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DIAGONAL NON-SEMICONTINUOUS VARIATIONAL PROBLEMS

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Abstract. We study the minimum problem for non sequentially weakly lower semicontinuos functionals of the form

$$\mathcal{F}(u) = \int_I f(x, u(x), u'(x))\,\mathrm{d}x,$$

defined on Sobolev spaces, where the integrand $f : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is assumed to be non convex in the last variable. Denoting by $\overline{f}$ the lower convex envelope of f with respect to the last variable, we prove the existence of minimum points of $\mathcal{F}$ assuming that the application $p \mapsto \overline{f}(\cdot, p, \cdot)$ is separately monotone with respect to each component p_i of the vector p and that the Hessian matrix of the application $\xi \mapsto \overline{f}(\cdot, \cdot, \xi)$ is diagonal. In the special case of functionals of sum type represented by integrands of the form $f(x, p, \xi) = g(x, \xi) + h(x, p)$, we assume that the separate monotonicity of the map $p \mapsto h(\cdot, p)$ holds true in a neighbourhood of the (unique) minimizer of the relaxed functional and not necessarily on its whole domain.

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1. Introduction

In this paper we consider the minimum problem for functionals of the form

$$\mathcal{F}(u) = \int_I f(x, u(x), u'(x))\,\mathrm{d}x.$$

The domain I is a bounded open interval of $\mathbb{R}$, the competing maps belong to the Sobolev space $W^{1,q}(I, \mathbb{R}^m)$, with prescribed boundary conditions, while the integrand $f : I \times \mathbb{R}^{2m} \to \mathbb{R}$ satisfies standard growth condition at infinity, so that the functional is coercive on $W^{1,q}(I, \mathbb{R}^m)$, but is assumed to be non-convex in the last variable, so that $\mathcal{F}$ is not (sequentially weakly lower) semicontinuous. This fact inhibits the use of the Direct Method, and, in order to prove the existence of minimizers, forces to look for more sophisticated techniques.

We introduce and apply a new approach to this class of problems based on Γ-convergence, in the aim of finding a general way to manage the lackness of (s.w.l.) semicontinuity. Indeed, many authors devoted themselves to the minimization of non (s.w.l) semicontinuous variational problems (for short: non semicontinuous problems) adopting various techniques, each of which needs to be adapted to the single considered class of functionals.

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Although these efforts improved the comprehension of the matter, a general approach is still lacking, and, as a consequence, a wide area of variational problems with relevant applicative interest is not covered by any theory. We mainly refer to vectorial problems, corresponding to functionals represented by integrands $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n$, which, in order to have semicontinuity, should be assumed to be quasiconvex. While the corresponding non-convex scalar case ($n > 1$, $m = 1$) has been investigated by various authors, very few results have been obtained in the non-quasiconvex one, although it exhibits interesting theoretical features and finds application in many important concrete problems. Even the one-dimensional vectorial case ($n = 1$, $m > 1$) studied in this paper is poor of consideration in the literature: actually we know and mention the results contained in chapter 16.7, page 472 of [4] (see also [3]) and its generalization appeared in [2].

While these existence results are based on Lyapunov theorem (see again Chap. 16 in [4]), in the present work we adopt another procedure which consists in the construction of a sequence of functionals $(\overline{\mathcal{F}}^n)$ $\varGamma$-converging to the relaxed functional $\overline{\mathcal{F}}$, represented by the lower convex envelope of f with respect to the last variable. The convex integrands F^n of the functionals $\overline{\mathcal{F}}^n$ satisfy suitable uniform growth properties at infinity ensuring the existence of a sequence (u_n) of minimizers of $\overline{\mathcal{F}}^n$ which turns out to be relatively compact in the strong Sobolev topology. Then, $\varGamma$-convergence implies that a cluster point of (u_n) is a minimizer of $\overline{\mathcal{F}}$ and, due to additional *structural conditions* imposed on the integrands F^n, and then, indirectly, on the original integrand f, we are able to show that it is a minimizer of $\mathcal{F}$ too. The mentioned structural conditions that we require, indirectly, on the pair of integrands $(f, \overline{f})$ are of this kind: the application $p \mapsto f(\cdot, p, \cdot)$ is separately monotone with respect to any single component p_i of the vector p, the application $\xi \mapsto \overline{f}(\cdot, \cdot, \xi)$ is affine on the set $\{f < \overline{f}\}$ and its Hessian matrix $\left(\overline{f}_{\xi\xi}\right)$ is diagonal (and this last property inspires the title of the article). Our result is not contained nor contains the results of the two works quoted above ([2, 4]), which concerns functionals of sum type with no diagonal assumption but, on the other hand, with stronger conditions on the u-dependence, namely linearity or concavity.

The paper is organized as follows. Section 2 contains the statement of the problem and the main assumptions, while in Section 3, in order to let the reader familiarize with our hypotheses, we exhibit a class of functionals which satisfy our assumptions. In Section 4 we prove our main result and in Section 5 we consider the special case of functional of sum type, represented by integrands of the form $f(x, p, \xi) = g(x, \xi) + h(x, p)$, showing that the separate monotonicity assumption on the map $p \mapsto h(\cdot, p)$ may be weakened, by imposing that it holds true not necessarily on the whole domain of h, but only on a neighbourhood of the map $\overline{u}$, where $\overline{u}$ is the unique minimizer of the relaxed functional represented by the integrand $\overline{f}(x, p, \xi) = \overline{g}(x, \xi) + h(x, p)$. This last condition is certainly verified whenever h is of class C^1 and we have $h_{p_i}(x, \overline{u}(x)) \neq 0$ for every $x \in \overline{I}$ and for every $i = 1, \dots, d$.

2. Statement of the problem, hypotheses and notations

In this paper $\mathbb{R}^d$ is the d-dimensional euclidean space and $|\cdot|$ is the euclidean norm in $\mathbb{R}^d$, while by $\xi \cdot \eta$ we mean the inner product of the vectors $\xi, \eta \in \mathbb{R}^d$. Given an interval I of $\mathbb{R}$, we use the spaces $C^k(I, \mathbb{R}^d)$, $L^r(I, \mathbb{R}^d)$, $W^{1,r}(I, \mathbb{R}^d)$, $W_0^{1,r}(I, \mathbb{R}^d)$, for $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $1 \leq r \leq \infty$, endowed with their usual (strong and weak) topologies. Given a function $\phi : \mathbb{R}^d \to \mathbb{R}$ we denote by ϕ_{y_i} and by $\phi_{y_i y_j}$, respectively, the first and the second derivative of ϕ with respect to the components y_i and y_j of the vector $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. The gradient of ϕ is written as ϕ_y, while the Hessian matrix is denoted by $\mathcal{H}_y(\phi)$. The symbol $\|\cdot\|_q$ stands for the norm in $L^q(I)$ and by E^c we denote the complement $A \setminus E$ of a subset $E \subseteq A$. By Q_R we mean the d-dimensional hypercube $[-R, R]^d \subseteq \mathbb{R}^d$.

We consider a continuous function $f : \overline{I} \times \mathbb{R}^{2d} \to \mathbb{R}$, where $I =]a, b[$ is a bounded open interval of $\mathbb{R}$ and $f = f(x, p, \xi)$ ($x \in I$, $p \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$) is assumed to be non-convex in the last variable ξ. We devote our study to the minimization of the functional

$$\mathcal{F}(u) = \int_I f(x, u(x), u'(x))\, \mathrm{d}x, \quad u \in \mathcal{W}, \tag{2.1}$$

where the set $\mathcal{W}$ of competing maps is defined by

$$\mathcal{W} \doteq \left\{u \in W^{1,q}(I,\mathbb{R}^d) : u(a) = u_a, u(b) = u_b\right\}, \tag{2.2}$$

the boundary values u_a and u_b are given vectors in $\mathbb{R}^d$ and $q \in]1,\infty[$ is an index related to the growth of f at infinity (see Hypothesis 1 below).

We call $\overline{f} = \overline{f}(x,p,\xi)$ the lower convex envelope of f with respect to ξ and introduce the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_I \overline{f}(x,u(x),u'(x))\,\mathrm{d}x, \quad u \in \mathcal{W}. \tag{2.3}$$

For expositive convenience, we set $\mathcal{F}(u) = \overline{\mathcal{F}}(u) = +\infty$ for every $u \in W^{1,q}(I,\mathbb{R}^d)\setminus\mathcal{W}$.

We look for minimizer of the functional $\mathcal{F}$ as strong limit in $W^{1,q}(I,\mathbb{R}^d)$ of the sequence of minimizers of functionals $\overline{\mathcal{F}}_n$ suitably constructed in order to overcome the non (s.w.l.) semicontinuity of $\mathcal{F}$. For this reason we formulate the assumptions of our procedure on a sequence of functions $(\overline{f}^n)$ approximating $\overline{f}$, so that, once given a specific functional, the application of our theory consists in the verification that its integrand may be approximated by a sequence satisfying the required conditions.

Hypothesis 1. There exists a sequence $(\overline{f}^n)$ in $C^2(\overline{I}\times\mathbb{R}^{2d},\mathbb{R})$ with the following properties.

(i) The sequence $(\overline{f}^n)$ approximates the convex envelope $\overline{f}$ in the following sense:

$$\overline{f}^n \longrightarrow \overline{f} \quad \text{uniformly on compact subsets of} \quad \overline{I}\times\mathbb{R}^{2d}. \tag{2.4}$$

(ii) There exist $\alpha > 0$, $\beta \geq 0$, $\gamma \in L^1(I)$, $q \in]1,+\infty[$ and $r \in [0,q[$ such that $\forall n \in \mathbb{N}$, $\forall (x,p,\xi) \in I\times\mathbb{R}^{2d}$ and $\forall i \in \{1,\dots,d\}$, the following growth conditions hold true:

$$\overline{f}^n(x,p,\xi) \geq \alpha|\xi|^q - \beta|p|^r - \gamma(x); \tag{2.5}$$

$$|\overline{f}^n_\xi(x,p,\xi)| \geq \alpha|\xi|^{q-1} - \beta. \tag{2.6}$$

(iii) There exists $\gamma_1 \in L^1(I)$ and, for every $R > 0$, there exist $D = D_R > 0$ such that $\forall n \in \mathbb{N}$, $\forall x \in I$ and $\forall (p,\xi) \in Q_R \times \mathbb{R}^d$ we have

$$|\overline{f}^n(x,p,\xi)|, |\overline{f}^n_\xi(x,p,\xi)|, |\overline{f}^n_p(x,p,\xi)| \leq \gamma_1(x) + D|\xi|^q. \tag{2.7}$$

(iv) $\forall n \in \mathbb{N}$ and $\forall (x,p,\xi) \in I\times\mathbb{R}^{2d}$ we have

$$\mathcal{H}_\xi(\overline{f}^n) = \operatorname{diag}\left(\overline{f}^n_{\xi_i\xi_i}(x,p,\xi)\right)_{i\in\{1,\dots,d\}}. \tag{2.8}$$

Setting

$$\eta_i \doteq \inf\left\{\overline{f}^n_{\xi_i\xi_i}(x,p,\xi) : (x,p,\xi) \in I\times\mathbb{R}^{2d})\right\}, \quad i \in \{1,\dots,d\}, \tag{2.9}$$

we assume

$$\eta_i \geq 0 \quad \forall i \in \{1,\dots,d\}, \tag{2.10}$$

so that, in particular, the map $\xi \mapsto \overline{f}^n(x,p,\xi)$ is convex.

(v) For every $i \in \{1,\dots,d\}$ and $n \in \mathbb{N}$ we introduce the family of subsets of $\mathbb{R}^d$ given by

$$\Xi^n_i(x,p) \doteq \left\{\xi \in \mathbb{R} : \overline{f}^n_{\xi_i\xi_i}(x,p,\xi) = 0\right\}. \tag{2.11}$$

Then, $\forall n \in \mathbb{N}$, $\forall i \in \{1,\dots,d\}$ for which $\eta_i = 0$ and $\forall (x,p,\xi) \in I\times\mathbb{R}^{2d}$ we impose

$$\text{either} \quad \overline{f}^n_{p_i}(x,p,\xi) \leq 0 \tag{2.12}$$

$$\text{or} \quad \overline{f}^n_{p_i}(x,p,\xi) \geq 0 \tag{2.13}$$

and set

$$\sigma_i \doteq \begin{cases} \operatorname{sign}(\overline{f}^n_{p_i}) & \text{if} \quad \eta_i = 0 \\ 0 & \text{if} \quad \eta_i > 0, \end{cases} \tag{2.14}$$

$$\sigma \doteq (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d. \tag{2.15}$$

(vi) $\forall n \in \mathbb{N}$, $\forall i \in \{1, \dots, d\}$ $\forall (x,p) \in I \times \mathbb{R}^d$ and $\forall \xi \in \Xi^n_i(x,p)$ we impose

$$\begin{cases} \overline{f}^n_{x\xi_i}(x,p,\xi) \geq 0 & \text{if} \quad \sigma_i = -1 \\ \overline{f}^n_{x\xi_i}(x,p,\xi) \leq 0 & \text{if} \quad \sigma_i = +1 \end{cases} \tag{2.16}$$

and, $\forall j \in \{1, \dots, d\}$,

$$\overline{f}^n_{\xi_i p_j}(x,p,\xi) = 0. \tag{2.17}$$

(vii) For every $R \geq 0$ there exists $C_R \geq 0$ such that, $\forall n \in \mathbb{N}$, $\forall i \in \{1, \dots, d\}$, $\forall (x,p) \in I \times \mathbb{R}^d$ and $\forall \xi \in \Xi^n_i(x,p)^c \doteq \mathbb{R}^d \setminus \Xi^n_i(x,p)$ such that $\xi \in Q_R$, we have

$$\frac{\left|\overline{f}^n_{x\xi_i}(x,p,\xi)\right|}{\overline{f}^n_{\xi_i\xi_i}(x,p,\xi)} \leq C_R \tag{2.18}$$

and, $\forall j \in \{1, \dots, d\}$,

$$\frac{\left|\overline{f}^n_{\xi_i p_j}(x,p,\xi)\right|}{\overline{f}^n_{\xi_i\xi_i}(x,p,\xi)} \leq C_R. \tag{2.19}$$

(viii) For every $\epsilon > 0$ there exist $n_\epsilon > 0$ such that $\forall n \geq n_\epsilon$ and $\forall (x,p,\xi) \in I \times \mathbb{R}^{2d}$ satisfying

$$\overline{f}^n(x,p,\xi) < f(x,p,\xi) - \epsilon, \tag{2.20}$$

there exists $k \in \{1, \dots, d\}$ with $\eta_k = 0$ such that

$$\overline{f}^n_{\xi_k\xi_k}(x,p,\xi) = 0. \tag{2.21}$$

Remark 2.1. It is immediate to see that properties in (2.4), (2.5) and in item (iv) imply that the functional $\overline{\mathcal{F}}$ is coercive and (s.w.l.) semicontinuous on $W^{1,q}(I)$, so that it admits at least one minimizer on $\mathcal{W}$.

Before explaining Hypothesis 1 (see Rem. 2.3 below), we define the approximating functionals.

Definition 2.2. We take a sequence (ϵ_n) in $\mathbb{R}^+$ such that $\epsilon_n \to 0$ and, for every $n \in \mathbb{N}$ and for every $(x,p,\xi) \in I \times \mathbb{R}^{2d}$, we set

$$\begin{aligned} F^n(x,p,\xi) &\doteq \frac{\epsilon_n^2}{2}\xi^2 + \overline{f}^n(x,p,\xi) + \epsilon_n \sigma \cdot p \\ &= \frac{\epsilon_n^2}{2}\sum_{i=1}^d \xi_i^2 + \overline{f}^n(x,p,\xi) + \epsilon_n, \sum_{i=1}^d \sigma_i p_i. \end{aligned} \tag{2.22}$$

where the vector σ is defined in (2.15). Then we set

$$\overline{\mathcal{F}}^n(u) \doteq \int_I F^n(x, u(x), u'(x))\,\mathrm{d}x, \quad u \in \mathcal{W} \cap H^1(I, \mathbb{R}^d), \quad n \in \mathbb{N}. \tag{2.23}$$

As usual, we set $\overline{\mathcal{F}}^n(u) = +\infty$ for every $u \in W^{1,q}(I, \mathbb{R}^d) \setminus \big(\mathcal{W} \cap H^1(I, \mathbb{R}^d)\big)$.

Remark 2.3. Hypothesis 1 needs some comments.

(1) The conditions expressed in (ii), (iii), (iv) are classical and ensure the well posedness of the minimum problems for the functional $\overline{\mathcal{F}}^n$ defined in (2.23). In addition, property (2.5) ensures the equi-coercivity of the functionals $\overline{\mathcal{F}}^n$, (2.6) implies the uniform boundedness of the derivative of their minimizers u_n, while item (2.7) guarantees the validity of Euler-Lagrange equations in weak form.
(2) The uniform convergence of $\overline{f}^n$ to $\overline{f}$ and the third inequality in (2.7) imply that the sequence of approximating functionals $\overline{\mathcal{F}}^n$ Γ-converges to the functional $\overline{\mathcal{F}}$ in both strong and weak topology of $W^{1,q}(I,\mathbb{R}^d)$. This will imply, in particular, the convergence of the minimum of $\overline{\mathcal{F}}^n$ to the minimum of $\overline{\mathcal{F}}$.
(3) Hypotheses in items (v)–(vii) are what we have called *structural conditions*: (2.12)–(2.13), (2.16) and (2.17) translate to the present case the properties used in the literature on non convex variational problems, while (2.18) and (2.19) are boundedness conditions which can be easily verified by direct computations on a given sequence of approximating integrands obtained by mollification. The diagonal assumption (2.8) is of technical nature and is necessary in order to make our argument work. Up to now we are not able to say if it can be weakened.
(4) Condition in item (viii) says that the set on which the function $\overline{f}^n$ is separately affine (*i.e.* $\overline{f}^n_{\xi_i\xi_i} = 0$) approaches, as $n \to \infty$, the detachment set $\{\overline{f} < f\}$.

Remark 2.4. While the integrands $\overline{f}^n$ inherit the structural properties of the integrands f and $\overline{f}$, the integrands F^n introduced in definition 2.2 are suitably constructed in order to make our procedure work.

(1) In definition (2.22) we add the perturbing term $\epsilon_n\sigma \cdot p$ in correspondence to the hypotheses $\overline{f}^n_{p_i} \leq 0$ or $\overline{f}^n_{p_i} \geq 0$ expressed in (2.12)–(2.13). This term forces the minimum point u_n of $\overline{\mathcal{F}}^n$ to maximize or minimize (respectively) the integrals $\int_I u_{n,i}\,\mathrm{d}x$. By this way we incorporate in the present theory the integro-extremization method, introduced and developed in papers, [9,10,12,14].
(2) The term $\epsilon_n^2\xi^2$ in (2.22) is necessary in order to ensure that a minimizer u_n of $\overline{\mathcal{F}}^n$ is of class C^2 and satisfies the Euler-Lagrange equation in classical form. It introduces a second order addendum in each row of the system of Euler equations which can be interpreted as a sort of "vanishing viscosity" term.
(3) The restriction $\mathcal{W} \cap H^1(I,\mathbb{R}^d)$ on the domain of the functionals $\overline{\mathcal{F}}^n$ is necessary only in the case $1 < q < 2$, while it is evident that it has no effect whenever $q \geq 2$. It is due to the introduction of the term $\epsilon_n\xi^2$ in definition (2.22) which forces to assume, *a priori*, that the derivative u' of maps u in the domain of definition of the approximating functionals belong to $L^2(I,\mathbb{R}^d)$. This fact is merely technical, since, as we will see, the minimizers u_n of the functionals $\overline{\mathcal{F}}^n$ turn out to ly in $W^{1,\infty}(I,\mathbb{R}^d)$.

3. Examples

In this section we exhibit a class of functionals for which the hypotheses of previous section may be directly verified.

We define an integrand $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ of the form

$$f(x,p,\xi) = [g_1(\xi_1) + g_2(\xi_2) + g_3(\xi_3)] + \alpha(x)h(p_1,p_3) + \beta(x)k(p_1,p_3), \tag{3.1}$$

where

$$g_1(\xi) = |\xi_1^2 - 1|, \quad g_2(\xi_2) = \left|\xi_2^2 - 1\right|. \quad g_3(\xi_3) = \xi_3^2, \tag{3.2}$$

The maps $\alpha, \beta : I \to \mathbb{R}$ are of class C^1 and for every $x \in I$ we impose $M \geq \alpha(x) \geq \alpha_0 > 0$, $M \geq \beta(x) \geq \beta_0 > 0$, for some positive M, $\alpha'(x) \geq 0$ and $\beta'(x) \leq 0$. For what concerns the p dependence we require that $h, k : \mathbb{R}^2 \to \mathbb{R}$ are continuous, satisfy standard growth conditions, and $p_1 \mapsto h(p_1,\cdot)$ is monotone decreasing on $\mathbb{R}$, while $p_2 \mapsto h(p_2,\cdot)$ is monotone increasing on $\mathbb{R}$. No more conditions are needed on the dependence on p_3 since, according to the notations of Hypothesis 1, we have $\eta_1 = \eta_2 = 0$ and $\eta_3 = 2$, so that, correspondingly, we set $\sigma_1 = -1$, $\sigma_2 = +1$ and $\sigma_3 = 0$.

The relaxed functional is represented by the integrand

$$\overline{f}(x,p,\xi)=[\overline{g}_1(\xi_1)+\overline{g}_2(\xi_2)+g_3(\xi_3)]+\alpha(x)h(p_1,p_2)+\beta(x)k(p_1,p_3), \tag{3.3}$$

where, for $i=1,2$,

$$\overline{g}_i(\xi_i)=\begin{cases}\xi_i^2-1 & |\xi_i|\geq 1\\ 0 & |\xi_i|\leq 1.\end{cases}$$

The approximating sequence $(\overline{f}^n)$ can be constructed by regularization, which is needed for the addenda g_1 and g_2 and for the maps α,β,h,k. By the properties of regularizing families and by easy computations, we see that all conditions in Hypothesis 1 are satisfied with $q=2$.

A further example is given by an integrand of the form

$$f(x,p,\xi)=[\alpha_1(p_1)g_1(\xi_1)+\alpha_2(p_2)g_2(\xi_2)+\alpha_3(p_3)g_3(\xi_3)]+\alpha(x)h(p_1,p_2)+\beta(x)k(p_1,p_3),$$

where the assumptions on the last two addenda are the same and the factors $\alpha_1,\alpha_2,\alpha_3$ are sufficiently regular, locally bounded, strictly positive and satisfy suitable monotonicity properties whose detail are left to the reader.

These simple leading examples allow the reader to familiarize with the list of assumptions of Hypothesis 1, so that he will be able to recognize the range of application of our results.

4. The existence result

We recall a well known relative compactness result in Sobolev spaces.

Lemma 4.1. *Let $I\subseteq\mathbb{R}$ be an open bounded interval and (v_n) a sequence in $C^2(I,\mathbb{R})\cap C^1(\overline{I},\mathbb{R})$ such that*

$$\|v_n'\|_\infty\leq M \quad \forall n\in\mathbb{N} \tag{4.1}$$

for some $M>0$. Assume in addition that

$$v_n''(x)\leq M \quad \text{or} \quad v_n''(x)\geq -M \quad \forall x\in I,\ \forall n\in\mathbb{N}. \tag{4.2}$$

Then the sequence (v_n) is relatively compact in $W^{1,r}(I,\mathbb{R})$ for every $r\in[1,\infty[$.

The first result concerns the minimization of the functionals $\overline{\mathcal{F}}^n$.

Theorem 4.2. *Assume Hypothesis 1. For every $n\in\mathbb{N}$ there exists $u_n\in\mathcal{W}\cap W^{1,\infty}(I,\mathbb{R}^d)$ which minimizes $\overline{\mathcal{F}}^n$ on $\mathcal{W}\cap H^1(I,\mathbb{R}^d)$. In addition $u_n\in C^2(I,\mathbb{R}^d)\cap C^1(\overline{I},\mathbb{R}^d)$ and there exists $R\geq 0$ such that*

$$\|u_n'\|_{W^{1,\infty}}\leq R \quad \forall n\in\mathbb{N}. \tag{4.3}$$

Finally, for every $n\in\mathbb{N}$, u_n satisfies the Euler-Lagrange equations in classical form:

$$\begin{aligned}\left(\epsilon_n^2+\overline{f}^n_{\xi_i\xi_i}(x,u_n,u_n')\right)u_{n,i}''&=\epsilon_n\sigma_i+\overline{f}^n_{p_i}(x,u_n,u_n')\\ &\quad-\overline{f}^n_{x\xi_i}(x,u_n,u_n')-\sum_{j=1}^d\overline{f}^n_{p_j\xi_i}(x,u_n,u_n')u_{n,j}',\\ &\quad i\in\{1,\dots,d\}.\end{aligned} \tag{4.4}$$

Proof. We sketch the proof, which is a collection of well known arguments which can be found in basic textbooks on Calculus of variations.

Step 1. First of all (see for example [7], Chap. 4) we observe that inequalities (2.5) imply that

$$\overline{\mathcal{F}}^n(u)\geq\epsilon_n^2\|u_n'\|_2^2+\tilde{\alpha}\|u'\|_q^q-\tilde{\beta} \quad \forall u\in\mathcal{W}\cap H^1(I,\mathbb{R}^d), \quad \forall n\in\mathbb{N} \tag{4.5}$$

for suitable $\tilde{\alpha} > 0$ and $\tilde{\beta} \in \mathbb{R}$ independent on n. Inequality (4.5) implies that the functionals $\overline{\mathcal{F}}^n$ are coercive on $W^{1,q}(I,\mathbb{R}^d) \cap H^1(I,\mathbb{R}^d)$. Assumption (iv) in Hypothesis 1 guarantees the convexity of the map

$$\xi \mapsto F^n(x,p,\xi)$$

and then we deduce the existence of a minimizer $u_n \in \mathcal{W} \cap H^1(I,\mathbb{R}^d)$ of the functional $\overline{\mathcal{F}}^n$.

By virtue of the uniform convergence (2.4) and of growth properties (2.7), it is easy to verify that the sequence $(\overline{\mathcal{F}}^n)$ Γ-converges to $\overline{\mathcal{F}}$ with respect to both strong and weak topology of $W^{1,q}(I,\mathbb{R}^d)$ (see Chaps. 4, 5 and 6 of [6] and, in particular, adapt the proof of Thm. 5.14). Consequently (see again [6], Chap. 7 or [1]), we may conclude that

$$\overline{\mathcal{F}}^n(u_n) \to \min_{\mathcal{W}} \overline{\mathcal{F}} = \inf_{\mathcal{W}} \mathcal{F}$$

and it follows, in particular, that there exists a constant $K > 0$ such that

$$\overline{\mathcal{F}}^n(u_n) \leq K \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Then inequalities (4.5) and (4.6) imply that there exist a positive $T > 0$ such that

$$\|u_n\|_\infty + \|u_n'\|_q \leq T \quad \forall n \in \mathbb{N}. \tag{4.7}$$

From this last inequality we obtain, in particular, that the sequence (u_n) admits a subsequence weakly converging in $W^{1,q}(I,\mathbb{R}^d)$.

Step 2. Inequalities (2.7) ensure that for every $n \in \mathbb{N}$ the map u_n satisfies Euler-Lagrange equation in weak form (see [5], Chap. 4):

$$\int_I \left[\overline{f}_\xi^n(x,u_n,u_n') \cdot \eta' + \overline{f}_p^n(x,u_n,u_n') \cdot \eta\right] \mathrm{d}x = 0 \quad \forall \eta \in W_0^{1,\infty}(I,\mathbb{R}^d). \tag{4.8}$$

By an integration by parts (see [4], Chap. 2.4, p. 42 and ff.), we deduce from (4.8) that for every $n \in \mathbb{N}$ there exists an absolutely continuous map $\psi_n : I \to \mathbb{R}^d$ such that

$$\psi_n(x) = \overline{f}_\xi^n(x,u_n(x),u_n'(x)) \quad \forall x \in I \quad \forall n \in \mathbb{N} \tag{4.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x) = \overline{f}_p^n(x,u_n(x),u_n'(x)) \quad \text{a.e. } x \in I \quad \forall n \in \mathbb{N}. \tag{4.10}$$

By the third inequality in (2.7), by (4.7) and integrating (4.10), we deduce that there exists a positive constant C such that

$$|\psi_n(x)| \leq C \quad \forall x \in I \quad \forall n \in \mathbb{N}. \tag{4.11}$$

Hence (2.6), (4.9) and (4.11) imply that the map u_n lies in $W^{1,\infty}(I,\mathbb{R}^d)$ for every $n \in \mathbb{N}$ and there exists a constant $\tilde{R} > 0$ such that $\|u_n'\|_\infty \leq \tilde{R}$ for every $n \in \mathbb{N}$ (see again [4] Chaps. 2.4, 2.5, 2.6). It follows that there exists $R > 0$ such that (4.3) holds true.

Step 3. By virtue of the term $\epsilon_n^2\xi^2$ in the definition (2.22) of $F^n(x,p,\xi)$, by assumptions (2.8), (2.9), (2.10) and by the properties of the elements u_n summarized up to now, it is immediate to verify that the hypotheses of Theorem 2.6. (iii) in [4] (p. 60) are satisfied; then the last item of the statement concerning Euler-Lagrange equation is proved. $\square$

Lemma 4.1 and Theorem 4.2 provide the tools to prove the strong relative compactness of the sequence (u_n), as stated by the following

Proposition 4.3. *Assume Hypothesis 1 and let (u_n) be the sequence of minimizers of the functionals $\overline{\mathcal{F}}^n$ provided by Theorem 4.2. Then there exist a subsequence (u_{n_k}) and a map $u \in \mathcal{W} \cap W^{1,\infty}(I, \mathbb{R}^d)$ such that $u_{n_k} \to u$ in $W^{1,q}(I, \mathbb{R}^d)$, $u_{n_k}(x) \to u(x)$ and $u'_{n_k}(x) \to u'(x)$ for almost every $x \in I$. In addition u is a minimizer of $\overline{\mathcal{F}}$.*

Proof. Recall that by (4.3) we have

$$\|u'_n\|_\infty \le R \quad \forall n \in \mathbb{N}. \tag{4.12}$$

Fix $i \in \{1, \dots, d\}$, $n \in \mathbb{N}$ and consider equations (4.4). Extracting the second derivative $u''_{n,i}$ we obtain

$$\begin{aligned} u''_{n,i} &= \frac{\epsilon_n \sigma_i + \overline{f}^n_{p_i}(x, u_n, u'_n)}{\epsilon_n^2 + \overline{f}^n_{\xi_i \xi_i}(x, u_n, u'_n)} - \frac{\overline{f}^n_{x\xi_i}(x, u_n, u'_n) + \sum_{j=1}^d \overline{f}^n_{p_j \xi_i}(x, u_n, u'_n) u'_{n,j}}{\epsilon_n^2 + \overline{f}^n_{\xi_i \xi_i}(x, u_n, u'_n)} \\ &= \mathcal{A} + \mathcal{B}. \end{aligned} \tag{4.13}$$

We introduce the set

$$J^i_n \doteq \left\{ x \in I : \overline{f}^n_{\xi_i \xi_i}(x, u_n(x), u'_n(x)) = 0 \right\}, \tag{4.14}$$

with the obvious remark that, whenever $\sigma_i = 0$, we have $J^i_n = \emptyset$ for every $n \in \mathbb{N}$.

Suppose first $\sigma_i = -1$, corresponding to the assumption $f_{p_i} \le 0$ (see (2.12)) and consider the term $\mathcal{B}$. By virtue of conditions (2.16)–(2.17) in (iv) of Hypothesis 1, we have

$$\mathcal{B} \le 0 \quad \text{on} \quad J^i_n. \tag{4.15}$$

On the other hand, (4.12) and conditions (2.18)–(2.19) imply that

$$|\mathcal{B}| \le C_R \quad \text{on} \quad (J^i_n)^c. \tag{4.16}$$

Turning our attention to $\mathcal{A}$, condition (2.12) implies immediately that

$$\mathcal{A} \le 0 \quad \text{on} \quad I, \tag{4.17}$$

while we observe, incidentally, that, recalling definition (4.14), we have

$$\mathcal{A} \le -\epsilon_n^{-1} \quad \text{on} \quad J^i_n. \tag{4.18}$$

Remarking that all estimates are independent on $n \in \mathbb{N}$, and collecting (4.15)–(4.17), we conclude that

$$u''_{n,i}(x) \le C_R \quad \forall x \in I \quad \forall n \in \mathbb{N}. \tag{4.19}$$

In case $\sigma_i = +1$ the above one-sided estimates (4.15), (4.17) and (4.18) turn out to be reversed, so that we obtain

$$u''_{n,i}(x) \ge C_R \quad \forall x \in I \quad \forall n \in \mathbb{N} \tag{4.20}$$

and

$$\mathcal{A} \ge \epsilon_n^{-1} \quad \text{on} \quad J^i_n. \tag{4.21}$$

We are left to treat the case $\sigma_i = 0$, which, recalling item (vi) in Hypothesis 1, corresponds to the case $\overline{f}^n_{\xi_i \xi_i} \ge \eta_i > 0$. We immediately deduce from (4.13) that there exists a positive C'_R such that

$$|u''_{n,i}(x)| \le \frac{C'_R}{\eta_i} \quad \forall x \in I \quad \forall n \in \mathbb{N}. \tag{4.22}$$

Hence Lemma 4.1 ensure the existence of the subsequence (u_{n_k}) of the statement. Since, as we have already seen, the sequence $\overline{\mathcal{F}}^n$ Γ-converges to $\overline{\mathcal{F}}$ with respect to the strong topology of $W^{1,q}(I, \mathbb{R}^d)$, we immediately conclude that the limit u of the sequence (u_{n_k}) is a minimizer of $\overline{\mathcal{F}}$ (see [6], Chap. 7). $\square$

We may state now the main result of this section.

Theorem 4.4. *Assume Hypothesis 1 and let $u \in \mathcal{W} \cap W^{1,\infty}(I, \mathbb{R}^d)$ be the minimizer of $\overline{\mathcal{F}}$ provided by Proposition 4.3. Then u is a minimizer of $\mathcal{F}$.*

Proof. We maintain the notations of previous statements and proofs and, for the sake of simplicity, we call (u_n) the converging subsequence provided by Proposition 4.3.

First of all we claim that for every index $i \in \{1, \dots, d\}$ we have

$$\mathrm{m}(J_n^i) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.23}$$

Assume, by contradiction, that for some $j \in \{1, \dots, d\}$ there exists some $\rho > 0$ such that

$$\mathrm{m}(J_n^j) \geq \rho > 0 \quad \forall n \in \mathbb{N}. \tag{4.24}$$

As we have seen, we have necessarily $\sigma_j \neq 0$ and suppose, to fix ideas, $\sigma_j = -1$. Recalling (4.15) and (4.18), observe that we have

$$u''_{n,j}(x) \leq -\epsilon_n^{-1} \quad \forall x \in J_n^j \ \forall n \in \mathbb{N}. \tag{4.25}$$

Formulas (4.24) and (4.25) imply that

$$\begin{aligned} u'_{n,j}(b) &= u'_{n,j}(a) + \int_{(J_n^j)^c} u''_{n,j}(t)\,\mathrm{d}t + \int_{J_n^j} u''_{n,j}(t)\,\mathrm{d}t \\ &\leq R + C_R \mathrm{m}(I) - \rho\epsilon_n^{-1} \quad \overset{n\to\infty}{\longrightarrow} -\infty. \end{aligned} \tag{4.26}$$

Since (4.26) contradicts (4.3), the claim (4.23) is proved (the case $\sigma_j = +1$ is analogous).

We know that $u'_n \to u'$ almost everywhere in I and that $u_n \to u$ uniformly on I, so that, by Hypothesis (2.4), we have

$$\overline{f}^n(x, u_n(x), u'_n(x)) \overset{n\to\infty}{\longrightarrow} \overline{f}(x, u(x), u'(x)) \quad \text{a.e.} \quad x \in I. \tag{4.27}$$

In order to obtain the result we need to show that

$$\overline{f}^n(x, u_n(x), u'_n(x)) \overset{n\to\infty}{\longrightarrow} f(x, u(x), u'(x)) \quad \text{a.e.} \quad x \in I. \tag{4.28}$$

Assume, by contradiction, that there exists a measurable subset $S \subseteq I$ such that $\mathrm{m}(S) > 0$ and

$$\overline{f}(x, u(x), u'(x)) < f(x, u(x), u'(x)) \quad \text{a.e.} \quad x \in S. \tag{4.29}$$

By Egorov theorem, (4.27) and (4.29) imply that there exist a positive $\epsilon > 0$, a subset $S_\epsilon \subseteq S$, with $\mathrm{m}(S_\epsilon) \geq \rho > 0$ and $n_\epsilon \in \mathbb{N}$ such that

$$\overline{f}^n(x, u_n(x), u'_n(x)) < f(x, u_n(x), u'_n(x)) - \epsilon \quad \text{a.e.} \quad x \in S_\epsilon \ \forall n \geq n_\epsilon. \tag{4.30}$$

Conditions (2.20) and (2.21) in item (viii) in Hypothesis 1 and (4.30) imply that there exists $\overline{n}_\epsilon \geq n_\epsilon$ and an index $k \in \{1, \dots, d\}$ (depending on x) with $\sigma_k \neq 0$, such that

$$\overline{f}^n_{\xi_k\xi_k}(x, u_n(x), u'_n(x)) = 0 \quad \text{a.e.} \quad x \in S_\epsilon \ \forall n \geq \overline{n}_\epsilon. \tag{4.31}$$

Recalling definition (4.14), formula (4.31) implies that

$$\sum_{j=1}^{d} \mathrm{m}(J_n^j) \geq \rho \quad \forall n \geq n_\epsilon, \tag{4.32}$$

and this contradicts (4.23). Hence (4.28) is proved. ∎

5. Removing the monotonicity in u

In the proof of Proposition 4.3, ensuring the strong compactness of the minimizing sequence, we have used condition (2.12), which says that the integrand $\overline{f}$ must be separately monotone in the variable u. It is worth to ask whether it is possible to prove the existence of minimizers of the non convex functional when such assumption is removed.

In this section we consider a functional of sum type for which the monotonicity assumption provided by condition (v) of Hypothesis 1 fails. We show that assuming that the relaxed functional $\overline{\mathcal{F}}$ admits a unique minimizer $\overline{u}$ and that the mentioned monotonicity assumption holds true in a neighbourhood of $\overline{u}$, then $\overline{u}$ minimizes also the non convex functional $\mathcal{F}$.

We adopt the notations of Section 2 and state the assumptions needed in the present case.

Hypothesis 2. We take a continuous function $g : \overline{I} \times \mathbb{R}^d \to \mathbb{R}$, a C^1 function $h : \overline{I} \times \mathbb{R}^d \to \mathbb{R}$ and, for every $(x, p, \xi) \in \overline{I} \times \mathbb{R}^{2d}$, set

$$f(x,p,\xi) = g(x,\xi) + h(x,p), \quad \overline{f}(x,p,\xi) = \overline{g}(x,\xi) + h(x,p), \tag{5.1}$$

where $\overline{g}$ is the lower convex envelope of g with respect to the second variable.

Then we consider the set $\mathcal{W}$ and the functionals $\mathcal{F}$ and $\overline{\mathcal{F}}$ as defined, respectively, in (2.2), (2.1) and (2.3).

We assume that there exist two sequences $(\overline{g}^n)$ and (h^n) in $C^2(\overline{I} \times \mathbb{R}^d, \mathbb{R})$ such that, as in Hypothesis 1,

$$\overline{g}^n \to \overline{g}, \quad h^n \to h, \quad \text{uniformly on compact subsets of} \quad \overline{I} \times \mathbb{R}^d.$$

and, in addition,

$$h^n_p \to h_p \quad \text{uniformly on compact subsets of} \quad \overline{I} \times \mathbb{R}^d. \tag{5.2}$$

Setting

$$\overline{f}^n(x,p,\xi) \doteq \overline{g}^n(x,\xi) + h^n(x,p) \quad \forall (x,p,\xi) \in I \times \mathbb{R}^{2d},$$

we assume that all conditions of Hypothesis 1 hold true except item *(v)*.

Remark 5.1. Clearly conditions and formulas in Hypothesis 1 should be re-written in terms of functions g^n and h^n. We leave this elementary operation to the reader, remarking, for example, that in this case the set Ξ^i_n introduced in item (2.11) depends only on x and that formula (2.19) vanishes.

We may state the result of this section.

Theorem 5.2. *Assume Hypothesis 2 and suppose that the functional $\overline{\mathcal{F}}$ admits a unique minimizer $\overline{u} \in \mathcal{W}$ such that*

$$h_{p_i}(x, \overline{u}(x)) \neq 0 \quad \forall x \in \overline{I} \quad \forall i \in \{1, \dots, d\} : \eta_i = 0. \tag{5.3}$$

Then $\overline{u}$ minimizes $\mathcal{F}$.

Proof. First of all we define the vector $\sigma = (\sigma_i, \dots, \sigma_d)$, as in (2.14), according to the sign of $h_{p_i}(\cdot, \overline{u}(\cdot))$, that is to say

$$\sigma_i \doteq \begin{cases} \operatorname{sign}(h_{p_i}(\cdot, \overline{u}(\cdot))) & \text{if} \quad \eta_i = 0 \\ 0 & \text{if} \quad \eta_i > 0, \end{cases} \tag{5.4}$$

Then we observe that condition (v) of Hypothesis 1 (the unique one which now is not valid) has been used in section 4 only in the proof of Proposition 4.3. Hence Theorem 4.2 holds true and we may assert that the functional $\overline{\mathcal{F}}$ admits at least one minimizer $\overline{u}$, which by assumption is unique, and that the sequence (u_n) of minimizers of the functionals $\overline{\mathcal{F}}^n$ satisfies all properties of Theorem 4.2. In particular, by virtue of (4.3), we may state that (u_n) is relatively compact in the weak topology of $W^{1,q}(I,\mathbb{R}^d)$, so that, extracting if necessary a subsequence and invoking again Γ-convergence (see Chap. 7 in [6]), we may assert that

$$u_n \rightharpoonup \overline{u} \quad \text{in } W^{1,q}(I,\mathbb{R}^d), \qquad u_n \to \overline{u} \quad \text{uniformly on} \quad \overline{I}. \tag{5.5}$$

and

$$\overline{\mathcal{F}}^n(u_n) \to \overline{\mathcal{F}}(\overline{u}).$$

In order to reproduce the existence result, we need the strong convergence in $W^{1,q}(I,\mathbb{R}^d)$ of the sequence u_n. To this aim we consider formula (4.13), which in the present case takes the form

$$u''_{n,i} = \frac{\epsilon_n \sigma_i + h^n_{p_i}(x,u_n)}{\epsilon_n^2 + \overline{g}^n_{\xi_i\xi_i}(x,u'_n)} - \frac{\overline{g}^n_{x\xi_i}(x,u'_n)}{\epsilon_n^2 + \overline{g}^n_{\xi_i\xi_i}(x,u_n,u'_n)},$$

where, by assumption, the equivalent of condition (2.16) holds true, that is to say, if $\sigma_i = -1$,

$$\overline{g}^n_{x\xi_i}(x,\xi) \geq 0 \quad \forall n \in \mathbb{N}, \quad \forall x \in I, \quad \forall \xi \in \Xi^i_n(x),$$

while the inequality is reversed when $\sigma_i = +1$.

With respect to Section 4 we do not have $h^n_{p_i} \leq 0$ on the whole domain, but, by (5.2) and (5.5), we may invoke the uniform convergence

$$h^n_p(\cdot,u_n) \to h_p(\cdot,\overline{u}) \quad \text{uniformy on} \quad \overline{I}. \tag{5.6}$$

Assume, to fix ideas, $\sigma_i = -1$ and recall that, by assumption (5.3), for some positive ρ we have

$$h_{p_i}(x,\overline{u}(x)) < -\rho \quad \forall x \in \overline{I}. \tag{5.7}$$

We deduce from (5.6) and (5.7) that there exists $\overline{n} \in \mathbb{N}$ such that

$$h^n_{p_i}(x,u_n(x)) \leq 0 \quad \forall x \in \overline{I} \quad \forall n \geq \overline{n}.$$

From this point on, the proof proceeds like the ones of Proposition 4.3 and of Theorem 4.4, providing then the claimed result. □

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