ESAIM: Control, Optimisation and Calculus of Variations January 2002, Vol. 7, 223–237
URL: http://www.emath.fr/cocv/ DOI: 10.1051/cocv:2002009

ON A VOLUME CONSTRAINED VARIATIONAL PROBLEM IN SBV$^2(\Omega)$: PART I

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Abstract. We consider the problem of minimizing the energy

$$E(u) := \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x + \int_{S_u \cap \Omega} (1 + |[u](x)|) \, \mathrm{d}H^{N-1}(x)$$

among all functions $u \in SBV^2(\Omega)$ for which two level sets $\{u = l_i\}$ have prescribed Lebesgue measure α_i. Subject to this volume constraint the existence of minimizers for $E(\cdot)$ is proved and the asymptotic behaviour of the solutions is investigated.

Mathematics Subject Classification. 49J45, 35R35, 76T05.

Received June 27, 2000. Revised January 12, July 27 and September 18, 2001.

1. Introduction

In this paper we study the existence of minimizers of a volume constrained variational problem. Precisely, given real numbers α_i and l_i, $i = 0, 1$, such that $\alpha_i > 0$, $\alpha_0 + \alpha_1 \le \mathcal{L}^N(\Omega)$ and $l_0 < l_1$, and defining

$$E(u) := \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x + \int_{S_u \cap \Omega} (1 + |[u](x)|) \, \mathrm{d}H^{N-1}(x)$$

and

$$K := \left\{ u \in SBV^2(\Omega) \, : \, \mathcal{L}^N(\{u = l_i\}) = \alpha_i, \ i = 0, 1 \right\},$$

we consider the problem

$$(P) \quad \min_{u \in K} E(u)$$

that is, to minimize the energy $E(\cdot)$ among all functions $u \in SBV^2(\Omega)$ whose level sets $\{u = l_i\}$ have prescribed Lebesgue measure α_i, for $i = 0, 1$.

The vector-valued case, $u : \Omega \to \mathbb{R}^d$, will be treated in a forthcoming paper.

Keywords and phrases: Special functions of bounded variation, level sets, lower semicontinuity, Γ-limit.

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The minimization of an energy under similar volume constraints was originally proposed in 1992 by Gurtin [11] who, motivated by a problem related to the interface between immiscible fluids, suggested the study of existence of minimizers and possible optimal designs for the energy

$$I(u) := \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x$$

where $u : \Omega \to \mathbb{R}$ satisfies

$$\mathcal{L}^N(\{u = 0\}) = \alpha \text{ and } \mathcal{L}^N(\{u = 1\}) = \beta,$$

and the constants $\alpha, \beta > 0$ are such that $\alpha + \beta < \mathcal{L}^N(\Omega)$.

Considering the constraints

$$\mathcal{L}^N(\{u = l_i\}) = \alpha_i, i = 0, \ldots, m, m \geq 1$$

where α_i are given positive real numbers such that

$$\sum_{i=0}^{m} \alpha_i < \mathcal{L}^N(\Omega)$$

and l_i are given vectors, this question was addressed by Ambrosio *et al.* in [2] who showed the existence of minimizers of $I(\cdot)$, in the vector-valued case, under the assumption that the vectors l_i are extremal points of their own convex hull. We refer also to [3] and [4] where related problems were treated.

A further step was taken by Tilli [12], in the scalar case, who established locally Hölder continuity of minimizers of $I(\cdot)$ and was able to drop the extremality assumption needed in [2] which, in the scalar case, is equivalent to the restriction $m = 1$, *i.e.* only two level sets are allowed. In our case we were unable to drop this assumption and that is why we only consider two level sets. We remark, however, that under the hypothesis $m = 1$ it was established by Tilli [12] that minimizers of $I(\cdot)$ are, in fact, locally Lipschitz continuous.

In the problem originally proposed by Gurtin the matter of regularity is crucial. Nonetheless, it is easily observed that when discontinuities in the admissible functions u are allowed, if $\mathcal{L}^N(\Omega) - (\alpha_0 + \alpha_1)$ is small enough, in order to avoid high gradients, minimizers of the energy might prefer to "jump" between the prescribed values l_i. This remark motivated our interest in problem (P) and our goal in this paper is twofold; on one hand to show existence of solutions of (P), and on the other hand to show that in some cases the solution with discontinuities is, in fact, preferred.

We organize the paper as follows. In Section 2 we introduce some notation and recall the main properties of the spaces $BV(\Omega)$, $SBV(\Omega)$, $SBV^2(\Omega)$ and $SBV_0(\Omega)$ which will be used in the sequel. In Section 3, following the arguments given in [2], we prove the existence of minimizers of problem (P), by first introducing a relaxed problem (P^*). Using a compactness theorem due to Ambrosio [1] and a lower semicontinuity result, we show existence of solutions to problem (P^*). Our main result of this section, Theorem 3.3, states that any solution of (P^*) is also a solution of (P). Section 4 is devoted to the case $N = 1$. There we obtain an explicit solution to our problem in dimension 1 and we show that, when $\mathcal{L}^N(\Omega) - (\alpha_0 + \alpha_1)$ is small enough, a discontinuous solution is obtained. Finally, in Section 5 we study the asymptotic behaviour of the solutions as $\alpha_0 + \alpha_1 \nearrow \mathcal{L}^N(\Omega)$.

2. Preliminaries and definitions

In what follows, $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected Lipschitz domain, $\mathcal{L}^N$ and H^{N-1} are, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, χ_A denotes the characteristic function of a set A and $\mathcal{B}(\mathbb{R}^N)$ represents the set of all Borel subsets of $\mathbb{R}^N$. Our functions u are real-valued and we use the standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$; $C_0^\infty(\Omega)$ stands for the space of real-valued smooth functions with compact support in Ω, $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and the letter C will be used to indicate a constant whose value might change from line to line.

Given an $L^1(\Omega)$ function u the *Lebesgue set* of u, Ω_u, is defined as the set of points $x \in \Omega$ such that there exists $\tilde{u}(x) \in \mathbb{R}$ satisfying

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{B(x,\varepsilon)} |u(y) - \tilde{u}(x)| \, \mathrm{d}y = 0.$$

The *Lebesgue discontinuity set* S_u of u is the set of points $x \in \Omega$ which are not Lebesgue points, that is $S_u := \Omega \setminus \Omega_u$. By Lebesgue's Differentiation theorem, S_u is $\mathcal{L}^N$-negligible and the function $\tilde{u} : \Omega \to \mathbb{R}$, which coincides with u $\mathcal{L}^N$-almost everywhere in Ω_u, is called the *Lebesgue representative* of u.

The *approximate upper* and *lower limits* of u are given by

$$u^+(x) := \inf \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N(\{y \in \Omega \cap B(x,\varepsilon) : u(y) > t\}) = 0 \right\}$$

and

$$u^-(x) := \sup \left\{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N(\{y \in \Omega \cap B(x,\varepsilon) : u(y) < t\}) = 0 \right\};$$

if $u^+(x) = u^-(x)$ then $x \in \Omega_u$ and $u^+(x) = u^-(x) = \tilde{u}(x)$. The *jump set* or *singular set* of u is defined as

$$J_u := \left\{ x \in \Omega : u^-(x) < u^+(x) \right\}$$

and we denote by $[u](x)$ the *jump* of u at x, *i.e.* $[u](x) := u^+(x) - u^-(x)$.

We recall briefly some facts on functions of bounded variation which will be used in the sequel. We refer to [9, 10] and [13] for a detailed exposition on this subject.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation*, $u \in BV(\Omega)$, if for all $j = 1, ..., N$, there exists a finite Radon measure μ_j such that

$$\int_\Omega u(x) \frac{\partial \phi}{\partial x_j}(x) \, \mathrm{d}x = - \int_\Omega \phi(x) \, \mathrm{d}\mu_j(x)$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative Du is the vector-valued measure μ with components μ_j.

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega),$$

where $|Du|(\Omega)$ represents the total variation of the measure Du.

If $u \in BV(\Omega)$ then the distributional derivative Du may be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u H^{N-1} \lfloor S_u + C_u, \tag{2.1}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure and C_u is the Cantor part of Du which vanishes on all Borel sets B with $H^{N-1}(B) < +\infty$. The three measures appearing in (2.1) are mutually singular.

If $u \in BV(\Omega)$ it is well known that S_u is countably $N-1$ rectifiable, *i.e.*

$$S_u = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $H^{N-1}(E) = 0$ and K_n are compact subsets of C^1 hypersurfaces. Furthermore, for H^{N-1} a.e. $x \in S_u$, $u^+(x) \neq u^-(x)$ and there exists a unit vector $\nu_u(x) \in S^{N-1}$, normal to S_u at x, such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in B(x,\varepsilon) : (y-x) \cdot \nu_u(x) > 0\}} |u(y) - u^+(x)| \, \mathrm{d}y = 0$$

and
$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{\{y\in B(x,\varepsilon):(y-x)\cdot\nu_u(x)<0\}}|u(y)-u^-(x)|\,\mathrm{d}y=0.$$
In particular, $H^{N-1}(S_u\setminus J_u)=0$.

The space of *special functions of bounded variation*, $SBV(\Omega)$, introduced by De Giorgi and Ambrosio in [7], is the space of functions $u\in BV(\Omega)$ such that $C_u=0$, *i.e.* for which
$$Du=\nabla u\mathcal{L}^N+(u^+-u^-)\otimes\nu_u H^{N-1}\lfloor S_u.$$

Definition 2.1. Given $p>1$, a function $u\in SBV(\Omega)$ is said to belong to $SBV^p(\Omega)$ if $\nabla u\in L^p(\Omega;\mathbb{R}^N)$ and $H^{N-1}(S_u\cap\Omega)<+\infty$.

In this paper we will be concerned with functions in $SBV^2(\Omega)$ and in this space we consider the following definition of weak convergence as introduced by Braides and Chiadò–Piat in [5].

Definition 2.2. Given $\{u_n\}\subset SBV^2(\Omega)$ and $u\in SBV^2(\Omega)$ we say that u_n converges weakly to u in $SBV^2(\Omega)$, $u_n\rightharpoonup u$ in $SBV^2(\Omega)$, if $u_n\to u$ in $L^1(\Omega)$, $\sup_n|Du_n|(\Omega)<+\infty$ and $\nabla u_n\rightharpoonup\nabla u$ weakly in $L^2(\Omega;\mathbb{R}^N)$.

The introduction of this kind of convergence was motivated by the following compactness theorem due to Ambrosio [1].

Theorem 2.3. *Let $\{u_n\}\subset SBV^2(\Omega)$ be such that*
$$\sup_n||u_n||_{BV(\Omega)}<+\infty$$
and
$$\sup_n\left\{\int_\Omega|\nabla u_n|^2\,\mathrm{d}x+H^{N-1}(S_{u_n}\cap\Omega)\right\}<+\infty.$$
Then there exists a subsequence $\{u_{n_j}\}\subset\{u_n\}$ converging weakly to a function u in $SBV^2(\Omega)$. Moreover,
$$H^{N-1}(S_u\cap\Omega)\le\liminf_{j\to\infty}H^{N-1}(S_{u_{n_j}}\cap\Omega).$$

Remark 2.4. Using this compactness result one can show the lower semicontinuity, with respect to L^1 convergence, of the functional
$$\int_\Omega|\nabla u(x)|^2\,\mathrm{d}x+\int_{S_u\cap\Omega}|[u](x)|\mathrm{d}H^{N-1}(x)+H^{N-1}(S_u\cap\Omega);$$
see, for instance [5].

An $\mathcal{L}^N$-measurable set $A\subset\Omega$ is said to be of *finite perimeter* in Ω if $\chi_A\in BV(\Omega)$. The *perimeter* of A in Ω is defined by
$$\mathrm{Per}_\Omega(A):=\sup\left\{\int_A\mathrm{div}\varphi(x)\,\mathrm{d}x:\varphi\in C_0^1(\Omega;\mathbb{R}^N),\ \|\varphi\|_\infty\le 1\right\}.$$
Given a set $A\subset\Omega$ of locally finite perimeter the *reduced boundary* of A in Ω, ∂^*A, consists of those points $x\in\Omega$ for which the following conditions hold:

i) $|D\chi_A|(B(x,r))>0$ for all $r>0$;

ii) the limit
$$\nu_A(x):=\lim_{r\to 0}\frac{D\chi_A(B(x,r))}{|D\chi_A|(B(x,r))}$$
exists and $|\nu_A(x)|=1$.

The function $\nu_A:\partial^*A\to S^{N-1}$ is called the generalised inner normal to A.

In Section 3 we will need the following result which can be found in [9].

Proposition 2.5. *Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then there exists a positive constant A, depending only on N, such that, for each $x_0 \in \partial^* E$*

$$\liminf_{r\to 0^+} \frac{H^{N-1}\left(\partial^* E \cap B(x_0,r)\right)}{r^{N-1}} \geq A > 0.$$

As in [5], we will use the symbol $SBV_0(\Omega)$ to denote the space

$$SBV_0(\Omega) = \left\{u \in SBV(\Omega) : H^{N-1}(S_u \cap \Omega) < +\infty, \nabla u = 0 \text{ a.e. in } \Omega\right\}\cdot$$

We say that a sequence (E_i) is a *Borel partition* of a given set $B \in \mathcal{B}(\mathbb{R}^N)$ if and only if

$$E_i \in \mathcal{B}(\mathbb{R}^N),\ \forall i \in \mathbb{N};\ E_i \cap E_j = \emptyset \text{ if } i \neq j;\ \bigcup_{i=1}^{\infty} E_i = B.$$

We say that (E_i) is a *Caccioppoli partition* if each E_i is a set of finite perimeter. The relation between Caccioppoli partitions and functions in $SBV_0(\Omega)$ is expressed in the following result, whose proof can be found in [6] (see Lem. 1.4, 1.10 and Rem. 1.5).

Lemma 2.6. *If $u \in SBV_0(\Omega)$ then there exist a Borel partition (E_i) of Ω, and a sequence (u_i) in $\mathbb{R}$ with $u_i \neq u_j$ for $i \neq j$, such that*

$$u = \sum_{i=1}^{\infty} u_i \chi_{E_i} \text{ a.e. in } \Omega,$$

$$H^{N-1}(S_u \cap \Omega) = \frac{1}{2}\sum_{i=1}^{\infty} H^{N-1}(\partial^* E_i \cap \Omega) = \frac{1}{2}\sum_{i\neq j} H^{N-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega)$$

$$(u^+, u^-, \nu_u) \sim (u_i, u_j, \nu_i)\ H^{N-1} \text{ a.e. on } \partial^* E_i \cap \partial^* E_j \cap \Omega$$

where ν_i is the inner normal to E_i.

3. Existence of solutions

Let α_i and l_i, $i = 0, 1$, be given real numbers satisfying $\alpha_i > 0$, $\alpha_0 + \alpha_1 \leq \mathcal{L}^N(\Omega)$ and $l_0 < l_1$. Define

$$E(u) := \int_{\Omega} |\nabla u(x)|^2 \,\mathrm{d}x + \int_{S_u\cap\Omega} (1 + |[u](x)|)\,\mathrm{d}H^{N-1}(x)$$

and

$$K := \left\{u \in SBV^2(\Omega)\ :\ \mathcal{L}^N(\{u = l_i\}) = \alpha_i,\ i = 0, 1\right\}\cdot$$

Our goal in this section is to prove existence of solutions of the problem

$$(P)\quad \min_{u\in K} E(u)\cdot$$

In order to do so we consider the auxiliary problem (P^*), which is to minimize the energy $E(\cdot)$ among all functions $u \in SBV^2(\Omega)$ satisfying the relaxed conditions

$$\mathcal{L}^N(\{u = l_i\}) \geq \alpha_i,\quad i = 0, 1.$$

The following simple application of Fatou's lemma, proved in [2], will enable us to prove existence of solutions of (P^*).

Proposition 3.1. *For any sequence $\{u_n\}$ of Lebesgue measurable functions converging a.e. to a Lebesgue measurable function u, and for any closed set $A \subset \mathbb{R}$ we have*

$$\mathcal{L}^N\left(\{x \in \Omega : u(x) \in A\}\right) \geq \limsup_{n \to +\infty} \mathcal{L}^N\left(\{x \in \Omega : u_n(x) \in A\}\right) \cdot$$

Proposition 3.2. *Problem (P^*) admits a solution. Moreover, if $u \in SBV^2(\Omega)$ is a minimum for (P^*), then $l_0 \leq u(x) \leq l_1$ for a.e. $x \in \Omega$.*

Proof. Let $\{u_n\}$ be a minimizing sequence for (P^*). Substituting, if necessary, u_n by $w_n = \max\{l_0, \min\{l_1, u_n\}\}$, we can assume, without loss of generality, that $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$ and hence also in $L^1(\Omega)$. Then, it is easy to check that $\{u_n\}$ satisfies the hypotheses of Theorem 2.3 and thus there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and a function u in $SBV^2(\Omega)$ such that $u_{n_j} \rightharpoonup u$ in $SBV^2(\Omega)$. It follows immediately from Proposition 3.1 that

$$\mathcal{L}^N(\{u = l_i\}) \geq \alpha_i, \quad i = 0, 1.$$

This, together with Remark 2.4, leads to the result. $\square$

Clearly the previous result holds true also in the case where more than two level sets are considered.

Our next step is to show that if u is a minimum for (P^*) then $u \in K$, thus proving that u is also a minimum for (P).

Theorem 3.3. *If u minimizes (P^*), then u also minimizes (P).*

Proof. It suffices to show that $\mathcal{L}^N(\{u = l_i\}) = \alpha_i$, for $i = 0, 1$.

Suppose that $\mathcal{L}^N(\{u = l_0\}) > \alpha_0$ and assume for simplicity that $l_0 = 0$. Let $r > 0$ be such that $\mathcal{L}^N(\{u = 0\}) - r > \alpha_0$ and consider a smooth cut-off function $\phi \in C_0^\infty(\mathbb{R}^N; [0,1])$ such that $\mathcal{L}^N(\operatorname{supp}\phi) < r$. Let $0 < \varepsilon < 1$ and consider the perturbations

$$u_\varepsilon := u + \varepsilon\phi(l_1 - u).$$

It is clear that

$$\mathcal{L}^N(\{u_\varepsilon = l_1\}) \geq \mathcal{L}^N(\{u = l_1\}) \geq \alpha_1.$$

On the other hand,

$$\mathcal{L}^N(\{u_\varepsilon = 0\}) \geq \mathcal{L}^N(\{u = 0\}) - \mathcal{L}^N(\operatorname{supp}\phi) > \mathcal{L}^N(\{u = 0\}) - r \geq \alpha_0$$

and so $u_\varepsilon \in SBV^2(\Omega)$ is admissible for (P^*). Therefore, the minimality of u yields $E(u) \leq E(u_\varepsilon)$, $\forall \varepsilon \in (0,1)$. Noticing that $S_{u_\varepsilon} \subseteq S_u$ and

$$|[u_\varepsilon(x)]| = |u_\varepsilon^+(x) - u_\varepsilon^-(x)| = |(1 - \varepsilon\phi(x))(u^+(x) - u^-(x))| \leq |[u(x)]|$$

for the prescribed values of ε, since ϕ takes values between 0 and 1, the comparison of the energies leads to

$$\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \leq \int_\Omega |\nabla u_\varepsilon(x)|^2 \, \mathrm{d}x$$

for $\varepsilon \in (0,1)$. Expanding ∇u_ε in the previous expression, dividing by ε and letting finally $\varepsilon \to 0^+$, one arrives at

$$2\int_\Omega \left(-|\nabla u(x)|^2\phi(x) + (l_1 - u)\nabla u(x) \cdot \nabla\phi(x)\right) \mathrm{d}x \geq 0. \tag{3.1}$$

Using a partition of unity argument, the function $\phi \equiv 1$ in Ω can be written as a finite sum of smooth cut-off functions, each of small compact support and for which (3.1) holds. We may therefore replace $\phi \equiv 1$ in (3.1) in order to obtain

$$-2 \int_{\Omega} |\nabla u(x)|^2 \, \mathrm{d}x \geq 0$$

and we conclude that $\nabla u = 0$ $\mathcal{L}^N$ a.e. in Ω, *i.e.* $u \in SBV_0(\Omega)$.

Next we use a characterization of such functions in order to derive a contradiction. Indeed, by Lemma 2.6, there exist a Caccioli partition $\{E_i\}$ of Ω and a sequence of real numbers $\{u_i\}$ with $u_i \neq u_j$ for $i \neq j$, such that

$$u = \sum_{i=0}^{\infty} u_i \chi_{E_i} \text{ a.e. in } \Omega.$$

Assume without loss of generality that $E_0 := \{u = 0\}$. Since E_0 is a Caccioli set, we know that $\overline{\partial^* E_0} = \partial E_0$ (*cf.* [10]), and so we can choose a point $x_0 \in \partial^* E_0 \cap \Omega$. Next, for ε, k, satisfying

$$0 < w_N \varepsilon^N < \mathcal{L}^N(\{u = 0\}) - \alpha_0, \;\; 0 < \frac{\varepsilon}{2k} < u^+(x_0), \tag{3.2}$$

where w_N denotes the volume of the unit ball in $\mathbb{R}^N$, and for $x \in B(x_0, \varepsilon)$, set

$$f_{\varepsilon,k}(x) := \begin{cases} \dfrac{\varepsilon}{2k} & \text{if } \; x \in B\left(x_0, \dfrac{\varepsilon}{2}\right) \\ \dfrac{\varepsilon - |x|}{k} & \text{if } \; x \in B(x_0, \varepsilon) \setminus B\left(x_0, \dfrac{\varepsilon}{2}\right) \cdot \end{cases}$$

and define

$$u_{\varepsilon,k}(x) := \begin{cases} u(x) & \text{in } \Omega \setminus B(x_0, \varepsilon) \\ \max\{u(x), f_{\varepsilon,k}(x)\} & \text{in } \; B\left(x_0, \varepsilon\right) \cdot \end{cases}$$

Clearly

$$u_{\varepsilon,k} \in SBV^2(\Omega),$$
$$S_{u_{\varepsilon,k}} = \{x \in S_u : u^+(x) > f_{\varepsilon,k}(x)\} \subseteq S_u,$$

and by (3.2)

$$\mathcal{L}^N(\{u_{\varepsilon,k} = 0\}) \geq \alpha_0, \;\; \mathcal{L}^N(\{u_{\varepsilon,k} = l_1\}) \geq \alpha_1.$$

Therefore, $u_{\varepsilon,k}$ is admissible for (P^*). Also, due to the definition of the approximate upper and lower limits and the continuity of $f_{\varepsilon,k}$, we have

$$[u_{\varepsilon,k}](x) = \begin{cases} [u](x) & \text{if } u^-(x) \geq f_{\varepsilon,k}(x) \\ u^+(x) - f_{\varepsilon,k}(x) & \text{if } u^-(x) < f_{\varepsilon,k}(x) < u^+(x) \\ 0 & \text{if } u^+(x) \leq f_{\varepsilon,k}(x). \end{cases} \tag{3.3}$$

Comparing the energies of u and $u_{\varepsilon,k}$ (which are equal outside $\overline{B(x_0, \varepsilon)}$), we claim that

$$E(u_{\varepsilon,k}) < E(u),$$

for some appropriate choice of ε, k, thus contradicting the minimality of u. Indeed,

$$\begin{aligned} E(u_{\varepsilon,k}) < E(u) \quad &\Leftrightarrow \quad \int_{B(x_0,\varepsilon)} |\nabla u_{\varepsilon,k}(x)|^2 \, \mathrm{d}x \\ &\leq \quad \int_{S_u \cap B(x_0,\varepsilon)} (1 + [u](x)) \, \mathrm{d}H^{N-1}(x) - \int_{S_{u_{\varepsilon,k}} \cap B(x_0,\varepsilon)} (1 + [u_{\varepsilon,k}](x)) \, \mathrm{d}H^{N-1}(x). \end{aligned}$$

Since $S_{u_{\varepsilon,k}} \subseteq S_u$ and

$$\int_{B(x_0,\varepsilon)} |\nabla u_{\varepsilon,k}(x)|^2 \, \mathrm{d}x \leq \frac{\varepsilon^N w_N}{k^2},$$

it suffices to show that

$$\begin{aligned}
\frac{\varepsilon^N w_N}{k^2} &\leq \int_{S_{u_{\varepsilon,k}} \cap B(x_0,\varepsilon)} ([u](x) - [u_{\varepsilon,k}](x)) \, \mathrm{d}H^{N-1}(x) + \int_{S_u \setminus S_{u_{\varepsilon,k}} \cap B(x_0,\varepsilon)} [u](x) \, \mathrm{d}H^{N-1}(x) \\
&= \int_{S_u \cap B(x_0,\varepsilon) \cap \{u^- < f_{\varepsilon,k} < u^+\}} (f_{\varepsilon,k}(x) - u^-(x)) \, \mathrm{d}H^{N-1}(x) + \int_{S_u \cap \{u^+ \leq f_{\varepsilon,k}\} \cap B(x_0,\varepsilon)} [u](x) \, \mathrm{d}H^{N-1}(x),
\end{aligned}$$

by (3.3). In fact, since the last integral is nonnegative and as $f_{\varepsilon,k} = \frac{\varepsilon}{2k}$ in $B(x_0, \frac{\varepsilon}{2})$ and $u^-(x) = 0$ for $x \in \partial^* E_0$, it is enough to prove that

$$\frac{\varepsilon^N w_N}{k^2} \leq \int_{\partial^* E_0 \cap B(x_0, \frac{\varepsilon}{2}) \cap \{u^+ > \frac{\varepsilon}{2k}\}} \frac{\varepsilon}{2k} \, \mathrm{d}H^{N-1}(x).$$

Therefore, if we can show that

$$\frac{w_N}{k} \leq \frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon}{2}) \cap \{u^+ > \frac{\varepsilon}{2k}\})}{2\varepsilon^{N-1}}, \tag{3.4}$$

for some appropriate choice of ε, k, the desired contradiction follows.

By Proposition 2.5, there exists $C > 0$, such that

$$\liminf_{r \to 0^+} \frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{r}{2}))}{r^{N-1}} > C > 0.$$

Hence we can fix ε_1 satisfying (3.2), and such that

$$0 < C < \frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}))}{\varepsilon_1^{N-1}}.$$

Choose k satisfying (3.2). Then, by the general properties of nested families of measurable sets,

$$\frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}))}{\varepsilon_1^{N-1}} = \lim_{\varepsilon \to 0^+} \frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon}{2k}\})}{\varepsilon_1^{N-1}}$$

and therefore, there exists $\varepsilon_2 \leq \varepsilon_1$ such that

$$\frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_2}{2k}\})}{\varepsilon_1^{N-1}} > C > 0.$$

Thus,

$$\begin{aligned}
0 < C\varepsilon_1^{N-1} &= H^{N-1}\left(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_2}{2k}\right\}\right) \\
&= H^{N-1}\left(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_1}{2k}\right\}\right) \\
&\quad + H^{N-1}\left(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{2k} < u^+ \leq \frac{\varepsilon_1}{2k}\right\}\right).
\end{aligned}$$

Since

$$H^{N-1}\Big(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{2k} < u^+ \leq \frac{\varepsilon_1}{2k}\right\}\Big) \leq H^{N-1}\Big(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{0 < u^+ \leq \frac{\varepsilon_1}{2k}\right\}\Big),$$

again, due to the fact that this is a nested family of measurable sets, we conclude that

$$\lim_{k\to+\infty} H^{N-1}\left(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{\frac{\varepsilon_2}{2k} < u^+ \leq \frac{\varepsilon_1}{2k}\right\}\right) = 0$$

and so we can choose k_1 sufficiently large so that

$$H^{N-1}\left(\partial^* E_0 \cap B\left(x_0, \frac{\varepsilon_1}{2}\right) \cap \left\{u^+ > \frac{\varepsilon_1}{2k_1}\right\}\right) \geq C\varepsilon_1^{N-1} > 0. \tag{3.5}$$

Finally, letting

$$k_2 > \max\left\{k_1, \frac{2w_N}{C}\right\},$$

where C is the constant appearing in (3.5), it follows from (3.5) that

$$\begin{aligned}\frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_1}{2k_2}\})}{2\varepsilon_1^{N-1}} &\geq \frac{H^{N-1}(\partial^* E_0 \cap B(x_0, \frac{\varepsilon_1}{2}) \cap \{u^+ > \frac{\varepsilon_1}{2k_1}\})}{2\varepsilon_1^{N-1}} \\ &\geq \frac{C}{2} > \frac{w_N}{k_2},\end{aligned}$$

and (3.4) is proved.

Since the assumption that $\mathcal{L}^N(\{u = 0\}) > \alpha_0$ yielded a contradiction we conclude, therefore, that $\mathcal{L}^N(\{u = 0\}) = \alpha_0$.

A similar argument allows us to show that $\mathcal{L}^N(\{u = l_1\}) = \alpha_1$. □

4. The case N = 1

This section is devoted to the characterization of solutions of problem (P) in the 1-dimensional case ($N = 1$), when Ω is an interval. We remark that explicit minimizers for the energy

$$I(u) := \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x$$

under the constraints

$$\mathcal{L}^N(\{u = l_0\}) = \alpha_0, \quad \mathcal{L}^N(\{u = l_1\}) = \alpha_1,$$

with $\alpha_0 + \alpha_1 < \mathcal{L}^N(\Omega)$, are only known when $N = 1$ and Ω is an interval, where each minimizer is a monotone and piecewise affine function and the minimal energy is given by

$$\frac{(l_1 - l_0)^2}{\mathcal{L}^N(\Omega) - (\alpha_0 + \alpha_1)}$$

(see [2]).

We begin by showing that if

$$\mathcal{L}^N(\Omega) - (\alpha_0 + \alpha_1) < \frac{(l_1 - l_0)^2}{1 + (l_1 - l_0)}, \tag{4.1}$$

then the minimum obtained in [2] is no longer a solution to our problem. In fact, it is easy to see that if (4.1) holds, the energy of such a piecewise affine function is larger than the energy associated with $w \in SBV^2(\Omega)$ which takes only the two prescribed values l_0 and l_1 and has only one discontinuity point (consequently, for at least one $i \in \{0,1\}$, $\mathcal{L}^N(\{w = l_i\}) > \alpha_i$). This does not contradict Theorem 3.3 since it is possible to construct a function u satisfying the constraints

$$\mathcal{L}^N(\{u = l_i\}) = \alpha_i, \quad i \in \{0,1\}$$

and such that $E(u) < E(w)$. Indeed, taking for simplicity $l_0 = 0$, $l_1 = 1$ and $\Omega = (0,1)$, and assuming that $\alpha_0 + \alpha_1 < 1$, define

$$u_h(x) = \begin{cases} 0 & \text{if } x \in]0, \alpha_0[\\ (x - \alpha_0)\dfrac{h}{l} & \text{if } x \in \left[\alpha_0, \dfrac{1+\alpha_0-\alpha_1}{2}\right[\\ 1 + (x - 1 + \alpha_1)\dfrac{h}{l} & \text{if } x \in \left[\dfrac{1+\alpha_0-\alpha_1}{2}, 1-\alpha_1\right[\\ 1 & \text{if } x \in [1-\alpha_1, 1[\end{cases}$$

where $l = \frac{1-(\alpha_0+\alpha_1)}{2}$ and $0 < h < \frac{1}{2}$ is to be determined. One can check easily that the energy associated with u_h is

$$E(u_h) = 2\frac{h^2}{l} + 2 - 2h$$

while the energy associated with the function taking only the two prescribed values $l_0 = 0$ and $l_1 = 1$, and having only one discontinuity point, equals 2. Hence, for $h < l$ the energy associated with u_h is less than 2, and the minimum is attained at $h_0 = \frac{l}{2}$, the energy in this case being

$$E(u_{h_0}) = 2 - \frac{l}{2} = 2 - \frac{1-(\alpha_0+\alpha_1)}{4}.$$

We could also attain the same values of $E(\cdot)$ with the functions v_h defined by

$$v_h(x) = \begin{cases} 0 & \text{if } x \in]0, \alpha_0[\\ (x-\alpha_0)\frac{h}{l} & \text{if } x \in [\alpha_0, 1-\alpha_1[\\ 1 & \text{if } x \in [1-\alpha_1, 1[\cdot \end{cases}$$

Actually, considering $\lambda \in (0,1]$, $\mu_1, \mu_2 > 0$ such that $1 + \lambda l \mu_2 - \lambda l \mu_1 - 2l\mu_2 > 0$, and the functions

$$u_{\lambda,\mu_1,\mu_2}(x) = \begin{cases} 0 & \text{if } x \in]0, \alpha_0[\\ (x-\alpha_0)\mu_1 & \text{if } x \in [\alpha_0, \alpha_0 + \lambda l[\\ 1 + (x - 1 + \alpha_1)\mu_2 & \text{if } x \in [\alpha_0 + \lambda l, 1-\alpha_1[\\ 1 & \text{if } x \in [1-\alpha_1, 1[\end{cases}$$

and minimizing $E(u_{\lambda,\mu_1,\mu_2})$ with respect to the parameter λ and to the slopes of the affine parts of u_{λ,μ_1,μ_2} we arrive at the same value, $2 - \frac{l}{2}$, attained by any $\lambda \in (0,1]$ and for $\mu_1 = \mu_2 = \frac{1}{2}$.

Is this the minimum of the energy? In fact, it is easy to see that minima for the energy are either of the form obtained in [2] or are attained at these functions. In the first case the minimal energy is given by $\frac{1}{1-(\alpha_0+\alpha_1)}$. If there are discontinuities in the solution, there is only one, since $E(u) > 2$ if more than one jump is allowed, and the energy attained with just one discontinuity point can be as low as $2 - \frac{1-(\alpha_0+\alpha_1)}{4}$, as seen before. Applying the reasoning given in [2] we see that this is in fact the minimum of the energy in the presence of a discontinuity and that the number of connected components of $\{y \in (0,1) : y = u(x)\}$ in this situation is at most 2. Hence, we have

$$E(u) \geq \min\left\{\frac{1}{1-(\alpha_0+\alpha_1)}, 2 - \frac{1-(\alpha_0+\alpha_1)}{4}\right\}.$$

We conclude that for $1 - (\alpha_0 + \alpha_1) < 4 - 2\sqrt{3}$, the solution with a jump is preferred.

In the case of a general interval Ω and levels l_0 and l_1 we have

$$E(u) \geq \min\left\{\frac{(l_1-l_0)^2}{\mathcal{L}^N(\Omega)-(\alpha_0+\alpha_1)}, 1+(l_1-l_0)-\frac{\mathcal{L}^N(\Omega)-(\alpha_0+\alpha_1)}{4}\right\}$$

so a discontinuous solution is obtained if

$$1+(l_1-l_0)-\frac{\mathcal{L}^N(\Omega)-(\alpha_0+\alpha_1)}{4} < \frac{(l_1-l_0)^2}{\mathcal{L}^N(\Omega)-(\alpha_0+\alpha_1)}.$$

5. Asymptotic behaviour of the solutions

Our goal in this section is to study the asymptotic behaviour of the solutions $u_{\alpha\beta}$ of the problem

$$(P_{\alpha\beta})\quad \min\left\{E(u) : u \in SBV^2(\Omega), \mathcal{L}^N(\{u=0\})=\alpha, \mathcal{L}^N(\{u=1\})=\beta\right\}$$

as $(\alpha+\beta) \nearrow \mathcal{L}^N(\Omega)$. We denote by $m_{\alpha\beta} := E(u_{\alpha\beta})$ and, for any constant $\gamma \in (0, \mathcal{L}^N(\Omega))$, we set

$$p_\gamma := \min\left\{\mathrm{Per}_\Omega(A) : A \subset \Omega, \mathcal{L}^N(A)=\gamma\right\}, \tag{5.1}$$

where $\mathrm{Per}_\Omega(A)$ denotes the perimeter of A in Ω.

Our first result identifies the Γ-limit of a suitable sequence of functionals. In order to prove it we need the following approximation lemma which can be found in [2].

Lemma 5.1. *Let $A \subset \Omega$ be a set of finite perimeter such that $0 < \mathcal{L}^N(A) < \mathcal{L}^N(\Omega)$. There exists a sequence of bounded, open sets $D_n \subset \mathbb{R}^N$ with smooth boundary in $\mathbb{R}^N$ such that $\mathcal{L}^N(A) = \mathcal{L}^N(D_n \cap \Omega)$, χ_{D_n} converges to χ_A in $L^1(\Omega)$, and*

$$\lim_{n\to+\infty} H^{N-1}(\partial D_n \cap \overline{\Omega}) = \mathrm{Per}_\Omega(A).$$

Theorem 5.2. *For any $u \in L^1(\Omega)$ and any $\alpha, \beta > 0$ with $\alpha+\beta < \mathcal{L}^N(\Omega)$, we define*

$$F_{\alpha\beta}(u) := \begin{cases} E(u) & \text{if } u \in SBV^2(\Omega),\ \mathcal{L}^N(\{u \leq 0\}) \geq \alpha,\ \mathcal{L}^N(\{u \geq 1\}) \geq \beta \\ +\infty & \text{otherwise} \end{cases}$$

and

$$G_\gamma(u) := \begin{cases} 2\mathrm{Per}_\Omega(A) & \text{if } u = \chi_A \text{ and } \mathcal{L}^N(A)=\gamma \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\Gamma(L^1) - \lim_{\substack{\alpha \to \mathcal{L}^N(\Omega)-\gamma \\ \beta \to \gamma \\ \alpha+\beta < \mathcal{L}^N(\Omega)}} F_{\alpha\beta}(u) = G_\gamma(u), \quad \forall u \in SBV^2(\Omega).$$

Proof. We adapt the proof of a similar result obtained in [2].

Without loss of generality we can assume that $\mathcal{L}^N(\Omega)=1$. We fix sequences $\{\alpha_n\}$ and $\{\beta_n\}$, converging to $(1-\gamma)$ and γ, respectively, and we denote by $F^+(u), F^-(u)$, the upper and lower Γ-limits

$$F^+(u) := \inf_{\{u_n\}}\left\{\limsup_{n\to+\infty} F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^1(\Omega)\right\}$$

and

$$F^-(u) := \inf_{\{u_n\}}\left\{\liminf_{n\to+\infty} F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^1(\Omega)\right\}.$$

We must prove that $F^- \geq G_\gamma \geq F^+$.

Step 1. We first establish the inequality $F^- \geq G_\gamma$ by showing that

$$\liminf_{n\to+\infty} F_{\alpha_n\beta_n}(u_n) \geq G_\gamma(u) \tag{5.2}$$

for any sequence $\{u_n\}$ converging to u in $L^1(\Omega)$. It is not restrictive to assume that the lim inf in (5.2) is a finite limit, and to assume, by a truncation argument, that $0 \leq u_n \leq 1$. We first prove that $u = \chi_A$ is a characteristic function and that $\mathcal{L}^N(A) = \gamma$. Indeed, by Proposition 3.1 applied to the closed sets $\{0\}$ and $\{1\}$, we deduce that

$$\mathcal{L}^N(\{u=0\}) \geq \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 0\}) \geq 1-\gamma$$

and

$$\mathcal{L}^N(\{u=1\}) \geq \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 1\}) \geq \gamma.$$

In particular, there exists a Borel set $A \subset \Omega$ such that $u = \chi_A$ and from the previous inequalities we obtain

$$\gamma \leq \mathcal{L}^N(\{u=1\}) \leq \mathcal{L}^N(A)$$

and

$$1-\gamma \leq \mathcal{L}^N(\{u=0\}) \leq \mathcal{L}^N(\Omega \setminus A) = 1 - \mathcal{L}^N(A)$$

so that $\mathcal{L}^N(A) = \gamma$ as claimed. We now show that

$$\liminf_{n\to+\infty} E(u_n) \geq 2\mathrm{Per}_\Omega(A)$$

thus establishing (5.2). Indeed, since $u_n \to u$ in $L^1(\Omega)$, by the lower semicontinuity of the total variation we have

$$\begin{aligned}
2\mathrm{Per}_\Omega(A) &= 2|D\chi_A|(\Omega) \leq 2\liminf_{n\to+\infty}|Du_n|(\Omega)\\
&= 2\liminf_{n\to+\infty}\left[\int_{L_n}|\nabla u_n(x)|\,\mathrm{d}x + \int_{S_{u_n}\cap\Omega}|[u_n](x)|\,\mathrm{d}H^{N-1}(x)\right],
\end{aligned}$$

where $L_n := \{0 < u_n < 1\}$. By Hölder's inequality, and as $|[u_n](x)| \leq 1$, it follows that

$$\begin{aligned}
2\mathrm{Per}_\Omega(A) &\leq \liminf_{n\to+\infty}\left[2\left(\int_\Omega |\nabla u_n(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\mathcal{L}^N(L_n)\right)^{\frac{1}{2}} + \int_{S_{u_n}\cap\Omega} 2|[u_n](x)|\,\mathrm{d}H^{N-1}(x)\right]\\
&\leq \liminf_{n\to+\infty}\left[\int_\Omega |\nabla u_n(x)|^2\,\mathrm{d}x + \mathcal{L}^N(L_n) + \int_{S_{u_n}\cap\Omega} 1 + |[u_n](x)|\,\mathrm{d}H^{N-1}(x)\right]\\
&\leq \liminf_{n\to+\infty}\left[E(u_n) + (1-(\alpha_n+\beta_n))\right]\\
&= \liminf_{n\to+\infty} E(u_n)\cdot
\end{aligned}$$

Step 2. We now prove that $F^+(u) \leq G_\gamma(u)$. It is not restrictive to assume that $u = \chi_A$ is a characteristic function, $\mathcal{L}^N(A) = \gamma$ and $\mathrm{Per}_\Omega(A) < +\infty$.

We first assume that $A = D \cap \Omega$ for some bounded, open set D with smooth boundary in $\mathbb{R}^N$, and we prove that

$$F^+(u) \leq 2H^{N-1}(\partial D \cap \overline{\Omega})\cdot \tag{5.3}$$

Let $d(x)$ be the signed-distance function from ∂D, *i.e.*

$$d(x) := \begin{cases} \mathrm{dist}(x, \partial D) & \text{if } x \notin D \\ -\mathrm{dist}(x, \partial D) & \text{if } x \in D. \end{cases}$$

Since $1 - (\alpha_n + \beta_n) \to 0$, for any $\sigma > 0$ and for n large enough,

$$\mathcal{L}^N(\{x \in \Omega : |d(x)| < \sigma\}) > 1 - (\alpha_n + \beta_n),$$

and hence we may find λ_n, $\mu_n \in (-\sigma, \sigma)$ such that $\lambda_n < 0 < \mu_n$ and

$$\mathcal{L}^N(\{x \in \Omega : d(x) \leq \lambda_n\}) = \beta_n, \qquad \mathcal{L}^N(\{x \in \Omega : d(x) \geq \mu_n\}) = \alpha_n.$$

By construction, the functions

$$u_n(x) := \begin{cases} \frac{1}{2}\min(d(x), \mu_n) - \frac{1}{2}\mu_n & \text{if } x \notin D \\ 1 - \frac{\lambda_n + \mu_n}{2} - \frac{1}{2}\max(d(x), \lambda_n) + \frac{1}{2}\mu_n + \lambda_n & \text{if } x \in D \end{cases}$$

converge to u in $L^1(\Omega)$ and satisfy the constraints

$$\begin{aligned} \mathcal{L}^N(\{u_n \leq 0\}) \geq \mathcal{L}^N(\{u_n = 0\}) \geq \alpha_n, \\ \mathcal{L}^N(\{u_n \geq 1\}) \geq \mathcal{L}^N(\{u_n = 1\}) \geq \beta_n. \end{aligned}$$

Thus, using the identity $|\nabla d| = 1$, we have

$$\begin{aligned} F^+(u) &\leq \limsup_{n\to\infty} E(u_n) \\ &\leq \limsup_{n\to\infty} \left(\int_\Omega |\nabla u_n(x)|^2 \, \mathrm{d}x + \int_{S_{u_n}\cap\Omega} 1 + |[u_n](x)| \, \mathrm{d}H^{N-1}(x) \right) \\ &= \limsup_{n\to\infty} \left(\frac{1}{4}\mathcal{L}^N(\{\lambda_n < d(x) < \mu_n\}) + \left(1 + \left|1 + \frac{\lambda_n + \mu_n}{2}\right|\right) H^{N-1}(\partial D \cap \overline{\Omega}) \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{4}(1 - \alpha_n - \beta_n) + 2H^{N-1}(\partial D \cap \overline{\Omega}) \right) \\ &= 2H^{N-1}(\partial D \cap \overline{\Omega}) \end{aligned}$$

and (5.3) is proved.

It remains to show that

$$F^+(u) \leq 2\mathrm{Per}_\Omega(A)$$

for a general set A of finite perimeter. By the previous lemma we can find a sequence of bounded, open sets D_n with smooth boundary in $\mathbb{R}^N$ such that $u_n := \chi_{D_n\cap\Omega}$ converge to $u = \chi_A$ in $L^1(\Omega)$, $\mathcal{L}^N(D_n \cap \Omega) = \mathcal{L}^N(A) = \gamma$ and

$$\lim_{n\to+\infty} H^{N-1}(\partial D_n \cap \overline{\Omega}) = \mathrm{Per}_\Omega(A).$$

Hence, by inequality (5.3) and using the lower semicontinuity of $u \mapsto F^+(u)$ (see, for instance [8]), we obtain

$$F^+(u) \leq \liminf_{n\to+\infty} F^+(u_n) \leq 2\liminf_{n\to+\infty} H^{N-1}(\partial D_n \cap \overline{\Omega}) = 2\mathrm{Per}_\Omega(A).$$

$\square$

From Theorem 5.2, recalling that Γ-convergence ensures that minimizers of $(P_{\alpha\beta})$ converge to minimizers of (5.1), and that minima for $(P_{\alpha\beta})$ tend to the minimum for the limit problem, follows the main result of this section.

Theorem 5.3. *For any* $\gamma \in (0, \mathcal{L}^N(\Omega))$,

$$\lim_{\substack{\alpha \to \mathcal{L}^N(\Omega) - \gamma \\ \beta \to \gamma \\ \alpha + \beta < \mathcal{L}^N(\Omega)}} m_{\alpha\beta} = 2p_\gamma.$$

Moreover, any limit point in the $L^1(\Omega)$ *topology of* $u_{\alpha\beta}$ *is the characteristic function of a minimizing set for (5.1).*

Proof. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences converging to $1-\gamma$ and γ, respectively, and let $u_n \in SBV^2(\Omega; [0,1])$ be the corresponding solutions of $(P_{\alpha_n\beta_n})$. As before, we may assume that $0 \le u_n \le 1$. By the general properties of Γ-convergence (see [8]), it suffices to show that the sequence $\{u_n\}$ is relatively compact in $L^1(\Omega)$.

Let $A \subset \Omega$ be a set of finite perimeter with $\mathcal{L}^N(A) = \gamma$, and, in view of Theorem 5.2, let $\{v_n\}$ be a sequence converging to χ_A in $L^1(\Omega)$ such that

$$\lim_{n \to +\infty} F_{\alpha_n\beta_n}(v_n) = 2\mathrm{Per}_\Omega(A).$$

Since u_n are minimizers of $E(\cdot)$, we have

$$\limsup_{n \to +\infty} F_{\alpha_n\beta_n}(u_n) \le 2\mathrm{Per}_\Omega(A).$$

Setting $L_n := \{0 < u_n < 1\}$, by Hölder's inequality it follows that

$$\begin{aligned}
|Du_n|(\Omega) &= \int_{L_n} |\nabla u_n(x)|\,\mathrm{d}x + \int_{S_{u_n}\cap\Omega} |[u_n(x)]|\,\mathrm{d}H^{N-1}(x) \\
&\le \left(\int_{L_n} |\nabla u_n(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\mathcal{L}^N(L_n)\right)^{\frac{1}{2}} + \int_{S_{u_n}\cap\Omega} 1 + |[u_n(x)]|\,\mathrm{d}H^{N-1}(x) \\
&\le \int_{\Omega} |\nabla u_n(x)|^2\,\mathrm{d}x + \mathcal{L}^N(L_n) + \int_{S_{u_n}\cap\Omega} 1 + |[u_n(x)]|\,\mathrm{d}H^{N-1}(x) \\
&= E(u_n) + \mathcal{L}^N(L_n).
\end{aligned}$$

Thus,

$$\limsup_n |Du_n|(\Omega) \le \limsup_n \left(E(u_n) + 1 - \alpha_n - \beta_n\right) \le 2\mathrm{Per}_\Omega(A).$$

On the other hand, $|u_n| \le 1$ and so u_n is uniformly bounded in $BV(\Omega)$. Since the embedding $BV(\Omega) \subset L^1(\Omega)$ is compact, the conclusion follows. □

The authors wish to thank L. Mascarenhas and J.P. Matos for their comments on the subject of this paper and also the referee, whose queries led to the improvement of the proof of Theorem 3.3. The research of Ana Cristina Barroso was partially supported by FCT, PRAXIS XXI and project PRAXIS 2/2.1/MAT/125/94. The research of José Matias was partially supported by a FCT grant through the Research Units Pluriannual Funding Program.

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