Remark 1.2. It is well known that if $F_j \equiv G$ for all $j \in \mathbb{N}$, then $\Gamma(d)\text{-}\lim_j F_j = \overline{G}$, where $\overline{G}$ is the d-lower semicontinuos envelope of G (or d-relaxed functional), given by

$$\overline{G}(x) := \inf\left\{\liminf_{j\to+\infty} G(x_j) \mid \{x_j\} \subset X\,,\quad d(x_j,x) \to 0\right\}\cdot$$

Since in general the Γ-limit may not exist, we introduce the $\Gamma(d)$-upper and lower limits.

Definition 1.3. Let $F_j : X \to \overline{\mathbb{R}}$ and $x \in X$. We define the $\Gamma(d)$-upper limit of $\{F_j\}$ at x as

$$\Gamma(d)\text{-}\limsup_{j\to+\infty} F_j(x) := \inf\left\{\limsup_{j\to+\infty} F_j(x_j) \mid \{x_j\} \subset X\,,\quad d(x_j,x) \to 0\right\}\cdot$$

Similarly, we define the $\Gamma(d)$-lower limit of $\{F_j\}$ at x as

$$\Gamma(d)\text{-}\liminf_{j\to+\infty} F_j(x) := \inf\left\{\liminf_{j\to+\infty} F_j(x_j) \mid \{x_j\}\subset X\,, \quad d(x_j,x)\to 0\right\}\cdot$$

Therefore, the upper and lower Γ-limits always exist and, since the two infima in the definition above are actually minima, the Γ-limit exists at x if and only if $\Gamma(d)\text{-}\liminf_j F_j(x) = \Gamma(d)\text{-}\limsup_j F_j(x)$. Anyway, for Γ-convergence the following compactness property holds:

Proposition 1.4. *If (X,d) is a separable metric space and $F_j : X \to \overline{\mathbb{R}}$, $j\in\mathbb{N}$, are given functions, there exists an increasing sequence $\{j_k\}$ such that $\Gamma(d)\text{-}\lim_k F_{j_k}(x)$ exists for all $x\in X$.*

We also recall the following facts about set functions:

Definition 1.5. A function $\alpha : \mathcal{A}\to[0,+\infty]$ is called an increasing set function if $\alpha(\emptyset)=0$ and $\alpha(A)\le\alpha(B)$ if $A\subseteq B$. An increasing set function α is said to be subadditive if

$$\alpha(A\cup B)\le\alpha(A)+\alpha(B)$$

for all $A,B\in\mathcal{A}$, and it is said to be superadditive if

$$\alpha(A\cup B)\ge\alpha(A)+\alpha(B)$$

for all $A,B\in\mathcal{A}$ with $A\cap B=\emptyset$; finally, α is said to be inner regular if for all $A\in\mathcal{A}$

$$\alpha(A)=\sup\{\alpha(B)\mid B\in\mathcal{A}\,,\quad B\subset\subset A\}\cdot$$

In this paper we will consider functionals defined on the Lebesgue space $L^1(\Omega;\mathbb{R}^N)$. We will first prove an integral representation theorem and then we will apply the direct method of Γ-convergence, which consists in proving general abstract compactness results, ensuring the existence of Γ-converging sequences, and then recovering enough information on the structure of the Γ-limits to obtain a representation in a suitable form. For both problems we make use of the localization method, which consists in considering at the same time the dependence of the Γ-limit on the function and on the open set.

More precisely, if $F_j : L^1(\Omega;\mathbb{R}^N)\times\mathcal{A}\to[0,+\infty]$ is a sequence of functionals, $u\in L^1(\Omega;\mathbb{R}^N)$ and $A\in\mathcal{A}$, we denote by $F'(u,A)$ and $F''(u,A)$ the lower and upper Γ-limits

$$F'(u,A) =: \inf\left\{\liminf_{j\to+\infty} F_j(u_j,A)\mid\{u_j\}\subset L^1(\Omega;\mathbb{R}^N)\,,\ u_j\to u \text{ in } L^1(\Omega;\mathbb{R}^N)\right\}\,,$$
$$F''(u,A) =: \inf\left\{\limsup_{j\to+\infty} F_j(u_j,A)\mid\{u_j\}\subset L^1(\Omega;\mathbb{R}^N)\,,\ u_j\to u \text{ in } L^1(\Omega;\mathbb{R}^N)\right\}\,,$$

so that with the previous notation d is the metric of $L^1(\Omega;\mathbb{R}^N)$ and $X=L^1(\Omega;\mathbb{R}^N)$.

The following compactness theorem holds for the Γ-limit of localized functionals, see [17]:

Proposition 1.6. *Let $F_j : L^1(\Omega;\mathbb{R}^N)\times\mathcal{A}\to[0,+\infty]$ be a sequence of functionals. Suppose that for every $u\in L^1(\Omega;\mathbb{R}^N)$ the lower and upper Γ-limits*

$$\alpha'(A) := F'(u,A) = \Gamma(L^1(\Omega;\mathbb{R}^N))\text{-}\liminf_{j\to+\infty} F_j(u,A)$$
$$\alpha''(A) := F''(u,A) = \Gamma(L^1(\Omega;\mathbb{R}^N))\text{-}\limsup_{j\to+\infty} F_j(u,A)$$

define inner regular increasing set functions. Then there exists a subsequence $\{j_k\}$ such that the Γ-limit

$$F(u,A) = \Gamma(L^1(\Omega;\mathbb{R}^N))\text{-}\lim_{k\to+\infty} F_{j_k}(u,A)$$

exists for all $A \in \mathcal{A}$ *and* $u \in L^1(\Omega;\mathbb{R}^N)$.

2. Functionals with $p(x)$-growth

In this paper we consider functionals $F : L^1(\Omega;\mathbb{R}^N)\times\mathcal{A} \to [0,+\infty]$ satisfying a growth condition of the form

$$0 \le F(u,A) \le \beta \int_A (a(x) + |Du|^{p(x)})\,\mathrm{d}x$$

for all $A \in \mathcal{A}$ and all $u \in L^1(\Omega;\mathbb{R}^N)$ such that $\int_\Omega (|u|^{p(x)} + |Du|^{p(x)})\,\mathrm{d}x < +\infty$, where β is a positive constant, $a(x) \in L^1(\Omega)$ and $p : \Omega \to \mathbb{R}$ are given functions with $p(x) \ge 1$.

It is natural to define, for every $A \in \mathcal{A}$, the sets

$$\begin{aligned} L^{p(x)}(A;\mathbb{R}^N) &:= \left\{ u : A \to \mathbb{R}^N \mid \int_A |u|^{p(x)}\,\mathrm{d}x < +\infty \right\}, \\ W^{1,p(x)}(A;\mathbb{R}^N) &:= \left\{ u \in L^{p(x)}(A;\mathbb{R}^N) \mid Du \in L^{p(x)}(A;\mathbb{R}^{nN}) \right\}, \\ W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) &:= \left\{ u : A \to \mathbb{R}^N \mid u_{|B} \in W^{1,p(x)}(B;\mathbb{R}^N) \quad \forall B \in \mathcal{A},\ B \subset\subset A \right\}, \end{aligned}$$

which coincide with the spaces $L^p(A;\mathbb{R}^N)$, $W^{1,p}(A;\mathbb{R}^N)$ and $W^{1,p}_{\mathrm{loc}}(A;\mathbb{R}^N)$ in case $p(x) \equiv p \ge 1$. In the sequel, the target space $\mathbb{R}^N$ will be omitted when it is clear from the context, for example within proofs.

Note that in general $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are not vector spaces: taking *e.g.* $p(x)$ so that $\int_\Omega 2^{p(x)}\,\mathrm{d}x = +\infty$, clearly $u \equiv 1 \in L^{p(x)}(\Omega)$ but $2u \notin L^{p(x)}(\Omega)$. It is easy to show that they become vector spaces if $\sup_\Omega p(x) < +\infty$, thus $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ is a vector space if $p(x)$ is locally bounded (as *e.g.* if $p(x)$ is continuous).

In any case, since $|\cdot|^{p(x)}$ is a convex function if $p(x) \ge 1$, it comes out that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are convex sets.

We will work with continuous exponent functions $p(x)$, or in the last section with a particular class of discontinuous functions. We introduce the following assumptions, which we shall use only when needed:

Definition 2.1. A family $\{\Omega_i\}$ is a locally finite regular partition of an open set Ω if each Ω_i is an open set with Lipschitz boundary and

$$\Omega = \Sigma \cup \bigcup_{i=1}^{+\infty} \Omega_i$$

where $|\Sigma| = 0$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \ne j$, and if $A \in \mathcal{A}_0$ yields $A \cap \Omega_i = \emptyset$ except for a finite number of indices.

Definition 2.2. A function $p : \Omega \to [1,+\infty)$ is a regular piecewise continuous exponent if there exist a locally finite regular partition $\{\Omega_i\}$ of Ω and, for every i, a uniformly continuous function $p_i : \Omega_i \to [1,+\infty)$ such that $p(x) = p_i(x)$ for every $x \in \Omega_i$.

Remark 2.3. If $p(x)$ is a regular piecewise continuous exponent satisfying

$$\inf_{x\in\Omega_i} p_i(x) > 1 \qquad \forall\, i \tag{2.1}$$

then for every $A \in \mathcal{A}_0$ there exist $\overline{p}$, $\overline{q}$ such that $1 < \overline{p} \le p(x) \le \overline{q} < +\infty$ a.e. in A.

Since we need to work with open sets A which do not satisfy the assumptions of the Sobolev embedding theorem, if $u \in L^1(A)$ and $\int_A |Du|^{p(x)}\,\mathrm{d}x < +\infty$ we cannot expect that $|u|^{p(x)}$ belongs to $L^1(A)$, see Example 2.5.

However, we can prove the following:

Lemma 2.4. *Let $p : \Omega \to [1,+\infty)$ be continuous in Ω. Then, for every function $u \in L^1(\Omega;\mathbb{R}^N)$ such that $\int_\Omega |Du|^{p(x)}\,\mathrm{d}x < +\infty$ and every $A \in \mathcal{A}_0$ we have*

$$\int_A |u|^{p(x)}\,\mathrm{d}x < +\infty\,. \tag{2.2}$$

In addition, if Ω has Lipschitz boundary and $p(x)$ is uniformly continuous in Ω, then (2.2) *holds for every $A \in \mathcal{A}$ and in particular $|u|^{p(x)} \in L^1(\Omega)$.*

Proof. Let us fix $A \in \mathcal{A}_0$; for every $x_0 \in \overline{A}$, choose $\varepsilon_{x_0} > 0$ so small that

$$1 \le p(x_0) - \varepsilon_{x_0} < p(x_0) + \varepsilon_{x_0} < (p(x_0) - \varepsilon_{x_0})^* \qquad \text{if} \quad p(x_0) > 1\,,$$

or just $p(x_0) + \varepsilon_{x_0} < 1^*$ if $p(x_0) = 1$. Now, by the continuity of $p(x)$ we can find $\delta = \delta(x_0, \varepsilon_{x_0}) > 0$ such that $B_\delta(x_0) \subset\subset \Omega$ and in $\overline{B}_\delta(x_0)$ either $|p(x) - p(x_0)| < \varepsilon_{x_0}$ if $p(x_0) \le n$, or $p(x) \ge n$ if $p(x_0) > n$. By the compactness of $\overline{A}$, we can extract a finite cover $\bigcup_{i=1}^m B_{\delta_i}(x_i)$ of A, where $x_i \in \overline{A}$ and $\delta_i = \delta(x_i, \varepsilon_{x_i})$ for each $i = 1, \ldots, m$. Denoting now

$$B_i := B_{\delta_i}(x_i)\,, \qquad p^i_{\min} := \min_{x \in \overline{B}_i} p(x)\,, \qquad p^i_{\max} := \max_{x \in \overline{B}_i} p(x)\,,$$

we have

$$\int_{B_i} |Du|^{p^i_{\min}}\,\mathrm{d}x \le \int_{B_i} (1 + |Du|^{p(x)})\,\mathrm{d}x < +\infty\,,$$

hence $u \in W^{1,p^i_{\min}}(B_i)$ for each $i = 1, \ldots, m$. Since by construction either $p^i_{\min} \ge n$ or $p^i_{\max} < (p^i_{\min})^*$, the Sobolev embedding theorem yields that $u \in L^{p^i_{\max}}(B_i)$ for each $i = 1, \ldots, m$. Then we get

$$\int_A |u|^{p(x)}\,\mathrm{d}x \le \sum_{i=1}^m \int_{B_i} (1 + |u|^{p^i_{\max}})\,\mathrm{d}x < +\infty\,,$$

which proves (2.2). Finally, if Ω has Lipschitz boundary and $p(x)$ is uniformly continuous in Ω, once $p(x)$ is extended in a continuous way to the closure $\overline{\Omega}$, to obtain $|u|^{p(x)} \in L^1(\Omega)$ we can argue exactly as above, simply replacing $B_{\delta_i}(x_i)$ with $B_{\delta_i}(x_i) \cap \overline{\Omega}$, provided the radii δ_i are chosen so that all these sets have Lipschitz boundary. □

Example 2.5. Remark that Lemma 2.4 may fail to hold if $p(x)$ is a generic discontinuous function. Indeed, if $x_0 \in \Omega$ and $p(x)$ is any function such that $\int_{B_\varepsilon(x_0)} 2^{p(x)}\,\mathrm{d}x = +\infty$ for every ε, then the function $u(x) \equiv 2$ is in $L^1(\Omega)$ and $\int_\Omega |Du|^{p(x)}\,\mathrm{d}x = 0$, but $|u|^{p(x)} \notin L^1_{\mathrm{loc}}(\Omega)$ since $\int_{B_\varepsilon(x_0)} |u|^{p(x)}\,\mathrm{d}x = +\infty$. However, the following result clarifies the situation:

Corollary 2.6. *If $p : \Omega \to [1,+\infty)$ is a regular piecewise continuous exponent, then from $u \in L^1(\Omega;\mathbb{R}^N)$ and $\int_\Omega |Du|^{p(x)}\,\mathrm{d}x < +\infty$ we deduce $\int_A |u|^{p(x)}\,\mathrm{d}x < +\infty$ for every $A \in \mathcal{A}_0$.*

To prove this result, it is enough to enclose A in an open set $A' \in \mathcal{A}_0$ such that $A' \cap \Omega_i$ has Lipschitz boundary for every i, and to apply Lemma 2.4.

Remark 2.7. It follows from Lemma 2.4 that if $p(x)$ is uniformly continuous in $A \in \mathcal{A}$, then

$$W^{1,p(x)}(A;\mathbb{R}^N) = \left\{ u \in L^1(A;\mathbb{R}^N) \mid \int_A |Du|^{p(x)}\,\mathrm{d}x < +\infty \right\}$$

if A has Lipschitz boundary, or at least the cone property (see [4]). In any case, if $p(x)$ is continuous in Ω we have that

$$W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) = \{u \in L^1_{\mathrm{loc}}(A;\mathbb{R}^N) \mid |Du|^{p(x)} \in L^1_{\mathrm{loc}}(A)\}\,. \tag{2.3}$$

In the sequel we will also need the following result:

Lemma 2.8. *Let $p : \Omega \to [1,+\infty)$ be continuous in Ω. If $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, then for every open set $A \in \mathcal{A}_0$ with Lipschitz boundary and for every sequence $\{u_j\}$ converging to u in $L^1(A;\mathbb{R}^N)$ with*

$$\sup_{j\in\mathbb{N}} \int_A |Du_j|^{p(x)}\,\mathrm{d}x < +\infty\,, \tag{2.4}$$

we have

$$\lim_{j\to+\infty} \int_A |u_j - u|^{p(x)}\,\mathrm{d}x = 0\,.$$

Proof. If $\bigcup_{i=1}^{m} B_{\delta_i}(x_i)$ is a finite cover of A defined as in Lemma 2.4, but with the radii δ_i chosen so that $A_i := B_{\delta_i}(x_i) \cap A$ has Lipschitz boundary, we deduce by (2.4)

$$\int_{A_i} |Du_j|^{p^i_{\min}}\,\mathrm{d}x \le C \qquad \forall\, i = 1,\dots,m\,, \quad \forall\, j \in \mathbb{N}\,,$$

thus $\{u_j\}$ is bounded in $W^{1,p^i_{\min}}(A_i)$, for each $i = 1,\dots,m$. By Rellich's theorem we obtain that $u_j \to u$ in $L^{p^i_{\max}}(A_i)$ for each i, hence $u \in L^{p(x)}(A)$ and

$$\lim_{j\to+\infty} \int_A |u_j - u|^{p(x)}\,\mathrm{d}x \le \lim_{j\to+\infty} \sum_{i=1}^{m} \int_{A_i} |u_j - u|^{p(x)}\,\mathrm{d}x \le \lim_{j\to+\infty} \sum_{i=1}^{m} \int_{A_i} (|u_j - u|^{p^i_{\min}} + |u_j - u|^{p^i_{\max}})\,\mathrm{d}x = 0$$

and the proof is complete. □

We have an analogous of this result for discontinuous exponents:

Corollary 2.9. *The result of Lemma 2.8 holds again for regular piecewise continuous exponents $p : \Omega \to [1,+\infty)$.*

Remark 2.10. By Remark 2.7 the sequence $\{u_j\}$ of Lemma 2.8 belongs to $W^{1,p(x)}(A;\mathbb{R}^N)$; in addition, by Proposition 2.13 below we also have $u \in W^{1,p(x)}(A;\mathbb{R}^N)$.

If A', A are open sets in $\mathcal{A}$, with $A' \subset\subset A$, a cut-off function between A' and A is a smooth function $\varphi \in C^\infty_0(\Omega)$ with $\operatorname{spt}\varphi \subset A$, $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on A'. In order to prove the fundamental estimate for a family of functionals with $p(x)$-growth (compare Prop. 4.4) we will also need the following:

Lemma 2.11. *Let $p : \Omega \to [1,+\infty)$ be continuous in Ω. Also, let $A', A, B \in \mathcal{A}$ with $A' \subset\subset A$, and let $u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N)$ and $v \in W^{1,p(x)}_{\mathrm{loc}}(B;\mathbb{R}^N)$. Then for every cut-off function φ between A' and A we have that $\varphi u + (1-\varphi)v \in W^{1,p(x)}_{\mathrm{loc}}(A' \cup B;\mathbb{R}^N)$.*

Proof. Clearly $\varphi u + (1-\varphi)v \in L^1_{\mathrm{loc}}(A' \cup B)$; in addition, by setting $K := \operatorname{spt}\varphi$ and $q := \sup_K p(x) < +\infty$, for every $C \in \mathcal{A}$ with $C \subset\subset A' \cup B$ we have

$$\begin{aligned}\int_C |D(\varphi u + (1-\varphi)v)|^{p(x)}\,\mathrm{d}x &= \int_C |\varphi Du + (1-\varphi)Dv + D\varphi \otimes (u-v)|^{p(x)}\,\mathrm{d}x \\ &= \int_{C\cap A'} |Du|^{p(x)}\,\mathrm{d}x + \int_{C\setminus K} |Dv|^{p(x)}\,\mathrm{d}x + \int_{(C\cap K)\setminus A'} |\cdot|^{p(x)}\,\mathrm{d}x =: I + II + III\,.\end{aligned}$$

Since $C \cap A' \subset\subset A$ and $C \setminus K \subset\subset B$, the terms I and II are finite; also,

$$III \le 4^{q-1} \int_{(C\cap K)\setminus A'} [(|Du|^{p(x)} + |Dv|^{p(x)}) + (1 + \|D\varphi\|_\infty^q)(|u|^{p(x)} + |v|^{p(x)})]\,\mathrm{d}x$$

which is finite, since $(C \cap K) \setminus A' \subset\subset A \cap B$, and by (2.3) the proof is complete. □

Corollary 2.12. *The result of Lemma* 2.11 *holds again for regular piecewise continuous exponents* $p : \Omega \to [1, +\infty)$.

In order to preserve the $p(x)$-growth condition in the Γ-limit (see Th. 4.1 and Prop. 4.3) we need to study some measure-theoretic and lower semicontinuity properties of the functionals $\Psi_{p(x)}, \widetilde{\Psi}_{p(x)} : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$ given by

$$\Psi_{p(x)}(u, A) := \begin{cases} \displaystyle\int_A |Du(x)|^{p(x)}\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N), \end{cases} \tag{2.5}$$

$$\widetilde{\Psi}_{p(x)}(u, A) := \begin{cases} \displaystyle\int_A (|u|^{p(x)} + |Du(x)|^{p(x)})\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N), \end{cases} \tag{2.6}$$

where $p : \Omega \to [1, +\infty)$ is a given function (possibly discontinuous).

It is easy to verify that $\Psi_{p(x)}(u, \cdot)$ and $\widetilde{\Psi}_{p(x)}(u, \cdot)$ are measures on $\mathcal{A}$ for every $u \in L^1(\Omega; \mathbb{R}^N)$. We remark that if we replace $W^{1,p(x)}_{\mathrm{loc}}(A)$ with $W^{1,p(x)}(A)$ in (2.5), then $\Psi_{p(x)}(u, \cdot)$ may fail to be inner regular, and hence a measure, even if $p(x) \equiv p > 1$, since we may find $u \in L^1(\Omega) \setminus L^p(A)$ with $Du \in L^p(A)$ when the boundary of A is not smooth. In the definition (2.6) of $\widetilde{\Psi}_{p(x)}$ we can replace $W^{1,p(x)}_{\mathrm{loc}}(A)$ with $W^{1,p(x)}(A)$ still obtaining exactly the same functional, due to the presence of the term $\int_A |u|^{p(x)}\,\mathrm{d}x$. In addition we are able to prove lower semicontinuity of the functionals $\Psi_{p(x)}$ and $\widetilde{\Psi}_{p(x)}$.

Proposition 2.13. *If* $p : \Omega \to (1, +\infty)$ *is continuous and such that* $p(x) \ge p > 1$ *on* Ω, *then for every* $A \in \mathcal{A}$ *the functional* $\Psi_{p(x)}(\cdot, A)$ *defined in* (2.5) *is lower continuous in* $L^1(\Omega; \mathbb{R}^N)$. *The same result holds for the functional* $\widetilde{\Psi}_{p(x)}(\cdot, A)$ *defined in* (2.6) *without any continuity assumption on* $p(x)$.

Proof. Let $u_j \to u$ in $L^1(\Omega)$ and $A \in \mathcal{A}$: we have to prove that

$$\Psi_{p(x)}(u, A) \le \liminf_{j\to+\infty} \Psi_{p(x)}(u_j, A). \tag{2.7}$$

We may assume that the right-hand side is finite and that we are working on a subsequence, which we still label $\{u_j\}$, such that the lower limit in (2.7) is a limit. Then $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(A)$ and $\int_A |Du_j|^{p(x)}\,\mathrm{d}x \le c$, which implies that $\{Du_j\}$ is bounded in $L^p(A)$ since

$$\int_A |Du_j|^p \le \int_A (1 + |Du_j|^{p(x)})\,\mathrm{d}x \le c\,.$$

Thus we may also suppose that $\{Du_j\}$ converges weakly in $L^p(A)$, but $u_j \to u$ in $L^1(\Omega)$, so it is easy to verify that $Du \in L^p(A)$ and $Du_j \rightharpoonup Du$ in $L^p(A)$. Now we may apply De Giorgi's lower semicontinuity theorem [7] (Th. 2.3.1), which holds without any regularity assumption on A and by which the functional $G(v, w) = \int_A |w|^{p(x)}\,\mathrm{d}x$, defined for $v \in L^1(A; \mathbb{R}^N)$, $w \in L^p(A; \mathbb{R}^{nN})$, is lower semicontinuous with respect to the strong topology in $L^1(A;\mathbb{R}^N)$ and the weak topology in $L^p(A;\mathbb{R}^{nN})$, obtaining that

$$\int_A |Du|^{p(x)}\,\mathrm{d}x \le \lim_{j\to+\infty}\int_A |Du_j|^{p(x)}\,\mathrm{d}x = \lim_{j\to+\infty}\Psi_{p(x)}(u_j,A) < +\infty\,.$$

Then $\int_A |Du|^{p(x)}\,\mathrm{d}x < +\infty$ and by the continuity assumption on $p(x)$ we may apply Lemma 2.4 to conclude that $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $\Psi_{p(x)}(u,A) = \int_A |Du|^{p(x)}\,\mathrm{d}x$, which yields (2.7) and completes the proof.

To prove lower semicontinuity of the functional $\widetilde{\Psi}_{p(x)}(\cdot,A)$, we apply De Giorgi's theorem to the functional $G(v,w) = \int_A(|v|^{p(x)} + |w|^{p(x)})\,\mathrm{d}x$ and we may conclude without applying Lemma 2.4, thus the continuity of $p(x)$ is not needed. □

Corollary 2.14. *The result of Proposition* 2.13 *holds again for regular piecewise continuous exponents, provided* (2.1) *holds.*

Example 2.15. Remark that $\Psi_{p(x)}(\cdot,A)$ may fail to be $L^1(\Omega)$-l.s.c. if $p(x)$ is not continuous. Indeed, if we take $p(x)$ as in Example 2.5, but with

$$\int_\Omega (2-\delta)^{p(x)}\,\mathrm{d}x < +\infty \qquad \forall\,\delta > 0\,,$$

then the sequence $u_j(x) := 2 - 1/j$ converges to $u(x) := 2$ in $L^1(\Omega)$ and $Du_j \equiv 0$, thus $\Psi_{p(x)}(u_j,\Omega) = 0$, but $\Psi_{p(x)}(u,\Omega) = +\infty$ since $u \notin W^{1,p(x)}_{\mathrm{loc}}(\Omega)$.

For our purposes, mainly to prove the integral representation Theorem 3.1, we need to introduce a suitable notion of strong convergence on the set $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$, which in particular coincides with the $W^{1,p}_{\mathrm{loc}}$ convergence in the constant case $p(x) \equiv p \ge 1$. More precisely, we give the following:

Definition 2.16. We say that a sequence $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ converges to $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ strongly in $W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ if

$$\lim_{j\to+\infty}\int_A (|u_j - u|^{p(x)} + |Du_j - Du|^{p(x)})\,\mathrm{d}x = 0 \tag{2.8}$$

for every $A \in \mathcal{A}_0$.

Remark 2.17. If $u_j \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, then $|Du_j|^{p(x)} \to |Du|^{p(x)}$ in $L^1_{\mathrm{loc}}(\Omega)$, as an easy consequence of the estimate $|Du_j|^{p(x)} \le 2^{p(x)-1}(|Du_j - Du|^{p(x)} + |Du|^{p(x)})$ and of the dominated convergence theorem.

We will make use of the following density result with respect to the convergence above, which is essentially due to Zhikov [30], compare also [1] (Lem. 4.5).

Proposition 2.18. *Let $p : \Omega \to [1,+\infty)$ be a continuous function satisfying the following local estimate about the modulus of continuity:*

$$\forall\,A \in \mathcal{A}_0 \quad \exists\,\gamma_A > 0 : \quad |p(x) - p(y)| \le \frac{\gamma_A}{|\log|x-y||} \quad \forall\,x,y \in A\,, \quad 0 < |x-y| < \frac{1}{2}\,\cdot \tag{2.9}$$

Then for every $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ there exists a sequence of smooth functions $\{u_j\} \subset C^\infty_0(\Omega;\mathbb{R}^N)$ such that $u_j \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. If in addition $u \in W^{1,p(x)}(\Omega;\mathbb{R}^N)$, then $u_j \to u$ also in $L^1(\Omega;\mathbb{R}^N)$.

Proof. Let $\rho(x)$ be a standard mollifier with support in the unit ball $B_1(0) \subset \mathbb{R}^n$, for every $j \in \mathbb{N}$ let $\rho_j(x) := j^n\rho(j\,x)$, and set

$$u_j(x) := (\overline{u}_j * \rho_j)(x)\,, \qquad \overline{u}_j(x) := \begin{cases} u(x) & \text{if } \operatorname{dist}(x,\partial\Omega) > 1/j \\ 0 & \text{otherwise}\,. \end{cases}$$

Then it is well known that $u_j \in C_0^\infty(\Omega)$, $u_j \to u$ in $W^{1,1}_{\rm loc}(\Omega)$, and $u_j \to u$ in $L^1(\Omega)$ if $u \in L^1(\Omega)$. To prove that $|Du_j - Du|^{p(x)} \to 0$ in $L^1_{\rm loc}(\Omega)$, fix $A \in \mathcal{A}_0$, choose $0 < \varepsilon_0 < 1$ so that $\mathrm{dist}(A, \partial\Omega) \geq 2\varepsilon_0$, call $q_0 := \sup_A p(x)$ and try to apply the Lebesgue dominated convergence theorem to the sequence $\{|Du_j - Du|^{p(x)}\}_{j>2/\varepsilon_0}$. Since a.e. in A we have $Du_j = Du * \rho_j \to Du$ and $|Du_j - Du|^{p(x)} \leq 2^{q_0-1}(|Du_j|^{p(x)} + |Du|^{p(x)})$, it is enough to estimate from above the sequence $\{|Du_j|^{p(x)}\}_{j>2/\varepsilon_0}$ with some sequence strongly convergent in $L^1(A)$. To do this, let us set

$$A' := \{x \in \Omega \mid \mathrm{dist}(x, A) < \varepsilon_0\}, \quad 1 \leq p := \inf_{x\in A'} p(x), \quad q := \sup_{x\in A'} p(x) < +\infty,$$
$$g(x) := |Du|^{p(x)}, \quad p_j(x) := \min\{p(y) \mid |x-y| \leq 1/j\} \quad \text{and} \quad \varphi_j(x,z) := |z|^{p_j(x)},\ z \in \mathbb{R}^{nN}.$$

Using the fact that $\varphi_j(x, Du(y)) \leq 1 + |Du(y)|^{p(y)}$ if $|x-y| \leq 1/j$, we are able to estimate $|Du_j|^{p_j(x)}$. Indeed, since φ_j is a Carathéodory function convex with respect to z, by Jensen inequality we have

$$\begin{aligned}\varphi_j(x, Du_j(x)) &\leq \int\limits_{|x-y|\leq 1/j} \rho_j(x-y)\varphi_j(x, Du(y))\,\mathrm{d}y \\ &\leq \int\limits_{|x-y|\leq 1/j} \rho_j(x-y)(1 + |Du(y)|^{p(y)})\,\mathrm{d}y = 1 + (g * \rho_j)(x).\end{aligned} \tag{2.10}$$

In addition, using Hölder inequality if $p > 1$, and simply estimating ρ_j if $p = 1$, we obtain

$$\begin{aligned}|Du_j(x)| &\leq \int\limits_{|x-y|\leq 1/j} |\rho_j(x-y)\,Du(y)|\,\mathrm{d}y \\ &\leq \|Du\|_{L^p(A')} \cdot j^{n-n(p-1)/p} \cdot \|\rho\|_{L^{p/(p-1)}(B_1(0))} = c(p)\,\|Du\|_{L^p(A')} \cdot j^{n/p}.\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11) we have

$$\begin{aligned}|Du_j(x)|^{p(x)} &= |Du_j(x)|^{p(x)-p_j(x)} \cdot \varphi_j(x, Du_j(x)) \\ &\leq (1 + c(p)^{q-p}\,\|Du\|^{q-p}_{L^p(A')}) \cdot (j^{n/p})^{p(x)-p_j(x)} \cdot (1 + (g * \rho_j)(x)).\end{aligned}$$

Finally, if $x \in A$, estimate (2.9) means that $0 \leq p(x) - p_j(x) \leq \gamma_A/\log j$, hence

$$(j^{n/p})^{p(x)-p_j(x)} \leq (j^{\gamma_A/\log j})^{n/p} = \mathrm{e}^{n\gamma_A/p}$$

and then we obtain

$$|Du_j(x)|^{p(x)} \leq c\,(1 + (g * \rho_j)(x))$$

for all $x \in A$, where $c > 0$ only depends on n, p, q, γ_A and u. Now, since $g * \rho_j \to g$ in $L^1(A)$, we can apply the Lebesgue dominated convergence theorem to conclude at the same time that $|Du_j(x)|^{p(x)} \to |Du(x)|^{p(x)}$ and $|Du_j(x) - Du(x)|^{p(x)} \to 0$ in $L^1(A)$, as required.

Finally, it follows directly from Lemma 2.8 that $\int_A |u_j - u|^{p(x)}\,\mathrm{d}x \to 0$, thus $u_j \to u$ in $W^{1,p(x)}_{\rm loc}(\Omega)$ and the proof is complete. □

We recall now that a function $u \in L^1(\Omega; \mathbb{R}^N)$ is piecewise affine in Ω if there exists a finite family $\{\Omega_i\}_{i\in I}$ of disjoint open subsets of Ω and a Borel subset N of Ω with $|N| = 0$ such that $\Omega = (\bigcup_{i\in I} \Omega_i) \bigcup N$ and $u_{|\Omega_i}$ is affine on each Ω_i. The following density result holds:

Proposition 2.19. *Under the assumptions of Proposition* 2.18*, for every* $u \in W^{1,p(x)}(\Omega; \mathbb{R}^N)$ *there exists a sequence* $\{u_j\} \subset W^{1,p(x)}(\Omega; \mathbb{R}^N)$ *of functions which are piecewise affine on* Ω *and such that* $u_j \to u$ *both in* $L^1(\Omega; \mathbb{R}^N)$ *and in* $W^{1,p(x)}_{\rm loc}(\Omega; \mathbb{R}^N)$.

Proof. By Proposition 2.18 for every $u \in W^{1,p(x)}(\Omega)$ there exists a sequence $\{u_j\} \subset C_0^\infty(\Omega)$ converging to u both in $L^1(\Omega)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. On the other side, by [19] (Ch. X, Prop. 2.1), given $u \in C_0^\infty(\Omega)$ there exists a sequence $\{u_j\}$ of functions in $C_0^0(\Omega)$ which are piecewise affine on Ω and such that $u_j \to u$ and $Du_j \to Du$ uniformly in Ω. Then $u_j \in W^{1,p(x)}(\Omega)$ and, since uniform convergence implies that $\int_\Omega (|u_j - u|^{p(x)} + |Du_j - Du|^{p(x)})\,\mathrm{d}x \to 0$, the desired sequence is obtained by a diagonal procedure. ☐

3. An integral representation result

In this section we state and prove an integral representation theorem for a class of functionals with $p(x)$-growth. More precisely, we have the following:

Theorem 3.1. *Let $p : \Omega \to [1, +\infty)$ be a continuous function satisfying* (2.9). *Let $F : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$ satisfy the following conditions:*

1. (locality) *F is local, i.e., $F(u, A) = F(v, A)$ for every $A \in \mathcal{A}$ and $u, v \in L^1(\Omega; \mathbb{R}^N)$ with $u = v$ a.e. on A;*
2. (measure property) *for all $u \in L^1(\Omega; \mathbb{R}^N)$ the set function $F(u, \cdot)$ is increasing, and is the trace on $\mathcal{A}$ of a Borel measure;*
3. (growth conditions) *there exist $\beta > 0$ and $a(x) \in L^1_{\mathrm{loc}}(\Omega)$ such that*
$$0 \le F(u, A) \le \beta \int_A (a(x) + |Du|^{p(x)})\,\mathrm{d}x$$
for all $u \in W^{1,p(x)}(\Omega; \mathbb{R}^N)$ and $A \in \mathcal{A}$;
4. (translation invariance in u) *$F(u + c, A) = F(u, A)$ for all $u \in L^1(\Omega; \mathbb{R}^N)$, $A \in \mathcal{A}$, $c \in \mathbb{R}^N$;*
5. (lower semicontinuity) *$F(\cdot, A)$ is sequentially lower semicontinuous with respect to the strong convergence in $L^1(\Omega; \mathbb{R}^N)$ for all $A \in \mathcal{A}$.*

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ such that
$$F(u, A) = \int_A f(x, Du(x))\,\mathrm{d}x \tag{3.1}$$
for every $A \in \mathcal{A}$ and for every $u \in L^1(\Omega; \mathbb{R}^N)$ such that $u_{|A} \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbb{R}^N)$; in addition, the function $f(x, \cdot)$ is quasi-convex on $\mathbb{R}^{nN}$ for a.e. $x \in \Omega$ and satisfies the growth condition
$$0 \le f(x, z) \le \beta\,(a(x) + |z|^{p(x)}) \tag{3.2}$$
for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$.

Proof. The proof of Theorem 3.1 works as in the classical case of Sobolev spaces, see [9] (Th. 1.1), with the exception of Steps 4, 5 and 7; for more details on the other steps see *e.g.* [14] (Th. 20.1) and [10] (Th. 9.1).

Step 1: definition of f.

For every $z \in \mathbb{R}^{nN}$ denote by $u_z : \Omega \to \mathbb{R}^N$ the linear function $u_z(x) = zx$. By ii), $F(u_z, \cdot)$ can be extended to a Borel measure on Ω which, by iii), is absolutely continuous with respect to the Lebesgue measure. Hence, there exists a density function $g_z \in L^1_{\mathrm{loc}}(\Omega)$ such that
$$F(u_z, A) = \int_A g_z(x)\,\mathrm{d}x$$
for all $A \in \mathcal{A}$. We set $f(x, z) := g_z(x)$ for all $x \in \Omega$ and $z \in \mathbb{R}^{nN}$. Then it is easy to verify that f is a Borel function satisfying (3.2) for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$.

Step 2: integral representation on piecewise affine functions.

By the measure property of F it is easy to show that (3.1) holds for all $A \in \mathcal{A}$ and all $u \in W^{1,p(x)}(\Omega)$ which are piecewise affine on Ω.

Step 3: rank-one-convexity of f.

It can be shown that for every fixed $x \in \Omega$ the function $f(x,\cdot)$ is rank-one-convex, that is

$$f(x, tz_2 + (1-t)z_1) \le tf(x,z_2) + (1-t)f(x,z_1)$$

for all $z_1, z_2 \in \mathbb{R}^{nN}$ with $\operatorname{rank}(z_1 - z_2) \le 1$ and for every $t \in (0,1)$. In particular it is well known that a rank-one-convex function with a growth condition of order p satisfies a local Lipschitz condition, hence $f(x,\cdot)$ is locally Lipschitz for every $x \in \Omega$ and then f is a Carathéodory function.

Step 4: the inequality $F(u,A) \le \int_A f(x,Du(x))\,\mathrm{d}x$ for $u \in W^{1,p(x)}(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}$.

As f is a Carathéodory function (see Step 3) satisfying the growth conditions (3.2), we have that for every $A' \in \mathcal{A}_0$ the functional

$$u \mapsto \int_{A'} f(x,Du(x))\,\mathrm{d}x \tag{3.3}$$

is continuous with respect to the $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ convergence of Definition 2.16 (use the dominated convergence theorem on subsequences).

Now, let $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}$. By Proposition 2.19 there exists a sequence $\{u_j\}$ of functions in $W^{1,p(x)}(\Omega)$ which are piecewise affine on Ω and such that $u_j \to u$ in $L^1(\Omega)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. Then by lower semicontinuity v) of F, Step 2 and the continuity of the functional (3.3) in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$, we obtain for every $A' \in \mathcal{A}_0$, $A' \subset\subset A$, that

$$F(u,A') \le \liminf_{j\to+\infty} F(u_j,A') = \lim_{j\to+\infty} \int_{A'} f(x,Du_j(x))\,\mathrm{d}x = \int_{A'} f(x,Du(x))\,\mathrm{d}x.$$

Since $F(u,\cdot)$ is a measure, taking the limit as $A' \nearrow A$ we get by the monotone convergence theorem

$$F(u,A) \le \int_A f(x,Du(x))\,\mathrm{d}x \tag{3.4}$$

for every $u \in W^{1,p(x)}(\Omega)$ and for every $A \in \mathcal{A}$.

Step 5: the equality $F(u,A) = \int_A f(x,Du(x))\,\mathrm{d}x$ for $u \in W^{1,p(x)}(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}$.

Fix $u \in W^{1,p(x)}(\Omega)$ and let $A, A' \in \mathcal{A}$ with $A' \subset\subset A$. We modify the function u in the following way: take $A'' \in \mathcal{A}_0$ such that $A' \subset\subset A'' \subset\subset \Omega$, let φ be a cut-off function between A' and A'' and set $\widetilde{u} := \varphi u$. It is easy to verify that $\widetilde{u} \in W^{1,p(x)}(\Omega)$ and that $\widetilde{u} + v \in W^{1,p(x)}(\Omega)$ for every $v \in W^{1,p(x)}(\Omega)$, since $\widetilde{u}$ has compact support.

Consider the functional $G : L^1(\Omega) \times \mathcal{A} \to [0,+\infty]$ defined by

$$G(v,B) := F(v + \widetilde{u}, B).$$

Then G satisfies all hypotheses of Theorem 3.1. Indeed, i), ii), iv) and v) are trivially satisfied, whereas for all $v \in W^{1,p(x)}(\Omega)$ and all $B \in \mathcal{A}$ we have

$$0 \le G(v,B) = F(v+\widetilde{u},B) \le \beta \int_B (a(x) + |D\widetilde{u} + Dv|^{p(x)})\,\mathrm{d}x \le \int_B (b(x) + |Dv|^{p(x)})\,\mathrm{d}x$$

where $\gamma = 2^{q-1}\beta$, with $q = \sup_{x\in A''} p(x) < +\infty$, and $b(x) = a(x) + |D\widetilde{u}(x)|^{p(x)} \in L^1_{\rm loc}(\Omega)$. Therefore from Steps 1–4 above it follows that there exists a Carathéodory function $g : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, satisfying (3.2) with γ and $b(x)$ instead of β and $a(x)$, such that

$$G(v,B) \le \int_B g(x,Dv(x))\,\mathrm{d}x \qquad \forall\, v \in W^{1,p(x)}(\Omega)\,, \quad \forall\, B \in \mathcal{A}\,, \tag{3.5}$$

with equality for v piecewise affine in Ω. In addition, arguing as for (3.3), we can prove that for every $B' \in \mathcal{A}_0$ the functional

$$v \mapsto \int_{B'} g(x,Dv(x))\,\mathrm{d}x \tag{3.6}$$

is continuous in $W^{1,p(x)}_{\rm loc}(\Omega)$. We now prove that

$$F(u,A') = \int_{A'} f(x,Du(x))\,\mathrm{d}x\,; \tag{3.7}$$

since $F(u,\cdot)$ is a measure, taking $A' \nearrow A$ we will obtain (3.1) for all $A \in \mathcal{A}$ and $u \in W^{1,p(x)}(\Omega)$.

By Proposition 2.19 there exists a sequence $\{u_j\}$ of functions in $W^{1,p(x)}(\Omega)$, piecewise affine in Ω, such that $u_j \to \widetilde{u}$ in $L^1(\Omega)$ and in $W^{1,p(x)}_{\rm loc}(\Omega)$. Then, using the locality i) of F, Step 4, Step 2, equation (3.5) and the continuity of the functionals (3.3) and (3.6), we obtain

$$\begin{aligned}
\int_{A'} g(x,0)\,\mathrm{d}x &= G(0,A') = F(\widetilde{u},A') = F(u,A') \le \int_{A'} f(x,Du)\,\mathrm{d}x \\
&= \int_{A'} f(x,D\widetilde{u})\,\mathrm{d}x = \lim_{j\to+\infty}\int_{A'} f(x,Du_j)\,\mathrm{d}x = \lim_{j\to+\infty} F(u_j,A') \\
&= \lim_{j\to+\infty} G(u_j-\widetilde{u},A') \le \lim_{j\to+\infty}\int_{A'} g(x,D(u_j-\widetilde{u}))\,\mathrm{d}x = \int_{A'} g(x,0)\,\mathrm{d}x
\end{aligned}$$

and (3.7) is proved.

Step 6: the equality $F(u,A) = \int_A f(x,Du(x))\,\mathrm{d}x$ for $u_{|A} \in W^{1,p(x)}_{\rm loc}(A;\mathbb{R}^N)$ and $A \in \mathcal{A}$.

If $u \in L^1(\Omega)$, $A \in \mathcal{A}$ and $u_{|A} \in W^{1,p(x)}_{\rm loc}(A)$, then for every $A' \in \mathcal{A}_0$, $A' \subset\subset A$, we can find a function $v \in W^{1,p(x)}(\Omega)$ such that $v_{|A'} = u_{|A'}$ (it suffices to take $v = \varphi u$, where $\varphi \in C^\infty_0(\Omega)$ is a cut-off function between A' and A'', with $A' \subset\subset A'' \subset\subset A$). Then, by the locality of F and Step 5 we have

$$F(u,A') = F(v,A') = \int_{A'} f(x,Dv(x))\,\mathrm{d}x = \int_{A'} f(x,Du(x))\,\mathrm{d}x$$

and we obtain the assertion as $A' \nearrow A$, by the measure property of F.

Step 7: quasi-convexity of f.

It is enough to prove that, for every $A \in \mathcal{A}_0$ with Lipschitz boundary, $f(x,\cdot)$ is quasi-convex on $\mathbb{R}^{nN}$ for a.e. $x \in A$. If $A \in \mathcal{A}_0$ is fixed and $q := \sup_A p(x) < +\infty$, the restriction $f : A \times \mathbb{R}^{nN} \to [0,+\infty)$ is a Carathéodory integrand with

$$0 \le f(x,z) \le \beta(a(x) + |z|^{p(x)}) \le \beta(1 + a(x) + |z|^q) \le b(x)(1+|z|^q)$$

where $b(x) := \beta(1+a(x)) \in L^1(A)$. In addition, by lower semicontinuity v) it is easy to verify that the functional (3.3) is sequentally weakly l.s.c. on $W^{1,q}(A)$; hence we can apply [7] (Th. 4.1.5) to obtain that $f(x,\cdot)$ is quasi-convex in $\mathbb{R}^{nN}$ for a.e. $x \in A$. □

4. A Γ-convergence result

In this section we state and prove the main result of the paper, which characterizes the Γ-limit in the $L^1(\Omega;\mathbb{R}^N)$ topology of a class of functionals with $p(x)$-growth. We will consider functions satisfying

$$\alpha|z|^{p(x)} \le f(x,z) \le \beta(a(x) + |z|^{p(x)}) \tag{4.1}$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$, where $0 < \alpha \le \beta$ are positive constants and $a(x) \in L^1(\Omega)$. More precisely, we are able to prove the following results:

Theorem 4.1. *Let $p : \Omega \to (1,+\infty)$ be a continuous function with $p(x) \ge p > 1$ for all $x \in \Omega$. Let $f_j : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$ be Borel functions satisfying* (4.1) *for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$. Finally let $F_j : L^1(\Omega;\mathbb{R}^N) \times \mathcal{A} \to [0,+\infty]$ be the variational functionals defined by*

$$F_j(u,A) := \begin{cases} \displaystyle\int_A f_j(x,Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbb{R}^N)\,. \end{cases} \tag{4.2}$$

Then there exists a subsequence $\{j_k\}$ such that the Γ-limit

$$F(u,A) := \Gamma(L^1(\Omega;\mathbb{R}^N))\text{-}\lim_{k\to+\infty} F_{j_k}(u,A) \tag{4.3}$$

exists for all $u \in L^1(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}$, with estimates

$$\alpha\,\Psi_{p(x)}(u,A) \le F(u,A) \le \beta\left(\int_A a(x)\,\mathrm{d}x + \Psi_{p(x)}(u,A)\right) \tag{4.4}$$

for all $u \in L^1(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}$, where the functional $\Psi_{p(x)}$ is defined in (2.5).

Theorem 4.2. *Let $F(u,A)$ be the as in* (4.3) *and, in addition to the hypotheses of Theorem* 4.1*, suppose that the function p satisfies* (2.9)*. Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, satisfying the growth estimate* (4.1) *for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$, with $f(x,\cdot)$ quasi-convex for a.e. $x \in \Omega$, such that for every $A \in \mathcal{A}$*

$$F(u,A) = \begin{cases} \displaystyle\int_A f(x,Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbb{R}^N)\,. \end{cases} \tag{4.5}$$

Proof of Theorem 4.1. The proof is divided in two steps: first we prove existence of the Γ-limit and then we show measure property.

Step 1: existence of the Γ-limit.

First of all, we prove a growth estimates for the upper and lower Γ-limits $F''(u,A)$ and $F'(u,A)$.

Proposition 4.3. *If $p : \Omega \to (1,+\infty)$ is a continuous function with $p(x) \ge p > 1$ in Ω, and $\{F_j\}$ is given by* (4.2)*, then for every $A \in \mathcal{A}$ and $u \in L^1(\Omega;\mathbb{R}^N)$ we have*

$$\alpha\,\Psi_{p(x)}(u,A) \le F'(u,A) \le F''(u,A) \le \beta\left(\int_A a(x)\,\mathrm{d}x + \Psi_{p(x)}(u,A)\right)\,. \tag{4.6}$$

Proof. Since the functional F_j satisfies (4.4), and since by Proposition 2.13 the functional $\Psi_{p(x)}(\cdot,A)$ is $L^1(\Omega)$-l.s.c. for every $A \in \mathcal{A}$, applying [14] (Prop. 6.7) we obtain (4.6) for every $u \in L^1(\Omega)$ and $A \in \mathcal{A}$. □

Remark that Proposition 4.3 yields that the Γ-limit $F(u, A)$ exists and is equal to $+\infty$ for every $A \in \mathcal{A}$ and every $u \in L^1(\Omega)$ such that $u \notin W^{1,p(x)}_{\rm loc}(A)$ or $u \in W^{1,p(x)}_{\rm loc}(A)$ but $\int_A |Du|^{p(x)}\,{\rm d}x = +\infty$.

To prove existence of the Γ-limit for every u and A, we want to apply the compactness result of Proposition 1.6; thus we need to show that for fixed $u \in L^1(\Omega)$ the upper and lower Γ-limits $F''(u,\cdot)$ and $F'(u,\cdot)$ define inner regular increasing set functions. Now, the increasing property is inherited by the same property of the integral functionals F_j. To show inner regularity, we use the following uniform fundamental $L^{p(x)}$ estimate for the sequence $\{F_j\}$:

Proposition 4.4. (Uniform fundamental estimate) *If $p : \Omega \to [1, +\infty)$ is continuous in Ω, for all open sets $A, A', B \in \mathcal{A}$, with $A' \subset\subset A$, and for every $\sigma > 0$, there exists a constant $M_\sigma > 0$, depending on α, β, $p(x)$ and $a(x)$, such that for every $u, v \in L^1(\Omega; \mathbb{R}^N)$ there exists a cut-off function φ between A' and A such that for any $j \in \mathbb{N}$*

$$F_j(\varphi u + (1-\varphi)v, A' \cup B) \le (1+\sigma)(F_j(u, A) + F_j(v, B)) + M_\sigma \int_{A\cap B} |u-v|^{p(x)}\,{\rm d}x + \sigma\,. \tag{4.7}$$

Proof. Fix $A, A', B \in \mathcal{A}$, with $A' \subset\subset A$, $\sigma > 0$ and choose $u, v \in L^1(\Omega)$ so that the right-hand side of (4.7) is finite. Then by (4.2) we have $u \in W^{1,p(x)}_{\rm loc}(A)$, $v \in W^{1,p(x)}_{\rm loc}(B)$, $|Du|^{p(x)} \in L^1(A)$ and $|Dv|^{p(x)} \in L^1(B)$. If $2\,\delta := {\rm dist}(A', \partial A)$, $0 < \nu < \delta$, $0 < r < \delta - \nu$, $d(x) := {\rm dist}(x, A')$, let φ be a cut-off function between $\{x \in A \mid d(x) < r\}$ and $\{x \in A \mid d(x) < \nu + r\}$, with $\|D\varphi\|_\infty \le 2/\nu$. Since by Lemma 2.11 the function $\varphi u + (1-\varphi)v$ belongs to $W^{1,p(x)}_{\rm loc}(A' \cup B)$, by (4.2) one has

$$\begin{aligned} MF_j(\varphi u + (1-\varphi)v, A' \cup B) &= \int_{A'\cup B} f_j(x, D(\varphi u + (1-\varphi)v))\,{\rm d}x \\ &\le F_j(u, A) + F_j(v, B) + \int_{B_r^\nu} f_j(x, D(\varphi u + (1-\varphi)v))\,{\rm d}x\,, \end{aligned} \tag{4.8}$$

where

$$B_r^\nu := \{x \in B \mid r < d(x) < r + \nu\}\cdot$$

We need to estimate the right-hand side in (4.8) independently of r and ν, which will be chosen at the end. To this purpose note that for any possible choice of r and ν we have $B_r^\nu \subset K_\delta := \{x \in A \cap B \mid 0 < d(x) < \delta\}$, with $K_\delta \in \mathcal{A}_0$. Therefore, if we set $q := \sup_{K_\delta} p(x)$, which is independent of r and ν, by (4.1) we obtain (for $2/\nu > 1$)

$$\begin{aligned} \int_{B_r^\nu} f_j(x, D(\varphi u + (1-\varphi)v))\,{\rm d}x &\le \beta\, 2^{q-1} \int_{B_r^\nu} (a(x) + |Du|^{p(x)} + |Dv|^{p(x)} + |D\varphi \otimes (u-v)|^{p(x)})\,{\rm d}x \\ &\le \mu(B_r^\nu) + \beta\, 2^{2q-1}\, \nu^{-q} \int_{A\cap B} |u-v|^{p(x)}\,{\rm d}x\,, \end{aligned}$$

where

$$\mu(E) := \beta\, 2^{q-1} \int_E (a(x) + |Du|^{p(x)} + |Dv|^{p(x)})\,{\rm d}x\,.$$

Now, if $N \in \mathbb{N}$ is large enough that

$$N \ge \max\left\{ \frac{\beta\, 2^{q-1}}{\sigma\alpha}\,,\ \frac{\beta\, 2^{q-1}}{\sigma} \int_{A\cap B} a(x)\,{\rm d}x \right\}\,,$$

there exists $k \in \{1, \ldots, N\}$ such that, for $r = \delta\,(k-1)/N$ and $\nu = \delta/N$, we have

$$\mu(B_r^\nu) \le \frac{1}{N}\,\mu(K_\delta) \le \frac{1}{N}\,\mu(A \cap B) \le \sigma\,\alpha \int_{A\cap B} (|Du|^{p(x)} + |Dv|^{p(x)})\,{\rm d}x + \sigma < +\infty\,.$$

In addition, by (4.1) we obtain

$$\alpha \int_{A\cap B} (|Du|^{p(x)} + |Dv|^{p(x)})\,\mathrm{d}x \le \int_{A\cap B} (f_j(x,Du) + f_j(x,Dv))\,\mathrm{d}x \le F_j(u,A) + F_j(v,B)\,.$$

Therefore we obtain (4.7) with $M_\sigma = \beta 2^{2q-1} N^q \delta^{-q}$, which only depends on α, β, $p(x)$, $a(x)$, A', A and B, as required. ☐

We are now able to prove the following:

Proposition 4.5. *For every $u \in L^1(\Omega;\mathbb{R}^N)$, the lower and upper Γ-limits $F'(u,\cdot)$ and $F''(u,\cdot)$ define inner regular set functions on $\mathcal{A}$.*

Proof. We prove inner regularity of the lower Γ-limit; the same proof works for the upper Γ-limit. For any fixed $u \in L^1(\Omega)$ and $C \in \mathcal{A}$, we have to show that

$$F'(u,C) \le \sup\{F'(u,A) \mid A \in \mathcal{A}\,, \quad A \subset\subset C\}\,, \tag{4.9}$$

since the other inequality follows immediately from the monotonicity of $F'(u,\cdot)$. If the right-hand side of (4.9) is equal to $+\infty$ there is nothing to prove; else, by (4.6) and (2.5) we have $\sup\{\int_A |Du|^{p(x)}\,\mathrm{d}x \mid A \in \mathcal{A},\ A \subset\subset C\} < +\infty$, thus $|Du|^{p(x)} \in L^1(C)$ and hence $u \in W^{1,p(x)}_{\mathrm{loc}}(C)$ by (2.3).

Now fix a compact subset $K \subset C$, choose $A', A \in \mathcal{A}_0$ such that $K \subset A' \subset\subset A \subset\subset C$, A with Lipschitz boundary, and define $B := C \setminus K$. By definition of $F'(u,A) < +\infty$ there exists a sequence $\{u_j\}$ converging to u in $L^1(\Omega)$ such that $F'(u,A) = \liminf_{j\to+\infty} F_j(u_j,A)$. As usual, we may assume that the lower limit is a limit and that

$$\sup_{j\in\mathbb{N}} \int_A |Du_j|^{p(x)}\,\mathrm{d}x < +\infty\,, \tag{4.10}$$

so $\{u_j\} \subset W^{1,p(x)}(A)$ by Lemma 2.4. By the fundamental estimate (4.7), applied to u_j on A and u on B, and by (4.1), for any $\sigma > 0$ we can find $M_\sigma > 0$ and a sequence $\{\varphi_j\}$ of cut-off functions between A' and A such that

$$\begin{aligned} F_j(\varphi_j u_j + (1-\varphi_j)u, A'\cup B) &\le (1+\sigma)(F_j(u_j,A) + F_j(u,B)) + M_\sigma \int_{A\cap B} |u_j - u|^{p(x)}\,\mathrm{d}x + \sigma \\ &\le (1+\sigma)\left(F_j(u_j,A) + \beta\int_B (a(x) + |Du|^{p(x)})\,\mathrm{d}x\right) + M_\sigma \int_A |u_j-u|^{p(x)}\,\mathrm{d}x + \sigma\,. \end{aligned}$$

Now, since $\varphi_j u_j + (1-\varphi_j)u \to u$ in $L^1(\Omega)$, using (4.10) and Lemma 2.8 we obtain

$$\begin{aligned} F'(u,C) &= F'(u,A'\cup B) \le \liminf_{j\to+\infty} F_j(\varphi_j u_j + (1-\varphi_j)u, A'\cup B) \\ &\le (1+\sigma)\left(F'(u,A) + \beta\int_{C\setminus K} (a(x) + |Du|^{p(x)})\,\mathrm{d}x\right) + \sigma \end{aligned}$$

that is, by the arbitrariness of $\sigma > 0$,

$$F'(u,C) \le F'(u,A) + \beta\int_{C\setminus K} (a(x) + |Du|^{p(x)})\,\mathrm{d}x\,. \tag{4.11}$$

Since $|Du|^{p(x)} \in L^1(C)$, the last term in (4.11) can be taken arbitrarily small as $|C\setminus K| \to 0$ and the proof is complete. ☐

Applying Proposition 1.6, we obtain a subsequence $\{j_k\}$ such that the Γ-limit $F(u, A)$ in (4.3) exists for all $u \in L^1(\Omega)$ and $A \in \mathcal{A}$; in addition $F(u, \cdot)$ is an increasing set function on $\mathcal{A}$ for every $u \in L^1(\Omega)$ and $F(u, A)$ satisfies the estimates (4.4) by Proposition 4.3.

Step 2: measure property of the Γ-limit.

We now show that for all $u \in L^1(\Omega; \mathbb{R}^N)$ the Γ-limit $F(u, \cdot)$ is the trace on $\mathcal{A}$ of a Borel measure.

In order to apply the De Giorgi-Letta criterion [18], we have to prove that for every fixed $u \in L^1(\Omega)$ the increasing set function $F(u, \cdot)$ is subadditive, superadditive and inner regular. Since we have just proved (Prop. 4.5) that both the upper and lower Γ-limits are inner regular, and it is well known that the lower Γ-limit $F'(u, \cdot)$ is superadditive (see *e.g.* [14], Prop. 16.12), we only have to show that the upper Γ-limit $F''(u, \cdot)$ is subadditive.

Proposition 4.6. *For every $u \in L^1(\Omega; \mathbb{R}^N)$ we have*

$$F''(u, A \cup B) \leq F''(u, A) + F''(u, B) \tag{4.12}$$

for every $A, B \in \mathcal{A}$.

The subadditivity (4.12) of $F''(u, \cdot)$ follows, by means of inner regularity, from the following weak subadditivity property:

$$F''(u, A' \cup B) \leq F''(u, A) + F''(u, B) \tag{4.13}$$

for every $A', A, B \in \mathcal{A}$ with $A' \subset\subset A$ and B with Lipschitz boundary.

We first prove (4.13) under the additional assumptions that A belongs to $\mathcal{A}_0$ and $A \cap B$ has Lipschitz boundary. To this purpose let us fix $u \in L^1(\Omega)$ and let $A', A, B \in \mathcal{A}$, with $A' \subset\subset A$, $A \in \mathcal{A}_0$, B and $A \cap B$ with Lipschitz boundary, such that the right-hand side of (4.13) is finite. By definition of upper Γ-limit, there exist two sequences $\{u_j\}$ and $\{v_j\}$, both converging to u in $L^1(\Omega)$, such that

$$F''(u, A) = \limsup_{j \to +\infty} F_j(u_j, A) < +\infty \quad \text{and} \quad F''(u, B) = \limsup_{j \to +\infty} F_j(v_j, B) < +\infty$$

and hence by (4.2) and (4.1)

$$\sup_{j \in \mathbb{N}} \int_A |Du_j|^{p(x)}\, \mathrm{d}x < +\infty \quad \text{and} \quad \sup_{j \in \mathbb{N}} \int_B |Dv_j|^{p(x)}\, \mathrm{d}x < +\infty\,. \tag{4.14}$$

By the fundamental estimate (Prop. 4.4) applied to u_j and v_j, for any $\sigma > 0$ we can find $M_\sigma > 0$ and a sequence $w_j := \varphi_j u_j + (1 - \varphi_j) v_j$, where the φ_j are cut-off functions between A' and A, such that

$$\begin{aligned} F_j(w_j, A' \cup B) &\leq (1+\sigma)(F_j(u_j, A) + F_j(v_j, B)) + M_\sigma \int_{A \cap B} |u_j - v_j|^{p(x)}\, \mathrm{d}x + \sigma \\ &\leq (1+\sigma)(F_j(u_j, A) + F_j(v_j, B)) + M_\sigma\, 2^{q-1} \int_{A \cap B} (|u_j - u|^{p(x)} + |v_j - u|^{p(x)})\, \mathrm{d}x + \sigma \end{aligned}$$

where $q := \sup_{x \in A \cap B} p(x) < +\infty$. Since $w_j \to u$ in $L^1(\Omega)$, $A \cap B$ belongs to $\mathcal{A}_0$, $A \cap B$ has Lipschitz boundary and (4.14) holds, by definition of upper Γ-limit and Lemma 2.8 we obtain

$$F''(u, A' \cup B) \leq \limsup_{j \to +\infty} F_j(w_j, A' \cup B) \leq (1+\sigma)(F''(u, A) + F''(u, B)) + \sigma$$

and hence (4.13) holds by the arbitrariness of $\sigma > 0$. To extend (4.13) to any $A \in \mathcal{A}$, it is enough to take $C \in \mathcal{A}_0$ such that $A' \subset\subset C \subset\subset A$ and $C \cap B$ has Lipschitz boundary, and by the monotonicity of $F''(u, \cdot)$ we obtain

$$F''(u, A' \cup B) \leq F''(u, C) + F''(u, B) \leq F''(u, A) + F''(u, B)\,.$$

Now, to prove (4.12) we fix $u \in L^1(\Omega)$ and $A, B \in \mathcal{A}$ such that the right-hand side of (4.12) is finite, which in particular gives $F''(u, A \cup B) < +\infty$ by (4.6). By inner regularity, it is clear that (4.13) yields (4.12) for any $A, B \in \mathcal{A}$, provided B has Lipschitz boundary. In fact, for any $C \in \mathcal{A}$ with $C \subset\subset A \cup B$, by enlarging a bit the subset $C \setminus \overline{B}$, we can find $A' \subset\subset A$ such that $C \subset A' \cup B$, which yields

$$F''(u, C) \leq F''(u, A' \cup B) \leq F''(u, A) + F''(u, B)$$

and by inner regularity (4.12). Finally, to prove (4.12) for any $B \in \mathcal{A}$, since $F''(u, A \cup B) < +\infty$, by inner regularity for each small $\varepsilon > 0$ we can find $C \subset\subset A \cup B$ such that

$$F''(u, C) \geq F''(u, A \cup B) - \varepsilon\,.$$

By enlarging a bit the set $C \setminus \overline{A}$, we can find an open subset $\widetilde{B}$ of B with Lipschitz boundary and such that $C \subset A \cup \widetilde{B}$. Then one has

$$F''(u, A \cup B) \leq F''(u, C) + \varepsilon \leq F''(u, A \cup \widetilde{B}) + \varepsilon \leq F''(u, A) + F''(u, \widetilde{B}) + \varepsilon \leq F''(u, A) + F''(u, B) + \varepsilon$$

and hence (4.12), by letting $\varepsilon \to 0^+$. □

Proof of Theorem 4.2. We can apply Theorem 3.1 to the Γ-limit functional $F(u, A)$ given by (4.3). Indeed the locality property is well known (see *e.g.* [14], Prop. 16.15), the measure property ii) was proved in Step 2 of Theorem 4.1, we saw in Step 1 that the Γ-limit F satisfies (4.4), which in particular gives the growth condition iii), and conditions iv) and v) are trivially satisfied. Therefore, Theorem 3.1 implies that (3.1) holds for all $u \in W^{1,p(x)}_{\text{loc}}(A)$ and $A \in \mathcal{A}$, with f quasi-convex. Finally, it is easy to conclude that (4.4) and (2.5) yield the growth estimate (4.1) for f and the integral representation (4.5) on all of $L^1(\Omega; \mathbb{R}^N)$. □

5. Integrands depending on u

In this section we extend the results previously obtained to the case of functionals with explicit dependence on u. We will then consider functions satisfying

$$\alpha(|u|^{p(x)} + |z|^{p(x)}) \leq f(x, u, z) \leq \beta(a(x) + |u|^{p(x)} + |z|^{p(x)}) \tag{5.1}$$

for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{nN}$, where $0 < \alpha \leq \beta$ and $a(x) \in L^1(\Omega)$. We are able to prove the following results:

Theorem 5.1. *Let $p : \Omega \to (1, +\infty)$ be a continuous function with $p(x) \geq p > 1$ for all $x \in \Omega$. Let $f_j : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0, +\infty)$ be Borel functions satisfying* (5.1) *for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{nN}$. Finally let $F_j : L^1(\Omega; \mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$ be the variational functionals defined by*

$$F_j(u, A) := \begin{cases} \displaystyle\int_A f_j(x, u(x), Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\text{loc}}(A; \mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbb{R}^N)\,. \end{cases}$$

Then there exists a subsequence $\{j_k\}$ such that the Γ-limit

$$F(u, A) := \Gamma(L^1(\Omega; \mathbb{R}^N))\text{-}\lim_{k \to +\infty} F_{j_k}(u, A) \tag{5.2}$$

exists for all $u \in L^1(\Omega; \mathbb{R}^N)$ and $A \in \mathcal{A}$, with estimates

$$\alpha\, \widetilde{\Psi}_{p(x)}(u, A) \leq F(u, A) \leq \beta \left(\int_A a(x)\,\mathrm{d}x + \widetilde{\Psi}_{p(x)}(u, A) \right), \tag{5.3}$$

where $\widetilde{\Psi}_{p(x)}$ is the local functional given by (2.6)*.*

Theorem 5.2. *Let $F(u,A)$ be as in* (5.2) *and, in addition to the hypotheses of Theorem* 5.1*, suppose that the function p satisfies* (2.9) *and that for every $A \in \mathcal{A}_0$*

$$|f_j(x,u_1,z) - f_j(x,u_2,z)| \le \omega_A(|u_1-u_2|)(b_A(x) + |z|^{p(x)}) \tag{5.4}$$

holds for a.e. $x \in A$, where $b_A(x) \in L^1(A)$ and $\omega_A : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function with $\omega_A(0) = 0$. Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0,+\infty)$, satisfying the growth estimate (5.1) *for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{nN}$, with $f(x,u,\cdot)$ quasi-convex in $\mathbb{R}^{nN}$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^N$, such that for every $A \in \mathcal{A}$*

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x,u(x),Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbb{R}^N)\,. \end{cases} \tag{5.5}$$

In proving Theorems 5.1 and 5.2, we will only outline the main differences from the case when there is no dependence on u of the integrands f_j. Condition (5.4) is required to rely on the classical integral representation theorem of Buttazzo and Dal Maso, see [8] (Th. 1.10). In fact, we will first write the Γ-limit $F(u,A)$ as an integral functional only for $u \in W^{1,q}(A;\mathbb{R}^N)$ and $A \in \mathcal{A}_0$, where $q = \sup_{x\in A} p(x)$, and then we will extend the integral representation.

Proof of Theorem 5.1. Since each of the local functionals F_j satisfies (5.3), and by Proposition 2.13 the functional $\widetilde{\Psi}_{p(x)}(\cdot,A)$ is $L^1(\Omega)$-l.s.c. for every $A \in \mathcal{A}$, we obtain

$$\alpha\,\widetilde{\Psi}_{p(x)}(u,A) \le F'(u,A) \le F''(u,A) \le \beta\left(\int_A a(x)\,\mathrm{d}x + \widetilde{\Psi}_{p(x)}(u,A)\right) \tag{5.6}$$

for every $u \in L^1(\Omega)$ and $A \in \mathcal{A}$. In particular, the Γ-limit $F(u,A)$ is equal to $+\infty$ if and only if $u \in L^1(\Omega) \setminus W^{1,p(x)}(A)$.

Moreover, the uniform fundamental $L^{p(x)}$ estimate of Proposition 4.4 holds again. In fact, the upper estimate of $F_j(\varphi u + (1-\varphi)v, (A' \cup B) \cap \{0 < \varphi < 1\})$, due to (5.1), contains the extra term

$$\beta \int_{B_r^\nu} |\varphi u + (1-\varphi)v|^{p(x)}\,\mathrm{d}x \le \beta \int_{B_r^\nu} (|u|^{p(x)} + |v|^{p(x)})\,\mathrm{d}x\,.$$

Then it suffices to choose

$$\mu(E) := \beta\, 2^{q-1} \int_E (a(x) + |u|^{p(x)} + |Du|^{p(x)} + |v|^{p(x)} + |Dv|^{p(x)})\,\mathrm{d}x$$

and remark that by (5.1) one has

$$\alpha \int_{A\cap B} (|u|^{p(x)} + |Du|^{p(x)})\,\mathrm{d}x \le \int_{A\cap B} f_j(x,u(x),Du(x))\,\mathrm{d}x\,.$$

Monotonicity of the lower and upper Γ-limits $F'(u,\cdot)$ and $F''(u,\cdot)$ is trivial. Moreover, to show inner regularity, we reduce to prove (4.9) in case the right-hand side is finite, which yields that $u \in W^{1,p(x)}(C)$ by (5.6) and (2.6). Following Proposition 4.5, by means of the fundamental $L^{p(x)}$ estimate we deduce, instead of (4.11),

$$F'(u,C) \le F'(u,A) + \beta \int_{C\setminus K} (a(x) + |u|^{p(x)} + |Du|^{p(x)})\,\mathrm{d}x$$

and again, as $K \nearrow C$, we obtain (4.9), and hence inner regularity. By Proposition 1.6, we then have existence of the Γ-limit $F(u,A)$ with the growth estimates (5.3) given by (5.6).

To show the measure property of the Γ-limit $F(u,\cdot)$, following Step 2 in Theorem 4.1, it suffices to recover subadditivity of the upper Γ-limit $F''(u,\cdot)$: this holds since the right-hand side of (4.13) is finite for $u \in W^{1,p(x)}(A \cup B)$, and the proof of Proposition 4.6 actually relies on the fundamental $L^{p(x)}$ estimate. □

Proof of Theorem 5.2. To avoid overcrowding the paper with indices, we relabel F_j the Γ-converging subsequence. To obtain the integral representation of the Γ-limit, take any sequence $\mathcal{O}_i \nearrow \Omega$ such that $\mathcal{O}_i \in \mathcal{A}_0$ and $\mathcal{O}_i$ has Lipschitz boundary for every i. Then (5.4) holds for a.e. $x \in \mathcal{O}_i$ whereas, by continuity of $p(x)$,

$$q_i := \sup_{x \in \mathcal{O}_i} p(x) < +\infty \qquad \forall\, i\,. \tag{5.7}$$

Denoting by $\mathcal{A}_i$ the family of open subsets of $\mathcal{O}_i$, we now consider the functional $G_i : W^{1,q_i}(\mathcal{O}_i) \times \mathcal{A}_i \to [0,+\infty]$ given by

$$G_i(u,A) := F(\overline{u},A)\,,$$

where $\overline{u} \in W^{1,p(x)}(\Omega)$ is any extension of u. From now on we drop all the indices i: we will recover the full notation later. By (5.3) and (5.7) we have

$$0 \le G(u,A) \le \beta \int_A (2 + a(x) + |u|^q + |Du|^q)\,\mathrm{d}x$$

for all $u \in W^{1,q}(\mathcal{O})$ and $A \in \mathcal{A}$, and hence G satisfies a growth condition of order q. Moreover, locality of F implies that also G is local on $\mathcal{A}$. Also, since $F(\cdot,A)$ is $L^1(\Omega)$-l.s.c., and $\mathcal{O}$ has Lipschitz boundary, for every $A \in \mathcal{A}$ the functional $u \mapsto G(u,A)$ is sequentially w^*-$W^{1,\infty}(\mathcal{O})$-l.s.c. and strongly $W^{1,q}(\mathcal{O})$-l.s.c.. Finally, since $G(u,\cdot)$ is a measure on $\mathcal{A}$ for every $u \in W^{1,q}(\mathcal{O})$, in order to apply [8] (Th. 1.10) we have to show that G also satisfies a weak condition ω. More precisely, we show the existence of a sequence of functions $\omega_m(x,r)$, integrable in x, increasing and continuous in r and with $\omega_m(x,0) \equiv 0$, such that

$$|G(u+s,A) - G(u,A)| \le \int_A \omega_m(x,|s|)\,\mathrm{d}x \tag{5.8}$$

for every $m \in \mathbb{N}$, $A \in \mathcal{A}$, $s \in \mathbb{R}^N$ and $u \in C^1(\overline{\mathcal{O}})$ with $|u|$, $|u+s|$, $|Du| \le m$ in $\mathcal{O}$. In fact, assume *e.g.* that $G(u+s,A) \ge G(u,A)$, let $\overline{u} \in W^{1,p(x)}(\Omega)$ be an extension of u and let $\{u_j\}$ be such that $u_j \to \overline{u}$ in $L^1(\Omega)$ and $F_j(u_j,A) \to F(\overline{u},A) = G(u,A)$. Then by (5.4) we have

$$\begin{aligned}
G(u+s,A) - G(u,A) &= F(\overline{u}+s,A) - F(\overline{u},A) \le \liminf_{j\to+\infty} (F_j(u_j+s,A) - F_j(u_j,A)) \\
&\le \liminf_{j\to+\infty} \int_A |f_j(x,u_j+s,Du_j) - f_j(x,u_j,Du_j)|\,\mathrm{d}x \\
&\le \liminf_{j\to+\infty} \int_A \omega_{\mathcal{O}}(|s|)(b_{\mathcal{O}}(x) + |Du_j|^{p(x)})\,\mathrm{d}x \\
&\le \liminf_{j\to+\infty} \omega_{\mathcal{O}}(|s|) \left(\int_A b_{\mathcal{O}}(x)\,\mathrm{d}x + \frac{1}{\alpha} F_j(u_j,A) \right) = \omega_{\mathcal{O}}(|s|) \left(\int_A b_{\mathcal{O}}(x)\,\mathrm{d}x + \frac{1}{\alpha} F(\overline{u},A) \right) \\
&\le \omega_{\mathcal{O}}(|s|) \int_A \left(b_{\mathcal{O}}(x) + \frac{\beta}{\alpha}(2 + a(x) + |u|^q + |Du|^q) \right) \mathrm{d}x\,,
\end{aligned}$$

where we used (5.3), and we obtain (5.8) with $\omega_m(x,r) := \omega(r)\left(b_{\mathcal{O}}(x) + \dfrac{\beta}{\alpha}(2 + a(x) + 2m^q)\right)$.

Then, by [8] (Th. 1.10), there exists a Carathéodory integrand $g(x,u,z) : \mathcal{O} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0,+\infty)$, quasi-convex in z for a.e. $x \in \mathcal{O}$ and all $u \in \mathbb{R}^N$, and satisfying a growth condition of order q, such that

$$G(u,A) = \int_A g(x,u(x),Du(x))\,\mathrm{d}x \tag{5.9}$$

holds for every $u \in W^{1,q}(\mathcal{O})$ and $A \in \mathcal{A}$. We now recover the notation with the index i. In particular, equation (5.9) holds for the restriction $u_{|\mathcal{O}_i}$ of every function $u \in C_0^\infty(\Omega)$, and thus for $i < j$ large enough that $\mathrm{spt}(u) \subset \mathcal{O}_i$ we have $G_i(u,\mathcal{O}_i) = F(u,\mathcal{O}_i) = G_j(u,\mathcal{O}_i)$, so that $g_i(\cdot,u,z) \equiv g_j(\cdot,u,z)$ for a.e. $x \in \mathcal{O}_i$. Setting $f(x,u,z) := g_i(x,u,z)$ if $x \in \mathcal{O}_i$, we have that

$$F(u,A) = \int_A f(x,u(x),Du(x))\,\mathrm{d}x \tag{5.10}$$

for every $u \in C_0^\infty(\Omega)$ and all $A \in \mathcal{A}_0$ and hence, by the measure property, for all $A \in \mathcal{A}$.

To extend the integral representation (5.10) to all $u \in W^{1,p(x)}(\Omega)$, we first remark that (5.3) and (5.10) yield that the integrand f satisfies the $p(x)$-growth condition (5.1). Consequently, since f is a Carathéodory function satisfying (5.1), we have that for every $A \in \mathcal{A}_0$ the functional $u \mapsto \int_A f(x,u,Du)\,\mathrm{d}x$ is continuous with respect to the $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ convergence, compare Step 4 in Section 3. Now fix $u \in W^{1,p(x)}(\Omega)$ and, by the density result in Proposition 2.18, let $\{u_j\} \subset C_0^\infty(\Omega)$ be a smooth sequence converging to u both in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ and in $L^1(\Omega)$: by the lower semicontinuity of F

$$F(u,A) \le \liminf_{j\to+\infty} F(u_j,A) = \lim_{j\to+\infty} \int_A f(x,u_j,Du_j)\,\mathrm{d}x = \int_A f(x,u,Du)\,\mathrm{d}x\,, \tag{5.11}$$

hence one inequality holds in (5.10) for all $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}_0$. To obtain equality, let $\overline{u} \in W^{1,p(x)}(\Omega)$ be an extension of u outside A, with compact support in Ω, see Step 5 of Section 3, let q_0 be the maximum of p on the support of $\overline{u}$, and consider the functional $G : L^1(\Omega) \times \mathcal{A} \to [0,+\infty]$ given by

$$G(v,B) := F(v+\overline{u},B) + \alpha\,\widetilde{\Psi}(\overline{u},B)\,.$$

It is easy to show that G satisfies hypotheses similar to (5.1), since in particular by (5.1)

$$\begin{aligned} G(v,B) &\ge \alpha \int_B (|\overline{u}+v|^{p(x)} + |\overline{u}|^{p(x)} + |D\overline{u}+Dv|^{p(x)} + |D\overline{u}|^{p(x)})\,\mathrm{d}x \\ &\ge \alpha \int_B 2^{1-p(x)}(|v|^{p(x)} + |Dv|^{p(x)})\,\mathrm{d}x \;\ge\; \alpha\,2^{1-q_0}\,\widetilde{\Psi}(v,B) \end{aligned}$$

and similarly

$$G(v,B) \le \beta \int_B (\widetilde{a}(x) + |v|^{p(x)} + |Dv(x)|^{p(x)})\,\mathrm{d}x$$

for all $v \in W^{1,p(x)}(\Omega)$ and $B \in \mathcal{A}$, with $\widetilde{a}(x) \in L^1(\Omega)$. Then, by the previous argument, there exists a Carathéodory function $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to [0,+\infty)$, satisfying $p(x)$-growth estimates similar to (5.1), such that

$$G(v,B) \le \int_B g(x,v(x),Dv(x))\,\mathrm{d}x\,, \qquad \forall\, v \in W^{1,p(x)}(\Omega)\,,\ B \in \mathcal{A}_0\,, \tag{5.12}$$

with equality holding for $v \in C_0^\infty(\Omega)$. As we remarked above for f, also the functional $v \mapsto \int_B g(x,u,Dv)\,\mathrm{d}x$ is strongly continuous in $W^{1,p(x)}_{\mathrm{loc}}$. If $\{u_j\} \subset C_0^\infty(\Omega)$ is the same sequence we chose to get (5.11), and

$\overline{\alpha} := \alpha\,\widetilde{\Psi}(\overline{u},A)$, we have

$$\begin{aligned}\int_A g(x,0,0)\,\mathrm{d}x - \overline{\alpha} = F(\overline{u},A) = F(u,A) \le \int_A f(x,u,Du)\,\mathrm{d}x = \lim_{j\to+\infty}\int_A f(x,u_j,Du_j)\,\mathrm{d}x \\ = \lim_{j\to+\infty} F(u_j,A) = \lim_{j\to+\infty} G(u_j-\overline{u},A) - \overline{\alpha} \le \lim_{j\to+\infty}\int_A g(x,u_j-\overline{u},D(u_j-\overline{u}))\,\mathrm{d}x - \overline{\alpha} \\ = \lim_{j\to+\infty}\int_A g(x,u_j-u,D(u_j-u))\,\mathrm{d}x - \overline{\alpha} = \int_A g(x,0,0)\,\mathrm{d}x - \overline{\alpha}\end{aligned}$$

and therefore equality in (5.10) holds for all $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}_0$ and hence, by the measure property of F, for all $A \in \mathcal{A}$.

Finally, we argue as in Step 6 in Section 3 to extend the integral representation (5.10) to all $u \in L^1(\Omega)$ with $u_{|A} \in W^{1,p(x)}(A)$; since by (5.3) we have $F(u,A) = +\infty$ if $u_{|A} \notin W^{1,p(x)}(A)$, we obtain (5.5) and the proof is complete. □

6. Integrands with discontinuous $p(x)$-growth exponent

In this section we extend the results previously obtained to the more general case of functionals with $p(x)$-growth where $p(x)$ is allowed to be discontinuous in a negligible set $\Sigma \subset \Omega$: henceforward, $p(x)$ is a regular piecewise continuous exponent, according to Definition 2.2.

In the sequel, if $A \in \mathcal{A}$ we will denote $A_i := A \cap \Omega_i$ the intersection of A with Ω_i. We will again obtain existence and integral representation of Γ-limits. More precisely, we are going to prove the following results.

Theorem 6.1. *Let $p : \Omega \to (1,+\infty)$ be a regular piecewise continuous exponent satisfying* (2.1)*, and let $f_j : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$ be Borel functions satisfying* (4.1) *for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$. If F_j is as in* (4.2) *there exists a subsequence $\{j_k\}$ such that the Γ-limit* (4.3) *exists for all $u \in L^1(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}$, with estimates* (4.4)*.*

Theorem 6.2. *Under the assumptions of Theorem* 6.1*, suppose that each function p_i satisfies* (2.9) *in Ω_i. Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, with $f(x,\cdot)$ quasi-convex for a.e. $x \in \Omega$, f satisfying the growth estimates* (4.1)*, such that the integral representation* (4.5) *holds for all $A \in \mathcal{A}$.*

In the particular case when $p(x)$ is constant on each region Ω_i, as in the case of standard p-growth we obtain the precise expression of the relaxed functional, compare [7] (4.4.5). To this aim, if $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$ is a Borel function satisfying the growth condition (4.1), and $\mathcal{F} : L^1(\Omega;\mathbb{R}^N) \times \mathcal{A} \to [0,+\infty]$ is the variational functional given by

$$\mathcal{F}(u,A) := \begin{cases} \displaystyle\int_A f(x,Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbb{R}^N) \end{cases}$$

for all $A \in \mathcal{A}$, we denote by $\overline{\mathcal{F}}(u,A)$ the relaxed functional of $\mathcal{F}(u,A)$ in the L^1 topology,

$$\overline{\mathcal{F}}(u,A) := \inf\left\{\liminf_{k\to+\infty}\mathcal{F}(u_k,A) \mid \{u_k\} \subset L^1(\Omega;\mathbb{R}^N), \quad u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^N)\right\}.$$

Also, we recall that if $g : \mathbb{R}^{nN} \to [0,+\infty)$ is a Borel function, the quasi-convex envelope of g, denoted by Qg, is the greatest quasi-convex function which is lower than or equal to g; if f is defined on $\Omega \times \mathbb{R}^{nN}$, we denote by Qf the quasi-convex envelope of $z \mapsto f(x,z)$. We will now prove the following:

Corollary 6.3. *Under the assumptions of Theorem* 6.1*, with $f_j \equiv f$, suppose in particular that there exist constants $p_i \ge 1$ such that $p(x) \equiv p_i$ for each $x \in \Omega_i$ and all i. Then, if $f(x,\cdot)$ is upper semicontinuous in $\mathbb{R}^{nN}$*

for a.e. $x \in \Omega$*, we have*

$$\overline{\mathcal{F}}(u, A) := \begin{cases} \displaystyle\int_A Qf(x, Du(x))\,\mathrm{d}x & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbb{R}^N) \end{cases} \tag{6.1}$$

for all $A \in \mathcal{A}$*, where* $Qf(x, \cdot)$ *is the quasi-convex envelope of* $f(x, \cdot)$*, for a.e.* $x \in \Omega$*.*

Proof of Corollary 6.3. By Theorem 6.2 (which still holds if $p_i(x) \equiv 1$ for some i) we obtain the integral representation (6.1) with a generic integrand $\varphi(x, z)$ quasi-convex in $\mathbb{R}^{nN}$ for a.e. $x \in \Omega$ and satisfying

$$\alpha|z|^{p_i} \le \varphi(x, z) \le \beta(a(x) + |z|^{p_i}) \tag{6.2}$$

a.e. in Ω_i, see Remark 6.4 below.

Now the integral functional $u \mapsto \int_A \varphi(x, Du)\,\mathrm{d}x$ is $L^1(\Omega)$-lower semicontinuous in $W^{1,p(x)}_{\mathrm{loc}}(A)$, for all $A \in \mathcal{A}$. In fact, if $\{u_k\} \subset W^{1,p(x)}_{\mathrm{loc}}(A)$ is such that $u_k \to u$ in $L^1(\Omega)$ with

$$\liminf_{k\to+\infty} \int_A \varphi(x, Du_k)\,\mathrm{d}x < +\infty\,, \tag{6.3}$$

equation (6.2) yields $\sup_k \|Du_k\|_{L^{p_i}(A_i)} < +\infty$. Then, passing to a subsequence which we relabel $\{u_j\}$ we can suppose that $u_j \rightharpoonup u$ in $W^{1,p_i}(A_i)$, or in the weak BV sense if $p_i = 1$, and that the lower limit in (6.3) is actually a limit.

Recall now that quasi-convexity of φ yields lower semicontinuity of the functional $u \mapsto \displaystyle\int_{A_i} \varphi(x, Du)\,\mathrm{d}x$ with respect to w-$W^{1,p_i}(A_i)$ convergence (weak BV if $p_i = 1$). We then deduce

$$\begin{aligned} \int_A \varphi(x, Du)\,\mathrm{d}x &= \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du)\,\mathrm{d}x \le \sum_{i=1}^{+\infty} \liminf_{j\to+\infty} \int_{A_i} \varphi(x, Du_j)\,\mathrm{d}x \\ &\le \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du_j)\,\mathrm{d}x = \lim_{j\to+\infty} \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du_j)\,\mathrm{d}x = \lim_{j\to+\infty} \int_A \varphi(x, Du_j)\,\mathrm{d}x \end{aligned}$$

and hence $L^1(\Omega)$-lower semicontinuity. Therefore, by definition of relaxed functional we obtain

$$\int_A Qf(x, Du(x))\,\mathrm{d}x \le \int_A \varphi(x, Du(x))\,\mathrm{d}x \le \int_A f(x, Du(x))\,\mathrm{d}x$$

for all $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $A \in \mathcal{A}$. Finally, since Qf and φ are Carathéodory integrands, and $f(x, \cdot)$ is upper semicontinuous, we have

$$Qf(x, z) \le \varphi(x, z) \le f(x, z)$$

for a.e. $x \in \Omega$ and all $z \in \mathbb{R}^{nN}$, and hence $\varphi = Qf$ by the quasi-convexity of φ. □

Remark 6.4. The estimate from below in (6.2) holds even if $p_i = 1$ for some i. In fact, if $u_k \to u_z(x) := zx$ in $L^1(\Omega)$ with $F(u_k, A) \to \overline{\mathcal{F}}(u_z, A) < +\infty$, $A \subset \Omega_i$ and $p_i = 1$, up to a subsequence we have $u_k \rightharpoonup u_z$ in the weak BV sense so that, by lower semicontinuity of the total variation, we have

$$\alpha \int_A |z|\,\mathrm{d}x \le \liminf_{k\to+\infty} \alpha \int_A |Du_k|\,\mathrm{d}x \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A) = \overline{\mathcal{F}}(u_z, A) = \int_A \varphi(x, z)\,\mathrm{d}x\,.$$

Proof of Theorem 6.1. We first remark that by Corollary 2.14 we again obtain (4.6), whereas the fundamental estimate (Prop. 4.4) holds too, since it relies on Corollary 2.12 and on the fact that $q := \sup_{x \in K_\delta \setminus \Sigma} p(x) < +\infty$, which holds again by Remark 2.3, as $K_\delta \in \mathcal{A}_0$.

Inner regularity of the upper and lower Γ-limits holds again, since Proposition 4.5 actually relies on (4.6, 2.3), which follows from Corollary 2.6, on Proposition 4.4 and Corollary 2.9. We then obtain Γ-convergence by Proposition 1.6, with the estimates (4.4).

Subadditivity (4.12) of the upper Γ-limit $F''(u, A)$ holds again, since Proposition 4.6 relies on inner regularity, on (4.4), on the fundamental estimate and on Corollary 2.9. Then, by the De Giorgi–Letta criterion, we obtain the measure property of the Γ-limit $F(u, \cdot)$. □

Proof of Theorem 6.2. Conditions i)–iv) and v) in Theorem 3.1 are easily verified. However, the growth exponent $p(x)$ satisfies the local estimate (2.9) for each open set $A \subset\subset \Omega_i$. Then for every i we obtain a quasi-convex function $f_i : \Omega_i \times \mathbb{R}^{nN} \to [0, +\infty)$, satisfying growth condition (3.2), such that (3.1) holds, with $f = f_i$, for each $u \in W^{1,p(x)}_{\text{loc}}(A)$ and for all open sets $A \in \mathcal{A}$ with $A \subset \Omega_i$.

To extend the integral representation to all open sets $A \in \mathcal{A}$, we define $f(x, z) := f_i(x, z)$ if $x \in \Omega_i$ and $z \in \mathbb{R}^{nN}$, and set

$$\Sigma_r := \{x \in \Omega \mid \text{dist}(x, \Sigma) < r\}\cdot$$

Then, for every $u \in W^{1,p(x)}_{\text{loc}}(A)$ and $A \in \mathcal{A}_0$, if $r > 0$ is small, by increasing property and subadditivity we have

$$F(u, A \setminus \overline{\Sigma}_r) \le F(u, A) \le F(u, A \setminus \overline{\Sigma}_r) + F(u, \Sigma_{2r} \cap A)\,. \tag{6.4}$$

Now we have

$$F(u, A \setminus \overline{\Sigma}_r) = \sum_{i=1}^{+\infty} F(u, A_i \setminus \overline{\Sigma}_r) = \sum_{i=1}^{+\infty} \int_{A_i \setminus \overline{\Sigma}_r} f_i(x, Du)\,\mathrm{d}x = \int_{A \setminus \overline{\Sigma}_r} f(x, Du)\,\mathrm{d}x\,.$$

Moreover we get

$$\lim_{r \to 0^+} |\Sigma_{2r} \cap A| = 0\,. \tag{6.5}$$

In fact, since $A \in \mathcal{A}_0$, if $0 < \delta < \text{dist}(A, \partial\Omega)$ is fixed and $A_\delta := \{x \in \Omega \mid \text{dist}(x, A) \le \delta\}$, then A_δ is a compact subset of Ω and hence Definition 2.1 yields that A_δ intersects finitely many Ω_i. Then we can select a finite set of indices I_δ such that for every $0 < r < \delta/2$

$$\Sigma_{2r} \cap A \subset \bigcup_{i \in I_\delta} (\partial\Omega_i)_{2r}\,, \tag{6.6}$$

where $(\partial\Omega_i)_{2r} := \{x \in \Omega \mid \text{dist}(x, \partial\Omega_i) < 2r\}$. Now, since Ω_i has smooth boundary we have that $|(\partial\Omega_i)_{2r}| \to 0^+$ as $r \to 0^+$ and hence by (6.6) we obtain (6.5).

By (6.5), by growth condition (4.4) and by absolute continuity we get

$$F(u, \Sigma_{2r} \cap A) \le \beta \int_{\Sigma_{2r} \cap A} (a(x) + |Du|^{p(x)})\,\mathrm{d}x \to 0$$

as $r \to 0^+$. Finally, since

$$\int_{A \setminus \overline{\Sigma}_r} f(x, Du)\,\mathrm{d}x \to \int_A f(x, Du)\,\mathrm{d}x\,,$$

letting $r \to 0^+$ in (6.4) we obtain (3.1) for $A \in \mathcal{A}_0$ and hence, by the measure property, for all $A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\text{loc}}(A)$. Finally, (4.6) yields growth condition (4.1) for f and integral representation (4.5) on all $L^1(\Omega)$. □

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