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THE DYNAMICAL LAME SYSTEM: REGULARITY OF SOLUTIONS, BOUNDARY CONTROLLABILITY AND BOUNDARY DATA CONTINUATION

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Abstract. The boundary control problem for the dynamical Lame system (isotropic elasticity model) is considered. The continuity of the "input → state" map in L_2-norms is established. A structure of the reachable sets for arbitrary $T > 0$ is studied. In general case, only the first component $u(\cdot, T)$ of the complete state $\{u(\cdot, T), u_t(\cdot, T)\}$ may be controlled, an approximate controllability occurring in the subdomain filled with the shear (slow) waves. The controllability results are applied to the problem of the boundary data continuation. If T_0 exceeds the time needed for shear waves to fill the entire domain, then the response operator ("input → output" map) R^{2T_0} uniquely determines R^T for any $T > 0$. A procedure recovering R^∞ *via* R^{2T_0} is also described.

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1. Introduction

1.1. About the paper

This paper deals with the issue of boundary approximate controllability and related unique continuation property for a system of dynamic elasticity governed by the Lame model.

Our goal is to provide a description of sets which are approximately reachable by the actuation on an arbitrary (possibly small) portion of the boundary within an arbitrary (possibly short) time – in the system which has variable (in space) coefficients. As such, this problem is very different from a large body of papers dealing with exact controllability for constant coefficient Lame models with seizable portion of the boundary accessible to control action for a sufficiently long time (see, *e.g.* [2]).

In 1993 Tataru extended the classical Holmgren–John theorem on uniqueness of the continuation across non-characteristic surfaces to solutions of PDE's with nonanalytic coefficients [20]. In particular, for the case of time independent coefficients the required smoothness of the coefficients in [20] is rather minimal (C^1). One of the corollaries of this remarkable result is that the boundary controllability of the dynamical system governed by the scalar wave equation: on any finite time interval such the system turns out to be approximately controllable

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in the subdomain filled by waves. This last property plays the key role in the BC-method that is an approach to inverse problems based upon their relations with the boundary control theory [4].

In 1998 the Holmgren–John–Tataru theorem was generalized to a class of hyperbolic systems which are principally weakly coupled [9]. This class includes Lame systems with C^3 coefficients which are space dependent.

Our paper draws on further consequences of this generalization in the context of boundary controllability of the Lame system. Indeed, the unique continuation result in [9] is one of the main tools used in providing characterization of reachable sets. These results should lead to a variant of the BC-method for inverse problems formulated in the context of dynamic elasticity theory.

We dedicate this paper to memory of J.-L. Lions who's contributions, impact and influence on the field has been and will be everlasting.

1.2. **The Lame system**

In a bounded domain $\Omega \subset \mathbf{R}^3$ with the smooth enough (say C^3) boundary Γ consider the dynamical system

$$u_{tt} - Lu = 0 \qquad \text{in } Q^T; \tag{1.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega; \tag{1.2}$$

$$u = f \qquad \text{on } \Sigma^T, \tag{1.3}$$

where $Q^T := \Omega \times (0,T)$; $\Sigma^T := \Gamma \times [0,T]$; L is an operator acting on $\mathbf{R}^3$-valued functions defined by:

$$(Lu)_i := \rho^{-1} \sum_{j,k,l=1}^{3} \partial_j c_{ijkl} \partial_l u_k \quad i = 1,2,3$$

($\partial_j := \frac{\partial}{\partial x^j}$); c_{ijkl} is the elasticity tensor of the Lame model:

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk});$$

ρ, λ, μ are smooth functions depending on spatial variable only and satisfying the usual ellipticity condition: $\rho > 0, \mu > 0, 3\lambda + 2\mu > 0$ in $\bar{\Omega}$. Vector function $f = \{f_i(\gamma,t)\}_{i=1}^3$ is the Dirichlet boundary control given on Σ^T; $u = u^f(x,t) = \{u_i^f(x,t)\}_{i=1}^3$ is the solution (*wave*).

1.3. **Metrics and domains of influence**

The hyperbolic system (1.1–1.3) has two families of the characteristics $\varphi(x,t) = \text{const}$ in Q^T determined by the equations

$$\left(\frac{\partial \varphi}{\partial t}\right)^2 - c_\alpha^2 |\nabla \varphi|^2 = 0, \quad \alpha = p, s, \tag{1.4}$$

where

$$c_p := \left(\frac{2\mu + \lambda}{\rho}\right)^{\frac{1}{2}}, \quad c_s := \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$$

are the velocities of pressure and shear waves, p – characteristics being ordinary whereas s – characteristics being of multiplicity 2.

The velocities determine two Riemannian metrics in $\bar{\Omega}$:

$$\mathrm{d}s^2 = \frac{|\mathrm{d}x|^2}{c_\alpha^2}, \quad \alpha = p, s;$$

we denote dist_α the corresponding distances.

For an open subset $\sigma \subseteq \Gamma$ define the domains of influence of σ

$$\Omega_\alpha^{\sigma,T} := \{x \in \Omega \,|\, \mathrm{dist}_\alpha(x, \sigma) < T\}, \quad T > 0$$

and the times of filling

$$T_\alpha^\sigma := \inf\{T > 0 \,|\, \Omega_\alpha^{\sigma,T} \supset \Omega\}\cdot$$

The relation $c_s < c_p$ implies $\Omega_s^{\sigma,T} \subset \Omega_p^{\sigma,T}, \quad T_s^\sigma > T_p^\sigma$.

Introduce the space $\mathcal{H} := L_{2,\rho}(\Omega; \mathbf{R}^3)$ (with measure $\rho \mathrm{d}x$) and its subspaces

$$\mathcal{H}_\alpha^{\sigma,T} := \{y \in \mathcal{H} \,|\, \mathrm{supp}\, y \subset \bar{\Omega}_\alpha^{\sigma,T}\}, \quad \alpha = p, s;$$

with the following obvious relation $\mathcal{H}_s^{\sigma,T} \subseteq \mathcal{H}_p^{\sigma,T}$.

1.4. The results

Our first result concerns the regularity of the "input $\rightarrow$ trajectory" map.

Theorem 1. *The map $f \rightarrow u^f$ is continuous from $L_2(\Sigma^T; \mathbf{R}^3)$ into $C([0,T]; L_2(\Omega; \mathbf{R}^3))$.*

We shall discuss next reachability properties. To this end we introduce: $\Sigma^{\sigma,T} := \bar{\sigma} \times [0, T]$. The sets of waves

$$\mathcal{U}^{\sigma,T} := \{u^f(\cdot, T) |\, f \in L_2(\Sigma^T; \mathbf{R}^3);\ \mathrm{supp} f \subseteq \Sigma^{\sigma,T}\};$$
$$\mathcal{U}_0^{\sigma,T} := \{u^f(\cdot, T) |\, f \in C^\infty(\Sigma^T;\ \mathbf{R}^3);\ \mathrm{supp} f \subset \sigma \times (0, T)\}$$

play the role of reachable sets. Due to hyperbolicity of the system (1.1–1.3) and the above regularity result one has

$$\mathcal{U}^{\sigma,T} \subset \mathcal{H}_p^{\sigma,T}. \tag{1.5}$$

In the space $\mathcal{H}$ introduce the projections:

$$X_\alpha^{\sigma,T} y := \begin{cases} y & \text{in } \Omega_\alpha^{\sigma,T}; \\ 0 & \text{in } \Omega \setminus \Omega_\alpha^{\sigma,T}; \end{cases}$$

so that $X_\alpha^{\sigma,T}\mathcal{H} = \mathcal{H}_\alpha^{\sigma,T}, \quad \alpha = p, s$. The following result clarifies the character of the embedding (1.5).

Theorem 2. *For any $T > 0$ the equality*

$$\mathrm{clos}_{\mathcal{H}} X_s^{\sigma,T} \mathcal{U}^{\sigma,T} = \mathcal{H}_s^{\sigma,T} \tag{1.6}$$

holds. In particular, when $T > T_s^\sigma$, $\mathrm{clos}_{\mathcal{H}} \mathcal{U}^{\sigma,T} = \mathcal{H}$.

This equality is interpreted as an approximate controllability of the system (1.1–1.3) in the subdomain $\Omega_s^{\sigma,T}$ filled by s–waves at the moment $t = T$. Simple examples (for small enough T) demonstrate that in the wider subdomain $\Omega_p^{\sigma,T}$ such the controllability does not occur. Notice, that the theorem deals with controllability with respect to only the first component of the complete state $\{u, u_t\}$. This is natural: as we show, for times $T < T_p^\sigma$ $u^f(\cdot, T)$ determines $u_t^f(\cdot, T)$.

The next result concerns a completeness of waves in the Sobolev classes for large enough T connected with geometry of Ω.

Theorem 3. *If $T > T_s^\sigma$ the relation*

$$\operatorname{clos}_{H^2} \mathcal{U}_0^{\sigma,T} = H^2(\Omega; \mathbf{R}^3) \bigcap H_0^1(\Omega; \mathbf{R}^3) \tag{1.7}$$

holds.

Our last result concerns to the response operator ("input $\to$ output" map) $R^{\sigma,T} : L_2(\Sigma^{\sigma,T}; \mathbf{R}^3) \to L_2(\Sigma^{\sigma,T}; \mathbf{R}^3)$; $\operatorname{Dom} R^{\sigma,T} = C_0^\infty(\Sigma^{\sigma,T}; \mathbf{R}^3)$,

$$(R^{\sigma,T} f)_i := \sum_{j,k,l=1}^{3} \nu_j c_{ijkl}\, \partial_l u^{\tilde f}{}_k \quad \text{on } \Sigma^{\sigma,T},$$

where $\{\nu_j\}_{j=1}^3$ are the components of the outward normal, the control $\tilde f$ in the right hand side is the continuation of f from $\Sigma^{\sigma,T}$ to Σ^T by zero.

Theorem 4. *The operator $R^{\sigma,2T}$ given for a finite fixed $T > T_s^\sigma$ determines the operators $R^{\sigma,T'}$ for all T'* $\in (0, \infty)$.

In other words, dynamical boundary measurements (data) given on $\Sigma^{\sigma,2T}$ with a *fixed* $T > T_s^\sigma$ may be uniquely continued onto $\Sigma^{\sigma,\infty}$ *without resorting to the evolution equation* (1.1). We also describe at the end of the paper an effective procedure for the continuation.

Remark. Just for simplicity it is assumed in this paper that the coefficients of the system are C^∞. A more detailed analysis shows that C^3-smoothness of Γ, λ, μ, ρ is enough to preserve all of the results. The last requirement is motivated by applicability of unique continuation results in [9]. On the other hand, other arguments in the paper do not require that much regularity (C^1 suffices). Thus, if any further progress is made in relaxing regularity for unique continuation results, our results will apply as well.

2. The Lame system

2.1. Spaces and subspaces

Unless otherwises stated, we assume Γ, λ, $\mu\,\rho$ to be C^∞–smooth. Everywhere in the paper $\sigma \subseteq \Gamma$ is a fixed open subset of the boundary.

The space of controls

$$\mathcal{F}^{\sigma,T} := \{f \in L_2(\Sigma^T; \mathbf{R}^3) |\, \operatorname{supp} f \in \Sigma^{\sigma,T}\}$$

plays the role of *external space* of the system (1.1–1.3); we denote

$$\mathcal{M}^{\sigma,T} := \{f \in \mathcal{F}^{\sigma,T} \bigcap C^\infty(\Sigma^T; \mathbf{R}^3) |\, \operatorname{supp} f \subset \sigma \times (0, T)\},$$

so that each $f \in \mathcal{M}^{\sigma,T}$ vanishes near $\Gamma \times \{t = 0\}$ and $\Gamma \times \{t = T\}$.

The *inner space* is $\mathcal{H} := L_{2,\rho}(\Omega; \mathbf{R}^3)$; as above, we select its subspaces

$$\mathcal{H}_\alpha^{\sigma,T} := \{y \in \mathcal{H} \,|\, \operatorname{supp} y \subset \bar\Omega_\alpha^{\sigma,T}\}, \quad \alpha = p, s.$$

Everywhere below, simplifying the notations, we omit σ in the case of $\sigma = \Gamma$: $\mathcal{F}^{\Gamma,T} := \mathcal{F}^T$; $\mathcal{M}^{\Gamma,T} := \mathcal{M}^T$ etc.

2.2. **Operator representation of Lame system**

We will find convenient to provide operator– in fact spectral– representation of solutions to Lame system:

$$u_{tt} - Lu = 0 \qquad \text{in } Q^T; \tag{2.1}$$

$$u|_{t=0} = u_0,\ u_t\,|_{t=0} = u_1 \qquad \text{in } \Omega; \tag{2.2}$$

$$u = f \qquad \text{on } \Sigma^T. \tag{2.3}$$

To accomplish this we introduce the following spaces and operators. $\mathcal{H}^2 := H^2(\Omega;\mathbf{R}^3)$, $\mathcal{H}_0^1 := H_0^1(\Omega;\mathbf{R}^3)$. $\mathcal{H}^{-1} := (\mathcal{H}_0^1)'$. The operator L_0; $\mathcal{H} \to \mathcal{H}$, $\mathrm{Dom}L_0 = \mathcal{H}^2 \cap \mathcal{H}_0^1$, $L_0 y := Ly$ is self-adjoint, negatively defined with $\mathrm{Dom}\,(-L_0)^{\frac{1}{2}} = \mathcal{H}_0^1$. Therefore, it generates sine and cosine operators [19] with the properties:

$$S(\cdot) \in \mathcal{L}(\mathcal{H} \to C([0,T];\mathcal{H}_0^1));\ C(t) := \frac{\mathrm{d}}{\mathrm{d}t} S(t) \in \mathcal{L}(\mathcal{H} \to \mathcal{H})$$
$$L_0 S(t) u = \frac{\mathrm{d}}{\mathrm{d}t} C(t) u,\ u \in \mathcal{H}_0^1;\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t) u = L_0 C(t) u;\ u \in \mathrm{Dom}\, L_0. \tag{2.4}$$

It is well known that $S(t), C(t)$ commute with L_0 and they obey the following trigonometric relations

$$S(t-s) = S(t)C(s) - C(t)S(s);\ C(t-s) = C(t)C(s) + L_0 S(t) S(s). \tag{2.5}$$

Since L_0 has discrete spectrum, we can write down spectral representation for sine and cosine operators: Let $\{\lambda_k\}_{k=1}^{\infty} : 0 > \lambda_1 \geq \lambda_2 \geq \ldots$ and $\{\varphi_k\}_{k=1}^{\infty}$ denote eigenvalues and eigenfunctions of L_0: $L_0 \varphi_k = \lambda_k \varphi_k$; $(\varphi_k, \varphi_l)_{\mathcal{H}} = \delta_{kl}$. The existence and continuity of L_0^{-1} follows from the fact that $\lambda_k \neq 0$. With the above notation we have

$$S(t)\, y = \sum_{k=1}^{\infty} \frac{\sin \sqrt{-\lambda_k}\, t}{\sqrt{-\lambda_k}}\ (y, \varphi_k)_{\mathcal{H}}\, \varphi_k\,, \quad 0 \leq t \leq T; \tag{2.6}$$

$$C(t)\, y = \sum_{k=1}^{\infty} \cos \sqrt{-\lambda_k}\, t\ \ (y, \varphi_k)_{\mathcal{H}}\, \varphi_k, \quad 0 \leq t \leq T. \tag{2.7}$$

We also introduce the so called Green's map G given by $G\psi := v$ iff $Lv = 0$ in Ω; $v = \psi$ on Γ. It is well known from standard elliptic theory that G is bounded from $L^2(\Gamma, \mathbf{R}^3)$ to $\mathcal{H}$.

With the help of Green's map we introduce the operator $W : \mathcal{F}^T \to L_2((0,T);\mathcal{H}^{-1})$ ("input → trajectory" map) defined by

$$(Wf)(t) := L_0 \int_0^t S(t-s) G f(s) \mathrm{d}s. \tag{2.8}$$

From (2.4) we clearly have: $W \in \mathcal{L}(\mathcal{F}^T \to C([0,T];\mathcal{H}^{-1}))$. Moreover, $W \in \mathcal{L}(H_0^1(\Sigma^T;\mathbf{R}^3) \to C([0,T];\mathcal{H}))$, where this last assertion follows from the representation of W which is valid for with any $f \in H_0^1(\Sigma^T;\mathbf{R}^3)$ $(Wf)(t) = -Gf(t) + \int_0^t C(t-s) G f_s(s) \mathrm{d}s$.

With the above notation, the solution $u(t)$ to Lame system (2.1–2.3) with $f \in H_0^2(\Sigma_T, \mathbf{R}^3)$ can be written as (see Sect. 3 in [13])

$$u(t) = C(t) u_0 + S(t) u_1 - (Wf)(t).$$

For $f \in \mathcal{F}^T$, $u_0 \in \mathcal{H}, u_1 \in \mathcal{H}^{-1}$ the formula above provides a definition for "ultra-weak" solution to (2.1–2.3), which resides in $C([0,T];\mathcal{H}^{-1})$.

Our first goal is to show that this "ultra–weak" solution is, in fact more regular by one spatial derivative. This is the regularity statement which amounts to proving that the operator $W: \mathcal{F}^T \to C([0,T];\mathcal{H})$ is bounded.

2.3. Regularity of solutions

If $f \in C^\infty(\Sigma^T; \mathbf{R}^3)$ vanishes near $\Gamma \times \{t=0\}$ then the problem (1.1–1.3) (subject to appropriate regularity of the coefficients) has the unique classical solution $u^f \in C^\infty(\bar{Q}^T; \mathbf{R}^3)$, the relations

$$u^{f_{tt}} = u^f_{tt} = Lu^f, \tag{2.9}$$

and

$$\operatorname{supp} u^f(\cdot,t) \subset \bar{\Omega}_p^{\sigma,t}, \quad 0 \le t \le T, \tag{2.10}$$

being valid due to time-invariant coefficients and hyperbolicity. In what follows we shall not need so much regularity and we shall show that $u^f \in C([0,T];\mathcal{H})$ for $f \in \mathcal{F}^T$, which in turn allows to define function u^f a.e. in Ω. Indeed, the above assertion follows from

Theorem 1. *The map W is continuous from $\mathcal{F}^T$ into $C([0,T];\mathcal{H})$. Moreover, the map $\frac{\mathrm{d}}{\mathrm{d}t}W$ is continuous from $\mathcal{F}^T$ into $C([0,T];\mathcal{H}^{-1})$.*

Proof.

Step 1: Preliminaries

We begin with some notation. An $\mathbf{R}^3$- valued function $y = \{y_j(x)\}_{j=1}^3$, $x \in \bar{\Omega}$ is said to be a *field*; the term "*function*" is reserved for scalar functions. We use the summation over repeating indexes and denote $y \cdot v := y_i v_i\,, \quad \langle \alpha, \beta \rangle := \alpha_{ij}\beta_{ij}$, the scalar products of vectors and matrices; $|y| := (y \cdot y)^{\frac{1}{2}}; \quad ||\alpha|| := \langle \alpha, \alpha \rangle^{\frac{1}{2}}$. The product $\alpha\, y$ is the vector $\alpha_{ij} y_j$. If y is a field we denote ∇y the matrix $(\nabla y)_{ij} := \partial_i y_j \quad (\partial_i := \frac{\partial}{\partial x_i})$ and set $(\operatorname{div} \alpha)_i := \partial_j \alpha_{ji}$. Finally, τ is the matrix conjugation: $(\alpha^\tau)_{ij} = \alpha_{ji}$. The elastic moduli tensor $\mathcal{C} = \{c_{ijkl}\}_{i,j,k,l=1}^3$ is considered as a "matrix $\to$ matrix" map: $(\mathcal{C}\alpha)_{ij} = c_{ijkl}\alpha_{kl}$, the moduli c_{ijkl} are smooth functions in $\bar{\Omega}$ satisfying the symmetry relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}\,. \tag{2.11}$$

The Lame model corresponds to

$$\mathcal{C}\alpha = \mu\,(\alpha + \alpha^\tau) + \lambda \operatorname{tr}\, \alpha\, I \tag{2.12}$$

where I is the unit matrix, $\operatorname{tr}\alpha := \alpha_{kk}$. Simple calculations with regard to $3\lambda + 2\mu > 0$, $\mu > 0$ allow to establish positivity of $\mathcal{C}$: for any $\alpha = \alpha^\tau$ one has

$$\langle \mathcal{C}\alpha, \alpha \rangle \ \ge c_0\, ||\alpha||^2 \tag{2.13}$$

with a constant $c_0 > 0$. Introducing the strain tensor

$$\varepsilon(y) := \frac{1}{2}\,[\nabla y + (\nabla y)^\tau]$$

the basic operator L may be written in the form

$$L = \rho^{-1} \operatorname{div} \mathcal{C}\varepsilon(\cdot) \tag{2.14}$$

or, by components, $(Ly)_i = \rho^{-1}\partial_j[2\mu\varepsilon_{ij}(y)+\lambda\varepsilon_{kk}(y)\delta_{ij}]$, where the density ρ is a positive function in $\bar{\Omega}$. Denote $\mathcal{H} := L_{2,\rho}(\Omega;\mathbf{R}^3)$ (with measure ρdx), $\mathcal{G} := L_2(\Gamma;\mathbf{R}^3)$ and recall Green's formula

$$(Lu,v)_{\mathcal{H}} - (u,Lv)_{\mathcal{H}} = (Nu,Dv)_{\mathcal{G}} - (Du,Nv)_{\mathcal{G}} \tag{2.15}$$

where D and N are the trace operators:

$$Du := u|_{\Gamma}; \quad Nu := \mathcal{C}\varepsilon(u)\nu|_{\Gamma},$$

$\nu = \{\nu_j\}_{j=1}^3$ is the outward normal, so that $(Nu)_i = c_{ijkl}\varepsilon_{kl}(u)\nu_j = c_{ijkl}\partial_l u_k \nu_j$.

By using formula (2.15) we easily establish the following representation

$$G^*L_0u = Nu\,, \quad u \in \operatorname{Dom} L_0. \tag{2.16}$$

Indeed, it suffices to apply formula (2.15) with $u \in \operatorname{Dom} L_0,\ v = G\psi,\ \psi \in \mathcal{G}$, which leads to

$$\begin{aligned}(G^*L_0u,\psi)_{\mathcal{G}} = (L_0u,G\psi)_{\mathcal{H}} &= (u,LG\psi)_{\mathcal{H}} + (Nu,DG\psi)_{\mathcal{G}} \\ &-(Du,NG\psi)_{\mathcal{G}} = (Nu,DG\psi)_{\mathcal{G}} = (Nu,\psi)_{\mathcal{G}}\end{aligned} \tag{2.17}$$

implying the conclusion in (2.16).

Step 2: The bound for $G^*L_0S(\cdot)$

We shall consider composition operators $G^*L_0S(\cdot)$, and $G^*(-L_0)^{\frac{1}{2}}C(\cdot)$ as (potentially) unbounded but densely defined operators : $\mathcal{H} \to \mathcal{F}^T$ given by:

$$(G^*L_0S(\cdot)u)(t) := G^*L_0S(t)u;\ u \in \operatorname{Dom} L_0$$
$$(G^*(-L_0)^{\frac{1}{2}}C(\cdot)u)(t) := G^*(-L_0)^{\frac{1}{2}}C(t)u\ u \in \operatorname{Dom} L_0.$$

Similarly, W^* which is originally defined as an element of $\mathcal{L}(L_2((0,T);\mathcal{H}_0^1) \to \mathcal{F}^T)$, can be considered as an unbounded operator (denoted by the same symbol): $L_2((0,T);\ \mathcal{H}) \to \mathcal{F}^T$.

The lemma stated below shows that these operators are in fact bounded.

Lemma 1. *The operators*

$$G^*L_0S(\cdot) : \mathcal{H} \to \mathcal{F}^T;\ G^*(-L_0)^{\frac{1}{2}}C(\cdot) : \mathcal{H} \to \mathcal{F}^T;\ W^* : L_2((0,T);\mathcal{H}) \to \mathcal{F}^T$$

are bounded.

Proof. To prove the lemma we shall follow the same strategy as in [13] with support of computations performed in [14] for the von Karman system.

The main task in proving the lemma is establishing the appropriate bound for elements in the domains of respective operators. By virtue of (2.16) the result of the Lemma 1 follows from the following proposition:

Proposition 1. *Let $a,b \in \operatorname{Dom} L_0, g \in L_2((0,T);\mathcal{H}_0^1)$ and let $w(t) := C(t)a + S(t)b + \int_0^t S(t-s)g(s)\,\mathrm{d}s$. Then the following estimate holds*

$$\int\limits_{\Sigma^T} \mathrm{d}\Gamma\,\mathrm{d}t\,||\varepsilon(w)||^2 \le c\left[||a||^2_{\mathcal{H}_0^1} + ||b||^2_{\mathcal{H}} + ||g||^2_{L_1((0,T);\mathcal{H})}\right]. \tag{2.18}$$

Proof. Follows the same strategy as in [13].

Step 1. Integrating by parts the equation $w_{tt} - Lw = g$ satisfied by w and accounting for boundary conditions, leads to the following variational equality

$$\int\limits_{Q^T} \mathrm{d}x\mathrm{d}t\,[\rho w_{tt}\cdot\eta - \rho g\cdot\eta + \langle \mathcal{C}\varepsilon(w), \varepsilon(\eta)\rangle] - \int\limits_{\Sigma^T} \mathrm{d}\Gamma\mathrm{d}t\,\mathcal{C}\varepsilon(w)\nu\cdot\eta = 0 \tag{2.19}$$

for any test field $\eta \in H^1(Q^T;\mathbf{R}^3)$. In what follows we choose and fix $\eta = (\nabla w)^\tau h$, so that $\eta_i = \partial_j w_i h_j$, with a smooth field $h = h(x)$.

Step 2. Introduce the tensors $e(w)$ and $m(w)$:

$$e_{lk}(w) = \frac{1}{2}[(\partial_i w_l)\partial_k h_i + (\partial_i w_k)\partial_l h_i];$$

$$m_{lk}(w) = \frac{1}{2}[(\partial^2_{ik} w_l)h_i + (\partial^2_{il} w_k)h_i]$$

where $\partial^2_{ik} := \partial_i\partial_k$. Using the symmetry of tensor $\mathcal{C}$ (2.11) it is straightforward to show that

$$\varepsilon((\nabla w)^T h) = e(w) + m(w), \tag{2.20}$$

and

$$\operatorname{div}[\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle h] = \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \operatorname{div} h + 2\langle \mathcal{C}\varepsilon(w), m(w)\rangle + \langle (\partial_k\mathcal{C})\varepsilon(w), \varepsilon(w)\rangle h_k. \tag{2.21}$$

Step 3. In order to simplify notations we omit differentials in integrals. Inserting chosen η in (2.19) yields:

$$\begin{aligned}
0 &= \int\limits_{Q^T} \{\rho(w_{tt} - g)\cdot(\nabla w)^\tau h + \langle \mathcal{C}\varepsilon(w), \varepsilon((\nabla w)^\tau h)\rangle\} \\
&\quad - \int\limits_{\Sigma^T} \mathcal{C}\varepsilon(w)\nu\cdot(\nabla w)^\tau h = \langle \text{see (2.20)}\rangle \\
&= \int\limits_{\Omega} \rho w_t\cdot(\nabla w)^\tau h\ \Big|_{t=0}^{t=T} - \frac{1}{2}\int\limits_{Q^T} \nabla|w_t|^2\cdot\rho h - \int_{Q_T} g\cdot(\nabla w)^\tau h \\
&\quad + \int\limits_{Q^T} \{\langle \mathcal{C}\varepsilon(w), e(w)\rangle + \langle \mathcal{C}\varepsilon(w), m(w)\rangle\} - \int\limits_{\Sigma^T} \mathcal{C}\varepsilon(w)\nu\cdot(\nabla w)^\tau h.
\end{aligned} \tag{2.22}$$

Applying the divergence theorem and taking into account $w|_\Gamma = 0$ one has

$$\int\limits_{Q^T} \nabla|w_t|^2\cdot\rho h = \int\limits_{Q^T} |w_t|^2 \operatorname{div}\rho h;$$

using (2.21) one obtains

$$\begin{aligned}\int\limits_{Q^T} \langle \mathcal{C}\varepsilon(w), m(w)\rangle &= \frac{1}{2}\int\limits_{Q^T} \{\operatorname{div}[\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle h] - \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \operatorname{div} h - \langle (\partial_k \mathcal{C})\varepsilon(w), \varepsilon(w)\rangle h_k\} \\ &= \frac{1}{2}\int\limits_{\Sigma^T} \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle h\cdot\nu - \frac{1}{2}\int\limits_{Q^T}\{\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \operatorname{div} h + \langle (\partial_k \mathcal{C})\varepsilon(w), \varepsilon(w)\rangle h_k\}\cdot\end{aligned}$$

Inserting in (2.22) gives

$$\begin{aligned}0 = &\int\limits_{\Omega} \rho w_t \cdot (\nabla w)^\tau h \;\Big|_{t=0}^{t=T} + \frac{1}{2}\int\limits_{Q^T} |w_t|^2 \operatorname{div} \rho h - \int_{Q_T} g\cdot(\nabla w)^\tau h \\ &+ \int\limits_{Q^T}\left\{\langle \mathcal{C}\varepsilon(w), e(w)\rangle - \frac{1}{2}\,[\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \operatorname{div} h + \langle (\partial_k \mathcal{C})\varepsilon(w), \varepsilon(w)\rangle h_k]\right\} \\ &+ \int\limits_{\Sigma^T}\left\{\frac{1}{2}\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle h\cdot\nu - \mathcal{C}\varepsilon(w)\nu\cdot(\nabla w)^\tau h\right\}\cdot \end{aligned}\tag{2.23}$$

Step 4. By exploiting the fact that w vanishes on Γ, we shall show that

$$\mathcal{C}\varepsilon(w)\nu\cdot(\nabla w)^\tau h = \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \nu\cdot h \quad \text{on } \Gamma. \tag{2.24}$$

Indeed, $w|_\Gamma = 0$ implies $\partial_j w_i = \partial_\nu w_i \nu_j$ on Γ, ($\partial_\nu := \frac{\partial}{\partial\nu}$); therefore, denoting $\mathcal{C}\varepsilon(w) =: \alpha$ one has

$$\mathcal{C}\varepsilon(w)\nu\cdot(\nabla w)^\tau h = \alpha_{ik}\nu_k(\partial_j w_i)h_j = \alpha_{ik}\nu_k(\partial_\nu w_i)\nu_j h_j. \tag{2.25}$$

On the other side, due to symmetry of α

$$\langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle = \alpha_{ij}\frac{1}{2}\left[\partial_i w_j + \partial_j w_i\right] = \alpha_{ij}\partial_j w_i = \alpha_{ij}(\partial_\nu w_i)\nu_j. \tag{2.26}$$

Comparing (2.25) with (2.26) leads to (2.24).

Step 5. Taking h parallel to ν on Γ, and using (2.24) one transforms (2.23) to

$$\begin{aligned}\int\limits_{\Sigma^T} \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle = &\int\limits_{\Omega} \rho w_t\cdot(\nabla w)^\tau h|_{t=0}^{t=T} - \int_{Q_T} g\cdot(\nabla w)^\tau h \\ &+ \int\limits_{Q^T}\left\{\frac{1}{2}\,|w_t|^2 \operatorname{div} \rho h + \langle \mathcal{C}\varepsilon(w), e(w) - \frac{1}{2}\,(\operatorname{div} h)\,\varepsilon(w)\rangle - \frac{1}{2}\langle (\partial_k \mathcal{C})\varepsilon(w), \varepsilon(w)\rangle h_k\right\}\cdot\end{aligned}$$

Since only the 1[st]-order derivatives of w enter the right hand side we easily get

$$\int\limits_{\Sigma^T} \langle \mathcal{C}\varepsilon(w), \varepsilon(w)\rangle \le c\,\sup{}_{t\in[0,T]}\;\left[||w(\cdot,t)||^2_{\mathcal{H}^1_0} + ||w_t(\cdot,t)||^2_{\mathcal{H}} + ||g||^2_{L_1((0,T);\mathcal{H})}\right]. \tag{2.27}$$

On the other hand, by (2.4)

$$||w(\cdot,t)||^2_{\mathcal{H}^1_0} + ||w_t(\cdot,t)||^2_{\mathcal{H}} \le c \left[||a||^2_{\mathcal{H}^1_0} + ||b||^2_{\mathcal{H}} + ||g||^2_{L_1((0,T);\mathcal{H})} \right],$$

whereas positivity (2.3) allows to transform (2.20) into (2.12). The Proposition is proved. □

Let $g = 0$. Since $\mathrm{Dom}\, L_0$ is dense in $\mathcal{H}$ and $\mathcal{H}^1_0$, the Proposition 1 yields continuity of the map $\{a, b\} \to \varepsilon(w)|_{\Sigma^T}$ from $\mathcal{H}^1_0 \times \mathcal{H}$ into $\mathcal{F}^T$. Thus, each $H^1(Q^T; \mathbf{R}^3)$-solution w of Lame system with finite energy initial data possesses the trace $\varepsilon(w)|_{\Sigma^T}$ in L_2 sense. The result of Proposition 1 when applied with $a = 0, b$ and then $a, b = 0$ implies, *via* principle of superposition and (2.16) the result stated in Lemma 1.

Step 3: Completion of the proof of Theorem 1

To complete the proof, we proceed as in [13] Section 3. We define the following operators as elements of $\mathcal{L}(\mathcal{F}^T \to L_\infty((0,T); \mathcal{H}^{-1}))$.

$$(J_0 f)(t) := L_0 \int_0^t S(s) G f(s)\, \mathrm{d}s;\ (J_1 f)(t) := (-L_0)^{\frac{1}{2}} \int_0^t C(s) G f(s)\, \mathrm{d}s.$$

From Lemma 1 and duality it follows that J_0 and J_1 are bounded: $\mathcal{F}^T \to L_\infty((0,T); \mathcal{H})$. Indeed, the above assertion follows *via* Riesz Representation theorem from the following estimate valid with an arbitrary test field $\phi \in L_1((0,T); \mathcal{H}^1_0)$:

$$\int_0^T ((J_0 f)(t), \phi(t))_{\mathcal{H}^{-1}, \mathcal{H}^1_0}\, \mathrm{d}t \le ||f||_{\mathcal{F}^T} ||G^* L_0 S(\cdot)||_{\mathcal{L}(\mathcal{H} \to \mathcal{F}^T)} \int_0^T ||\phi(t)||_{\mathcal{H}}\, \mathrm{d}t.$$

Analogous estimate applies to the second operator J_1.

Moreover, by standard density argument one shows that J_0 and J_1 are bounded: $\mathcal{F}^T \to C([0,T]; \mathcal{H})$ see Corollary 3.2 in [13]. To complete the proof it suffices to use trigonometric identities in (2.5) along with properties of sine and cosine operators in (2.4)

$$\begin{aligned}(Wf)(t) &= L_0 S(t) \int_0^t C(s) G f(s) \mathrm{d}s - L_0 C(t) \int_0^t S(s) G f(s) \mathrm{d}s \\ &= -C(t) L_0 \int_0^t S(s) G f(s) \mathrm{d}s + (-L_0)^{\frac{1}{2}} S(t) (-L_0)^{\frac{1}{2}} \int_0^t C(s) G f(s) \mathrm{d}s \\ &= (-L_0)^{\frac{1}{2}} S(t) (J_1 f)(t) - C(t)(J_0 f)(t). \end{aligned} \tag{2.28}$$

The requisite boundedness of W follows now from the continuity of J_0, J_1 as operators: $\mathcal{F}^T \to C([0,T]; \mathcal{H})$, and continuity of $(-L_0)^{\frac{1}{2}} S(t)$ and $C(t)$ as operators: $\mathcal{H} \to C([0,T]; \mathcal{H})$. The same argument applies to the velocity component:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} W f \right)(t) = -L_0 \int_0^t C(t-s) G f(s) \mathrm{d}s = -(-L_0)^{\frac{1}{2}} [C(t)(J_1 f)(t) + (-L_0)^{\frac{1}{2}} S(t)(J_0 f)(t)] \tag{2.29}$$

where, again, we have a composition of continuous operators acting on respective spaces such that the product is in $C([0,T]; [\mathrm{Dom}\,(-L_0)^{\frac{1}{2}}]')$- as desired of the final conclusion. The proof of Theorem 1 is thus completed.

Remark. An alternative way of completing the proof of Theorem 1 is to use the fact that W^* *a-priori* in $\mathcal{L}(L_2((0,T); \mathcal{H}^1) \to \mathcal{F}^T)$ is bounded: $L_2((0,T); \mathcal{H}) \to \mathcal{F}^T$. (This last assertion follows directly from Lem. 1.) The above implies, *via* duality (see Sect. 3 in [13]), the boundedness of $W : \mathcal{F}^T \to L_2((0,T); \mathcal{H})$. Final conclusion can be then reached by appealing to "lifting theorem" in [16] which allows to boost L_2 time regularity to C for time reversible dynamics.

Remark. The $L_2 \to L_2$ regularity of solutions of the Lame system is the same as of the scalar wave equation [13]. This is in contrast to the Maxwell system where the map $f \to u^f$ is not continuous in L_2-norms.

Remark. Extending the map $f \to u^f$ onto $\mathcal{F}^T$ by continuity we consider its images in $C([0,T];\mathcal{H})$ as generalized solutions of Lame system.

Theorem 1 implies the continuity of the *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}$,

$$W^T f := (Wf)(T) = L_0 \int_0^T S(t-s)Gf(s)\mathrm{d}s = u^f(\cdot,T).$$

Its reduction $W^{\sigma,T} := W^T \mid_{\mathcal{F}^{\sigma,T}}$ is understood as an operator from $\mathcal{F}^{\sigma,T}$ into $\mathcal{H}$. Also, taking the adjoint of W^T (see Sect. 3 [13]) one obtains $(W^T)^* \in \mathcal{L}(\mathcal{H} \to \mathcal{F}^T)$ is given by:

$$\left(W^T\right)^* y = G^* L_0 S(T-\cdot)\, y. \tag{2.30}$$

2.4. **Boundary control problem**

Introduce the *reachable set*

$$\mathcal{U}^{\sigma,T} := \mathrm{Ran} W^{\sigma,T} = W^{\sigma,T}\mathcal{F}^{\sigma,T}.$$

Taking into account the relation (2.10) the following statement of the boundary control problem (BCP) appears relevant: given $T > 0$ and $a \in \mathcal{H}_p^{\sigma,T}$, find $f \in \mathcal{F}^{\sigma,T}$ such that

$$u^f(\cdot,T) = a \tag{2.31}$$

holds. In the lemma below we shall show that, at least for small T, the reachable set is rather poor, and the BCP is not solvable in general.

Lemma 2. *Let $T < T_p^\sigma$ and nonzero $a \in \mathcal{H}_p^{\sigma,T}$ are such that the set $\Omega \setminus \mathrm{supp}\, a$ is connected and contains a neighborhood of σ in Ω. Then $a \notin \mathcal{U}^{\sigma,T}$.*

Proof. We are going to show that the conjecture $a \in \mathcal{U}^{\sigma,T}$ leads to a contradiction.

Step 1: Let $f \in \mathcal{F}^{\sigma,T}$ be such that $u^f(\cdot,T) = a$. As one can check, the field

$$u(x,t) := \begin{cases} 0 & \text{in} \quad \bar{\Omega} \times (-\infty, 0]; \\ u^f(x,t) & \text{in} \quad \bar{\Omega} \times (0,T]; \\ -u^f(x, 2T-t) & \text{in} \quad \bar{\Omega} \times (T, 2T]; \\ 0 & \text{in} \quad \bar{\Omega} \times (2T, +\infty) \end{cases}$$

belongs to the class $C(\mathbf{R} \setminus \{t = T\};\mathcal{H})$ and satisfies (in the sense of distributions $\mathcal{D}'(\Omega \times R)$) the equation

$$u_{tt} - Lu = -2{\delta'}_T\, u^f(\cdot,T) = -2{\delta'}_T\, a \quad \text{in } \Omega \times \mathbf{R}.$$

Here ${\delta'}_T(t) := \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-T)$ is the dipole supported at $t = T$. This easily implies that, for each k the field

$$\tilde{u}(x,k) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-ikt} u(x,t)\, \mathrm{d}t, \quad (x,k) \in \Omega \times \mathbf{R}$$

satisfies the equation

$$(L+k^2)\tilde{u}(\cdot,k) = 2ik\,\mathrm{e}^{-ikT}\,a \quad \text{in } \Omega$$

as a $\mathcal{D}'(\Omega))$–distribution.

Since $\operatorname{supp} u^f(\cdot,t) \subset \bar{\Omega}_p^{\sigma,T}$ for all $t \in [0,T]$, one has $\operatorname{supp} u(\cdot,t) \subset \bar{\Omega}_p^{\sigma,T}$ for all $t \in \mathbf{R}$. The latter obviously implies that its Fourier transform satisfies

$$\tilde{u}(\cdot,k) = 0 \quad \in \Omega \setminus \bar{\Omega}_p^{\sigma,T}. \tag{2.32}$$

Step 2: Thus, the field $\tilde{u}$ satisfies the homogeneous equation $(L+k^2)\tilde{u}(\cdot,k) = 0$ in the open set $\Omega \setminus \operatorname{supp} a$ and vanishes on its open subset (2.32). By the well known unique continuation principle for the Lame operator L [21] (see also [17]) one has $\tilde{u}(\cdot,k) = 0$ everywhere in $\Omega \setminus \operatorname{supp} a$, and, in particular, in a neighbourhood $\omega^\sigma \subset \Omega$ of σ.

Step 3: The last fact implies

$$u^f(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{ikt}\,\tilde{u}(x,k)\,\mathrm{d}k = 0, \quad (x,t) \in \omega^\sigma \times [0,T];$$

hence $f = u^f\,|_{\sigma\times[0,T]} = 0$. Consequently, $a = u^f(\cdot,T) = 0$, which fact contradicts $a \neq 0$.

Remark. Taking $a = 0$ in (2.31), and repeating all the steps of the proof we can easily obtain $f = 0$ that is equivalent to $\operatorname{Ker} W^{\sigma,T} = \{0\}$. Therefore, in the case of $T < T_p^\sigma$ the BCP can have at most one solution. The injectivity of W^T means that on the time interval $(0, T_p^\sigma)$ velocity is determined by displacement. Indeed, consider the set of pairs $\{\{u^f(\cdot,T), u_t^f(\cdot,T)\} \,|\, f \in \mathcal{F}^T\}$; if $u^f(\cdot,T) = W^{\sigma,T} f = 0$ then $f = 0$ by injectivity of $W^{\sigma,T}$, that yields $u_t^f(\cdot,T) = 0$. Thus, the set is the graph of a linear operator expressing $u_t^f(\cdot,T)$ trough $u^f(\cdot,T)$. This means that only one component of the complete state $\{u^f; u_t^f\}$ may be controlled at times $t < T_p^\sigma$. This phenomenon should be contrasted with the case when the time is large enough (see *e.g.* [2]).

2.5. **Comments**

- An analog of the result stated in Theorem 1 in the case of the scalar wave equation with constant coefficients was first shown in [15] and later generalized to various topological levels in [13]. Thus the L_2-regularity of "control → state" map is extended from the scalar case to the case of systems of dynamic elasticity. We note, that as in the case of the wave equation, the regularity result has no analogue for the Neumann problem. This is to say that L_2 tractions prescribed on the boundary do not produce H^1 solutions.
- By using the result in Theorem 1 along with semigroup methods one can prove higher (or lower) level optimal regularity of solutions with respect to various levels of regularity of controls. In fact, this can be done in the same way as in [13] where scalar wave equations are treated.
- The proof of Lemma 2 follows the scheme of the paper [1] (see also [7]).

3. Controllability in subdomain $\Omega_s^{\sigma,T}$ – proof of Theorem 2

3.1. **The dual system**

We recall from (2.30) and (2.16) that with $y \in \mathcal{H}$ we have

$$\left(W^T\right)^* y = G^* L_0 S(T-\cdot) y = N S(T-\cdot) y \in \mathcal{F}^T$$

where we recall $(Nu)_i = \nu_j c_{ijkl}\partial_l u_k \mid_\Gamma$; PDE interpretation of this is that for a solution $v = v^y(x,t), \quad y \in \mathcal{H}$ of the dynamical system

$$v_{tt} - Lv = 0 \qquad \text{in } Q^T; \tag{3.1}$$
$$v|_{t=T} = 0,\ v_t|_{t=T} = y \quad \text{in } \Omega; \tag{3.2}$$
$$v = 0 \qquad \text{on } \Sigma^T; \tag{3.3}$$

we have the following representation

$$\left(W^{\sigma,T}\right)^* y = Nv^y \mid_{\Sigma^{\sigma,T}} . \tag{3.4}$$

The rest of Section 3 is devoted to the proof of Theorem 2.

3.2. **Odd continuation and Cauchy data**

We are going to show that existence of $h \in \mathcal{H}$ satisfying:

$$(\alpha) \quad h \perp \mathcal{U}^{\sigma,T};\ (\beta) \quad \mathrm{supp} h \cap \Omega_s^{\sigma,T} \neq \{\emptyset\}$$

which statements will lead to a contradiction.

Let v^h be the solution of (3.1–3.3) with $y = h$; in accordance with (α) one has

$$0 = (W^T f, h)_{\mathcal{H}} = \langle \text{see}\,(3.4)\rangle = (f, Nv^h)_{\mathcal{F}^T}$$

for any $f \in \mathcal{F}^{T,\sigma}$; hence

$$Nv^h = 0 \quad \text{on } \sigma \times [0,T]. \tag{3.5}$$

Fix $\gamma \in \Gamma$ and choose the coordinates so that $\nu = \nu(\gamma) = \{0, 0, -1\}$. From (3.3) we have $\partial_j v_i^h = 0$ for $j = 1, 2\, i = 1, 2, 3$, whereas equality (3.5) takes the form of the system

$$c_{i3k3}\, \partial_3 v_k^h = 0, \quad i = 1, 2, 3$$

with diagonal matrix $c_{i3k3} = \mu\delta_{ik} + (\lambda + \mu)\delta_{i3}\delta_{3k}$. Solving this system, one gets $\partial_3 v_k^h = 0$ and, finally,

$$v^h = 0,\ \ \partial_\nu v^h = 0 \quad \text{on } \sigma \times [0,T],$$

so that the solution v^h has zero Cauchy data on $\Sigma^{\sigma,T}$.

As is easy to verify, due to $v|_{t=T} = 0$ the odd continuation

$$w(x,t) := \begin{cases} v^h(x,t) & \Omega \times (0,T); \\ -v^h(x, 2T - t) & \Omega \times [T, 2T) \end{cases}$$

turns out to be an H^1–solution of the problem

$$w_{tt} - Lw = 0 \qquad \text{in } \Omega \times (0, 2T); \tag{3.6}$$
$$w = 0,\ \ \partial_\nu w = 0 \quad \text{on } \sigma \times [0, 2T]. \tag{3.7}$$

3.3. The extension Ω'

Let Ω' be a domain with smooth boundary such that $\Omega' \supset \bar{\Omega}$. Continue the Lame parameters in Ω' preserving smoothness and conditions $\rho > 0, \mu > 0, 3\lambda + 2\mu > 0$ in $\bar{\Omega}'$.

Continue the solution w in $\Omega' \times (0, 2T)$ putting $w = 0$ in $[\Omega' \setminus \Omega] \times (0, 2T)$. Due to (3.6, 3.7) this continuation turns out to be an H^1– solution of the equation

$$w_{tt} - Lw = 0 \quad \text{in } \Omega' \times (0, 2T) \tag{3.8}$$

vanishing near one side of the smooth surface $\sigma \times (0, 2T)$. Since the surface is noncharacteristic with respect to both speeds s, p (time-like), by virtue of the uniqueness theorem [9] one has $w = 0$ in an open set $K_0^{\sigma,2T} \subset \Omega \times (0, 2T)$ such that $\partial K_0^{\sigma,2T} \supset \sigma \times [0, 2T]$.

Let us show that the solution w is continued by zero from $K_0^{\sigma,2T}$ onto the domain

$$K^{\sigma,2T} := \{(x, t) \in \Omega \times (0, 2T) |\, |t - T| < T - \text{dist}_s(x, \sigma)\}$$

bounded by characteristics of equation (3.6).

3.4. Paraboloids

Fix a point $x_0 \in \Omega_s^{\sigma,T}$; choose a point $\gamma_0 \in \sigma$. Let $l \subset \Omega$ be a smooth curve connecting γ_0 with x_0, of the s-length less than T, transversal to σ. Such a choice is possible due to $\text{dist}_s(x_0, \sigma) < T$.

Continue l smoothly across beyond σ in Ω'. On this continuation fix a point a close enough to σ so that the s-length of the segment $[a, x_0]$ is less than T. Parametrize this interval putting $s(m)$ equal to the s-length of $[a, m]$, so that $s(x_0) =: T_0 < T$.

Consider a (small) tube neighborhood ω of the segment $[a, x_0] \subset l$ in Ω' determined by the condition: each $x \in \omega$ is connected with l by unique shortest s-geodesic orthogonal to l. Let m_x be the point on l s-nearest to $x \in \omega$.

In ω introduce the "cylinder coordinates" $r(x) := \text{dist}_s(x, l)$, $s(x) := s(m_x)$; notice the inequality $s(x) \geq \text{dist}_s\,(x, \sigma)$, and the relations

$$\nabla s \cdot \nabla r = 0 \quad \text{in } \omega; \quad |\nabla s|^2 = \frac{1}{c_s^2} \quad \text{on } l \tag{3.9}$$

easily following from the definitions.

Taking $\varepsilon > 0$ small enough one can ensure in the "tube" $\omega_\varepsilon := \{x \in \omega |\, r(x) < \varepsilon\}$ the inequality

$$\left(\frac{T_0 + T}{2T_0}\right)^2 c_s^2(x)\, |\nabla s(x)|^2 > 1, \quad x \in \omega_\varepsilon \tag{3.10}$$

which is valid due to $T_0 < T$ and the second of the relations in (3.9).

Choose $r_0 > 0$ so small that the subdomain (*paraboloid*)

$$\pi_{r_0,\kappa}^{x_0,\gamma_0,l} := \left\{x \in \omega_\varepsilon \cap \Omega \,|\, \kappa - \frac{s(x)}{T_0} - \left(\frac{r(x)}{r_0}\right)^2 > 0\right\}$$

for any $\kappa \in (0, 1]$ is contained in $\Omega_s^{\sigma,T}$ whereas the components $\partial\pi_{r_0,\kappa}^{x_0,\gamma_0,l} \cap \Omega$ and $\partial\pi_{r_0,\kappa}^{x_0,\gamma_0,l} \cap \sigma$ of its boundary are smooth surfaces. We call these paraboloids admissible; notice that $\pi_{r_0,\kappa}^{x_0,\gamma_0,l}$ extends as κ grows.

Varying parameters of admissible paraboloids one can verify that they exhaust the subdomain $\Omega_s^{\sigma,T}$.

3.5. Lense-shaped domains

Denote

$$\varphi(x,t) := \left[\left(\frac{t}{T}-1\right)^2\right]^{\frac{T}{T_0+T}} - \kappa + \frac{s(x)}{T_0} + \left(\frac{r(x)}{r_0}\right)^2$$

and consider the "lense-shaped" domain

$$\Pi^{x_0,\gamma_0,l}_{r_0,\kappa} := \{(x,t)\in\Omega\times(0,2T)|\,\varphi(x,t)<0\}\cdot$$

The component of its boundary

$$\Sigma^{x_0,\gamma_0,l}_{r_0,\kappa} := \{(x,t)\in\Omega\times(0,2T)|\,\varphi(x,t)=0\}$$

is a smooth surface; calculating its characteristic form with regard to (3.9) we have

$$\left(\frac{\partial\varphi}{\partial t}\right)^2 - c_s^2|\nabla\varphi|^2 = \frac{4}{(T_0+T)^2}\left(\frac{t}{T}-1\right)^{\frac{2(T-T_0)}{T+T_0}} - \frac{c_s^2}{T_0^2}|\nabla s|^2 - 4c_s^2\left(\frac{r}{r_0}\right)^2|\nabla r|^2 \le \frac{4}{(T_0+T)^2} - \frac{c_s^2}{T_0^2}|\nabla s|^2 < 0.$$

Thus, $\Sigma^{x_0,\gamma_0,l}_{r_0,\kappa}$ is noncharacteristic (time–like).

The cross-sections of $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}$ by planes $t=\text{const}$ are admissible paraboloids. Let $(x,t)\in\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}$. By virtue of the inequality $s(x)\ge \operatorname{dist}_s(x,\sigma)$ mentioned in Section 3.4 one has the estimates

$$\begin{aligned}|t-T| &< T\left[\kappa - \frac{s(x)}{T_0} - \left(\frac{r(x)}{r_0}\right)^2\right]^{\frac{T_0+T}{2T}} \le T\left[1-\frac{s(x)}{T_0}\right]^{\frac{T_0+T}{2T}} \le T\left[1-\left(\frac{T_0+T}{2TT_0}\right)s(x)\right]\\ &= T-\left(\frac{T_0+T}{2T_0}\right)s(x) \le T-s(x)\le T-\operatorname{dist}_s(x,\sigma)\end{aligned}$$

which imply the inclusion $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}\subset K^{\sigma,2T}$. It is easy to show that the domains $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}$ extend as κ grows and exhaust the entire $K^{\sigma,2T}$.

3.6. Completing the proof of Theorem 2

Choose $x_0\in\operatorname{supp}h\cap\Omega_s^{\sigma,T}$. Since $w(\cdot,T)=h$ one has $(x_0,T)\in K^{\sigma,2T}\cap\operatorname{supp}w$.

Construct the domains $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}$, $\kappa\in(0,1]$. For small enough κ the obvious inclusion $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}\subset K_0^{\sigma,2T}$ holds, so that $w=0$ in $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}$. At the same time, if $\kappa=1$ then $\bar\pi^{x_0,\gamma_0,l}_{r_0,1}\ni x_0$; hence, by necessity $\operatorname{supp}w\cap\bar\Pi^{x_0,\gamma_0,l}_{r_0,1}\ne\{\emptyset\}$.

Define

$$\kappa_0 := \sup\{\kappa>0|\,\operatorname{supp}w\cap\bar\Pi^{x_0,\gamma_0,l}_{r_0,\kappa}=\{\emptyset\}\},$$

so that $\kappa_0\le 1$. The surface $\Sigma^{x_0,\gamma_0,l}_{r_0,\kappa_0}$ contains points of $\operatorname{supp}w$. On the other hand, it bounds the domain $\Pi^{x_0,\gamma_0,l}_{r_0,\kappa_0}$ where $w=0$. Since this surface is noncharacteristic, we reach contradiction with the theorem on uniqueness of continuation [9].

Thus, the conjectures (α), (β) (see Sect. 3.2) lead to contradiction that proves Theorem 2.

If $T>T_s^\sigma$ one has $\Omega_s^{\sigma,T}=\Omega$, $\mathcal{H}_s^{\sigma,T}=\mathcal{H}$, and the equality (1.6) takes the form

$$\operatorname{clos}_{\mathcal{H}}\mathcal{U}^{\sigma,T}=\mathcal{H}. \tag{3.11}$$

3.7. Comments

- The scheme of the proof of Theorem 2 is taken from Russell's paper [18] who first used the Holmgren–John uniqueness theorem in a study of controllability for hyperbolic problems. Otherwise, the author mentions "the germinal idea" of J.-L. Lions. The trick with "lense-shaped" surfaces comes from the classical paper [11]: our paraboloids is just a modification of John's construction.
- Theorem 2 shows that approximate controllability always occurs in $\Omega_s^{\sigma,T}$. At the same time, if $T < T_s^\sigma$, a simple example of $\Omega = \mathbf{R}^3_+$ with constant λ, μ, ρ demonstrates that it does not occur in the subdomain $\Omega_p^{\sigma,T} \setminus \Omega_s^{\sigma,T}$. Therefore the components u_i^f of solution are connected (not independent) in this subdomain. The interesting and important to inverse problems question is to clarify this connection.
- One more question coming from inverse problems is the following. Assume $T < T_s^\sigma$, let $e_j\,(\cdot)$ be the field in Ω identically equal to the coordinate vector $e_j = \{\delta_{jk}\}_{k=1}^3 \quad (j = 1, 2, 3)$. Does the cut off field $X_s^{\sigma,T} e_j$ belong to the reachable set $\mathcal{U}^{\sigma,T}$? Notice that the corresponding question is still open even in the case of the scalar wave equation in dimension $n > 1$.
- Due to density of $\mathcal{U}_0^{\sigma,T}$ in $\mathcal{F}^{\sigma,T}$ and the boundness of $W^{\sigma,T}$ one can replace $\mathcal{U}^{\sigma,T}$ by $\mathcal{U}_0^{\sigma,T}$ in (1.6, 3.11).

4. Controllability for times $T > T_s^\sigma$–proof of Theorem 3

4.1. Regularization

Recall that the classes of fields $\mathcal{H}^2$, $\mathcal{H}_0^1$, the operator L_0, the corresponding sine- and cosine-operator functions, and the relations between them were introduced in Section 2.2.

The solution of the dual problem (3.1–3.3) has the well-known representation in Ω:

$$v^y(\cdot, t) = S(t - T)\, y, \quad 0 \le t \le T. \tag{4.1}$$

In what follows it is convenient to extend v^y from $[0, T]$ to the entire time axis defining v^y *via* the right hand side of (4.1) for all $t \in \mathbf{R}$. Thus we have $v^y \in C(\mathbf{R}; \mathcal{H})$.

Choose and fix an even function $\eta \in C_0^\infty(\mathbf{R})$ such that $\eta \ge 0$, $\operatorname{supp} \eta \subset [-1, 1]$; $\int\limits_{-\infty}^{\infty} \eta(t) \mathrm{d}t = 1$. For $\varepsilon > 0$ define

$$\eta_\varepsilon(t) := \frac{1}{\varepsilon}\, \eta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbf{R}.$$

Due to evenness of $\eta(t)$ and oddness of $S(t)$ one has

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \eta_\varepsilon(s)\, v^y(\cdot, t - s)\, \mathrm{d}s &= \langle \text{see}\,(4.1)\rangle = \int\limits_{-\infty}^{\infty} \eta_\varepsilon(s)\, S(t - T - s)\, y\, \mathrm{d}s \\ &= \langle \text{see}\,(2.5)\rangle = S(t - T) \int\limits_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s)\, C(s)\, y\, \mathrm{d}s = v^{y_\varepsilon},\end{aligned} \tag{4.2}$$

where

$$y_\varepsilon := \int\limits_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s)\, C(s)\, y\, \mathrm{d}s.$$

The properties of η_ε easily lead to the corresponding properties of y_ε:

(i) $||y_\varepsilon||_{\mathcal{H}} \le ||y||_{\mathcal{H}}$; $\lim\limits_{\varepsilon\to 0} y_\varepsilon = y \quad$ in $\mathcal{H}$;

(ii) $y_\varepsilon \in \bigcap\limits_{j\ge 1} \operatorname{Dom} L_0^j$.

The latter implies $y_\varepsilon \in C^\infty(\bar\Omega; \mathbf{R}^3)$, $v^{y_\varepsilon} \in C^\infty(\bar\Omega \times \mathbf{R}; \mathbf{R}^3)$. It is also easy to see that $v^{y_\varepsilon} \to v^y$ as $\varepsilon \to 0$ in $C([0,T]; \mathcal{H})$ with any $T > 0$. Thus y_ε and v^{y_ε} can be viewed as special regularizations of y and v^y.

4.2. **A lemma**

Fix a positive $\varepsilon < T$, choose $\delta \in (\varepsilon, T)$ and denote

$$\mathcal{F}_\delta^{\sigma,T} := \{f \in \mathcal{F}^{\sigma,T} \,|\, \operatorname{supp} f \subset \bar\sigma \times [\delta, T]\}\cdot$$

For $f \in \mathcal{F}_\delta^{\sigma,T}$ the control $f_\varepsilon \in \mathcal{F}^{\sigma,T}$,

$$f_\varepsilon(\cdot,t) := \frac{1}{2}\int_0^T [\eta_\varepsilon(t-s) - \eta_\varepsilon(2T-t-s)] f(\cdot,s)\,\mathrm{d}s, \quad 0 \le t \le T$$

is well defined. As is easy to check, $f_\varepsilon(\cdot,t)$ vanishes near $t = 0$. Note the equality $(f_\varepsilon)_{tt} = (f_{tt})_\varepsilon$ which holds for sufficiently smooth $f \in \mathcal{F}_\delta^{\sigma,T}$.

Lemma 3. *For $y \in \mathcal{H}$ and $f \in \mathcal{F}_\delta^{\sigma,T}$ the following relation takes place:*

$$(W^T f_\varepsilon, y)_{\mathcal{H}} = (W^T f, y_\varepsilon)_{\mathcal{H}}. \tag{4.3}$$

Proof. Let $y \in \bigcap\limits_{j\ge 1} \operatorname{Dom} L_0^j$ so that v^y is smooth. Since η is even whereas v^y is odd with respect to $t = T$ one has

$$v^{y_\varepsilon}(\cdot,t) = \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) v^y(\cdot,s)\,\mathrm{d}s = \frac{1}{2}\int_{-\infty}^{T} [\eta_\varepsilon(t-s) - \eta_\varepsilon(2T-t-s)]\, v^y(\cdot,s)\,\mathrm{d}s,$$

which gives

$$N v^{y_\varepsilon}(\cdot,t) = \frac{1}{2}\int_{-\infty}^{T} [\eta_\varepsilon(t-s) - \eta_\varepsilon(2T-t-s)]\, N v^y(\cdot,s)\,\mathrm{d}s, \quad t \in \mathbf{R}. \tag{4.4}$$

The equalities

$$
\begin{aligned}
(W^T f_\varepsilon, y)_{\mathcal H} &= (f_\varepsilon, (W^T)^* y)_{\mathcal H} = \langle \text{see (3.4)} \rangle = \int\limits_{\sigma\times[0,T]} \mathrm d\Gamma\, \mathrm dt\, f_\varepsilon(\gamma,t)\cdot N v^y(\gamma,t)\\
&= \int\limits_{\sigma\times[0,T]} \mathrm d\Gamma\, \mathrm dt \left\{ \frac12 \int\limits_0^T [\eta_\varepsilon(t-s) - \eta_\varepsilon(2T-t-s)] f(\gamma,s)\,\mathrm ds \right\} \cdot N v^y(\gamma,t)\\
&= \int\limits_{\sigma\times[\delta,T]} \mathrm d\Gamma\, \mathrm ds\, f(\gamma,s)\cdot \frac12 \int\limits_0^T [\eta_\varepsilon(s-t) - \eta_\varepsilon(2T-s-t)] N v^y(\gamma,t)\,\mathrm dt\\
&= \langle \text{taking into account of the supports of } f,\ f_\varepsilon,\ \eta_\varepsilon \rangle\\
&= \int\limits_{\sigma\times[\delta,T]} \mathrm d\Gamma\, \mathrm ds ... \int\limits_{-\infty}^T ... \mathrm dt = \int\limits_{\sigma\times[0,T]} \mathrm d\Gamma\, \mathrm ds ... \int\limits_{-\infty}^T ... \mathrm dt = \langle \text{see (4.4)} \rangle\\
&= (f, N v^{y_\varepsilon})_{\mathcal F^T} = \langle \text{see (3.4)} \rangle = (W^T f, y_\varepsilon)_{\mathcal H}
\end{aligned}
$$

establish (4.3) for a given y. By virtue of density of chosen y's in $\mathcal H$ and continuity of the map $y \mapsto y_\varepsilon$ (see (i), Sect. 4.1) this result is extended on all $y \in \mathcal H$.

4.3. **Completing the proof**

Recall that $\mathcal U_0^{\sigma,T} = W^T \mathcal M^{\sigma,T}$. We argue by contradiction. Let $T > T_s^\sigma$, and assume that equality (1.7) does not hold. Since L_0 is an isomorphism from $\mathcal H^2 \cap \mathcal H_0^1$ (with H^2–metric) onto $\mathcal H$, the violation of (1.7) is equivalent to existence of a *nonzero* $y \in \mathcal H$ orthogonal to the set $L_0 \mathcal U_0^{\sigma,T}$ in $\mathcal H$.

Choose $\varepsilon, \delta : 0 < \varepsilon < \delta < T$, $f \in \mathcal F_\delta^{\sigma,T} \cap \mathcal M^{\sigma,T}$; and let f_ε be defined as in Section 4.2. The equality $f_\varepsilon(\cdot,T) = 0$ implies $u^{f_\varepsilon}(\cdot,T)\,|_\Gamma = 0$; the latter shows that $u^{f_\varepsilon}(\cdot,T) \in \operatorname{Dom} L_0$.

For taken y and f one has the relations:

$$
\begin{aligned}
0 &= (L_0 u^{f_\varepsilon}(\cdot,T), y)_{\mathcal H} = \langle \text{see (2.9)} \rangle = (W^T (f_\varepsilon)_{tt}, y)_{\mathcal H}\\
&= (W^T (f_{tt})_\varepsilon, y)_{\mathcal H} = \langle \text{see (4.3)} \rangle\\
&= (W^T f_{tt}, y_\varepsilon)_{\mathcal H} = (L_0 W^T f, y_\varepsilon)_{\mathcal H}\\
&= \langle \text{see (ii), Sect. 4.1} \rangle = (W^T f, L_0 y_\varepsilon)_{\mathcal H}.
\end{aligned} \tag{4.5}
$$

Now let us take ε, δ such that $0 < \varepsilon < \delta < T - T_s^\sigma$; in this case, due to time invariancy of the Lame system, we have

$$
W^T \mathcal F_\delta^{\sigma,T} = W^{T-\delta} \mathcal F^{\sigma,T-\delta} = \mathcal U^{\sigma,T-\delta}.
$$

Since $T - \delta > T_s^\sigma$, by virtue of density of controls $f \in \mathcal F_\delta^{\sigma,T} \cap \mathcal M^{\sigma,T}$ in $\mathcal F_\delta^{T,\sigma}$ and continuity of $W^T : \mathcal F^T \to \mathcal H$ one obtains

$$
\operatorname{clos}_{\mathcal H} W^T \{\mathcal F_\delta^{\sigma,T} \cap \mathcal M^{\sigma,T}\} = \operatorname{clos}_{\mathcal H} \mathcal U^{\sigma,T-\delta} = \langle \text{see (3.11)} \rangle = \mathcal H. \tag{4.6}
$$

Therefore, by (4.5) and (4.6) we conclude that $L_0 y_\varepsilon = 0$; hence $y_\varepsilon = 0$ because $\operatorname{Ker} L_0 = 0$. Passing through the limit $\varepsilon \to 0$; property (i), Section 4.1 implies $y = 0$ which contradicts the conjecture $y \neq 0$ proving Theorem 3.

4.4. **The Lame operator on waves**

Here we discuss a corollary of Theorem 3 which will be used later. We denote by $\bar{A}$ the closure of operator A.

For $T > 0$ introduce the operator $L_u^{\sigma,T} : \mathcal{H} \to \mathcal{H}$; $\operatorname{Dom} L_u^{\sigma,T} = \mathcal{U}_0^{\sigma,T}, L_u^{\sigma,T} y := Ly$ and notice the evident relation $L_u^{\sigma,T} \subset L_0$. If $T > T_s^\sigma$ this operator is densely defined in $\mathcal{H}$ whereas existence and continuity of $(L_u^{\sigma,T})^{-1} \subset L_0^{-1}$ easily lead to the following result:

$$\bar{L}_u^{\sigma,T} = L_0, \qquad T > T_s^\sigma. \tag{4.7}$$

5. Continuation of the response operator

5.1. **Free dynamics**

Consider the system

$$w_{tt} - Lw = 0 \qquad \text{in } Q^\infty; \tag{5.1}$$

$$w|_{t=0} = a,\ w_t|_{t=0} = b \quad \text{in } \Omega; \tag{5.2}$$

$$w = 0 \qquad \text{on } \Sigma^\infty \tag{5.3}$$

with $a, b \in \mathcal{H}$; let $w = w^{a,b}(x,t)$ be its solution. Our goal here is to obtain a representation of the trace $Nw^{a,b}$ on Σ^∞.

Evolution of the system (5.1–5.3) is governed by the operator L_0:

$$\begin{aligned} w^{a,b}(\cdot,t) &= \left[\cos t(-L_0)^{\frac{1}{2}}\right] a + (-L_0)^{-\frac{1}{2}} \left[\sin t(-L_0)^{\frac{1}{2}}\right] b \\ &= C(t)\, a \,+\, S(t)\, b; \quad w_t^{a,b}(\cdot,t) = C_t(t)\, a \,+\, S_t(t)\, b; \quad t \geq 0. \end{aligned} \tag{5.4}$$

Introduce the inversion $Y^T : \mathcal{F}^T \to \mathcal{F}^T$, $(Y^T f)(\cdot,t) := f(\cdot, T-t), 0 \leq t \leq T$ and take in (5.2) $a, b \in \bigcap_{j\geq 0} \operatorname{Dom} L_0^j$, so that $w^{a,b}$ is a classical solution. Comparing (3.1–3.3) with (5.1–5.3) one easily obtains

$$w^{a,b}(\cdot,t) = -\frac{\partial}{\partial t}[v^a(\cdot,T-t)] - v^b(\cdot,T-t)] \quad \text{in } \Omega, \quad 0 \leq t \leq T,$$

implying

$$Nw^{a,b}(\cdot,t) = -\frac{\partial}{\partial t} Nv^a(\cdot,T-t) - Nv^b(\cdot,T-t) \quad \text{on } \Gamma, \quad 0 \leq t \leq T$$

that is equivalent to

$$Nw^{a,b} = Y^T \left[\frac{\partial}{\partial t} \left(W^T\right)^* a - (W^T)^* b\right] \quad \text{on } \Sigma^T \tag{5.5}$$

in line with (3.4).

Denote $\Sigma_j^T := \Gamma \times [jT, (j+1)T]$, $j = 0, 1, ...$ (so that $\Sigma_0^T = \Sigma^T$) and introduce the operators $\mathcal{T}_j$, transferring functions on Σ_0^T into functions on Σ_j^T by the rule

$$(\mathcal{T}_j f)(\cdot,t) := f(\cdot, t - jT), \quad jT \leq t \leq (j+1)T.$$

The relation (5.5) and a shift with respect to time lead to the representation

$$Nw^{a,b}|_{\Sigma_j^T} = \mathcal{T}_j Y^T \left[\frac{\partial}{\partial t} \left(W^T\right)^* w^{a,b}(\cdot, jT) - \left(W^T\right)^* w_t^{a,b}(\cdot, jT)\right], \quad j = 0, 1, ... \tag{5.6}$$

In order to write it in a final form, let us insert (5.4) into (5.6). After simple transformations, we get

$$
\begin{aligned}
Nw^{a,b}|_{\Sigma_j^T} = \mathcal{T}_j Y^T \Bigg\{ & \left[\frac{\partial}{\partial t}(W^T)^* C(jT) - (W^T)^* C_t(jT)\right] a \\
& + \left[\frac{\partial}{\partial t}(W^T)^* S(jT) - (W^T)^* S_t(jT)\right] b \Bigg\}, \quad j = 0, 1, ... \qquad (5.7)
\end{aligned}
$$

5.2. The Lame system with $T = \infty$

Consider the system

$$u_{tt} - Lu = 0 \qquad \text{in } Q^\infty; \qquad (5.8)$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega; \qquad (5.9)$$

$$u = f \qquad \text{on } \Sigma^\infty \qquad (5.10)$$

with controls $f \in \mathcal{F}^\infty := L_2^{\mathrm{loc}}(\Sigma^\infty; \mathbf{R}^3)$ and the response operator

$$R^\infty f := N u^f \,|_{\Sigma^\infty}$$

defined on smooth controls vanishing near $\Gamma \times \{t = 0\}$. Notice the relation

$$(R^\infty f) \,|_{\Sigma^T} = R^T (f \,|_{\Sigma^T}) \qquad T > 0, \qquad (5.11)$$

following from the definitions.

Fix a finite $T > 0$; let $f \in C_0^\infty(\Gamma \times (0, \infty); \mathbf{R}^3)$, $\operatorname{supp} f \subset \Gamma \times [0, T]$. For times $t \geq T$ the control vanishes and we have free evolution described by (5.1, 5.2), so that

$$u^f(\cdot, t) = w^{a,b}(\cdot, t - T), \quad t \geq T \qquad (5.12)$$

with $a = u^f(\cdot, T) = W^T f$ and $b = u_t^f(\cdot, T) = W^T f_t$. Applying the trace operator N in (5.12), using representation (5.7), and denoting

$$
\begin{aligned}
P_j^T &:= \frac{\partial}{\partial t} \left(W^T\right)^* C((j-1)T) W^T - \left(W^T\right)^* C_t((j-1)T) W^T; \\
Q_j^T &:= \left[\frac{\partial}{\partial t} \left(W^T\right)^* S((j-1)T) W^T - \left(W^T\right)^* S_t((j-1)T) W^T\right] \frac{\partial}{\partial t},
\end{aligned}
$$

with regard to (5.11) one obtains the representation

$$R^\infty f = \begin{cases} R^{2T} f & \text{on } \Sigma_0^T \cup \Sigma_1^T; \\ \mathcal{T}_j Y^T \{P_j^T + Q_j^T\} f & \text{on } \Sigma_j^T, \ \ j = 2, 3, ... \end{cases} \qquad (5.13)$$

Considering a control with arbitrary $\operatorname{supp} f$, represent $f = \sum_p f_p$ with smooth f_p such that $\operatorname{supp} f_p \subset \Gamma \times [t_p, t_p + T]$ with some $t_1, t_2, ...$ Denote $\tilde{f}_p(\cdot, t) := f_p(\cdot, t + t_p)$, so that $\operatorname{supp} \tilde{f}_p \subset \Sigma_0^T$. Each $R^\infty \tilde{f}_p$ may be written in the form of (5.13); then, combining

$$R^\infty f(\cdot, t) = \sum_p (R^\infty \tilde{f}_p)(\cdot, t - t_p)$$

one can obtain a representation of the form (5.13) which determines R^∞ through R^{2T} and the operators P_j^T, Q_j^T. In what follows we'll show that for T is large enough both of the operators P_j^T, Q_j^T are also determined by R^{2T}.

5.3. The connecting operator

With reference to the system (2.1–2.3); the map $C^T:\mathcal{F}^T\to\mathcal{F}^T$,

$$C^T := (W^T)^* W^T$$

is called the *connecting operator*. This is a continuous nonnegative operator determined by the relation

$$(C^T f,g)_{\mathcal{F}^T} = (u^f(\cdot,T),u^g(\cdot,T))_{\mathcal{H}}. \tag{5.14}$$

The connecting operator may be simply and explicitly expressed *via* the response operator. Introduce auxiliary operators $\mathcal{S}^T:\mathcal{F}^T\to\mathcal{F}^{2T}$,

$$(\mathcal{S}^T f)(\cdot,t) := \begin{cases} f(\cdot,t), & 0\le t<T;\\ -f(\cdot,2T-t), & T\le t\le 2T;\end{cases}$$

and $I^{2T}:\mathcal{F}^{2T}\to\mathcal{F}^{2T}$,

$$(I^{2T}f)(\cdot,t) := \int_0^t f(\cdot,s)\,\mathrm{d}s, \quad 0\le t\le 2T.$$

Note the inclusion $\mathcal{S}^T C_0^\infty(\Gamma\times(0,T);\mathbf{R}^3)\subset \operatorname{Dom} R^{2T}$ and the relation

$$((\mathcal{S}^T)^* f)(\cdot,t) = f(\cdot,t)-f(\cdot,2T-t), \quad 0\le t\le T.$$

Lemma 4. *The representation*

$$C^T = \frac{1}{2}(\mathcal{S}^T)^* I^{2T} R^{2T}\mathcal{S}^T \tag{5.15}$$

holds on $C_0^\infty(\Gamma\times(0,T);\mathbf{R}^3)$.

Proof. Take $f,g\in C_0^\infty(\Gamma\times(0,T);\mathbf{R}^3)$ and denote $f_-:=\mathcal{S}^T f$; let u^{f_-} and u^g be the solutions of the problem (1.1–1.3) with the corresponding final times $2T$ and T.

The Blagovestchenskii function

$$\beta(s,t) := (u^{f_-}(\cdot,s),u^g(\cdot,t))_{\mathcal{H}}, \quad (s,t)\in[0,2T]\times[0,T]$$

satisfies the relations

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}\right)\beta(s,t) &= \int_\Omega \mathrm{d}x\,\rho\,[u^{f_-}(x,s)\cdot u^g_{tt}(x,t)-u^{f_-}_{tt}(x,s)\cdot u^g(x,t)]\\ &= \int_\Omega \mathrm{d}x\,[u^{f_-}(x,s)\cdot Lu^g(x,t)-Lu^{f_-}(x,s)\cdot u^g(x,t)]\\ &= \langle \text{see}\,(2.15)\rangle\\ &= \int_\Gamma \mathrm{d}\Gamma\,[Du^{f_-}(\gamma,s)\cdot Nu^g(\gamma,t)-Nu^{f_-}(\gamma,s)\cdot Du^g(\gamma,t)]\\ &= \int_\Gamma \mathrm{d}\Gamma\,[f_-(\gamma,s)\cdot(R^T g)(\gamma,t)-(R^{2T}f_-)(\gamma,s)\cdot g(\gamma,t)] \end{aligned} \tag{5.16}$$

and the conditions

$$\beta\,(s,0)=\beta_t\,(s,0)=0,\quad 0\le s\le 2T. \tag{5.17}$$

Integrating the wave equation (5.16) by the D'Alembert formula, taking into account (5.17), and putting $s=t=T$ one gets

$$\begin{aligned}\beta\,(T,T) &= \frac{1}{2}\int_0^T \mathrm{d}\eta \int_\eta^{2T-\eta} \mathrm{d}\xi \int_\Gamma \mathrm{d}\Gamma\,[f_-(\gamma,\xi)\cdot(R^T g)(\gamma,\eta)\\ &\quad -(R^{2T}f_-)(\gamma,\xi)\cdot g(\gamma,\eta)] = \langle\text{by oddness of } f_-\rangle\\ &= -\frac{1}{2}\int_{\Gamma\times[0,T]} \mathrm{d}\Gamma\,\mathrm{d}\eta \left[\int_\eta^{2T-\eta}\mathrm{d}\xi (R^{2T}f_-)(\gamma,\xi)\right]\cdot g(\gamma,\eta)\\ &= \left(\frac{1}{2}(\mathcal{S}^T)^* I^{2T} R^{2T}\mathcal{S}^T f, g\right)_{\mathcal{F}^T}. \end{aligned} \tag{5.18}$$

On the other hand, the definition of β gives

$$\begin{aligned}\beta\,(T,T) &= (u^{f_-}(\cdot,T),u^g(\cdot,T))_{\mathcal{H}} = (u^f(\cdot,T),u^g(\cdot,T))_{\mathcal{H}}\\ &= \langle\text{see}\,(5.14)\rangle = (C^T f,g)_{\mathcal{F}^T}. \end{aligned}\tag{5.19}$$

Comparing (5.18) with (5.19) one obtains (5.15). □

So, possessing R^{2T} one can recover C^T on smooth controls and then extend C^T onto $\mathcal{F}^T$ by continuity.

The connecting operator enters the polar decomposition (see *e.g.* [8]) of the control operator:

$$W^T = E^T\left[\left(W^T\right)^* W^T\right]^{\frac{1}{2}} = E^T (C^T)^{\frac{1}{2}}$$

where $E^T:\mathcal{F}^T\to\mathcal{H}$ is an isometry mapping $\operatorname{Ran}(C^T)^{\frac{1}{2}}$ onto $\operatorname{Ran}W^T=\mathcal{U}^T$.

In the case of controls acting from $\sigma\subset\Gamma$, define the operator $C^{\sigma,T}:\mathcal{F}^{\sigma,T}\to\mathcal{F}^{\sigma,T}$,

$$C^{\sigma,T} := \left(W^{\sigma,T}\right)^* W^{\sigma,T}$$

and recall that the "partial" response operator $R^{\sigma,2T}$ was introduced in Section 1.4. The representation

$$C^{\sigma,T} = \frac{1}{2}\left(\mathcal{S}^T\right)^* I^{2T} R^{\sigma,2T}\mathcal{S}^T \tag{5.20}$$

easily follows from (5.15) and the definitions.

For the control operator $W^{\sigma,T}$ one has the decomposition

$$W^{\sigma,T} = E^{\sigma,T}\left(C^{\sigma,T}\right)^{\frac{1}{2}} \tag{5.21}$$

with an isometry $E^{\sigma,T}:\mathcal{F}^{\sigma,T}\to\mathcal{H}$ mapping $\operatorname{Ran}\left(C^{\sigma,T}\right)^{\frac{1}{2}}$ onto $\operatorname{Ran}W^{\sigma,T}=\mathcal{U}^{\sigma,T}$. The adjoint operator $\left(W^{\sigma,T}\right)^*:\mathcal{H}\to\mathcal{F}^{\sigma,T}$ takes the form

$$\left(W^{\sigma,T}\right)^* = \left(C^{\sigma,T}\right)^{\frac{1}{2}}\left(E^{\sigma,T}\right)^*. \tag{5.22}$$

5.4. Operator $\tilde{L}_u^{\sigma,T}$

Recall that the Lame operator on waves $L_u^{\sigma,T}$ was introduced in Section 4.4. As is easy to see, the set of pairs

$$\{\{W^{\sigma,T}f, W^{\sigma,T}f_{tt}\} \mid f \in \mathcal{M}^{\sigma,T}\} \subset \mathcal{H}\times\mathcal{H} \tag{5.23}$$

forms its graph.

Introduce the operator $\tilde{L}_u^{\sigma,T} : \mathcal{F}^{\sigma,T} \to \mathcal{F}^{\sigma,T}$, $\operatorname{Dom}\tilde{L}_u^{\sigma,T} = (C^{\sigma,T})^{\frac{1}{2}}\mathcal{M}^{\sigma,T}$,

$$\tilde{L}_u^{\sigma,T} := (E^{\sigma,T})^* L_u^{\sigma,T} E^{\sigma,T};$$

where by (5.21–5.23) its graph is

$$\{\{\,(C^{\sigma,T})^{\frac{1}{2}}f, (C^{\sigma,T})^{\frac{1}{2}}f_{tt}\,\} \mid f \in \mathcal{M}^{\sigma,T}\} \subset \mathcal{F}^{\sigma,T}\times\mathcal{F}^{\sigma,T}. \tag{5.24}$$

In the case of $T > T_s^{\sigma}$, by virtue of the controllability (3.11) one has

$$\operatorname{clos}_{\mathcal{H}}\operatorname{Ran}E^{\sigma,T} = \operatorname{clos}_{\mathcal{H}}\mathcal{U}^{\sigma,T} = \mathcal{H}$$

so that $E^{\sigma,T}$ may be extended up to isometry from $\mathcal{F}^{\sigma,T}$ *onto* $\mathcal{H}$. The relation (4.7) and the definition of $\tilde{L}_u^{\sigma,T}$ easily lead to the equality

$$L_0 = E^{\sigma,T}\,\bar{\tilde{L}}_u^{\sigma,T}\,(E^{\sigma,T})^*. \tag{5.25}$$

5.5. Completion of the proof of Theorem 4

Let $\sigma \subseteq \Gamma$ be a given open subset. The corresponding "partial" response operator of system (5.8–5.10) $R^{\sigma,\infty} : \mathcal{F}^{\sigma,\infty} \to \mathcal{F}^{\sigma,\infty}$,

$$R^{\sigma,\infty}f = Nu^f\,|_{\Sigma^{\sigma,\infty}}$$

is defined on smooth controls supported on $\Sigma^{\sigma,\infty}$ and vanishing near $\Gamma\times\{t=0\}$. Considering $R^{\sigma,\infty}$ as a reduction of R^{∞} one can easily obtain the direct analog of representation (5.13):

$$R^{\sigma,\infty}f = \begin{cases} R^{\sigma,2T}f & \text{on } \Sigma_0^{\sigma,T}\cup\Sigma_1^{\sigma,T};\\ \mathcal{T}_jY^T\{P_j^{\sigma,T}+Q_j^{\sigma,T}\}f & \text{on } \Sigma_j^{\sigma,T},\ j=2,3,... \end{cases} \tag{5.26}$$

where

$$P_j^{\sigma,T} := \frac{\partial}{\partial t}\left(W^{\sigma,T}\right)^* C((j-1)T)W^{\sigma,T} - \left(W^{\sigma,T}\right)^* C_t((j-1)T)W^{\sigma,T};$$
$$Q_j^{\sigma,T} := \left[\frac{\partial}{\partial t}\left(W^{\sigma,T}\right)^* S((j-1)T)W^{\sigma,T} - \left(W^{\sigma,T}\right)^* S_t((j-1)T)W^{\sigma,T}\right]\frac{\partial}{\partial t}.$$

Fix $T > T_s^{\sigma}$. Introduce the operator-valued functions

$$\tilde{C}(t) := \cos t\left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{\frac{1}{2}};\ \ \tilde{S}(t) := \left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{-\frac{1}{2}}\sin t\left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{\frac{1}{2}};$$
$$\tilde{C}_t(t) := -\left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{\frac{1}{2}}\sin t\left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{\frac{1}{2}};\ \ \tilde{S}_t(t) := \cos t\left(-\bar{\tilde{L}}_u^{\sigma,T}\right)^{\frac{1}{2}}, \tag{5.27}$$

and

$$\tilde{P}_j^{\sigma,T} := \frac{\partial}{\partial t}\left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{C}\left((j-1)\,T\right)\left(C^{\sigma,T}\right)^{\frac{1}{2}} - \left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{C}_t\left((j-1)\,T\right)\left(C^{\sigma,T}\right)^{\frac{1}{2}};$$
$$\tilde{Q}_j^{\sigma,T} := \left[\frac{\partial}{\partial t}\left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{S}\left((j-1)\,T\right)\left(C^{\sigma,T}\right)^{\frac{1}{2}} - \left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{S}_t\left((j-1)\,T\right)\left(C^{\sigma,T}\right)^{\frac{1}{2}}\right]\frac{\partial}{\partial t}. \tag{5.28}$$

Under the assumption $T > T_s^\sigma$ we have the equalities

$$\begin{aligned}\left(W^{\sigma,T}\right)^* C(t) W^{\sigma,T} &= \langle \text{see } (5.22)\rangle = \left(C^{\sigma,T}\right)^{\frac{1}{2}}\left(E^{\sigma,T}\right)^* \cos t(-L_0)^{\frac{1}{2}} E^{\sigma,T}\left(C^{\sigma,T}\right)^{\frac{1}{2}}\\ &= \left(C^{\sigma,T}\right)^{\frac{1}{2}} \cos t\left[-\left(E^{\sigma,T}\right)^* L_0 E^{\sigma,T}\right]^{\frac{1}{2}}\left(C^{\sigma,T}\right)^{\frac{1}{2}}\\ &= \langle \text{see } (5.25)\rangle = \left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{C}(t)\left(C^{\sigma,T}\right)^{\frac{1}{2}};\end{aligned}$$

and, analogously,

$$\begin{aligned}\left(W^{\sigma,T}\right)^* C_t(t) W^{\sigma,T} &= \left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{C}_t(t)\left(C^{\sigma,T}\right)^{\frac{1}{2}};\\ (W^{\sigma,T})^* S(t) W^{\sigma,T} &= (C^{\sigma,T})^{\frac{1}{2}} \tilde{S}(t) (C^{\sigma,T})^{\frac{1}{2}};\\ \left(W^{\sigma,T}\right)^* S_t(t) W^{\sigma,T} &= \left(C^{\sigma,T}\right)^{\frac{1}{2}} \tilde{S}_t(t)\left(C^{\sigma,T}\right)^{\frac{1}{2}},\end{aligned} \tag{5.29}$$

implying

$$P_j^{\sigma,T} = \tilde{P}_j^{\sigma,T}; \quad Q_j^{\sigma,T} := \tilde{Q}_j^{\sigma,T};$$

and, finally, with regard to (5.26), leading to the required representation:

$$R^{\sigma,\infty} f = \begin{cases} R^{\sigma,2T} f & \text{on } \Sigma_0^{\sigma,T} \cup \Sigma_1^{\sigma,T};\\ \mathcal{T}_j Y^T\{\tilde{P}_j^{\sigma,T} + \tilde{Q}_j^{\sigma,T}\} f & \text{on } \Sigma_j^{\sigma,T}, \ \ j = 2, 3, \ldots \end{cases} \tag{5.30}$$

The operator $R^{\sigma,\infty}$ is determined by the operator $R^{\sigma,2T}$. Indeed, if the latter is given, one can

(i) find $C^{\sigma,T}$ by (5.20);

(ii) determine the operator $\tilde{L}_u^{\sigma,T}$ through its graph (5.24) and find its closure $\bar{\bar{\tilde{L}}}{}_0^{\sigma,T}$; find the operator-functions (5.27), and then (5.28);

(iii) for control f with $\operatorname{supp} f \subset \sigma \times (0, T)$ recover $R^{\sigma,\infty} f$ by the representation (5.30).

In the case of arbitrary $\operatorname{supp} f$, by means of the trick described at the end of Section 5.2, the reconstruction of $R^{\sigma,\infty} f$ may be reduced to the same procedure (i–iii).

Recalling (5.11) we conclude that $R^{\sigma,2T}$, $T > T_s^\sigma$ determines $R^{\sigma,T}$ with any $T > 0$ that proves the theorem.

5.6. Comments

- The continuation $R^{2T} \to R^\infty$ goes back to the classical problem of the extension of the Hermitian positive functions [12]. The positivity of the connecting operator $C^T = (W^T)^* W^T$ holding in our case plays the role of a natural analog of the positivity by Krein–Bochner in the case of scalar functions.
- The procedure of continuation (i–iii) in fact repeats the scheme of the paper [5] which, in its turn, follows the approach using the models of dynamical systems [3, 6].
- The use of the boundary data continuation is a well known device in inverse problems (see, *e.g.* [10]). In his talk at the conference (IMA, Minneapolis, Minnesota; July 2001) Isakov announced an approach to the problem of recovering the parameters of the Lame system based upon the continuation $R^{2T} \to R^\infty$ described above.

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