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LOCAL EXACT BOUNDARY CONTROLLABILITY OF ENTROPY SOLUTIONS TO LINEARLY DEGENERATE QUASILINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS *, **

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Abstract. In this paper, we study the local exact boundary controllability of entropy solutions to linearly degenerate quasilinear hyperbolic systems of conservation laws with characteristics of constant multiplicity. We prove the two-sided boundary controllability, the one-sided boundary controllability and the two-sided boundary controllability with fewer controls, by applying the strategy used in [T. Li and L. Yu, *J. Math. Pures et Appl.* **107** (2017) 1–40; L. Yu, *Chinese Ann. Math., Ser. B* (To appear)]. Our constructive method is based on the well-posedness of semi-global solutions constructed by the limit of ε-approximate front tracking solutions to the mixed initial-boundary value problem with general nonlinear boundary conditions, and on some further properties of both ε-approximate front tracking solutions and limit solutions.

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1. Introduction and main results

In this paper, we study the local exact boundary controllability of entropy solutions to $n\times n$ linearly degenerate quasilinear hyperbolic systems of conservation laws in one space dimension:

$$\partial_t H(u) + \partial_x G(u) = 0, \qquad t>0,\ 0<x<L, \tag{1.1}$$

where u is an n-vector valued unknown function of (t,x), G and H are smooth n-vector valued functions of u, defined on a ball $B_r(0)$ centered at the origin in $\mathbb{R}^n$ with suitable small radius r. Roughly speaking, we will consider the following question: For any given admissible initial and final states which are both close to the equilibrium state, is it possible to find suitable boundary controls such that the solution to the corresponding forward mixed initial-boundary value problem with general nonlinear boundary conditions reaches the given final state in a finite time.

Keywords and phrases. Linearly degenerate quasilinear hyperbolic systems of conservation laws, local exact boundary controllability, semi-global entropy solutions, ε-approximate front tracking solutions.

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Most results on the exact boundary controllability for 1-D quasilinear hyperbolic systems have been obtained in the framework of classical solutions. In particular, Li and Rao have systematically studied the local exact boundary controllability for general quasilinear hyperbolic systems based on the well-posedness of semi-global classical solutions (see [17–20] and the references therein).

It is well known that classical solutions to quasilinear hyperbolic conservation laws usually blow up in a finite time even if the initial data are sufficiently smooth. Thus it is natural to consider the entropy solution containing shocks, which are more important in many physical phenomena. The study on the exact boundary controllability for entropy solutions to nonlinear hyperbolic conservation laws was initiated in the scalar case by various methods, for instance, the generalized characteristics method applied in [3, 23] and the uniform control in vanishing viscosity in [10, 16]. However, the corresponding study in the system case is still quite open. In [7] Bressan and Coclite have shown that for a class of 2×2 systems (including DiPerna's system) there are initial data with small total variation, such that the corresponding entropy solutions can not reach a constant state in a finite time by any boundary control if the entropy solutions remain to have a small total variation. It implies that unlike in the case of classical solutions, one can not expect an analogous result on the exact boundary controllability for general quasilinear hyperbolic system of conservation laws in the framework of entropy solutions. Therefore, in the system case, there were only some results concerning special models of hyperbolic conservation laws, for example, for Temple system [2] and Euler equations [11, 12]. Recently, in [21, 24], the one-sided exact boundary null controllability of entropy solutions was firstly studied for a class of general quasilinear hyperbolic systems of conservation laws under the assumption that all negative (or positive) characteristic families are linearly degenerate.

In the present paper, following a similar strategy used in [21, 24], we study the local exact boundary controllability of entropy solutions to general linearly degenerate quasilinear hyperbolic systems of conservation laws (1.1) with characteristic of constant multiplicity. More precisely, we consider system (1.1) under the following assumptions:

(H1) System (1.1) is hyperbolic, that is, for any given $u \in B_r(0)$, the matrix $DH(u)$ is non-singular and the matrix $(DH(u))^{-1}DG(u)$ has n real eigenvalues $\lambda_i(u)$ $(i = 1, \dots, n)$ and a complete set of left (resp. right) eigenvectors $\{l_1(u), \dots, l_n(u)\}$ (resp. $\{r_1(u), \dots, r_n(u)\}$).

(H2) For any given $u \in B_r(0)$, each eigenvalue of $(DH(u))^{-1}DG(u)$ has a constant multiplicity. To fix the idea, we suppose that

$$\lambda_1(u) < \ldots < \lambda_k(u) < \lambda_{k+1}(u) \equiv \ldots \equiv \lambda_{k+p}(u) < \lambda_{k+p+1}(u) < \ldots < \lambda_n(u),$$

where $\lambda(u) := \lambda_{k+1}(u) \equiv \ldots \equiv \lambda_{k+p}(u)$ is an eigenvalue with constant multiplicity $p \geq 1$. When $p = 1$, system (1.1) is *strictly hyperbolic*.

(H3) There are no zero eigenvalues, that is, either all eigenvalues are negative (or positive), or there exist an $m \in \{1, \dots, n-1\}$ and a constant $c > 0$, such that

$$\lambda_m(u) < -c < 0 < c < \lambda_{m+1}(u), \qquad \forall u \in B_r(0). \tag{1.2}$$

Under this assumption $DG(u)$ is also a non-singular matrix. Without loss of generality, we assume that the eigenvalue $\lambda(u)$ is negative. In what follows we only consider the situation (1.2). Other situations can be treated similarly.

(H4) All eigenvalues $\lambda_i(u)$ $(i = 1, \dots, n)$ are linearly degenerate in the sense of Lax [15], that is,

$$D\lambda_i(u) \cdot r_i(u) \equiv 0, \qquad \forall u \in B_r(0).$$

In fact, the eigenvalue $\lambda(u)$ with constant multiplicity $p \geq 2$ must be linearly degenerate (see Lem. 2.1).

(H5) Assume that system (1.1) possesses an ***entropy-entropy flux pair***. Recall that (η, ζ) is an entropy-entropy flux pair of system (1.1) if $\eta : B_r(0) \to \mathbb{R}$ is a continuously differentiable convex function and $\zeta : B_r(0) \to \mathbb{R}$ is a continuously differentiable function, satisfying

$$D\eta(u)(DH(u))^{-1}DG(u) = D\zeta(u), \quad \forall u \in B_r(0).$$

By (H3), the boundaries $x = 0$ and $x = L$ are non-characteristic. We prescribe the following general nonlinear boundary conditions:

$$x = 0 : \; b_1(u) = g_1(t), \tag{1.3}$$

$$x = L : \; b_2(u) = g_2(t), \tag{1.4}$$

where $g_1 : \mathbb{R}^+ \to \mathbb{R}^{n-m}$, $g_2 : \mathbb{R}^+ \to \mathbb{R}^m$ are given boundaries data, and $b_1 \in \mathbf{C^1}(B_r(0);\, \mathbb{R}^{n-m})$, $b_2 \in \mathbf{C^1}(B_r(0); \mathbb{R}^m)$. Here the value of $u(t, 0)$ and $u(t, L)$ should be understood as the inner trace of the function $u(t, x)$ on the boundaries $x = 0$ and $x = L$, respectively.

In order to guarantee the well-posedness for the forward mixed initial-boundary value problem of system (1.1), we assume that

(H6) b_1 and b_2 satisfy the following conditions, respectively (see [22]):

$$\det\left[Db_1(u) \ldots r_{m+1}(u), \ldots, Db_1(u) \cdot r_n(u)\right] \neq 0,$$

$$\det\left[Db_2(u) \cdot r_1(u), \ldots, Db_2(u) \cdot r_m(u)\right] \neq 0, \qquad \forall u \in B_r(0).$$

Remark 1.1. In this paper, we concentrate our attention on the linearly degenerate case which covers some important physical situations, for example, the system of isentropic gas dynamics satisfying Chaplygin pressure law, the system for time-like extremal surfaces in (1+n)-dimensional Minkowski time-space, some kind of elastic spring system and some kind of hyperbolic conservation laws with rotational symmetry, etc (see [13, 14]).

Now we can write the forward mixed initial-boundary value problem of system (1.1) as follows:

$$\begin{cases} \partial_t H(u) + \partial_x G(u) = 0, & t > 0,\ 0 < x < L, \\ t = 0 : \; u = \bar{u}(x), & 0 < x < L, \\ x = 0 : \; b_1(u) = g_1(t), & t > 0, \\ x = L : \; b_2(u) = g_2(t), & t > 0. \end{cases} \tag{1.5}$$

Definition 1.2. For any given $T > 0$, $u = u(t, x) \in \mathbb{L}^1((0, T) \times (0, L))$ is an *entropy solution to system* (1.1) on the domain $\mathbb{D}_T := \{\ 0 < t < T,\ 0 < x < L\}$ if:

(1) u is a weak solution to (1.1) on the domain $\mathbb{D}_T$ in the sense of distributions, that is, for every $\phi \in C_c^1(\mathbb{D}_T)$ we have

$$\int_0^T \int_0^L \partial_t \phi(t, x) H(u(t, x)) + \partial_x \phi(t, x) G(u(t, x)) \mathrm{d}x\mathrm{d}t = 0.$$

(2) u is entropy admissible in the sense that there exists an entropy-entropy flux pair (η, ζ) of system (1.1), such that for every non-negative function $\phi \in C_c^1(\mathbb{D}_T)$ we have

$$\int_0^T \int_0^L \partial_t \phi(t, x) \eta(u(t, x)) + \partial_x \phi(t, x) \zeta(u(t, x)) \mathrm{d}x\mathrm{d}t \geq 0. \tag{1.6}$$

Moreover, if u also satisfies the initial-boundary conditions, that is, $\lim_{t\to 0+} u(t,x) = \bar{u}(x)$ for a.e. $x \in (0,L)$ and

$$\lim_{x\to 0+} b_1(u(t,x)) = g_1(t), \qquad \lim_{x\to L-} b_2(u(t,x)) = g_2(t) \quad \text{for a.e. } t\in(0,T),$$

then we say that u is an *entropy solution to the mixed initial-boundary value problem* (1.5) on the domain $\mathbb{D}_T$.

The main results of this paper are given by the following three theorems.

Theorem 1.3 (Two-sided boundary controllability)**.** *Under the Assumptions* (H1)−(H6)*, if*

$$T > L\max\left\{\frac{1}{|\lambda_m(0)|}, \frac{1}{\lambda_{m+1}(0)}\right\}, \tag{1.7}$$

then for any given initial data $\bar{u}$ *and final data* $\bar{\bar{u}}$ *with* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{u}) + |\bar{u}(0+)|$ *and* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ *sufficiently small, there exist boundary controls* g_1 *and* g_2 *with* $\operatorname*{Tot.Var.}_{0<t<T}(g_1) + |g_1(0+)|$ *and* $\operatorname*{Tot.Var.}_{0<t<T}(g_2) + |g_2(0+)|$ *sufficiently small, such that the forward mixed initial-boundary value problem* (1.5) *admits an entropy solution* $u = u(t,x)$ *on the domain* $\mathbb{D}_T$*, satisfying exactly the final condition*

$$t = T: \quad u = \bar{\bar{u}}, \qquad 0 < x < L. \tag{1.8}$$

Theorem 1.4 (**One-sided boundary controllability**)**.** *Under the assumptions* (H1)−(H6)*, suppose furthermore that*

$$\bar{m} := n - m \le m \tag{1.9}$$

and

$$\operatorname{rank}\big([Db_1(u)\cdot r_1(u), \dots, Db_1(u)\cdot r_m(u)]\big) = \bar{m}, \quad \forall u \in B_r(0). \tag{1.10}$$

If

$$T > L\left\{\frac{1}{|\lambda_m(0)|} + \frac{1}{\lambda_{m+1}(0)}\right\}, \tag{1.11}$$

then for any given initial data $\bar{u}$ *and final data* $\bar{\bar{u}}$ *with* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{u}) + |\bar{u}(0+)|$ *and* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ *sufficiently small, and for any given boundary data* g_1 *on* $x = 0$ *with* $\operatorname*{Tot.Var.}_{0<t<T}(g_1) + |g_1(0+)|$ *sufficiently small, there exists a boundary control* g_2*, acting on the boundary* $x = L$*, such that the forward mixed initial-boundary value problem* (1.5) *admits an entropy solution* $u = u(t,x)$ *on the domain* $\mathbb{D}_T$*, satisfying exactly the final condition* (1.8)*.*

Theorem 1.5 (**Two-sided boundary controllability with fewer controls**)**.** *Under the assumptions* (H1)−(H)*, suppose furthermore that* (1.9) *holds. Let* $\tilde{b}_2 : B_r(0) \to \mathbb{R}^{\bar{m}}$ *be the vector-value function consists of the first* $\bar{m}$ *components of* b_2*. Without loss of generality, suppose that*

$$\operatorname{rank}\left(\left[D\tilde{b}_2(u)\cdot r_{m+1}(u), \dots, D\tilde{b}_2(u)\cdot r_n(u)\right]\right) = \bar{m}, \quad \forall u \in B_r(0).$$

If $T > 0$ *satisfies* (1.11)*, then for any given initial data* $\bar{u}$ *and final data* $\bar{\bar{u}}$ *with* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{u}) + |\bar{u}(0+)|$ *and* $\operatorname*{Tot.Var.}_{0<x<L}(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ *sufficiently small, and for any given boundary data* $\tilde{g}_2 : (0,T) \to \mathbb{R}^{\bar{m}}$ *with* $\operatorname*{Tot.Var.}_{0<t<T}(\tilde{g}_2) + |\tilde{g}_2(0+)|$ *sufficiently small, there exists boundary control* g_1 *on* $x = 0$ *and boundary control* $\hat{g}_2 : (0,T) \to \mathbb{R}^{m-\bar{m}}$ *on* $x = L$*, such that the forward mixed initial-boundary value problem* (1.5) *associated with* $g_2 = (\tilde{g}_2, \hat{g}_2)^t$ *admits an entropy solution* $u = u(t,x)$ *on the domain* $\mathbb{D}_T$*, satisfying exactly the final condition* (1.8)*.*

Remark 1.6. In [21], we have proved the one-sided exact boundary null controllability for a class of hyperbolic conservation laws with linearly degenerate negative (or positive) characteristics and gave the sharp controllability time. In the present situation, although the corresponding two-sided boundary controllability can be obtained by using the result in [21] twice, however, the estimate of controllability time is no longer optimal in that way. In this paper, under the assumption that all eigenvalues are linearly degenerate, we obtain systematically three kinds of exact boundary controllability for system (1.1), where the final state can be any given small BV function close to the equilibrium $u = 0$, and the estimates (1.7) and (1.11) on the controllability time are sharp, respectively. Moreover, in the present results, compared with the previous ones, we still have some advantages such as the number n of equations in system (1.1) can be any integer ≥ 1 and the general nonlinear boundary conditions are taken into account.

Remark 1.7. In [13, 14], the authors obtained the exact two-sided boundary controllability for 2×2 linearly degenerate hyperbolic system, with the assumption $\lambda_1(u) < 0 < \lambda_2(u)$ in the framework of C^1-solutions. In fact, for any given initial data

$$\bar{u}(x) = \begin{cases} (\bar{u}_1^-(x), \bar{u}_2^-(x)) \in C^1([0, L/2]) \times C^1([0, L/2]), \\ (\bar{u}_1^+(x), \bar{u}_2^+(x)) \in C^1([L/2, L]) \times C^1([L/2, L]) \end{cases}$$

and final data

$$\bar{\bar{u}}(x) = \begin{cases} (\bar{\bar{u}}_1^-(x), \bar{\bar{u}}_2^-(x)) \in C^1([0, L/2]) \times C^1([0, L/2]), \\ (\bar{\bar{u}}_1^+(x), \bar{\bar{u}}_2^+(x)) \in C^1([L/2, L]) \times C^1([L/2, L]), \end{cases}$$

there exists a time $T > 0$ and control inputs $g_1(t), g_2(t)$ in the class of piecewise C^1 functions defined on the domain $\mathbb{D}_T$, such that the corresponding initial-boundary value problem (1.5) ($n = 2$) possesses a piecewise C^1-solution $u = u(t, x)$ containing contact discontinuities and satisfying the final condition (1.8).

As in [21], throughout this paper, the *solution* to a mixed initial-boundary value problem (1.5) means the limit of a convergent sequence of corresponding ε-solutions (which are usually called ε-approximate front tracking solutions as in [21]). This kind of solutions are actually entropy solutions, provided that system (1.1) possesses an entropy-entropy flux pair.

Following the strategy given in [21], we can obtain various kinds of exact boundary controllability as in the framework of classical solutions (see [17]). In fact, in order to apply the constructive method introduced in [17], we need to establish the following three basic ingredients:

(1) The well-posedness of semi-global solutions to the mixed initial-boundary value problem for linearly degenerate quasilinear hyperbolic systems of conservation laws with characteristics with constant multiplicity.
(2) The fact that the solution $u = u(t, x)$ to the forward mixed problem of system (1.1) on the domain $\mathbb{D}_T := \{0 < t < T,\ 0 < x < L\}$ is also a solution to the backward problem of system (1.1) on the domain $\mathbb{D}_T$. Moreover, u is also a solution to the corresponding leftward or rightward problem of system

$$\partial_x G(u) + \partial_t H(u) = 0 \tag{1.12}$$

on the same domain, where the role of t and x is exchanged such that x is regarded as the "time" variable and t is regarded as the "space" variables.
(3) The determinate domain of solution to the one-sided mixed initial-boundary value problem.

We have already proved the facts (1) and (3) in [21, 24] for quasilinear hyperbolic systems of conservation laws whose positive (resp. negative) characteristic families are all linearly degenerate. For that class of systems, we also obtained the equivalence between the solution to the forward problem for system (1.1) and the rightward (resp. leftward) problem for system (1.12). In this paper, under a stronger assumption that all characteristic families are linear degenerate, we can prove the stronger fact (2). The key idea is that under the linearly degenerate assumption the entropy inequality (1.6) is actually an equality, therefore we can solve the backward problem for system (1.1) in the same way as for the forward problem, and the solutions are equivalent in both senses.

The paper is organized as follows. In Section 2 we recall the results of well-posedness of semi-global solutions as the limits of ε-solutions , which are mainly proved in [21, 24]. In Section 3, based on the properties of semi-global solutions obtained in Section 2, we give the proof of Theorems 1.3–1.5.

2. Semi-global solutions

In this section, we collect some results about the well-posedness of semi-global solutions to the forward mixed initial-boundary value problem (1.5), which are proved in [21,24]. In fact, all the results (except some in Sect. 2.3) hold for more general systems whose characteristic families are either genuinely nonlinear or linearly degenerate.

Throughout this paper, in order to avoid abusively using constants, we denote by the notation C a positive constant which depends only on system (1.1), constant L and functions b_1, b_2, but is independent of the special choice of initial data $\bar{u}$, boundary data g_1, g_2 and time T. Moreover, we denote by $C(T)$ a positive constant which depends also on time T.

2.1. Preliminaries

For system (1.1), we normalize the left and right eigenvectors $l_i(u)$ and $r_i(u)$ $(i = 1, \dots, n)$ of $(DH)^{-1}DG(u)$, so that

$$l_i(u) \cdot r_j(u) \equiv \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \quad i, j = 1, \dots, n, \ \forall u \in B_r(0).$$

For any given $u \in B_r(0)$, when the eigenvalue $\lambda_i(u)$ is simple, we know that the i-rarefaction curve and the i-shock curve coincide if $\lambda_i(u)$ is linearly degenerate. Let $\sigma \mapsto R_i(\sigma)[u]$ denote the i-rarefaction curve passing through u. When the eigenvalue $\lambda(u)$ has constant multiplicity $p \geq 2$, we have the following

Lemma 2.1 [5,9]**.** *The eigenvalue $\lambda(u)$ with constant multiplicity $p \geq 2$ must be linearly degenerate, namely,*

$$\nabla\lambda(u) \cdot r_j(u) \equiv 0 \quad (j = k+1, \dots, k+p), \quad \forall u \in B_r(0).$$

Moreover, for any given $u^- \in B_r(0)$, there exists a p-dimensional connected smooth manifold $\Sigma(u^-)$ in a neighborhood of u^- with $u^- \in \Sigma(u^-)$, where $\Sigma(u^-)$ can be expressed by the following smooth parametric representation

$$u = \Psi_{k+1}(\sigma_{k+p}, \dots, \sigma_{k+1})[u^-], \quad \sigma_j \in [-\sigma_0, \sigma_0] \quad (j = k+1, \dots k+p)$$

for some small σ_0, such that

$$\frac{\partial}{\partial \sigma_j} u(0, .., 0)[u^-] = r_j, \quad (j = k+1, \dots, k+p).$$

In other words, for any given $u^+ \in \Sigma(u^-)$, there exist uniquely small numbers $\sigma_{k+1}, \dots, \sigma_{k+p}$ such that $u^+ = \Psi_{k+1}(\sigma_{k+p}, \dots, \sigma_{k+1})[u^-]$, and any discontinuity associated with the eigenvalue $\lambda(u)$

$$u_{k+1} = \begin{cases} u^+, & x > st, \\ u^-, & x < st \end{cases} \tag{2.1}$$

is always a contact discontinuity, i.e., we have

$$\begin{cases} G(u^+) - G(u^-) = s(H(u^+) - H(u^-)), \\ s = \lambda(u^-) = \lambda(u^-). \end{cases} \tag{2.2}$$

On the other hand, if u^+ is sufficiently close to u^-, then the solution (2.1) is a contact discontinuity implies that

$$\begin{cases} u^+ \in \Sigma(u^-), \\ s = \lambda(u^-), \end{cases} \tag{2.3}$$

which means that on the contact discontinuity (2.1) with small amplitude, condition (2.2) is equivalent to condition (2.3).

2.2. Solutions as the limit of ε-solutions

We first give the definition of ε-solutions in the linearly degenerate case .

Definition 2.2. Under the assumptions (H1)$-$(H6), for any given time $T > 0$ and any fixed $\varepsilon > 0$, we say that a continuous map

$$t \mapsto u^\varepsilon(t, \cdot) \in \mathbb{L}^1(0, L), \quad \forall t \in (0, T)$$

is an ε*-solution to system* (1.1) on the domain $\mathbb{D}_T$ if

(1) $u^\varepsilon = u^\varepsilon(t, x) \in B_r(0)$ for all $(t, x) \in \mathbb{D}_T$, and is piecewise constant with discontinuities occurring along finitely many straight lines with non-zero slope on the domain $\mathbb{D}_T$. Jumps can be of two types: physical fronts (contact discontinuities) and non-physical fronts, denoted by $\mathcal{P}$ and $\mathcal{NP}$, respectively.

(2) Along each i-physical front $x = x_\alpha(t)$ $(\alpha \in \mathcal{P})$, the left and right limits $u^{\mathrm{L}} := u^\varepsilon(t, x_\alpha(t)-)$ and $u^{\mathrm{R}} := u^\varepsilon(t, x_\alpha(t)+)$ of $u^\varepsilon(t, \cdot)$ on it are connected by

$$u^{\mathrm{R}} = R_i(\sigma_\alpha)[u^{\mathrm{L}}], \qquad \text{if} \quad i \in \{1, \ldots, k, k+p+1, \ldots, n\},$$
$$u^{\mathrm{R}} = \Psi_i(\sigma_{\alpha,p}, \ldots, \sigma_{\alpha,1})[u^{\mathrm{L}}], \qquad \text{if} \quad i = k+1,$$

where σ_α or $(\sigma_{\alpha,p}, \ldots, \sigma_{\alpha,1})$ denotes the wave amplitude. Moreover, the speed of the front approximately satisfies the Rankine-Hugoniot relation, that is,

$$|\dot{x}_\alpha(t) - \lambda_i(u^{\mathrm{L}})| \le C\varepsilon, \quad \forall t \in (0, T).$$

(3) All non-physical fronts $x = x_\alpha(t)$ $(\alpha \in \mathcal{NP})$ have the constant speed $\dot{x}_\alpha(t) \equiv \hat{\lambda}$ with either $\hat{\lambda} > \sup_{\substack{u \in B_r(0) \\ 1 \le i \le n}} |\lambda_i(u)|$ or $0 < \hat{\lambda} < c$, where c is given by (1.2). Moreover, the total strength of all non-physical waves in $u^\varepsilon(t, \cdot)$ is uniformly bounded by ε, namely,

$$\sum_{\alpha \in \mathcal{NP}} |u^\varepsilon(t, x_\alpha(t)+) - u^\varepsilon(t, x_\alpha(t)-)| \le \varepsilon, \qquad \forall t \in (0, T).$$

Moreover, if u^ε also satisfies approximatively the initial-boundary conditions, that is,

$$\|u^\varepsilon(0, \cdot) - \bar{u}(\cdot)\|_{\mathbb{L}^1(0,L)} \le \varepsilon,$$

$$\|b_1\big(u^\varepsilon(\cdot, 0+)\big) - g_1(\cdot)\|_{\mathbb{L}^1(0,T)} \le \varepsilon, \quad \|b_2\big(u^\varepsilon(\cdot, L-)\big) - g_2(\cdot)\|_{\mathbb{L}^1(0,T)} \le \varepsilon,$$

then $u^\varepsilon = u^\varepsilon(t, x)$ is called the ε*-solution to the mixed initial-boundary value problem* (1.5).

In [21, 24], the existence and stability of ε-solutions to the mixed initial-boundary value problem (1.5) on $\mathbb{D}_T$ have been proved. More precisely, for any given $T > 0$, any given initial-boundary data $(\bar{u}, g_1, g_2)$ and any given $\varepsilon > 0$ small enough, if $\Lambda(\bar{u}, g_1, g_2)$ is sufficiently small, where $\Lambda(\bar{u}, g_1, g_2) := \underset{0<x<L}{\text{Tot. Var.}}\,(\bar{u}) + |\bar{u}(0+)| + \sum_{i=1,2} \underset{0<t<T}{\text{Tot. Var.}}\,(g_i) + |b_1(\bar{u}(0+)) - g_1(0+)| + |b_2(\bar{u}(L-)) - g_2(0+)|$, we can construct an ε-solution $u^\varepsilon(t,x)$ to problem (1.5) on the domain $\mathbb{D}_T$ *via* an algorithm given in [21], such that for all $\varepsilon > 0$ small, the maps $t \mapsto u^\varepsilon(t,\cdot)$ are uniformly Lipschitz continuous in $\mathbb{L}^1$ norm with respect to t, and $\underset{0<x<L}{\text{Tot. Var.}}\,(u^\varepsilon(t,\cdot))$ remain uniformly sufficiently small for all $t \in (0,T)$. Moreover, since the stability holds for u^ε on the triangle domains

$$\mathfrak{L}(x_1) := \{(t,x) \mid 0 < t < \hat{\tau}_1(x_1), 0 < x < x_1(\hat{\tau}_1(x_1) - t)/\hat{\tau}_1(x_1)\}$$

and

$$\mathfrak{R}(x_0) := \{(t,x) \mid 0 < t < \hat{\tau}_2(x_0),\ (L - x_0)t/\hat{\tau}_2(x_0) + x_0 < x < L\}$$

for any given $x_1 \in (0,L]$ and $x_0 \in [0,L)$, where

$$\hat{\tau}_1(x_1) = x_1 \min_{u\in B_r(0)} \{|\lambda_1(u)|^{-1}\} \quad \text{and} \quad \hat{\tau}_2(x_0) = (L - x_0) \min_{u\in B_r(0)} \{\lambda_n(u)^{-1}\},$$

by induction, we obtain the stability of ε-solutions on the domain $\mathbb{D}_T$.

Now, fixing a sequence $\{\varepsilon^\nu\}$ converging to zero as $\nu \to +\infty$, by Helly's (Thm. [6], Thm. 2.3), we can extract a subsequence of $\{u^\nu\}$ with $u^\nu = u^{\varepsilon^\nu}$, still denoted by $\{u^\nu\}$, which converges to a limit function $u = u(t,x)$ in $\mathbb{L}^1((0,T)\times(0,L))$. Moreover, under the assumption that system (1.1) possesses an entropy-entropy flux pair, the limit function $u = u(t,x)$ is actually an entropy solution to problem (1.5) on the domain $\mathbb{D}_T$, and the equality holds in (1.6) (see [6], Sect. 7.4). In fact, we have the following theorem.

Proposition 2.3. *For any fixed $T > 0$, there exist positive constants δ and $C(T)$ such that problem (1.5) associated with any given initial-boundary data $(\bar{u}, g_1^u, g_2^u)$ with $\Lambda(\bar{u}, g_1^u, g_2^u) \le \delta$ admits a solution $u = u(t,x)$ on the domain $\mathbb{D}_T = \{0 < t < T,\ 0 < x < L\}$ as the limit of a sequence of ε-solutions, which satisfies*

$$\underset{0<x<L}{\text{Tot. Var.}}\,(u(t,\cdot)) \le C(T)\Lambda(\bar{u}, g_1^u, g_2^u), \quad \forall t \in (0,T),$$
$$\|u(t,\cdot) - u(s,\cdot)\|_{\mathbb{L}^1(0,L)} \le C(T)|t - s|, \quad \forall t,s \in (0,T),$$

and $u(t,x) \in B_r(0)$ for a.e. $(t,x) \in \mathbb{D}_T$.

Moreover, if $v = v(t,x)$ is a solution as the limit of a sequence of ε-solutions of problem (1.5) associated with the initial-boundary data $(\bar{v}, g_1^v, g_2^v)$ with $\Lambda(\bar{v}, g_1^v, g_2^v) \le \delta$, then for any given $x_0 \in [0,L)$ and $x_1 \in (0,L]$, there exist a positive constant C independent of x_0 and x_1, such that

$$\begin{aligned}&\|u(t,\cdot) - v(t,\cdot)\|_{\mathbb{L}^1(\mathfrak{L}_t(x_1))}\\ \le& C\left(\|\bar{u} - \bar{v}\|_{\mathbb{L}^1(0,x_1)} + \int_0^t |g_1^u(s) - g_1^v(s))|\mathrm{d}s\right), \quad \forall t \in [0, \hat{\tau}_1(x_1)]\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\|u(t,\cdot) - v(t,\cdot)\|_{\mathbb{L}^1(\mathfrak{R}_t(x_0))}\\ \le& C\left(\|\bar{u} - \bar{v}\|_{\mathbb{L}^1(x_0,L)} + \int_0^t |g_2^u(s) - g_2^v(s)|\mathrm{d}s\right), \quad \forall t \in [0, \hat{\tau}_2(x_0)],\end{aligned} \tag{2.5}$$

where

$$\mathfrak{L}_t(x_1) := \{x \mid 0 < x < x_1(\hat{\tau}_1(x_1) - t)/\hat{\tau}_1(x_1)\},$$
$$\mathfrak{R}_t(x_0) := \{x \mid (L - x_0)t/\hat{\tau}_2(x_0) + x_0 < x < L\},$$

and there exists a positive constant $C(T)$ depending on time T, such that

$$\|u(t,\cdot)-v(t,\cdot)\|_{\mathbb{L}^1(0,L)} \leq C(T)\left(\|\bar{u}-\bar{v}\|_{\mathbb{L}^1(0,L)} + \sum_{i=1,2}\int_0^t \left|g_i^u(s)-g_i^v(s)\right|\mathrm{d}s\right), \quad \forall t\in(0,T). \tag{2.6}$$

From (2.6), the solution provided by Proposition 2.3 is actually independent of different choices of the convergent sequence of ε-solutions.

Remark 2.4. The existence and stability of semi-global entropy solutions shown in Proposition 2.3 is a modification of the corresponding results from global in time entropy solutions to the one-sided initial-boundary value problem of system (1.1) proved in [1, 8] or to the initial-boundary value problem of system (1.1) with dissipative boundary conditions proved in [4], to semi-global entropy solutions to the initial boundary value problem without dissipative boundary conditions. As we mentioned in [21], according to (2.4) (resp. (2.5)), the triangle domain $\mathfrak{L}(x_1)$ (resp. $\mathfrak{R}(x_0)$) is the *determinate domain* of the solution to one-sided initial-boundary value problem (1.1) with the initial data on the interval $(0,x_1)$ (resp. (x_0,L)) and the boundary condition (1.3) (resp. (1.4)) on $x=0$ (resp. $x=L$). In particular, let $u=u(t,x)$ be the solution to problem (1.5) on the domain $\mathbb{D}_T$ given by Proposition 2.3 with $\Lambda(\bar{u},g_1,g_2)$ sufficiently small, for any given $x_0\in(0,L)$, if $\bar{u}\equiv 0$ on (x_0,L) and $g_2\equiv 0$ on the interval $(0,\hat{\tau}_2(x_0))$, then $u\equiv 0$ on the domain $\mathfrak{R}(x_0)\cap\mathbb{D}_T$.

2.3. Some further properties of ε-solutions and limit solutions

In order to prove Theorems 1.3–1.5, besides of the well-posedness of semi-global solution to problem (1.5), we also need to prove that an ε-solution to the forward problem is also an ε-solution in the leftward, rightward or backward sense. In fact, we have the following.

Lemma 2.5. *If u^ε is an ε-solution to system (1.1) in the forward sense, then:*

(i) *u^ε is also an ε-solution to the system*

$$\partial_x G(u)+\partial_t H(u)=0 \tag{2.7}$$

in the leftward or rightward sense.

(ii) *u^ε is also an ε-solution to system (1.1) in the backward sense.*

Proof. In [21, 24], we have proved that under the assumptions (H1)−(H3) and the assumption that all negative (resp. positive) eigenvalues are linearly degenerate, an ε-solution to system (1.1) in the forward sense is also an ε-solution to system (2.7) in the rightward (resp. leftward) sense, Under Assumption (H4), the negative and positive characteristics are all linearly degenerate. This yields the conclusion (i) immediately. Since a backward problem of (1.1) can be regarded as a rightward problem of (2.7), the conclusion (i) implies the conclusion (ii) immediately. □

Remark 2.6. For linearly degenerate hyperbolic systems of conservation laws, all jumps in entropy solutions are contact discontinuities. This means that the entropy inequality (1.6) is actually an equality. Therefore, the solution is reversible with respect to time t, which is in general not true for systems with genuinely nonlinear characteristic families.

Now, by passing to the limit, we obtain the corresponding results for semi-global solutions.

Proposition 2.7. *Suppose that u is a solution to system (1.1) in the forward sense, then u is also a solution to (1.1) in the backward sense. And vice versa. Moreover, u is also a solution to system (2.7) in the leftward or rightward sense.*

Remark 2.8. Since system (2.7) does not possess an entropy-entropy flux pair in general, even if system (1.1) possesses an entropy-entropy flux pair, the solution in the leftward or rightward sense of system (2.7) is not necessary to be an entropy solution, but it gives no influence to our consideration and final results.

Noting that the smallest and largest eigenvalues of system (2.7) are $\frac{1}{\lambda_m(u)}$ and $\frac{1}{\lambda_{m+1}(u)}$, respectively, by Remark 2.4 and applying the same argument in [21, 24], we can obtain the following proposition in which the initial-boundary condition is involved.

Proposition 2.9. *Suppose that $u = u(t,x)$ is a solution to forward problem (1.5) given by Proposition 2.3 on the domain $\{0 < t < T^*,\ 0 < x < L\}$ with $T^* \geq L \max_{u \in B_r(0)} \{\frac{1}{|\lambda_m(u)|}, \frac{1}{\lambda_{m+1}(u)}\}$. Then on the triangle domain $\{0 < t < T^*(L-x)/L,\ 0 < x < L\}$, u coincides with the solution $\tilde{u}$ to system (2.7) in the rightward sense, given by Proposition 2.3, associated with the initial condition*

$$x = 0: \ \tilde{u} = u(t, 0+), \quad 0 < t < T^*$$

and the following boundary condition reduced from the original initial data $\bar{u}$:

$$t = 0: \ \tilde{b}_1(\tilde{u}) = \tilde{b}_1(\bar{u}(x)), \quad x \in (0, L),$$

where $\tilde{b}_1 \in \mathbf{C^1}(B_r(0); \mathbb{R}^{n-m})$ is arbitrarily given, satisfying the same Assumption (H6) *as for b_1.*

Moreover, on the triangle domain $\{0 < t < T^* x/L,\ 0 < x < L\}$, u coincides with the solution $\tilde{u}$ to system (2.7) in the leftward sense, given by Proposition 2.3, associated with the initial condition

$$x = L: \ \tilde{u} = u(t, L-), \quad t \in (0, T^*)$$

and the boundary condition

$$t = 0: \ \tilde{b}_2(\tilde{u}) = \tilde{b}_2(\bar{u}(x)), \quad x \in (0, L),$$

where $\tilde{b}_2 \in \mathbf{C^1}(B_r(0); \mathbb{R}^m)$ is arbitrarily given, satisfying the same Assumption (H6) *as for b_2.*

Similar results hold for a solution $u = u(t,x)$ to the backward mixed initial-boundary value problem for system (1.1).

In the proof of Theorem 1.3, we need to consider two solutions obtained by solving system (2.7) leftward or rightward from $x = L/2$, respectively. We should prove that the combination of these two solutions gives a solution to system (1.1) in the forward sense. In fact, we have

Proposition 2.10. *Suppose that $u = u_l(t,x)$ (resp. $u = u_r(t,x)$) is a solution to system (2.7) in the leftward (resp. rightward) sense on the domain $\mathbb{D}_l := \{0 < t < T,\ 0 < x < L/2\}$ (resp. $\mathbb{D}_r := \{0 < t < T, L/2 < x < L\}$) with the initial condition*

$$x = L/2: \ u = a(t), \quad 0 < t < T.$$

Let

$$u(t,x) = \begin{cases} u_l(t,x), & (t,x) \in \mathbb{D}_l, \\ u_r(t,x) & (t,x) \in \mathbb{D}_r. \end{cases}$$

Then $u = u(t,x)$ is a solution to system (1.1) *in the forward sense.*

Proof. Suppose u_l (resp. u_r) is the limit of a sequence of ε^ν-solutions $u_l^\nu := u_l^{\varepsilon^\nu}$ (resp. $u_r^\nu := u_r^{\varepsilon^\nu}$). Since the limit solution is independent of choice of ε-solutions, we may assume without loss of generality that both u_l and

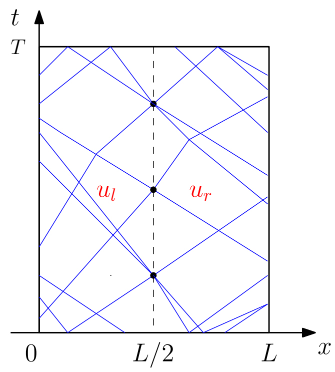

FIGURE 1. Fonts generated by approximate Riemann solver on the segment $x = L/2$.

u_r satisfy the same approximate initial data $a^\nu = a^\nu(t)$ which is a piecewise constant approximation of function $a = a(t)$. For each $\nu \geq 1$, let

$$u^\nu(t,x) = \begin{cases} u_l^\nu(t,x) & (t,x) \in \mathbb{D}_l, \\ u_r^\nu(t,x) & (t,x) \in \mathbb{D}_r. \end{cases}$$

By Lemma 2.5, in the interior of $\mathbb{D}_l$ (resp. $\mathbb{D}_r$), u_l^ν (resp. u_r^ν) is an ε^ν-solution to system (1.1) in the forward sense. Then, it suffices to clarify the situation near the segment $\mathfrak{S} := \{0 < t < T\} \times \{x = L/2\}$. In a small leftward (resp. rightward) neighborhood of $\mathfrak{S}$, u_l^ν (resp. u_r^ν) is obtained by an approximate Riemann solver at each jump point of $a^\nu = a^\nu(t)$ (see [21,24]). Since the speed of all fronts generated by the approximate Riemann solver is non-zero, there is no jump discontinuity for u^ν in a small neighborhood of the segment $\mathfrak{S}$ except for those jump points of a^ν (see Fig. 1). Moreover, since all these fronts are physical fronts, it is easy to check that they satisfy (2) of Definition 2.2 in the forward sense by the same argument in [21,24]. Therefore, u^ν must be an ε^ν-solution to system (1.1) in the forward sense. By passing to the limit, we obtain that $u = u(t,x)$ as the limit of sequence $u^\nu = u^\nu(t,x)$ is a solution to system (1.1) in the forward sense. □

3. LOCAL EXACT BOUNDARY CONTROLLABILITY

Now we are ready to apply the well-posedness of semi-global solutions obtained in the previous section to prove Theorems 1.3–1.5, respectively, namely, to realize the local exact boundary controllability for general linearly degenerate quasilinear hyperbolic systems of conservation laws with characteristics with constant multiplicity.

3.1. Two sided boundary controllability–proof of Theorem 1.3

In order to get Theorem 1.3, it suffices to establish the following

Lemma 3.1. *Under the same assumptions of Theorem 1.3, let $T > 0$ satisfy (1.7). For any given initial data $\bar{u}$ and final data $\bar{\bar{u}}$ with $\underset{0<x<L}{\text{Tot.Var.}}(\bar{u}) + |\bar{u}(0+)|$ and $\underset{0<x<L}{\text{Tot.Var.}}(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ sufficiently small, system (1.1) admits a solution $u = u(t,x)$ on the domain $\mathbb{D}_T$, satisfying simultaneously the initial condition*

$$t = 0: \ u = \bar{u}(x), \quad 0 < x < L \tag{3.1}$$

and the final condition.

$$t = T: \ u = \bar{\bar{u}}(x), \quad 0 < x < L. \tag{3.2}$$

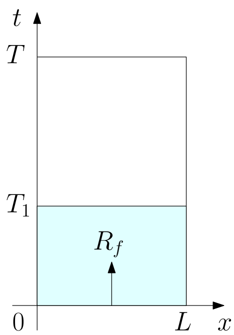

FIGURE 2. forward solution.

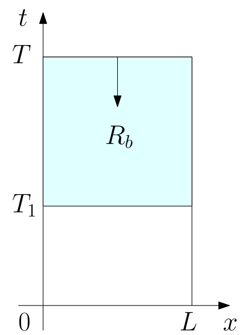

FIGURE 3. backward solution.

In fact, let $u = u(t,x)$ be a solution given by Lemma 3.1. Taking the boundary control as

$$g_1(t) := b_1(u(t,0+)), \quad g_2(t) := b_2(u(t,L-)), \quad \forall t \in (0,T),$$

we obtain the local exact two-sided boundary controllability desired by Theorem 1.3.

Proof of Lemma 3.1. Noting (1.7), for $r > 0$ sufficiently small we have

$$T > L \max_{u \in B_r(0)} \left\{ \frac{1}{|\lambda_m(u)|}, \frac{1}{\lambda_{m+1}(u)} \right\}. \tag{3.3}$$

Let

$$T_1 := \frac{L}{2} \max_{u \in B_r(0)} \left\{ \frac{1}{|\lambda_m(u)|}, \frac{1}{\lambda_{m+1}(u)} \right\}. \tag{3.4}$$

Step 1. Artificially choosing functions $\bar{g}_1 : (0,T_1) \to \mathbb{R}^{n-m}$ and $\bar{g}_2 : (0,T_1) \to \mathbb{R}^m$ with $\underset{0<t<T_1}{\text{Tot.Var.}}(\bar{g}_i) + |\bar{g}_i(0+)|$ $(i = 1,2)$ sufficiently small, we consider the forward mixed initial-boundary value problem of system (1.1) with the initial condition (3.1) and the following boundary conditions:

$$\begin{cases} x = 0: & b_1(u) = \bar{g}_1(t), \\ x = L: & b_2(u) = \bar{g}_2(t), \end{cases} \qquad t \in (0,T).$$

By Proposition 2.3 there exists a unique solution $u = u_f(t,x)$ on the domain $R_f = \{0 < t < T_1, 0 < x < L\}$ (see Fig. 2).

Step 2. We consider the backward mixed initial-boundary value problem of system (1.1) with the final condition (3.2) and the artificial boundary conditions

$$\begin{aligned} x = 0: &\quad l_r(u)u = \bar{\bar{g}}_r(t) \qquad (r = 1,\dots,m), \\ x = L: &\quad l_s(u)u = \bar{\bar{g}}_s(t) \qquad (s = m+1,\dots,n), \end{aligned} \tag{3.5}$$

where $\bar{\bar{g}}_i : (T_1,T) \to \mathbb{R}$ $(i = 1,\dots,n)$ are artificially given functions of t with $\underset{T_1<t<T}{\text{Tot.Var.}}(\bar{\bar{g}}_i) + |\bar{\bar{g}}_i(0+)|$ $(i = 1,2)$ sufficiently small. It is easy to see that the boundary condition (3.5) satisfies the corresponding Assumption (H6) in the backward sense. By Proposition 2.3, there exists a solution $u = u_b(t,x)$ on the domain $R_b := \{T_1 < t < T,\ 0 < x < L\}$ (see Fig. 3).

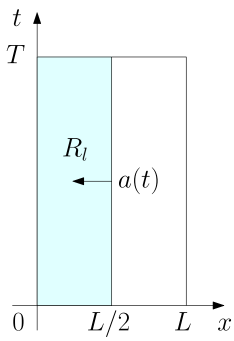

FIGURE 4. leftward solution.

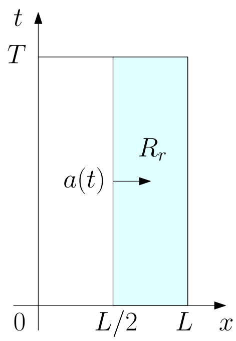

FIGURE 5. rightward solution.

Step 3. Define a function $a = a(t) : (0,T) \to \mathbb{R}^n$ by

$$a(t) = \begin{cases} u_f(t, L/2), & 0 < t < T_1, \\ u_b(t, L/2), & T_1 < t < T. \end{cases}$$

Now we exchange the role of variables t and x and consider the leftward problem for system (2.7) on the domain $R_l := \{0 < t < T,\ 0 < x < L/2\}$ with the final condition

$$x = L/2 : \ u = a(t), \quad 0 < t < T \tag{3.6}$$

and the following boundary conditions reduced from the initial data $\bar{u}$ and the finial data $\bar{\bar{u}}$, respectively,

$$t = 0 : \ l_r(u)u = l_r(\bar{u})\bar{u}, \quad r = 1, \ldots, m, \quad 0 < x < L/2, \tag{3.7}$$

$$t = T : \ l_s(u)u = l_s(\bar{\bar{u}})\bar{\bar{u}}, \quad s = m+1, \ldots, n, \quad 0 < x < L/2, \tag{3.8}$$

where $l_i(u)$ $(i = 1, \ldots, n)$ are the left eigenvectors of $(DH(u))^{-1}DG(u)$, which are also the left eigenvectors of $(DG(u))^{-1}DH(u)$. It is easy to see that these boundary conditions satisfy the corresponding Assumption (H6) for the leftward problem. Still by Proposition 2.3, the leftward problem admits a solution $u_l = u_l(t,x)$ on the domain R_l as the limit of a sequence of ε-solutions (see Fig. 4).

Step 4. Similarly, this rightward mixed initial-boundary value problem for system (2.7) with the initial condition (3.6) and the following reduced boundary conditions:

$$t = 0 : \ l_s(u)u = l_s(\bar{u})\bar{u}, \quad s = m+1, \ldots n, \quad L/2 < x < L, \tag{3.9}$$

$$t = T : \ l_r(u)u = l_r(\bar{\bar{u}})\bar{\bar{u}}, \quad r = 1, \ldots, m, \quad L/2 < x < L \tag{3.10}$$

admits a solution $u_r = u_r(t,x)$ on the domain $R_r(T) = \{0 < t < T,\ L/2 < x < L\}$ (see Fig. 5).

Step 5. Let

$$u(t,x) = \begin{cases} u_l(t,x), & (t,x) \in \mathbb{D}_l(T), \\ u_r(t,x) & (t,x) \in \mathbb{D}_r(T). \end{cases} \tag{3.11}$$

By Proposition 2.10, $u = u(t,x)$ is a solution to system (1.1).

Now it remains to show that u verifies the initial condition (3.1) and the final condition (3.2).

By Proposition 2.7, both u_f and u_l (resp. u_r) are solutions to system (2.7) in the leftward (resp. rightward) sense with the same final (resp. initial) condition

$$x = L/2 : \quad u = a(t), \quad 0 < t < T_1$$

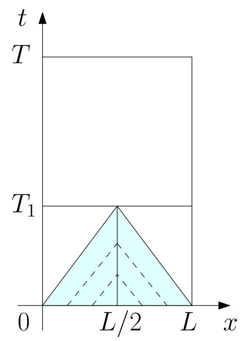

FIGURE 6. checking the initial condition.

and the same boundary condition (3.7) (resp. (3.9)). By Proposition 2.9 and Remark 2.4 for the leftward problem (resp. rightward problem) and noting (3.4), u_f coincides with u_l (resp. u_r) on the triangle domain

$$\{0 < t < 2T_1x/L,\ 0 < x < L/2\}$$
$$(\text{resp. } \{0 < t < 2T_1(L-x)/L,\ L/2 < x < L\})$$

(see Fig. 6). Then, since $u = u_f(t,x)$ satisfies the initial condition (3.1), $u = u(t,x)$ given by (3.11) verifies (3.1).

Similarly, by comparing u_l (resp. u_r) with u_b on the triangle domain

$$\{T - 2T_1x/L < t < T,\ 0 < x < L/2\}$$
$$(\text{resp. } \{2T_1(x-L)/L + T < t < T,\ L/2 < x < L\}),$$

we can obtain that $u = u(t,x)$ verifies (3.2) . Thus $u = u(t,x)$ is a desired solution and the proof of Lemma 3.1 is complete. □

3.2. One-sided boundary controllability–proof of Theorem 1.4

In order to get Theorem 1.4, it suffices to establish the following

Lemma 3.2. *Under the same assumptions of Theorem 1.4, let $T > 0$ satisfy (1.11). For any given initial data $\bar{u}$, final data $\bar{\bar{u}}$ and boundary data g_1 with $\underset{0<x<L}{\text{Tot. Var.}}(\bar{u}) + |\bar{u}(0+)|$, $\underset{0<x<L}{\text{Tot. Var.}}(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ and $\underset{0<t<T}{\text{Tot. Var.}}(g_1) + |g(0+)|$ sufficiently small, system (1.1) together with the boundary condition*

$$x = 0 :\ b_1(u) = g_1(t), \qquad t \in (0,T) \tag{3.12}$$

admits a solution $u = u(t,x)$ on the domain $\mathbb{D}_T$, satisfying simultaneously the initial condition (3.1) and the final condition (3.2).

In fact, let $u = u(t,x)$ be a solution given by Lemma 3.2. Taking the boundary control as

$$g_2(t) := b_2(u(t, L-)), \quad \forall t \in (0,T),$$

we obtain the local exact one-sided boundary controllability desired by Theorem 1.4.

Proof of Lemma 3.2. Noting (1.11), for $r > 0$ sufficiently small we have

$$T > L \max_{u \in B_r(0)} \left\{ \frac{1}{|\lambda_m(u)|} + \frac{1}{\lambda_{m+1}(u)} \right\}. \tag{3.13}$$

Step 1. Let

$$T_1 := L \max_{u \in B_r(0)} \frac{1}{|\lambda_m(u)|}. \tag{3.14}$$

Choosing a function $g_f : (0,T) \to \mathbb{R}^m$ with $\underset{0<t<T_1}{\text{Tot. Var.}}(g_f) + |g_f(0+)|$ sufficiently small, we consider the forward problem of system (1.1) with the initial condition (3.1) and the following boundary conditions:

$$\begin{cases} x = 0: \quad b_1(u) = g_1(t), \\ x = L: \quad b_2(u) = g_f(t), \end{cases} \qquad t \in (0, T_1).$$

By Proposition 2.3 there exists a solution $u = u_f(t,x)$ on the domain $\{0 < t < T_1,\ 0 < x < L\}$.

Step 2. Noting (1.10), without loss of generality, we assume that

$$\det \left[Db_1(u) \cdot r_1(u), \ldots, Db_1(u) \cdot r_{\bar{m}}(u) \right] \neq 0.$$

Then, by Proposition 2.3, there is a solution $u = u_b(t,x)$ on the domain $R_b = \{T_1 < t < T,\ 0 < x < L\}$ to the backward initial-boundary value problem of system (1.1) with the final condition (3.2), the boundary condition (1.3) and the following artificial boundary condition

$$x = 0: \quad l_p(u)u = g_p(t) \quad p = \bar{m}, \ldots, m,$$

$$x = L: \quad l_s(u)u = g_s(t) \quad s = m+1, \ldots, n,$$

where $g_i : (T_1, T) \to \mathbb{R}$ $(i = \bar{m}, \ldots, n)$ are any given functions of t with $\underset{T_1<t<T}{\text{Tot. Var.}}(g_i) + |g_i(0+)|$ sufficiently small.

Step 3. Let

$$a(t) = \begin{cases} u_f(t, 0+) & 0 < t < T_1, \\ u_b(t, 0+) & T_1 < t < T. \end{cases}$$

Now we exchange the role of variables t and x and consider the rightward problem for system (2.7) on domain $\mathbb{D}_T$ (see Fig. 7) with the initial condition

$$x = 0: \ u = a(t), \quad 0 < t < T$$

and the following boundary conditions reduced from the initial data $\bar{u}$ and the final data $\bar{\bar{u}}$:

$$t = 0: \ l_s(u)u = l_s(\bar{u})\bar{u}, \quad s = m+1, \ldots n,$$

$$t = T: \ l_r(u)u = l_r(\bar{\bar{u}})\bar{\bar{u}}, \quad r = 1, \ldots, m.$$

Still by Proposition 2.3, this rightward problem admits a solution $u = u(t,x)$ on the domain $\mathbb{D}_T$. By Proposition 2.7, u is also a solution of system (1.1) in the forward sense on $\mathbb{D}_T$. Since $u(t,0) = a(t)$ for a.e. $t \in (0,T)$, we have

$$b_1(u(t, 0+)) = g_1(t), \qquad \text{a.e. } t \in (0,T).$$

Step 4. Now it remains to show that u verifies the initial condition (3.1) and the final condition (3.2).

By Proposition 2.7, both u_f and u are solutions in the rightward sense. Then by Proposition 2.9 and Remark 2.4 for the rightward problem, and noting (3.14), u_f coincides with u on the triangle domain $\{0 < t < T_1(L-x)/L,\ 0 < x < L\}$ (see Fig. 8). This implies (3.1).

Similarly, Let

$$T_2 = L \max_{u \in B_r(0)} \frac{1}{\lambda_{m+1}(u)}. \tag{3.15}$$

Comparing u_b and u on the triangle domain $\{T_2 x/L + (T - T_2) < t < T,\ 0 < x < L\}$, and noting that $T - T_1 > T_2$, we can get (3.2).

Thus $u = u(t,x)$ is a desired solution and the proof of Lemma 3.2 is complete. □

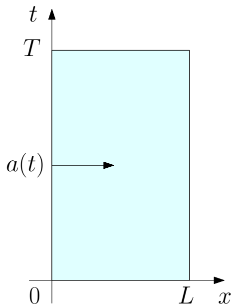

FIGURE 7. rightward solution.

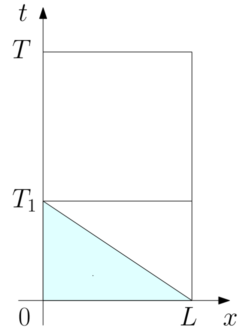

FIGURE 8. checking the initial condition.

3.3. Two-sided boundary controllability with fewer controls–proof of Theorem 1.5

In order to get Theorem 1.5, it suffices to establish the following

Lemma 3.3. *Under the same assumptions of Theorem 1.5, for any given initial data $\bar{u}$, any given final data $\bar{\bar{u}}$ and any given parts of boundary data $\tilde{g}_2 : (0,T) \to \mathbb{R}^{\bar{m}}$ with $\underset{0<x<L}{\text{Tot. Var.}}\,(\bar{u}) + |\bar{u}(0+)|$, $\underset{0<x<L}{\text{Tot. Var.}}\,(\bar{\bar{u}}) + |\bar{\bar{u}}(0+)|$ and $\underset{0<t<T}{\text{Tot. Var.}}\,(\tilde{g}_2) + |\tilde{g}_2(0+)|$ sufficiently small, system (1.1) together with the boundary condition*

$$x = L: \;\; \tilde{b}_2(u) = \tilde{g}_2(t), \qquad t \in (0,T) \tag{3.16}$$

admits a solution $u = u(t,x)$ on the domain $\mathbb{D}_T$, satisfying simultaneously the initial condition (3.1) and the final condition (3.2).

In fact, let $u = u(t,x)$ be a solution given by Lemma 3.2. Taking the boundary control as

$$g_1(t) := b_1(u(t,0+)), \quad \hat{g}_2(t) := \hat{b}_2(u(t,L-)), \quad \forall t \in (0,T),$$

where $\hat{b}_2$ is the vector function consisting of the last $(m - \bar{m})$ components of b_2, we obtain the local exact two-sided boundary controllability with fewer controls desired by Theorem 1.4.

Proof of Lemma 3.3. Noting (1.11), for $r > 0$ sufficiently small, (3.13) holds.

Step 1. Let T_1 be still given by (3.14). Then, artificially choosing boundary data $\bar{g}_1$ and $\hat{\bar{g}}_2 : (0,T_1) \to \mathbb{R}^{m-\bar{m}}$ with $\underset{0<t<T_1}{\text{Tot. Var.}}\,(\bar{g}_1) + |\bar{g}_1(0+)|$ and $\underset{0<t<T_1}{\text{Tot. Var.}}\,(\hat{\bar{g}}_2) + |\hat{\bar{g}}_2(0+)|$ sufficiently small, we consider the forward problem of system (1.1) with the initial condition (3.1), the boundary condition (3.16) on $x = L$ and the following boundary conditions:

$$\begin{aligned} x = 0: &\;\; b_1(u) = \bar{g}_1(t), \\ x = L: &\;\; \hat{b}_2(u) = \hat{\bar{g}}_2(t). \end{aligned}$$

By Proposition 2.3 there exists a unique solution $u = u_f(t,x)$ on the domain $R_f = \{0 < t < T_1, 0 < x < L\}$.

Step 2. Choose a function $\bar{\bar{g}} : (T_1,T) \to \mathbb{R}^m$ with $\underset{T_1<t<T}{\text{Tot. Var.}}\,(\bar{\bar{g}}) + |\bar{\bar{g}}(0+)|$ sufficiently small. By Proposition 2.3, there exist a solution $u = u_b(t,x)$ to the backward mixed initial boundary problem of system (1.1) on the domain $R_b = \{T_1 < t < T,\; 0 < x < L\}$ with the final condition (3.2), the boundary condition (3.16) and the artificial boundary condition

$$x = 0: \quad l_r(u)u = \bar{\bar{g}}_r(t) \;(r = 1,\dots,m), \quad t \in (T - T_2, T).$$

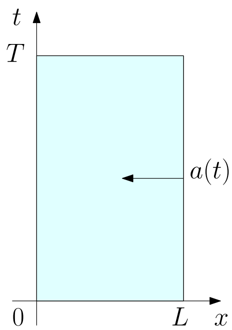

FIGURE 9. leftward solution.

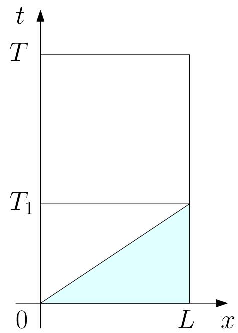

FIGURE 10. checking the initial condition.

Step 3. Define a function $a = a(t)$ by

$$a(t) = \begin{cases} u_f(t, L-), & 0 < t < T_1, \\ u_b(t, L-), & T_1 < t < T. \end{cases}$$

Now we exchange the role of variables t and x and consider the leftward problem for system (2.7) on the domain $\mathbb{D}_T$ with the final condition

$$x = L: \ u = a(t), \quad 0 < t < T$$

(see Fig. 9) and the following boundary conditions reduced from the initial data $\bar{u}$ and the finial data $\bar{\bar{u}}$:

$$t = 0: \ l_r(u)u = l_r(\bar{u})\bar{u}, \quad r = 1, \dots m,$$

$$t = T: \ l_s(u)u = l_s(\bar{\bar{u}})\bar{\bar{u}}, \quad s = m+1, \dots, n.$$

It is easy to see that these boundary conditions satisfy the corresponding Assumption (H6) in the leftward sense.

Still by Proposition 2.3, this leftward problem admits a solution $u = u(t, x)$ on the domain $\mathbb{D}_T$. By Proposition 2.7, u is also a solution of system (1.1) in the forward sense on $\mathbb{D}_T$.

Step 4. It remains to show that u verifies the initial condition (3.1) and the final condition (3.2).

By Proposition 2.7, both u_f and u are solutions in the leftward sense. Then by Proposition 2.9 and Remark 2.4 for the leftward problem, and noting (3.14), u_f coincides with u on the triangle domain $\{0 < t < T_1 x/L, \ 0 < x < L\}$ (see Fig. 10). This implies (3.1).

Similarly, taking T_2 as (3.15) and comparing u and u_b on the triangle domain $\{T - T_2 x/L < t < T, \ 0 < x < L\}$, we can get (3.2).

Thus $u = u(t, x)$ is a desired solution and the proof of Lemma 3.2 is complete. □

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