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WELL-POSEDNESS FOR SYSTEMS REPRESENTING ELECTROMAGNETIC/ACOUSTIC WAVEFRONT INTERACTION

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Abstract. In this paper we consider dispersive electromagnetic systems in dielectric materials in the presence of acoustic wavefronts. A theory for existence, uniqueness, and continuous dependence on data is presented for a general class of systems which include acoustic pressure-dependent Debye polarization models for dielectric materials.

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1. Introduction and problem formulation

Electromagnetic interrogation techniques have numerous useful applications, including locating mines or bunkers beneath the ground and detecting abnormal tissue noninvasively within the body. (See [1, 4, 11, 12], and [13] for some examples of the applications of electromagnetic interrogation.) In one such class of electromagnetic interrogation techniques, one uses a superconductive (also referred to as supraconductive or perfectly conductive) backing, such as metal, as a reflector for an oncoming electromagnetic wave. These electromagnetic wave reflections are then used to identify dielectric properties (conductivity and polarization) of the target materials. However, there are many applications (for example finding a nonmetal object underground or detecting a brain tumor) in which this is impossible or even harmful. In these cases, it would be beneficial for a traveling acoustic wave, perhaps even one occurring naturally, to serve as a virtual interface. In [4] the authors describe models and applications for techniques which employ superconductive metal backings and *standing* acoustic waves as reflectors for the electromagnetic waves. In addition, they propose a configuration in which a *traveling* acoustic wave might be used as a virtual interface to reflect an oncoming electromagnetic wave.

Electromagnetic interrogation techniques such as these may be modeled, under certain assumptions, by the one dimensional Maxwell's equations with temporally and spatially varying coefficients. Here we present a general model that describes, as a special case, the interaction between the electromagnetic wave and a traveling acoustic wave, formulated as pressure-dependent Debye polarization. We note that the models derived in [4] to describe the cases where perfectly conductive backings or standing acoustic waves are used as reflectors are also special (simpler) cases of this model.

The main purpose of this paper is to provide a general theoretical foundation for a class of electromagnetic/acoustic interaction problems. Specifically, in Section 2 we consider the well-posedness of a general variational form of the model. We consider the equation in the domain $0 \le z \le 1$ and assume that the boundary

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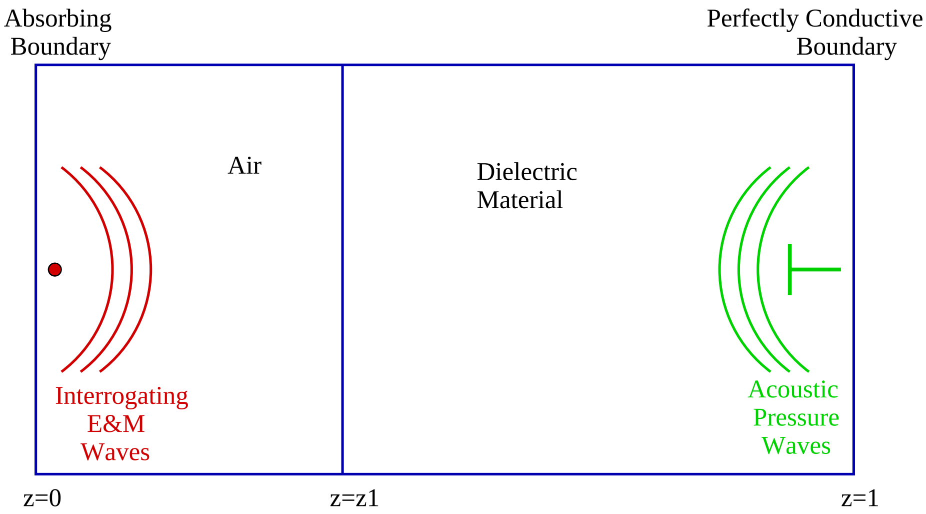

FIGURE 1. Schematic of geometry.

conditions are absorbing on the left ($z = 0$) and superconductive on the right ($z = 1$). We use general initial conditions for the electric field, but without loss of generality we assume that the polarization, present in the dielectric material region $(z_1, 1]$ with $0 < z_1 < 1$, and its first time derivative are initially zero. The geometry of the problem is shown in Figure 1. We shall show in Section 3 that we may use the same general form of the equation for the pressure-dependent polarization terms based on the Debye models. Finally, in Section 4 we present some sample numerical simulations for the system incorporating Debye polarization.

We first motivate the general variational form of the model. We note that under certain assumptions, including material homogeneity in directions perpendicular to the direction of electromagnetic wave propagation and the use of a polarized planar electromagnetic wave, Maxwell's equations can be written

$$\frac{\partial}{\partial z}E = -\mu_0 \frac{\partial}{\partial t}H \tag{1}$$

$$\frac{\partial}{\partial z}H = \frac{\partial}{\partial t}D + \sigma E + J_s \tag{2}$$

where E and H are the electric and magnetic fields, D is the electric flux density, and J_s is the source current density. We relate the electric flux density to the electric field by

$$D = \epsilon_0 \epsilon_r(t, z)E + P \tag{3}$$

where $\epsilon_r(t, z)$ may be used to represent effects of instantaneous polarization and P is the macroscopic polarization. We then take a spatial derivative of (1) and a time derivative of (2) and use equation (3) to obtain

$$\mu_0 a(t, z)\ddot{E} + \mu_0 \ddot{P} + \mu_0 \sigma \dot{E} - E'' = -\mu_0 \dot{J}_s. \tag{4}$$

Here and throughout we use $\dot{E}$ to denote $\frac{\partial}{\partial t}E$ and E' to denote $\frac{\partial}{\partial z}E$. To write (4) in variational form, we begin by formulating the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$, where $H = L^2(0,1)$ and $V = H^1_R(0,1) \equiv \{\phi \in H^1(0,1) : \phi(1) = 0\}$. We let $\langle \cdot, \cdot \rangle$ denote the usual L^2 inner product. Then (4) may be written as a variational form of Maxwell's equation in second order form for a general polarization term

$$\left\langle a\ddot{E}, \phi \right\rangle_{V^*,V} + \left\langle b\dot{E}, \phi \right\rangle + \left\langle e\ddot{P}, \phi \right\rangle + c\dot{E}(t,0)\phi(0) + \sigma_1(E, \phi) = \langle F, \phi \rangle_{V^*,V} \tag{5}$$

for all $\phi \in V$. (This is the same equation given in [4], Chap. 2.) Here the sesquilinear form σ_1 is defined by

$$\sigma_1(\phi,\psi) = c^2 \langle \phi', \psi' \rangle , \tag{6}$$

where $c^2 = \frac{1}{\epsilon_0\mu_0}$ is a positive constant and the parameter functions a, b, and e depend on geometry as well as conductivity and the instantaneous polarization of the dielectric medium. The absorbing boundary condition $\dot{E} - cE' = 0$ at $z = 0$ is a natural condition and is thus incorporated into this variational formulation of the equation, but the superconductive boundary condition at $z = 1$ is an essential boundary condition and is imposed in the definition of V.

We note that σ_1 is V-continuous and V-elliptic, *i.e.*, there exist positive constants c_1, c_2 such that

$$\sigma_1(\phi,\psi) = c^2 \langle \phi', \psi' \rangle \;\le c^2|\phi'|_H|\psi'|_H \le c_1|\phi|_V|\psi|_V \tag{7}$$

$$\sigma_1(\phi,\phi) = c^2 \langle \phi', \phi' \rangle = c^2|\phi'|_H^2 \ge c_2|\phi|_V^2, \tag{8}$$

since $|\phi|_V^2$ is equivalent to $|\phi'|_H^2 + |\phi(1)|^2 = |\phi'|_H^2$.

The model (5) is a very general Maxwell system that can be used with numerous polarization models. As an example, we show that (5) can be specialized to include a Debye polarization model with pressure-dependent coefficients. The pressure-dependent Debye polarization model we consider (see [5] and [2] for physics-based discussions) is given by

$$\dot{P} = -\frac{1}{(\tau_0 + \kappa_\tau p)}P + \frac{\epsilon_0\left(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p\right)}{(\tau_0 + \kappa_\tau p)}E \tag{9}$$

where $\tau = \tau(p) = \tau_0 + \kappa_\tau p$ is the pressure-dependent decay parameter, $\epsilon_s = \gamma(p) = \gamma_0 + \kappa_\gamma p$ and $\epsilon_\infty = \zeta(p) = \zeta_0 + \kappa_\zeta p$ are pressure-dependent dielectric parameters, and $p = p(t,z)$ is the acoustic pressure in the Debye material.

The solution to (9), for $P(0,z) = 0$, can be written

$$P(t,z) = \int_0^t \exp\left(\int_s^t \frac{-\mathrm{d}\xi}{\tau_0 + \kappa_\tau p(\xi,z)}\right) \frac{\epsilon_0\left(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(s,z)\right)}{(\tau_0 + \kappa_\tau p(s,z))} E(s,z)\;\mathrm{d}s.$$

We may use (9), its derivative, and its solution to replace $\ddot{P}$ in (5). These substitutions lead to the following variational form of the system

$$\left\langle a\ddot{E},\phi\right\rangle_{V^*,V} + \left\langle b\dot{E},\phi\right\rangle + \langle hE,\phi\rangle + \left\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\;\mathrm{d}s, \phi\right\rangle + c\dot{E}(t,0)\phi(0) + \sigma_1(E,\phi) = \langle F,\phi\rangle_{V^*,V}$$
$$E(0,z) = E_0(z) \quad \dot{E}(0,z) = E_1(z) \tag{10}$$

where $E_0 \in V$ and $E_1 \in H$ with coefficients, kernel and forcing functions, and sesquilinear form defined by

$$\begin{aligned}
a(t,z) &= 1 + (\epsilon_\infty - 1) I_{(z_1,1)} = 1 + (\zeta_0 + \kappa_\zeta p(t,z) - 1) I_{(z_1,1)} \\
b(t,z) &= \left(\frac{\sigma}{\epsilon_0} + \frac{1}{\epsilon_0} \frac{\epsilon_0 \left(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta) p(t,z)\right)}{(\tau_0 + \kappa_\tau p(t,z))} \right) I_{(z_1,1)} \\
h(t,z) &= \frac{1}{\epsilon_0} \left(\frac{\epsilon_0 (\kappa_\gamma - \kappa_\zeta) \dot{p}(t,z)}{(\tau_0 + \kappa_\tau p(t,z))} - \frac{(1 + \kappa_\tau \dot{p}(t,z))\, \epsilon_0 \left(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta) p(t,z)\right)}{(\tau_0 + \kappa_\tau p(t,z))^2} \right) I_{(z_1,1)} \\
G(t,s,z) &= \frac{1}{\epsilon_0} \frac{(1 + \kappa_\tau \dot{p}(t,z))\, \epsilon_0 \left(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta) p(s,z)\right)}{(\tau_0 + \kappa_\tau p(t,z))^2 (\tau_0 + \kappa_\tau p(s,z))} \exp\left(\int_s^t \frac{-\mathrm{d}\xi}{\tau_0 + \kappa_\tau p(\xi,z)} \right) I_{(z_1,1)} \qquad (11) \\
c^2 &= \frac{1}{\epsilon_0 \mu_0} \\
F(t,z) &= -\frac{1}{\epsilon_0} \dot{J}_s(t) \\
\sigma_1(\phi,\psi) &= c^2 \left\langle \phi', \psi' \right\rangle \cdot
\end{aligned}$$

(Here I_Ω is the indicator or characteristic function for a set Ω.)

As we shall see, under appropriate assumptions on the coefficients, kernel, and forcing function in (11), we can give general arguments that establish the well-posedness of the Debye-based system, as well as any other system that satisfies the general assumptions listed in the next section for the generalized system (12) below. We remark that pressure-dependent Lorentz-based models as well as those based on more general polarization models can be shown to be special cases of (12).

2. Well-posedness of solutions to the general variational form

Motivated by the Debye example and the wide range of applications mentioned in Section 1, we consider the general variational form

$$\left\langle a\ddot{E}, \phi \right\rangle_{V^*,V} + \left\langle b\dot{E}, \phi \right\rangle + \langle hE, \phi \rangle + \left\langle \int_0^t G(t,s,\cdot) E(s,\cdot)\, \mathrm{d}s, \phi \right\rangle + c\dot{E}(t,0)\phi(0) + c^2 \left\langle E', \phi' \right\rangle = \langle F, \phi \rangle_{V^*,V}, \quad \phi \in V$$

$$E(0,z) = E_0(z) \quad \dot{E}(0,z) = E_1(z), \qquad (12)$$

where $E_0 \in V$ and $E_1 \in H$. As introduced previously, we take $H = L^2(0,1)$ and $V = H^1_R(0,1) \equiv \{\phi \in H^1(0,1) : \phi(1) = 0\}$ which, with V^*, are Hilbert spaces that form a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. Specifically, we note that there is a value $k > 0$ such that for all $\phi \in V$, we have

$$|\phi|_H \le k |\phi|_V.$$

The usual duality product is denoted by $\langle \cdot, \cdot \rangle_{V^*,V}$; it is the extension by continuity of the H inner product from $H \times V$ to $V^* \times V$. Both inner products $\langle \cdot, \cdot \rangle$ and norms $|\cdot|$ denoted without subscripts are assumed to be in H. In addition, motivated by (11), we make the following assumptions:

A1) the coefficient a along with its derivatives $\dot{a}$ and $\ddot{a}$ are in $L^\infty(0,T;L^\infty[0,1])$, and for all $z \in [0,1]$, $a(z) \ge a_0$, for some $1 \ge a_0 > 0$;

A2) the coefficient b and its time derivative $\dot{b}$ are in $L^\infty(0,T;L^\infty[0,1])$ and $b(t,z) \ge 0$ for all $(t,z) \in [0,T] \times [0,1]$;

A3) the coefficient h is in $L^\infty(0,T;L^\infty[0,1])$;

A4) the kernel function G is in $L^\infty([0,T]\times[0,T];L^\infty[0,1])$;
A5) the sequilinear form σ_1 is given by $\sigma_1(\phi,\psi)=c^2\langle\phi',\psi'\rangle$ for $\phi,\psi\in V$ with $c>0$;
A6) the forcing function F is in $H^1(0,T,V^*)$.

We recall that the sesquilinear form $\sigma_1 : V\times V\to\mathbb{R}$ is V-continuous and V-elliptic, so that (7) and (8) are satisfied.

Under the above hypotheses, we seek solutions $t\to E(t)$ where $E(t)\in V$ and (12) is satisfied in the $L^2(0,T)$ sense for all $\phi\in V$. We begin by showing that such solutions exist.

To this end, we follow the arguments in [4,9]. We choose a linearly independent subset $\{w_i\}_{i=0}^\infty$ that spans V which is dense in H. We let $V^m\equiv\text{span}\{w_0,w_1,\dots,w_m\}$ and choose $E_{0m},E_{1m}\in V^m$ such that as $m\to\infty$, $E_{0m}\to E_0$ in V, and $E_{1m}\to E_1$ in H. Then we let $E_m=\sum_{i=0}^m\eta_i(t)w_i(z)$ be the unique solution on $0<t<T$ to the integrodifferential equation system (for existence we will use Ths. 1 and 2 of [3] combined with the arguments below)

$$\begin{aligned}\left\langle a\ddot{E}_m(t),w_j\right\rangle_{V^*,V}+\left\langle b\dot{E}_m(t),w_j\right\rangle+\langle hE_m(t),w_j\rangle+\left\langle\int_0^t G(t,s,\cdot)E_m(s,\cdot)\,\mathrm{d}s,w_j\right\rangle+c\dot{E}_m(t,0)w_j(0)\\+\sigma_1(E_m(t),w_j)=\langle F(t),w_j\rangle_{V^*,V}\\E_m(0)=E_{0m}\quad\dot{E}_m(0)=E_{1m}\end{aligned}\tag{13}$$

where $j=0,1,\dots,m$.

We note here that for any $i=0,1,\dots,m$, $w_i\in V=H^1_R(0,1)$, and thus w_i is absolutely continuous. From this we also have that $w_iw_j\in C(0,1)$. Thus, all products involving E_m and its time derivatives are spatially continuous functions on the interval $[0,1]$. Moreover, inner product terms containing coefficients in L^∞, *e.g.* $\left\langle b(t,\cdot)\dot{E}_m(t,\cdot),w_j\right\rangle$, are well-defined.

Proposition 1. *Under assumptions A1)–A6), the system (13) with $E_{0m},E_{1m}\in V^m$ has a unique solution with absolutely continuous $(E_m,\dot{E}_m)$.*

Proof. We may write (13) in the form

$$M_1(t)\ddot{\eta}(t)+M_2(t)\dot{\eta}(t)+M_3(t)\eta(t)+G_1(t,\eta(\cdot))=D_1(t)$$

where

$$\begin{aligned}\eta(t)&=[\eta_0(t)\ \eta_1(t)\ \dots\ \eta_m(t)]^T\\ [G_1(t,\eta(\cdot))]_j&=\sum_{i=1}^m\left\langle\int_0^t G(t,s,\cdot)\eta_i(s)\,\mathrm{d}s\ w_i,w_j\right\rangle,\\ [D_1(t)]_j&=\langle F(t),w_j\rangle\\ [M_1(t)]_{ij}&=\langle a(t)w_j,w_i\rangle\\ [M_2(t)]_{ij}&=\langle b(t)w_j,w_i\rangle+cw_i(0)w_j(0)\\ [M_3(t)]_{ij}&=\langle h(t)w_j,w_i\rangle+c^2\left\langle w_i',w_j'\right\rangle\cdot\end{aligned}$$

Since the w_i are linearly-independent and a satisfies the lower bound of A1), $M_1(t)$ is positive definite for each t, hence invertible. Then the above linear system may be written

$$\begin{aligned}\dot{\mathcal{Y}}(t)&=\mathcal{M}(t)\left(\mathcal{A}(t)\mathcal{Y}(t)+\mathcal{G}(t,\mathcal{Y}(\cdot))+\mathcal{D}(t)\right),\\&=\mathcal{F}(t,\mathcal{Y}(\cdot))+\mathcal{M}(t)\mathcal{D}(t)\end{aligned}\tag{14}$$

with

$$\mathcal{Y}(t) = \begin{bmatrix} \eta(t) \\ \dot{\eta}(t) \end{bmatrix}$$

$$\mathcal{M}(t) = \begin{bmatrix} I & 0 \\ 0 & M_1(t) \end{bmatrix}^{-1}$$

$$\mathcal{A}(t) = \begin{bmatrix} 0 & I \\ -M_3(t) & -M_2(t) \end{bmatrix}$$

$$[\mathcal{G}(t,\mathcal{Y}(\cdot))]_j = \begin{cases} 0, \quad j = 0,\dots,m \\ -\left[G(t,\eta(\cdot))\right]_j, \quad j = m+1,\dots,2m+1 \end{cases}$$

$$[\mathcal{D}(t)]_j = \begin{cases} 0, \quad j = 0,\dots,m \\ \left[D_1(t)\right]_j, \quad j = m+1,\dots,2m+1. \end{cases}$$

We point out that the notation $\mathcal{F}(t,\mathcal{Y}(\cdot))$ implies that for each $t \in [0,T]$, $\mathcal{F}$ depends on t and on the past history of $\mathcal{Y}$ in an interval $[0,t]$. We now want to argue that (14) does, in fact, have a unique solution that is absolutely continuous in t in $[0,T]$. Theorem 2 in [3] states that a unique solution exists for a system in the form (14) which satisfies the following conditions:

Y1) for a fixed $\mathcal{Y}$, $\mathcal{F}$ is measurable in t;
Y2) for almost every fixed $t \in [0,T]$, $\mathcal{F}$ is continuous in $\mathcal{Y}$;
Y3) there is an $L^1(0,T)$ function m_F such that

$$|\mathcal{F}(t,\mathcal{Y}(\cdot))| \le m_F(t) \sup_{s\in[0,T]} |\mathcal{Y}(s)| \qquad (t,\mathcal{Y}) \in [0,T]\times C[0,T]; \tag{15}$$

Y4) there is an $L^1(0,T)$ function k such that

$$\begin{gathered} |\mathcal{F}(t,\mathcal{Y}(\cdot)) - \mathcal{F}(t,\mathcal{X}(\cdot))| \le k(t) \sup_{s\in[0,T]} |\mathcal{Y}(s) - \mathcal{X}(s)| \\ \text{for } (t,\mathcal{Y}),(t,\mathcal{X}) \in [0,T]\times C[0,T]; \end{gathered} \tag{16}$$

Y5) The function $\mathcal{D}$ is in $L^1(0,T)$.

We now show that Y1)–Y5) are indeed satisfied for our system. Since the components of $\mathcal{M},\mathcal{A}$ and $\mathcal{G}(\cdot,\mathcal{Y}(\cdot))$ are in $L^\infty(0,T)$, we have that $\mathcal{F}(\cdot,\mathcal{Y}(\cdot))$ is measurable in t for a fixed $\mathcal{Y}$ and Y1) holds.

In order to verify Y2), we must show that both $\mathcal{M}(t)\mathcal{A}(t)\mathcal{Y}$ and $\mathcal{M}(t)\mathcal{G}(t,\mathcal{Y})$ are continuous in $\mathcal{Y}$. It is clear that this is true for $\mathcal{M}(t)\mathcal{A}(t)\mathcal{Y}$ and the continuity of $\mathcal{G}(t,\mathcal{Y})$ follows readily from A4) – see [6] for detailed arguments.

We next observe that there is an $L^1(0,T)$ function m_F such that (15) and, hence Y3), holds.

We first note that

$$|\mathcal{F}(t,\mathcal{Y}(\cdot))| \le |\mathcal{M}(t)||\mathcal{A}(t)| \sup_{s\in[0,T]} |\mathcal{Y}(s)| + |\mathcal{M}(t)||\mathcal{G}(t,\mathcal{Y}(\cdot))|.$$

Since the components of $\mathcal{M}$ and $\mathcal{A}$ are in $L^\infty(0,T)$ and hence

$$|\mathcal{M}| \le \bar{M} \quad \text{and} \quad |\mathcal{A}(t)| \le \bar{A}$$

for any $t \in [0,T]$, we need only show that there is a function $m_G \in L^1(0,T)$ such that

$$|\mathcal{G}(t,\mathcal{Y}(\cdot))| \le m_G(t) \sup_{s\in[0,T]} |\mathcal{Y}(s)|.$$

For this, one once again uses A4) – again see [6] for details.

In order to verify Y4), we note that the mapping $\mathcal{Y} \to \mathcal{F}(t, \mathcal{Y}(\cdot))$ is linear. Then the verification of Y4) follows immediately from Y3).

In verifying Y5), we need only note that the components of $\mathcal{D}$ are in $H^1(0,T)$.

Having argued that assumptions Y1)-Y5) hold, we may use Theorem 2 in [3] to conclude that (14) and hence (13) has a unique, absolutely continuous solution $(E_m, \dot{E}_m)$ for which we next derive *a priori* bounds.

Proposition 2. *There exists a subsequence* $\{E_{m_k}\}$ *and a limit function* E *satisfying* $E \in L^2(0,T;V)$, $\dot{E} \in L^2(0,T;H)$, *and* $\dot{E}(\cdot,0) \in L^2(0,T)$.

Proof. We multiply (13) by $\dot{\eta}_j(t)$ and sum over j to obtain

$$\begin{gathered}
\left\langle a\ddot{E}_m, \dot{E}_m \right\rangle_{V^*,V} + \left\langle b\dot{E}_m, \dot{E}_m \right\rangle + \left\langle hE_m, \dot{E}_m \right\rangle + \left\langle \int_0^t G(t,s,\cdot)E_m(s,\cdot)\ \mathrm{d}s, \dot{E}_m \right\rangle \\
+c\dot{E}_m(t,0)\dot{E}_m(t,0) + \sigma_1(E_m, \dot{E}_m) = \left\langle F, \dot{E}_m \right\rangle_{V^*,V} \\
E_m(0) = E_{0m} \quad \dot{E}_m(0) = E_{1m}.
\end{gathered} \tag{17}$$

We note that

$$2\left\langle a\ddot{E}_m, \dot{E}_m \right\rangle_{V^*,V} = \frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{a}\dot{E}_m|_H^2 - \left\langle \dot{a}\dot{E}_m, \dot{E}_m \right\rangle$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(E_m, E_m) = 2\sigma_1(E_m, \dot{E}_m),$$

so that (17) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(|\sqrt{a}\dot{E}_m|_H^2 + \sigma_1(E_m,E_m)\right) &+ \left\langle (2b-\dot{a})\dot{E}_m, \dot{E}_m \right\rangle + 2\left\langle hE_m, \dot{E}_m \right\rangle \\
&+2\left\langle \int_0^t GE_m(s,\cdot)\ \mathrm{d}s, \dot{E}_m \right\rangle + 2c|\dot{E}_m(t,0)|^2 = 2\left\langle F, \dot{E}_m \right\rangle_{V^*,V}.
\end{aligned}$$

Then

$$\begin{gathered}
|\sqrt{a}\dot{E}_m(t)|_H^2 + \sigma_1(E_m(t),E_m(t)) + \int_0^t \left\langle (2b-\dot{a})\dot{E}_m, \dot{E}_m \right\rangle\ \mathrm{d}\xi + \int_0^t 2\left\langle hE_m, \dot{E}_m \right\rangle\ \mathrm{d}\xi \\
+ \int_0^t 2\left\langle \int_0^\xi GE_m(s,\cdot)\ \mathrm{d}s, \dot{E}_m(\xi,\cdot) \right\rangle\ \mathrm{d}\xi + \int_0^t 2c|\dot{E}_m(\xi,0)|^2\ \mathrm{d}\xi \\
= |\sqrt{a}(0)\dot{E}_m(0)|_H^2 + \sigma_1(E_m(0),E_m(0)) + \int_0^t 2\left\langle F, \dot{E}_m \right\rangle_{V^*,V}\ \mathrm{d}\xi.
\end{gathered} \tag{18}$$

Using the V-continuity and V-ellipticity of σ_1 and the fact that $2ab \le a^2 + b^2$, we have

$$|\sqrt{a}\dot{E}_m(t)|_H^2 + c_2|E_m(t)|_V^2 + \int_0^t 2c|\dot{E}_m(\xi,0)|^2 \, \mathrm{d}\xi \le \int_0^t 2\left\langle -hE_m, \dot{E}_m \right\rangle \, \mathrm{d}\xi + \int_0^t \left\langle (\dot{a}-2b)\dot{E}_m, \dot{E}_m \right\rangle \, \mathrm{d}\xi$$

$$+ \int_0^t 2\left\langle -\int_0^\xi GE_m(s,\cdot)\,\mathrm{d}s, \dot{E}_m(\xi,\cdot) \right\rangle \, \mathrm{d}\xi + |\sqrt{a}\dot{E}_m(0)|_H^2 + \sigma_1(E_m(0),E_m(0)) + \int_0^t 2\left\langle F, \dot{E}_m \right\rangle_{V^*,V} \, \mathrm{d}\xi$$

$$\le \int_0^t \left\{ |hE_m|_H^2 + |\dot{E}_m|_H^2 \right\} \mathrm{d}\xi + \int_0^t \left\{ \frac{1}{2}|\dot{E}_m|_H^2 + \frac{1}{2}|(\dot{a}-2b)\dot{E}_m|_H^2 \right\} \mathrm{d}\xi + \int_0^t \left\{ \left| \int_0^\xi GE_m(s,\cdot)\,\mathrm{d}s \right|_H^2 + |\dot{E}_m|_H^2 \right\} \mathrm{d}\xi$$

$$+ |\sqrt{a}(0)\dot{E}_m(0)|_H^2 + c_1|E_m(0)|_V^2 + \left| \int_0^t 2\left\langle F, \dot{E}_m \right\rangle_{V^*,V} \, \mathrm{d}\xi \right|.$$

For $F \in H^1(0,T;V^*)$, we find

$$\left| \int_0^t 2\left\langle F, \dot{E}_m \right\rangle_{V^*,V} \, \mathrm{d}\xi \right| = \left| \int_0^t \left(2\frac{\mathrm{d}}{\mathrm{d}\xi}\left\langle F, E_m \right\rangle_{V^*,V} - 2\left\langle \dot{F}, E_m \right\rangle_{V^*,V} \right) \mathrm{d}\xi \right|$$

$$= \left| 2\left\langle F(t), E_m(t) \right\rangle_{V^*,V} - 2\left\langle F(0), E_m(0) \right\rangle_{V^*,V} - \int_0^t 2\left\langle \dot{F}, E_m \right\rangle_{V^*,V} \, \mathrm{d}\xi \right|$$

$$\le \frac{1}{\epsilon}|F(t)|_{V*}^2 + \epsilon|E_m(t)|_V^2 + |F(0)|_{V*}^2 + |E_m(0)|_V^2 + \int_0^t \left\{ |\dot{F}|_{V*}^2 + |E_m|_V^2 \right\} \mathrm{d}\xi.$$

Thus, from (18) we find

$$|\sqrt{a}\dot{E}_m(t)|_H^2 + c_2|E_m(t)|_V^2 + \int_0^t 2c|\dot{E}_m(\xi,0)|^2 \, \mathrm{d}\xi \le \int_0^t \left\{ |hE_m|_H^2 + |\dot{E}_m|_H^2 \right\} \mathrm{d}\xi$$

$$+ \int_0^t \left\{ \frac{1}{2}|\dot{E}_m|_H^2 + \frac{1}{2}|(\dot{a}-2b)\dot{E}_m|_H^2 \right\} \mathrm{d}\xi + \int_0^t \left\{ \left| \int_0^\xi GE_m(s,\cdot)\,\mathrm{d}s \right|_H^2 + |\dot{E}_m|_H^2 \right\} \mathrm{d}\xi$$

$$+ |\sqrt{a}(0)\dot{E}_m(0)|_H^2 + c_1|E_m(0)|_V^2 + \frac{1}{\epsilon}|F(t)|_{V*}^2 + \epsilon|E_m(t)|_V^2 + |F(0)|_{V*}^2 + |E_m(0)|_V^2$$

$$+ \int_0^t \left\{ |\dot{F}|_{V*}^2 + |E_m|_V^2 \right\} \mathrm{d}\xi.$$

Using the fact that $E_m(0) = E_{0m}$ and $\dot{E}_m(0) = E_{1m}$ and combining like terms, we have finally

$$|\sqrt{a}\dot{E}_m(t)|_H^2 + (c_2 - \epsilon)|E_m(t)|_V^2 + \int_0^t 2c|\dot{E}_m(\xi,0)|^2 \, \mathrm{d}\xi$$

$$\le \int_0^t \left\{ |hE_m|_H^2 + \frac{5}{2}|\dot{E}_m|_H^2 + \frac{1}{2}|(\dot{a}-2b)\dot{E}_m|_H^2 + \left| \int_0^\xi GE_m(s,\cdot)\,\mathrm{d}s \right|_H^2 + |\dot{F}|_{V*}^2 + |E_m|_V^2 \right\} \mathrm{d}\xi$$

$$+ |\sqrt{a}(0)E_{1m}|_H^2 + (c_1+1)|E_{0m}|_V^2 + \frac{1}{\epsilon}|F(t)|_{V*}^2 + |F(0)|_{V*}^2.$$

We next use the assumptions on the coefficients and kernel function to establish some bounds.

Hypothesis A1) implies that there exists $\bar{a} > 1$ such that

$$|\sqrt{a(0,\cdot)}E_{1m}(\cdot)|_H^2 \leq \bar{a}|E_{1m}(\cdot)|_H^2.$$

Moreover, there exists $a_0 > 0$ such that

$$a_0|\dot{E}_m(t,\cdot)|_H^2 \leq |\sqrt{a(\cdot)}\dot{E}_m(t,\cdot)|_H^2.$$

Hypotheses A1) and A2) allow us to show that there exists $\bar{b} > 0$ such that

$$\int_0^t \frac{1}{2}|(\dot{a}(\xi,\cdot) - 2b(\xi,\cdot))\dot{E}_m(\xi,\cdot)|_H^2 \ \mathrm{d}\xi \leq \bar{b}\int_0^t |\dot{E}_m(\xi,\cdot)|_H^2 \ \mathrm{d}\xi.$$

We use hypothesis A3) to claim that there exists $\bar{h} \geq 0$ such that

$$\int_0^t |h(\xi,\cdot)E_m(\xi,\cdot)|_H^2 \ \mathrm{d}\xi \leq \int_0^t\int_0^1 |h(\xi,z)|^2|E_m(\xi,z)|^2 \ \mathrm{d}z \ \mathrm{d}\xi \leq \bar{h}\int_0^t |E_m(\xi,\cdot)|_H^2 \ \mathrm{d}\xi.$$

Using A4), we have that there exists $\bar{G} \geq 0$ such that

$$\begin{aligned}\int_0^t \left|\int_0^\xi G(\xi,s,\cdot)E_m(s,\cdot) \ \mathrm{d}s\right|_H^2 \ \mathrm{d}\xi &= |G|_{L^\infty}^2 \int_0^t\int_0^1 \left(\int_0^\xi |E_m(s,z)| \ \mathrm{d}s\right)^2 \ \mathrm{d}z \ \mathrm{d}\xi \\ &\leq |G|_{L^\infty}^2 \int_0^t\int_0^1 \left(T^{\frac{1}{2}}|E_m(z)|_{L^2(0,\xi)}\right)^2 \ \mathrm{d}z \ \mathrm{d}\xi \\ &\leq T|G|_{L^\infty}^2 \int_0^t\int_0^1\int_0^t |E_m(s,z)|^2 \ \mathrm{d}s \ \mathrm{d}z \ \mathrm{d}\xi = T^2\bar{G}\int_0^t |E_m(\xi,\cdot)|_H^2 \ \mathrm{d}\xi.\end{aligned}$$

Using these bounds, we have

$$\begin{aligned}&a_0|\dot{E}_m(t)|_H^2 + (c_2-\epsilon)|E_m(t)|_V^2 + 2c\int_0^t |\dot{E}_m(\xi,0)|^2 \ \mathrm{d}\xi \\ &\leq \int_0^t \left\{(\bar{h}+T^2\bar{G})|E_m|_H^2 + \left(\frac{5}{2}+\bar{b}\right)|\dot{E}_m|_H^2 + |\dot{F}|_{V*}^2 + |E_m|_V^2\right\} \ \mathrm{d}\xi \\ &+\bar{a}|E_{1m}|_H^2 + (c_1+1)|E_{0m}|_V^2 + \frac{1}{\epsilon}|F(t)|_{V*}^2 + |F(0)|_{V*}^2.\end{aligned}$$

Letting

$$H(t) = \bar{a}|E_{1m}|_H^2 + (c_1+1)|E_{0m}|_V^2 + \frac{1}{\epsilon}|F(t)|_{V*}^2 + |F(0)|_{V*}^2 + \int_0^t |\dot{F}|_{V*}^2 \ \mathrm{d}\xi,$$

and recalling that

$$|E_m|_V^2 \geq |E_m|_H^2,$$

we find

$$a_0|\dot{E}_m(t)|_H^2 + (c_2-\epsilon)|E_m(t)|_V^2 + 2c\int_0^t |\dot{E}_m(\xi,0)|^2 \mathrm{d}\xi \leq H(t) + \int_0^t \left\{(1+\bar{h}+T^2\bar{G})|E_m|_V^2 + \left(\frac{5}{2}+\bar{b}\right)|\dot{E}_m|_H^2\right\} \ \mathrm{d}\xi.$$

We note that $\left(1+\bar{h}+T^2\bar{G}\right) \geq 1$. Moreover, we can choose ϵ such that $0 < c_2 - \epsilon \leq 1$, and

$$\frac{1+\bar{h}+T^2\bar{G}}{c_2-\epsilon} \geq 1.$$

Similarly, $\frac{5}{2}+\bar{b} > 1$ and $a_0 \leq 1$ so that

$$\frac{\frac{5}{2}+\bar{b}}{a_0} > 1.$$

Then,

$$\begin{aligned} a_0|\dot{E}_m(t)|_H^2 + (c_2-\epsilon)|E_m(t)|_V^2 + 2c\int_0^t |\dot{E}_m(\xi,0)|^2 \mathrm{d}\xi \leq H(t) + \int_0^t \left(1+\bar{h}+T^2\bar{G}\right)|E_m|_V^2 + (\frac{5}{2}+\bar{b})|\dot{E}_m|_H^2 \ \mathrm{d}\xi \\ \leq H(t) + \int_0^t \frac{\frac{5}{2}+\bar{b}}{a_0}\left(1+\bar{h}+T^2\bar{G}\right)|E_m|_V^2 + \left(\frac{1+\bar{h}+T^2\bar{G}}{c_2-\epsilon}\right)\left(\frac{5}{2}+\bar{b}\right)|\dot{E}_m|_H^2 \ \mathrm{d}\xi \\ \leq H(t) + \int_0^t \frac{(\frac{5}{2}+\bar{b})\left(1+\bar{h}+T^2\bar{G}\right)}{a_0\left(c_2-\epsilon\right)}\left((c_2-\epsilon)|E_m|_V^2 + a_0|\dot{E}_m|_H^2\right) \ \mathrm{d}\xi. \end{aligned} \tag{19}$$

We recall that the convergence of E_{0m} in V and the convergence of E_{1m} in H imply the boundedness of the sequences in their respective spaces. This, along with A6), yields that $H(t)$ is bounded. Hence we can use Gronwall's inequality to show that $\{E_m\}$ is bounded in $C(0,T;V)$ and $\{\dot{E}_m\}$ is bounded in $C(0,T;H)$. We can thus conclude that $\{\dot{E}_m(\cdot,0)\}$ is bounded in $L^2(0,T)$. It follows that there exist a subsequence $\{E_{m_k}\}$ and limits $E \in L^2(0,T;V)$, $\tilde{E} \in L^2(0,T;H)$, and $E_L \in L^2(0,T)$ such that

$$\begin{aligned} E_{m_k} &\to E \text{ weakly in } L^2(0,T;V) \\ \dot{E}_{m_k} &\to \tilde{E} \text{ weakly in } L^2(0,T;H) \\ \dot{E}_{m_k}(\cdot,0) &\to E_L \text{ weakly in } L^2(0,T). \end{aligned}$$

Since $E_{m_k} \in C(0,T;V)$ and $\dot{E}_{m_k} \in C(0,T;V)$, we have

$$E_{m_k}(t) - E_{m_k}(0) - \int_0^t \dot{E}_{m_k}(\xi) \ \mathrm{d}\xi = 0$$

in the V norm for all $t \in [0,T)$, and we note that this, of course, holds in the H norm as well.

We also have

$$E_{m_k}(0) = E_{0m_k} \to E_0$$

in the V sense, and

$$\int_0^t \dot{E}_{m_k}(\xi) \ \mathrm{d}\xi \to \int_0^t \tilde{E}(\xi) \ \mathrm{d}\xi$$

weakly in H for each $t \in [0,T)$.

We take weak limits in H to obtain

$$E(t) = E_0 + \int_0^t \tilde{E}(\xi) \ \mathrm{d}\xi$$

$$E(t,0) = E_0(0) + \int_0^t E_L(\xi) \ \mathrm{d}\xi$$

in the H sense. Thus $\dot{E}(t)$ exists almost everywhere in H with $\dot{E} = \tilde{E} \in L^2(0,T;H)$, while $E(0) = E_0$ and $\dot{E}(t,0) = E_L(t)$ almost everywhere.

Proposition 3. *The limit function E of the previous proposition is a solution of the equation (12).*

Proof. We let $\psi \in C^1(0,T)$ with $\psi(T) = 0$ be arbitrary and let $\psi_j = \psi(t)w_j$ where the $\{w_i\}_{i=0}^{\infty}$ are selected as before. For a fixed j, we have

$$\int_0^T \left\{ \left\langle a\ddot{E}_m, \psi_j \right\rangle_{V^*,V} + \left\langle b\dot{E}_m, \psi_j \right\rangle + \langle hE, \psi_j \rangle + \left\langle \int_0^t G(t,s,\cdot)E_m(s,\cdot)\ \mathrm{d}s, \psi_j \right\rangle \right.$$
$$\left. + c\dot{E}_m(t,0)\psi_j(0) + \sigma_1(E_m, \psi_j) \right\}\ \mathrm{d}t = \int_0^T \langle F, \psi_j \rangle_{V^*,V}\ \mathrm{d}t.$$

Then we integrate by parts in the first term to obtain

$$\int_0^T \left\{ - \left\langle \dot{E}_m, a\dot{\psi}_j + \dot{a}\psi_j \right\rangle_{V^*,V} + \left\langle b\dot{E}_m, \psi_j \right\rangle + \langle hE_m, \psi_j \rangle + \left\langle \int_0^t G(t,s,\cdot)E_m(s,\cdot)\ \mathrm{d}s, \psi_j \right\rangle \right.$$
$$\left. + c\dot{E}_m(t,0)\psi_j(0) + \sigma_1(E_m, \psi_j) \right\}\ \mathrm{d}t = \int_0^T \langle F, \psi_j \rangle_{V^*,V}\ \mathrm{d}t + \langle a(0)E_{1m}, \psi_j(0) \rangle$$

for each ψ_j.

We would like to be able to take weak limits as $m \to \infty$ in the previous equation, but first we must verify that this is possible, particularly in the integral term. We know that $E_m \to E$ weakly in $L^2(0,T;V)$ and $\psi \in C^1(0,T)$. Then for a function $g \in L^\infty(0,T;L^\infty(0,1))$ and any function $w \in V$, we have $g(s,\cdot)w(\cdot) \in H \subset V$ and $g(\cdot,z) \in L^2(0,T)$. So we may conclude that for each t

$$\int_0^T \langle g(s,\cdot)E_m(s), \psi(t)w \rangle\ \mathrm{d}s \to \int_0^T \langle g(s,\cdot)E(s), \psi(t)w \rangle\ \mathrm{d}s.$$

We next use $G \in L^\infty([0,T] \times [0,T]; L^\infty[0,1])$. Since $E_m \to E$ weakly in $L^2(0,T;V)$, we have that for any $t \in [0,T]$

$$\int_0^T \left\langle G(t,s,\cdot)I_{(0,t)}(s)E_m(s), \psi(t)w \right\rangle\ \mathrm{d}s \to \int_0^T \left\langle G(t,s,\cdot)I_{(0,t)}(s)E(s), \psi(t)w \right\rangle\ \mathrm{d}s.$$

This implies

$$\begin{aligned} \left\langle \int_0^t G(t,s,\cdot)E_m(s)\ \mathrm{d}s, \psi(t)w \right\rangle &= \int_0^t \langle G(t,s,\cdot)E_m(s), \psi(t)w \rangle\ \mathrm{d}s \\ &= \int_0^T \left\langle G(t,s,\cdot)I_{(0,t)}(s)E_m(s), \psi(t)w \right\rangle\ \mathrm{d}s \\ &\to \int_0^T \left\langle G(t,s,\cdot)I_{(0,t)}(s)E(s), \psi(t)w \right\rangle\ \mathrm{d}s \\ &= \left\langle \int_0^t G(t,s,\cdot)E(s)\ \mathrm{d}s, \psi(t)w \right\rangle \end{aligned}$$

for each $t \in [0,T]$ and thus by boundedness we have convergence in $L^1(0,T)$.

This convergence, as well as the fact that $\sigma_1(\cdot,\psi_j(t))\in V^*$, show that we are indeed able to take weak limits in the previous equation. As $m\to\infty$, we have for each j

$$\int_0^T \Big\{ -\Big\langle \dot{E}, a\dot{\psi}_j + \dot{a}\psi_j \Big\rangle_{V^*,V} + \Big\langle b\dot{E},\psi_j\Big\rangle + \langle hE,\psi_j\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,\psi_j\Big\rangle + c\dot{E}_m(t,0)\psi_j(0) + \sigma_1(E,\psi_j)\Big\}\,\mathrm{d}t = \int_0^T \langle F,\psi_j\rangle_{V^*,V}\,\mathrm{d}t + \langle a(0)E_1,\psi_j(0)\rangle\cdot \quad (20)$$

We restrict ψ to lie in $C_0^\infty(0,T)$ and write

$$\int_0^T -\dot{\psi}\Big\langle a\dot{E},w_j\Big\rangle_{V^*,V} - \psi\Big\langle \dot{a}\dot{E},w_j\Big\rangle \mathrm{d}t + \int_0^T \Big\{\Big\langle b\dot{E},w_j\Big\rangle + \langle hE,w_j\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,w_j\Big\rangle + c\dot{E}(t,0)w_j(0) + \sigma_1(E,w_j)\Big\}\,\psi\,\mathrm{d}t = \int_0^T \langle F,w_j\rangle_{V^*,V}\,\psi\,\mathrm{d}t$$

for each w_j.

Then we can interpret the first term in the sense of distributions as follows

$$\int_0^T \psi\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle a\dot{E},w_j\Big\rangle\,\mathrm{d}t + \int_0^T \Big\{-\Big\langle \dot{a}\dot{E},w_j\Big\rangle + \Big\langle b\dot{E},w_j\Big\rangle + \langle hE,w_j\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,w_j\Big\rangle + c\dot{E}(t,0)w_j(0) + \sigma_1(E,w_j)\Big\}\,\psi\,\mathrm{d}t = \int_0^T \langle F,w_j\rangle_{V^*,V}\,\psi\,\mathrm{d}t$$

for each w_j.

Thus for each j, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle a\dot{E},w_j\Big\rangle - \Big\langle \dot{a}\dot{E},w_j\Big\rangle + \Big\langle b\dot{E},w_j\Big\rangle + \langle hE,w_j\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,w_j\Big\rangle + c\dot{E}(t,0)w_j(0) + \sigma_1(E,w_j) = \langle F,w_j\rangle_{V^*,V} \quad (21)$$

holds in the $L^2(0,T)$ sense.

Since $\{w_j\}$ is total in V, this implies that $\ddot{E}\in L^2(0,T;V^*)$. Furthermore, upon observing that

$$\Big\langle a\ddot{E},\phi\Big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle a\dot{E},\phi\Big\rangle - \Big\langle \dot{a}\dot{E},\phi\Big\rangle,$$

we have for all $\phi\in V$

$$\Big\langle a\ddot{E},\phi\Big\rangle_{V^*,V} + \Big\langle b\dot{E},\phi\Big\rangle + \langle hE,\phi\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,\phi\Big\rangle + c\dot{E}(t,0)\phi(0) + \sigma_1(E,\phi) = \langle F,\phi\rangle_{V^*,V}$$

which is our original equation (12).

Now we have that $E(0,z)=E_0(z)$, but we need to show $\dot{E}(0,z)=E_1(z)$. We recall that (20) holds for all $\psi_j=\psi w_j$ with $\psi\in C^1(0,T)$ and $\psi(T)=0$. Then if we integrate by parts in the first term, we have

$$\int_0^T \Big\langle a\ddot{E},\psi_j\Big\rangle\,\mathrm{d}t - \Big\langle a\dot{E},\psi_j\Big\rangle\Big|_{t=0}^{t=T} + \int_0^T \Big\{\Big\langle b\dot{E},\psi_j\Big\rangle + \langle hE,\psi_j\rangle + \Big\langle \int_0^t G(t,s,\cdot)E(s,\cdot)\,\mathrm{d}s,\psi_j\Big\rangle + c\dot{E}(t,0)\phi_j(0) + \sigma_1(E,\psi_j)\Big\}\,\mathrm{d}t = \int_0^T \langle F,\psi_j\rangle_{V^*,V}\,\mathrm{d}t + \langle a(0)E_1,\psi_j(0)\rangle\cdot$$

Recalling (21), we can thus conclude that

$$-\left\langle a\dot{E},\psi_j\right\rangle|_{t=0}^{t=T}=\langle a(0)E_1,\psi_j(0)\rangle\,,$$

or, since $a(z)\geq a_0>0$ and $\psi_j(T)=0$,

$$\begin{aligned}\left\langle \dot{E}(0),\psi_j(0)\right\rangle &= \langle E_1,\psi_j(0)\rangle \text{ or}\\ \left\langle \dot{E}(0),w_j\right\rangle\psi(0) &= \langle E_1,w_j(0)\rangle\,\psi(0).\end{aligned}$$

Since this holds for all j and $\psi(0)$ is arbitrary, we have $\dot{E}(0)=E_1$, and E is in fact a solution of the system (12).

Proposition 4. *The solution to (2) established above is unique.*

Proof. It suffices to show that $E=0$ is the only solution that corresponds to the zero initial conditions $E_0=E_1=0$ and zero forcing function $F=0$. We begin by assuming E is a solution corresponding to zero initial data and zero forcing function. For all $\phi\in V$, this solution E satisfies

$$\left\langle a\ddot{E},\phi\right\rangle_{V^*,V}+\left\langle b\dot{E},\phi\right\rangle+\langle hE,\phi\rangle+\left\langle\int_0^t G(t,s,\cdot)E(s,\cdot)\;\mathrm{d}s,\phi\right\rangle+c\dot{E}(t,0)\phi(0)+\sigma_1(E,\phi)=0.$$

We define $\psi_s(t)$ for $t,s\in[0,T]$ by

$$\psi_s(t)=\begin{cases}-\int_t^s E(\xi)\;\mathrm{d}\xi, & t<s\\ 0, & t\geq s\end{cases}$$

and note that $\dot{\psi}_s(t)=E(t)$ and $\psi_s(T)=\psi_s(s)=0$. Since $\psi_s(t)\in V$, we can choose $\phi=\psi_s(t)$ to obtain

$$\left\langle a\ddot{E},\psi_s\right\rangle_{V^*,V}+\left\langle b\dot{E},\psi_s\right\rangle+\langle hE,\psi_s\rangle+\left\langle\int_0^t G(t,\xi,\cdot)E(\xi,\cdot)\;\mathrm{d}\xi,\psi_s\right\rangle+c\dot{E}(t,0)\psi_s(t)(0)+\sigma_1(E,\psi_s)=0. \quad (22)$$

Integrating this equation and considering some of the terms separately, we find

$$\begin{aligned}2\int_0^s\left\langle a\ddot{E},\psi_s\right\rangle\;\mathrm{d}t &= -2\textstyle\int_0^s\left(\left\langle a\dot{E},E\right\rangle+\left\langle \dot{a}\dot{E},\psi_s\right\rangle\right)\;\mathrm{d}t\\ &= \int_0^s\left(-\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{a}E|_H^2+\langle\dot{a}E,E\rangle-2\left\langle\dot{a}\dot{E},\psi_s\right\rangle\right)\;\mathrm{d}t\\ &= -|\sqrt{a}(s)E(s)|_H^2+\int_0^s\left(\langle\dot{a}E,E\rangle+2\left\langle E,\frac{\mathrm{d}}{\mathrm{d}t}(\dot{a}\psi_s)\right\rangle\right)\;\mathrm{d}t\\ &= -|\sqrt{a}(s)E(s)|_H^2+\int_0^s\left(3\left\langle\dot{a}E,E\right\rangle+2\left\langle\ddot{a}E,\psi_s\right\rangle\right)\;\mathrm{d}t,\end{aligned}$$

$$2\int_0^s\sigma_1(E,\psi_s)\;\mathrm{d}t=\int_0^s\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(\psi_s,\psi_s)\;\mathrm{d}t=-\sigma_1(\psi_s(0),\psi_s(0)),$$

$$\int_0^s\left(\dot{E}(t,0)\psi_s(t)(0)+|E(t,0)|^2\right)\;\mathrm{d}t=\int_0^s\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t,0)\psi_s(t)(0)\right)=0,$$

and

$$\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \langle bE, \psi_s \rangle \ \mathrm{d}t = \int_0^s \left(\left\langle b\dot{E}, \psi_s \right\rangle + \left\langle \dot{b}E, \psi_s \right\rangle + \langle bE, E \rangle \right) \ \mathrm{d}t$$
$$= \int_0^s \left(\left\langle b\dot{E}, \psi_s \right\rangle + \left\langle \dot{b}E, \psi_s \right\rangle + |\sqrt{b}E|_H^2 \right) \ \mathrm{d}t$$
$$= 0.$$

Using these relationships, we obtain from the integrated form of (22)

$$|\sqrt{a}E(s)|_H^2 + \sigma_1(\psi_s(0), \psi_s(0)) + \int_0^s 2c|E(t,0)|^2 \ \mathrm{d}t = \int_0^s \Big(3 \langle \dot{a}E, E \rangle + 2 \langle \ddot{a}E, \psi_s \rangle + 2 \left\langle -\dot{b}E, \psi_s \right\rangle$$
$$-2|\sqrt{b}E|_H^2 + 2 \langle hE, \psi_s \rangle + 2 \left\langle \int_0^t G(t,\xi,\cdot)E(\xi,\cdot) \ \mathrm{d}\xi, \psi_s \right\rangle \Big) \ \mathrm{d}t. \quad (23)$$

We may next use some of the previous assumptions on our coefficients to make the following estimates.
We note that

$$\int_0^s 3 \langle \dot{a}E, E \rangle \ \mathrm{d}t \leq 3|\dot{a}|_{L^\infty} \int_0^s |E|_H^2 \ \mathrm{d}t$$

and

$$\int_0^s 2 \langle \ddot{a}E, \psi_s \rangle \ \mathrm{d}t \leq |\ddot{a}|_{L^\infty}^2 \int_0^s |E|_H^2 \ \mathrm{d}t + \int_0^s |\psi_s|_H^2 \ \mathrm{d}t.$$

Thus from hypothesis A1), we have that there exists an $\alpha > 0$ such that for $s < T$

$$\int_0^s 3 \langle \dot{a}E, E \rangle + 2 \langle \ddot{a}E, \psi_s \rangle \ \mathrm{d}t \leq \int_0^s \left\{ \alpha|E|_H^2 + |\psi_s|_H^2 \right\} \ \mathrm{d}t.$$

Moreover, we note that

$$\int_0^s -2|\sqrt{b(t,\cdot)}E(t,\cdot)|_H^2 \ \mathrm{d}t \leq 2|b|_{L^\infty} \int_0^s |E(t,\cdot)|_H^2 \ \mathrm{d}t$$

and

$$\int_0^s 2 \left\langle -\dot{b}E, \psi_s \right\rangle \ \mathrm{d}t \leq |\dot{b}|_{L^\infty}^2 \int_0^s |E(t,\cdot)|_H^2 \ \mathrm{d}t + \int_0^s |\psi_s|_H^2 \ \mathrm{d}t.$$

Then, as a consequence of A2), there exists a $\beta > 0$ such that for $s \leq T$ we have

$$\int_0^s \left\{ -2|\sqrt{b(t,\cdot)}E(t,\cdot)|_H^2 + 2 \left\langle -\dot{b}E, \psi_s \right\rangle \right\} \ \mathrm{d}t \leq \int_0^s \beta|E(t,\cdot)|_H^2 + |\psi_s|_H^2 \ \mathrm{d}t.$$

We next substitute these bounds, as well as some of those established previously, into (23) to obtain the inequality

$$|\sqrt{a}E(s)|_H^2 + \sigma_1(\psi_s(0), \psi_s(0)) \leq \int_0^s \left\{ (\alpha + \beta + \bar{h} + T^2\bar{G})|E|_H^2 + 4|\psi_s|_H^2 \right\} \ \mathrm{d}t.$$

Furthermore, we note that

$$\begin{aligned}\int_0^s |\psi_s|_H^2 \, \mathrm{d}t &= \int_0^s \int_0^1 \left(\int_t^s E(\xi, z) \, \mathrm{d}\xi \right)^2 \, \mathrm{d}z \, \mathrm{d}t \\ &\le \int_0^s \int_0^1 \left(T^{\frac{1}{2}} |E(\cdot, z)|_{L^2(t,s)} \right)^2 \, \mathrm{d}z \, \mathrm{d}t \\ &\le T \int_0^s \int_0^1 \int_0^s |E(\xi, z)|^2 \, \mathrm{d}\xi \, \mathrm{d}z \, \mathrm{d}t \\ &\le T^2 \int_0^s |E(t)|_H^2 \, \mathrm{d}t.\end{aligned}$$

Then, we have

$$|\sqrt{a} E(s)|_H^2 + \sigma_1(\psi_s(0), \psi_s(0)) \le \int_0^s (\alpha + \beta + \bar{h} + T^2 \bar{G} + 4T^2) |E(t)|_H^2 \, \mathrm{d}t,$$

from which it follows that

$$a_0 |E(s)|_H^2 \le \int_0^s (\alpha + \beta + \bar{h} + T^2 \bar{G} + 4T^2) |E(t)|_H^2 \, \mathrm{d}t.$$

Finally, using Gronwall's inequality, we have $|E(s)|_H^2 = 0$ for all $s \in [0, T]$.

This establishes uniqueness of solutions.

Proposition 5. *The solution of (12) depends continuously on the initial conditions and forcing function.*

Proof. To begin, we let

$$H_m = \bar{a} |E_{1m}|_H^2 + (c_1 + 1) |E_{0m}|_V^2 + \frac{1}{\epsilon} |F|_{L^\infty(0,T;V^*)}^2 + |F(0)|_{V^*}^2 + |\dot{F}|_{L^2(0,T;V^*)}^2.$$

Then from (19) we have

$$a_0 |\dot{E}_m(t)|_H^2 + (c_2 - \epsilon) |E_m(t)|_V^2 \le H_m + \frac{(\frac{5}{2} + \bar{b}) (1 + \bar{h} + T^2 \bar{G})}{a_0 (c_2 - \epsilon)} \int_0^t \left((c_2 - \epsilon) |E_m|_V^2 + a_0 |\dot{E}_m|_H^2 \right) \, \mathrm{d}\xi.$$

By use of Gronwall's inequality, we have

$$a_0 |\dot{E}_m(t)|_H^2 + (c_2 - \epsilon) |E_m(t)|_V^2 \le H_m \exp \left(\frac{(\frac{5}{2} + \bar{b}) (1 + \bar{h} + T^2 \bar{G})}{a_0 (c_2 - \epsilon)} T \right) = H_m K_1.$$

If we integrate over the interval $(0, T)$, we have

$$a_0 |\dot{E}_m|_{L^2(0,T;H)}^2 + (c_2 - \epsilon) |E_m|_{L^2(0,T;V)}^2 \le H_m K_2.$$

We next use the fact that as $m \to \infty$, $E_{0m} \to E_0$ and $E_{1m} \to E_1$, the weak convergences established previously, and the weak lower semicontinuity of norms to conclude that

$$a_0 |\dot{E}|_{L^2(0,T;H)}^2 + (c_2 - \epsilon) |E|_{L^2(0,T;V)}^2 \le \left(\bar{a} |E_1|_H^2 + (c_1 + 1) |E_0|_V^2 + \left(1 + \frac{1}{\epsilon} \right) |F|_{L^\infty(0,T;V^*)}^2 + |\dot{F}|_{L^2(0,T;V^*)}^2 \right) K_2.$$

Since the mapping $(E_0, E_1, F, \dot{F}) \to (E, \dot{E})$ is linear, we thus have continuous dependence on the initial data and forcing function. Summarizing, we have proven the result:

Theorem. Under assumptions A1)–A6), the system (12) possesses a unique solution and $(E, \dot{E})$ depends continuously on initial data (E_0, E_1) and forcing function F from $(E_0, E_1, F) \in V \times H \times H^1(0, T; V^*)$ to $(E, \dot{E}) \in L^2(0, T; V) \times L^2(0, T; H)$.

3. Well-posedness of solutions to the system with pressure-dependent Debye polarization

In this section we apply the results of Section 2 to establish the well-posedness of the Debye polarization model with pressure-dependent coefficients. We consider (10) and (11) using the definitions of V, H, V^* and σ_1 given in Section 1. We recall that V,H, and V^* form a Gelfand triple as described in Section 2. Moreover, we note that σ_1 as defined in (6) is V-continuous and V-elliptic. The following discussion establishes the validity of hypotheses A1)–A6).

We first outline some assumptions about our parameters and pressure wave.

- P1) The pressure wave p in (9) is in $H^1(0, T; V)$ so that p is in the space $C(0, T; C[0, 1])$ and hence in $L^\infty(0, T; L^\infty[0, 1])$.
- P2) The derivatives of the pressure wave, $\dot{p}$ and $\ddot{p}$, are in $L^\infty(0, T; L^\infty[0, 1])$. (See [8] and [7] for details regarding the regularity of p.)
- P3) The parameters $\kappa_\gamma \geq 0$, $\kappa_\zeta \geq 0$, and $\kappa_\tau \geq 0$ are chosen such that there exists a value $\delta > 0$ such that $\gamma_0 + \kappa_\gamma p(t, z)$, $\zeta_0 + \kappa_\zeta p(t, z)$, and $\tau_0 + \kappa_\tau p(t, z)$ are greater than δ for all $z \in [0, 1]$ and $t \in [0, T]$.

We use these assumptions to verify that A1)–A6) hold for the Debye example.

- A1) The coefficient a is in $L^\infty(0, T; L^\infty[0, 1])$, as are its derivatives $\dot{a}$ and $\ddot{a}$, and for all $(t, z) \in [0, T] \times [0, 1]$, $a(t, z) \geq a_0$, for some $a_0 > 0$.

Proof. Recall that

$$a(t, z) = 1 + (\epsilon_\infty - 1)I_{(z_1,1)} = 1 + (\zeta_0 + \kappa_\zeta p(t, z) - 1)I_{(z_1,1)}.$$

We note that p, $\dot{p}$, and $\ddot{p}$ are all assumed to be $L^\infty(0, T; L^\infty[0, 1])$ functions. From the form of a, we therefore have a, $\dot{a}$, and $\ddot{a}$ in $L^\infty(0, T; L^\infty[0, 1])$.

We also have for $(t, z) \in [0, T] \times [0, z_1)$, $a(t, z) = 1 > 0$. Moreover, for $(t, z) \in [0, T] \times [z_1, 1]$, $a(t, z) = \zeta_0 + \kappa_\zeta p(t, z) > \delta > 0$.

- A2) The coefficient b is in $L^\infty(0, T; L^\infty[0, 1])$ and $b(t, z) \geq 0$ for all $(t, z) \in [0, T] \times [0, 1]$. Additionally, the time derivative of b, $\dot{b}$, exists and is in $L^\infty(0, T; L^\infty[0, 1])$.

Proof. Again recall

$$b(t, z) = \left(\frac{\sigma}{\epsilon_0} + \frac{(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(t, z))}{(\tau_0 + \kappa_\tau p(t, z))} \right) I_{(z_1,1)}.$$

We recall the restrictions placed on $\kappa_\gamma, \kappa_\zeta$, and κ_τ and conclude that b is strictly positive in $[0, T] \times (z_1, 1]$. Thus, $b(t, z) \geq 0$ for all $(t, z) \in [0, T] \times [0, 1]$.

Since $p \in L^\infty(0, T; L^\infty[0, 1])$, we have that $\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p \in L^\infty(0, T; L^\infty[0, 1])$. Moreover, since $\tau_0 + \kappa_\tau p > \delta$ for all $(t, z) \in [0, T] \times [0, 1]$, we know $(\tau_0 + \kappa_\tau p)^{-1} \in L^\infty(0, T; L^\infty[0, 1])$.

Hence b is bounded, and $b \in L^\infty(0, T; L^\infty[0, 1])$.

Next, we have

$$\dot{b}(t, z) = \left(\frac{((\kappa_\gamma - \kappa_\zeta)\dot{p}(t, z))\,(\tau_0 + \kappa_\tau p(t, z)) - (\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(t, z))\,(\kappa_\tau \dot{p}(t, z))}{(\tau_0 + \kappa_\tau p(t, z))^2} \right) I_{(z_1,1)}.$$

Since $(\tau_0 + \kappa_\tau p)^{-2}$, $\tau_0 + \kappa_\tau p, \gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p, (\kappa_\gamma - \kappa_\zeta)\dot{p}$, and $\kappa_\tau \dot{p}$ are all in $L^\infty(0,T;L^\infty[0,1])$, we can conclude that $\dot{b} \in L^\infty(0,T;L^\infty[0,1])$.

A3) The coefficient h is in $L^\infty(0,T;L^\infty[0,1])$.

Proof. From the definition

$$h(t,z) = \left(\frac{(\kappa_\gamma - \kappa_\zeta)\dot{p}(t,z)}{(\tau_0 + \kappa_\tau p(t,z))} - \frac{(1 + \kappa_\tau \dot{p}(t,z))\,(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(t,z))}{(\tau_0 + \kappa_\tau p(t,z))^2} \right) I_{(z_1,1)}.$$

Since $(\tau_0 + \kappa_\tau p)^{-1}$ and $(\kappa_\gamma - \kappa_\zeta)\dot{p}$ are in $L^\infty(0,T;L^\infty[0,1])$, we have $(\kappa_\gamma - \kappa_\zeta)\dot{p}\,(\tau_0 + \kappa_\tau p)^{-1}$ is in $L^\infty(0,T;L^\infty[0,1])$.

In the same way, since $\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p$ and $\kappa_\tau \dot{p}$ are in $L^\infty(0,T;L^\infty[0,1])$, and $(\tau_0 + \kappa_\tau p)^{-2} < \delta^{-2}$, we have $(1 + \kappa_\tau \dot{p})\,(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p)\,(\tau_0 + \kappa_\tau p)^{-2}$ is in $L^\infty(0,T;L^\infty[0,1])$. Therefore, we have h in $L^\infty(0,T;L^\infty[0,1])$.

A4) The kernel function G is in $L^\infty([0,T] \times [0,T];L^\infty[0,1])$.

Proof. The Debye kernel is given by

$$G(t,s,z) = \frac{(1 + \kappa_\tau \dot{p}(t,z))\,(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(s,z))}{(\tau_0 + \kappa_\tau p(t,z))^2\,(\tau_0 + \kappa_\tau p(s,z))} \exp\left(\int_s^t \frac{-\mathrm{d}\xi}{\tau_0 + \kappa_\tau p(\xi,z)} \right) I_{(z_1,1)}.$$

First, we note that

$$\frac{(1 + \kappa_\tau \dot{p}(t,z))\,(\gamma_0 - \zeta_0 + (\kappa_\gamma - \kappa_\zeta)p(s,z))}{(\tau_0 + \kappa_\tau p(t,z))^2\,(\tau_0 + \kappa_\tau p(s,z))} \in L^\infty([0,T] \times [0,T];L^\infty[0,1]).$$

Since $(\tau_0 + \kappa_\tau p(\cdot,z))^{-1} \in C(0,T)$, we know that

$$\int_s^t \frac{-\mathrm{d}\xi}{\tau_0 + \kappa_\tau p(\xi,z)}$$

is absolutely continuous in t and in s, hence

$$\exp\left(\int_s^t \frac{-\mathrm{d}\xi}{\tau_0 + \kappa_\tau p(\xi,z)} \right)$$

is in $L^\infty([0,T] \times [0,T])$.

Hence we may conclude that $G \in L^\infty([0,T] \times [0,T];L^\infty[0,1])$.

It is clear from the definition $c^2 = 1/\epsilon_0\mu_0$ that c^2 satisfies A5). Furthermore, by an appropriate choice of the source current J_s, we may guarantee that A6) holds and the forcing function defined by $F(t,z) = -\frac{1}{\epsilon_0}\dot{J}_s(t)$ is in $H^1(0,T,V^*)$.

With A1)–A6) satisfied by the pressure-dependent Debye polarization model, we may apply the theory in Section 2 and conclude that the system is well-posed.

4. Sample numerical results and conclusions

The main goal of the previous sections is to outline theoretical results for a general system that arises in electromagnetic interrogation problems. These results are important not only because they establish the well-posedness of a certain class of partial differential equations, but also because they provide a basis for constructing numerical approximations to their solutions, as well as a foundation for considering related parameter estimation

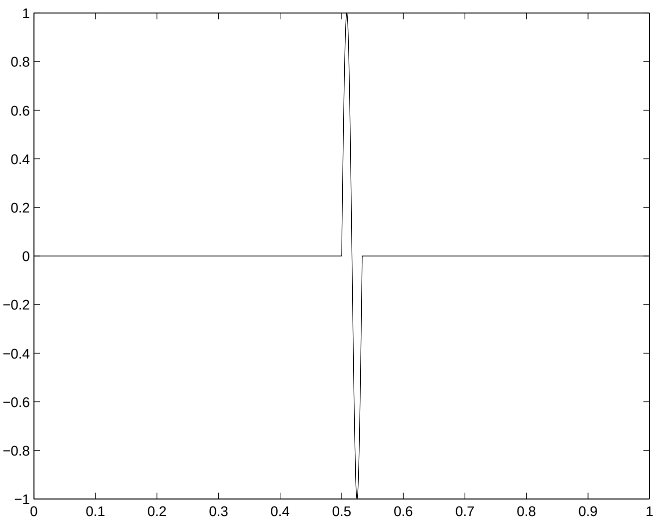

FIGURE 2. Pressure wave *vs.* depth.

problems (see, for example [4,9]). The ability to compute numerical solutions and subsequently estimate system parameters is critical in the study of interrogation problems.

In this section we present numerical solutions to (10) with coefficients defined by (11) to illustrate computations for systems that can be included in the theory presented in Section 2. By extending the given arguments, one can establish the convergence of a finite element approximation to the solution of (10). We use such approximation methods to compute the numerical solutions shown here. The numerical methodology used is similar to that outlined in Section 6.1.1 of [4] and incorporates ideas presented in [10].

Figures 3a–k are time snapshots of an electromagnetic wave pulse traveling through a layered medium. (The geometry is as shown in Fig. 1.) The pulse is initiated at $z = 0$ and travels to the right. Passing through the medium, it interacts with the air/dielectric interface at $z = z_1$ and an oncoming pressure wave (depicted in Fig. 2) traveling through the dielectric. As a result of each of these interactions, some of the electromagnetic wave energy is reflected, and wave reflections travel back toward $z = 0$. It is these reflections which return to $z = 0$ that would be observed and used as data in a parameter estimation problem. Figure 3c illustrates both the air/dielectric reflection which is returning to $z = 0$ and the original wave pulse traveling to the right; in Figure 3d one can see the air/dielectric reflection being absorbed at the boundary. Likewise, Figure 3g clearly shows the reflection from the pressure wave and the original pulse propagating in opposite directions; Figure 3j depicts the absorption of the electromagnetic reflection from the pressure wave at the boundary. It is interesting to note that when this reflection traveling toward $z = 0$ crosses the air/dielectric interface, some of its energy is again reflected; this phenomena can be seen by looking closely at Figures 3j and 3k.

Although the pressure wave is traveling to the left, its speed is extremely slow relative to that of the electromagnetic wave. Thus, Figure 2 can be thought of as the time snaphot of the pressure wave corresponding to *each* of the snapshots in Figure 3.

Taken together, these plots provide a picture of the behavior of the electromagnetic wave propagation and acoustic interaction described by the model equations (10, 11). Moreover, they serve as a visual example of the relevance in interrogation methodology of systems included in the theory presented in this paper.

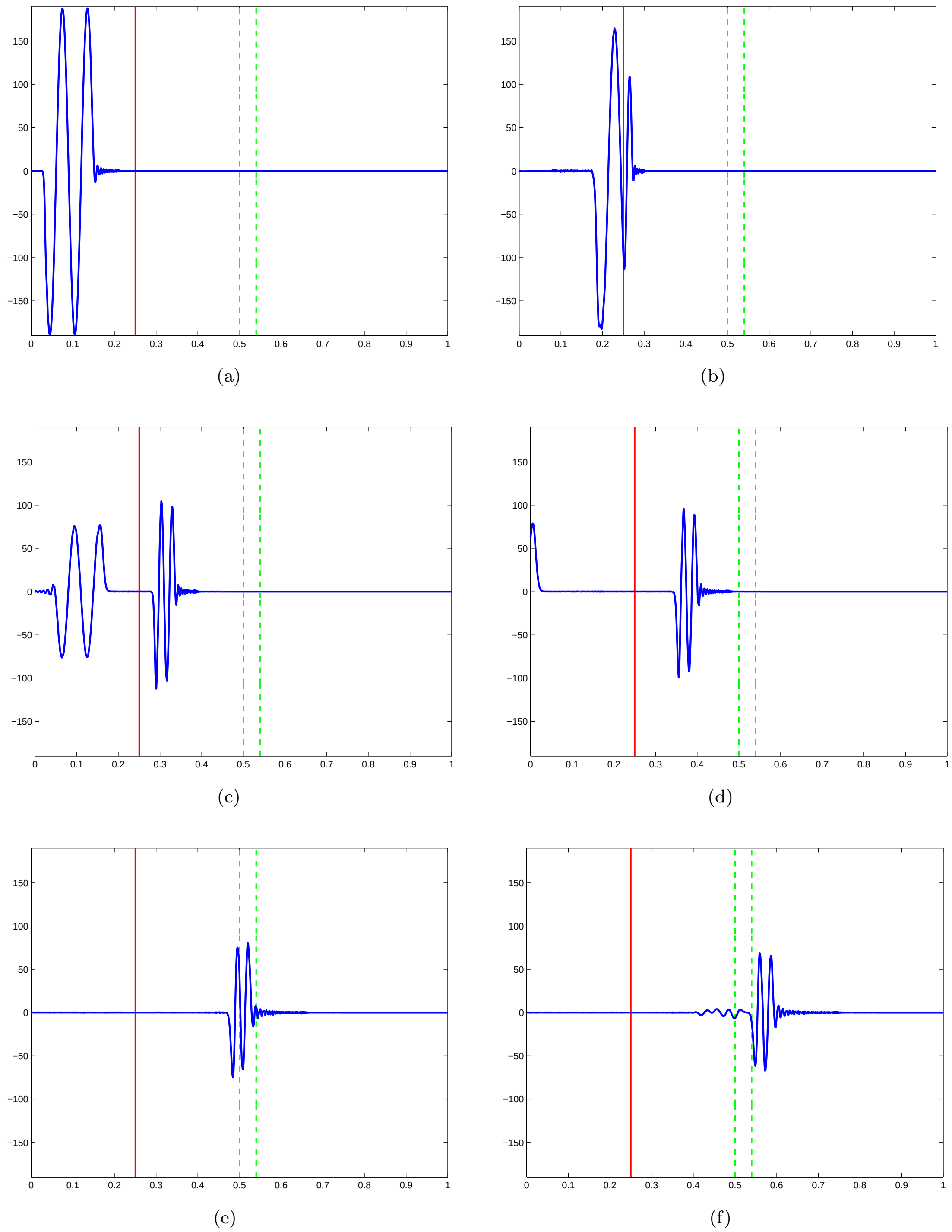

FIGURE 3. (a) E field *vs.* depth – $t = 5.0025e$-10; (b) E field *vs.* depth – $t = 1.00025e$-9; (c) E field *vs.* depth – $t = 1.50025e$-9; (d) E field *vs.* depth – $t = 2.00025e$-9; (e) E field *vs.* depth – $t = 3.00025e$-9; (f) E field *vs.* depth – $t = 3.50025e$-9.

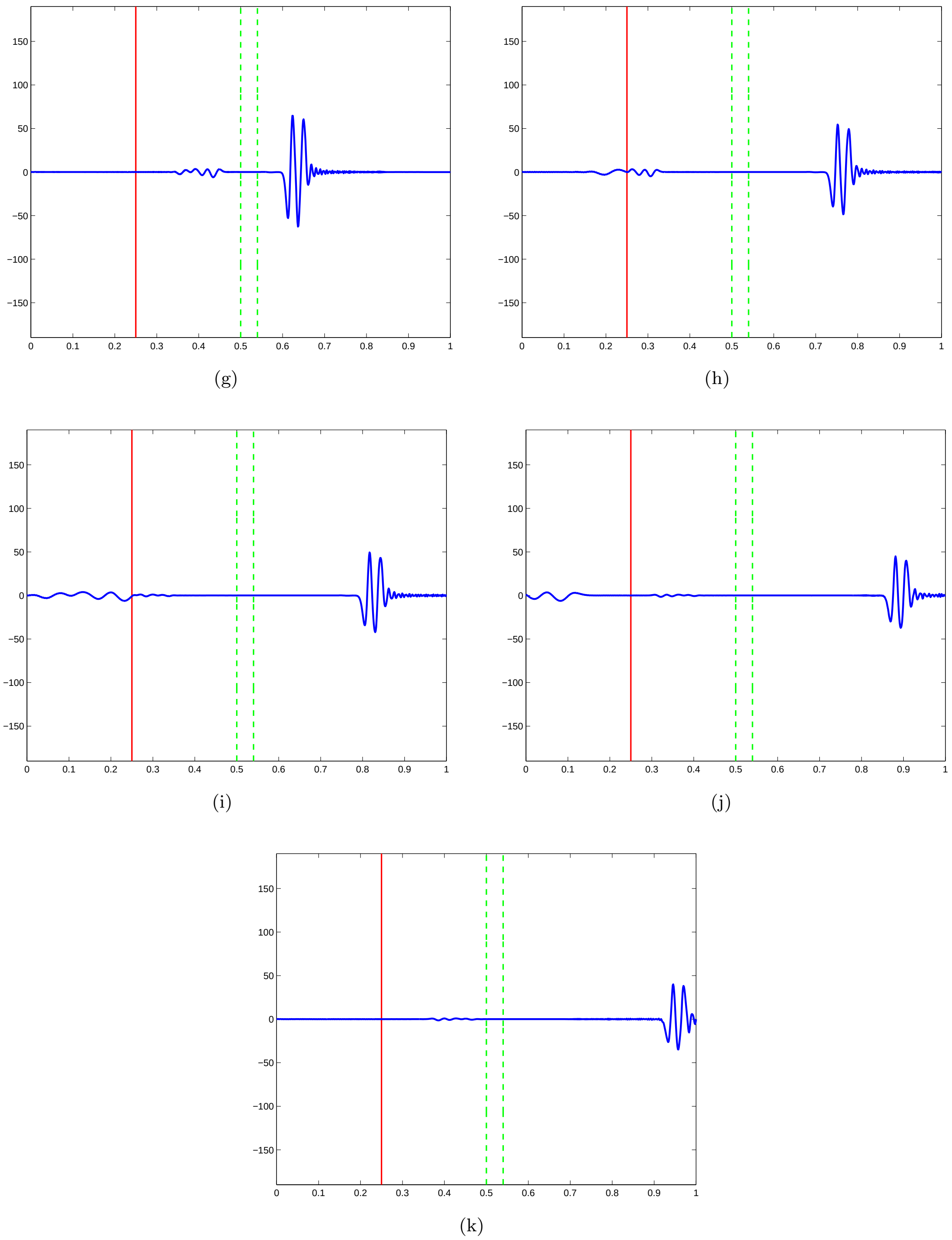

FIGURE 3. (g) E field *vs.* depth – $t = 4.00025e$-9; (h) E field *vs.* depth – $t = 5.00025e$-9; (i) E field *vs.* depth – $t = 5.50025e$-9; (j) E field *vs.* depth – $t = 6.00025e$-9; (k) E field *vs.* depth – $t = 6.50025e$-9.

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