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BANG-BANG CONTROL OF A THERMOSTAT WITH NONCONSTANT COOLING POWER

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Abstract. A control system describing the dynamic behavior of a car thermostat is considered. The cooling power of the car's radiator is allowed to depend on the ambient temperature. This physically natural assumption presents some challenges to mathematical investigation of the model. The existence and some properties of solutions of the control system are established.

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1. Introduction

Let T be an interval of the real line $\mathbb{R}$. By $C(T,\mathbb{R}^n)$, $n=1,2$, we will denote the space of continuous functions from T to $\mathbb{R}^n$ equipped with the sup–norm. Given a finite *delay* $r>0$, let $\mathcal{C}_0 := C([-r,0],\mathbb{R})$ and $|\cdot|_\infty$ be the norm on $\mathcal{C}_0$. For $x\in C([-r,1],\mathbb{R})$ define

$$x_t(\tau) := x(t+\tau), \quad \tau\in[-r,0].$$

This paper is concerned with the existence and some properties of solutions of the control system:

$$a_1\dot{v}(t)+a_2\dot{w}(t) = g(v_t,w_t)u(t) \qquad \text{for a.e.} \qquad t\in[0,1], \tag{1.1}$$

$$-\dot{w}(t)\in\partial I_{K(v(t))}(w(t)) \qquad \text{for a.e.} \qquad t\in[0,1], \tag{1.2}$$

$$v(\tau)=v_0(\tau), \qquad w(\tau)=w_0(\tau), \quad \tau\in[-r,0], \tag{1.3}$$

subject to the control constraint:

$$u(t)\in[u_\alpha,u_\beta] \qquad \text{for a.e.} \qquad t\in[0,1]. \tag{1.4}$$

Here, $K(v)$, $v\in\mathbb{R}$ is a given set, $I_{K(v)}$ is its indicator function, *i.e.* $I_{K(v)}(w)=0$ if $w\in K(v)$ and $I_{K(v)}(w)=+\infty$ otherwise. The operator $\partial I_{K(v)}$ is the subdifferential in the sense of the convex analysis of $I_{K(v)}$. The

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constants a_i, $i = 1, 2$, are given. The scalar function $g(\cdot,\cdot)$ defined on $\mathcal{C}_0 \times \mathcal{C}_0$ is *not* necessarily (globally) *Lipschitz continuous* and of a *sublinear growth*. Furthermore, $v_0, w_0 \in \mathcal{C}_0$ are given functions and $[u_\alpha, u_\beta] \subset \mathbb{R}$.

We note that equation (1.2) may alternatively be written in the equivalent form:

$$w(t) = \mathcal{H}[w(0), v](t) \qquad \text{for a.e.} \qquad t \in [0, 1], \tag{1.5}$$

where $\mathcal{H}$ represents a *hysteresis* operator of either stop or generalized play type with bounded generating curves (*cf.*, *e.g.*, [8,14]). Depending on which of the two operators is considered, we have two choices for the set $K(v)$ in (1.2) (see Hypotheses $H(K)a$ and $H(K)b$ of the next section).

Our work is mainly motivated by the study of automotive thermostat models appeared recently in the mathematical literature [3,15]. The thermostat models considered in these references are designed to naturally exhibit both delay and hysteresis behavior. In spite of the apparent importance for applications investigations of evolution systems, let alone control systems, featuring these two phenomena are still very few in number [5,6].

In [3,15] the authors considered two models describing the dynamic behavior of thermostats in cars controlling the operating temperature of the engine. The first model is given by the system:

$$\dot{\theta}(t) = q_e - q_r \omega(t - \tau), \quad t \geq 0, \tag{1.6}$$

$$\omega(t) = H_\beta(\theta(t)), \quad t \geq -\tau, \tag{1.7}$$

$$\theta(t) = \theta_0(t), \quad -\tau \leq t \leq 0, \quad \omega(-\tau) = \omega_0, \tag{1.8}$$

for the unknown temperature of the coolant fluid $\theta(t)$ and the fractional thermostat opening $\omega(t)$. Here, q_e is the engine heat generation, q_r is the *cooling power of the radiator* assumed in this model to be *constant*, θ_0 is the initial condition for the temperature over the interval $[-\tau, 0]$, ω_0 is the initial value of the thermostat opening, τ is the delay. H_β is a *generalized play* hysteresis operator.

The thermostat operates in such a way that excess heat produced by the engine is carried away by the coolant until the latter reaches a prescribed upper temperature threshold T_L when the thermostat starts opening $(0 < \omega < 1)$ to divert a fraction of flow to the radiator in order to cool it down. The thermostat continues to open until fully open $(\omega = 1)$ when the lower temperature threshold T_R is reached. When the engine coolant temperature falls below this threshold the thermostat starts closing and is fully closed $(\omega = 0)$ when the temperature reaches the value T_L.

The thermo-mechanical information characterizing the thermostat is given by two prescribed curves f_R and f_L and the (hysteresis) region lying between them. A hysteretic behavior assuming to follow the "generalized play model" pattern occurs since the way the thermostat opens (along f_R) when the temperature rises differs from the way it closes (along f_L) when the temperature falls. The appearance of delay is explained by the time the cooling flow takes to run from the radiator to the engine.

If the cooling power of the radiator q_r is allowed to depend on the air flow and the air ambient temperature (as in a real radiator) the corresponding changes to system (1.6)–(1.8) give rise to the second thermostat model considered in [3,15]: Find a pair $\{\theta, \omega\}$ such that

$$\dot{\theta}(t) = q_e - h\,\omega(t - \tau)\,\theta(t - \tau), \quad t \geq 0, \tag{1.9}$$

$$\omega(t) = H_\beta(\theta(t)), \quad t \geq -\tau, \tag{1.10}$$

$$\theta(t) = \theta_0(t), \quad -\tau \leq t \leq 0, \quad \omega(-\tau) = \omega_0, \tag{1.11}$$

where h is the radiator coefficient of heat exchange. We remark that mathematical analysis of this new system becomes more delicate as now the right-hand side is not more (globally) Lipschitz continuous, neither is it of a sublinear growth. Note also that given the equivalence of (1.2) and (1.5) system (1.9)–(1.11) is a special case of our evolution system (1.1)–(1.3) with a fixed control u. In particular, a possible optimization of the engine cooling process can be achieved in a natural way by a partial control of the heat exchange in the radiator in the energy balance equation (1.9).

When it comes to the control system (1.1)–(1.4) our interest lies in showing that its solutions are close, in a prescribed sense, to the extreme solutions, *i.e.* solutions of (1.1)–(1.3) with $u(t)$ equals either u_α or u_β, the extreme points of the control constraint interval (this is the so-called *bang-bang principle*).

We note that a control system rooted in the first thermostat model (1.6)–(1.8) (with globally Lipschitz right-hand sides of sub-linear growth) has recently been studied in [10]. The existence of solutions and a relaxation type result were obtained for this control system subjected to a nonconvex state-dependent constraint.

2. Preliminaries

In this section, we introduce some notions we use throughout the paper and state some known results which we need for our proofs.

A function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is called proper if its effective domain $\operatorname{dom}\varphi = \{x \in \mathbb{R};\ \varphi(x) < +\infty\}$ is nonempty. Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then, its subdifferential at a point $x \in \mathbb{R}$ is the set

$$\partial\varphi(x) = \{h \in \mathbb{R};\ h(y - x) \le \varphi(y) - \varphi(x) \quad \forall y \in \operatorname{dom}\varphi\}. \tag{2.1}$$

In case of the indicator function of an interval $[a, b]$ we have

$$\partial I_{[a,b]}(w) = \begin{cases} \emptyset & \text{if} \quad w \notin [a,b], \\ [0,+\infty) & \text{if} \quad w = b > a, \\ \{0\} & \text{if} \quad a < w < b, \\ (-\infty, 0] & \text{if} \quad w = a < b, \\ (-\infty,+\infty) & \text{if} \quad w = a = b. \end{cases} \tag{2.2}$$

We say that a sequence of proper convex lower semicontinuous functions $\varphi_n : \mathbb{R} \to \mathbb{R},\ n \ge 1$, *Mosco-converges* [12] to a proper convex lower semicontinuous function $\varphi : \mathbb{R} \to \mathbb{R}$, denoted $\varphi_n \overset{M}{\to} \varphi$, if:

(1) for any $x \in \mathbb{R}$ there exists a sequence $x_n \to x$ such that $\varphi_n(x_n) \to \varphi(x)$;
(2) for any $x \in \mathbb{R}$ and any sequence $x_n \in \mathbb{R},\ n \ge 1$, converging to x we have

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n).$$

The norm on the Euclidean space $\mathbb{R}^2$ we denote by $\|\cdot\|$.

The following assumptions on the data describing our problem (1.1)–(1.4) hold throughout the paper:

H(a). The constants $a_1 > 0$, $a_2 \ge 0$.
H(h). *The function* $g : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathbb{R}$ *has the properties*:

(1) *there exist a positive constant* K_0 *and a function* $k : \mathbb{R} \to \mathbb{R}^+$ *bounded from above on bounded sets such that*

$$|g(v, w)| \le K_0 + k(|w|_\infty)|v|_\infty, \qquad v, w \in \mathcal{C}_0; \tag{2.3}$$

(2) *g is locally Lipschitz continuous in the sense that*

$$|g(v_1, w_1) - g(v_2, w_2)| \le L(|v_1 - v_2|_\infty + |w_1 - w_2|_\infty) \tag{2.4}$$

provided (v_i, w_i), $i = 1, 2$, *lie in a ball* $\{(v, w) \in \mathcal{C}_0 \times \mathcal{C}_0;\ \|(v, w)\|_\infty \le \rho\}$, $\rho > 0$, *for some* $L > 0$ *depending on* ρ. *Here,* $\|\cdot\|_\infty$ *is the* sup-*norm on the space* $C([-r, 0], \mathbb{R}^2)$;

In addition, one of the following is true:

K(a). $K(v) = [h_L, h_R]$, $v \in \mathbb{R}$, $h_L < h_R$; or
K(b). $K(v) = [f_L(v), f_R(v)]$, $v \in \mathbb{R}$, *where* f_L, f_R *are such that* $f_L(v) \leq f_R(v)$, $v \in \mathbb{R}$, *and*

(1) f_L, f_R *are nondecreasing and Lipschitz continuous on* $\mathbb{R}$;
(2) *there exist* k_1, k_2 *such that* $f_L(v) = f_R(v)$ *for* $v \in (-\infty, k_1] \cup [k_2, +\infty)$ *and*

$$|f_L(v)|, |f_R(v)| \leq 1 \qquad \text{for all } v \in \mathbb{R}. \tag{2.5}$$

Remark 2.1.

(1) The situation H(K)a corresponds to the choice of the stop operator in equation (1.5), with the hysteresis region being an infinite strip in the plane (v, w). When H(K)b is assumed, the operator $\mathcal{H}$ in (1.5) is the generalized play, in which case the hysteresis region is the domain comprised between the curves f_L and f_R.
(2) The values k_1 and k_2 are counterparts of the thresholds T_L and T_R, respectively, in the thermostat model (1.9)–(1.11) considered in Introduction, while Assumption (2.5) is in accordance with the requirement

$$0 \leq \omega \leq 1$$

of this model.

Let $\Lambda \subset C([0,1], \mathbb{R})$ be a compact set and

$$\varphi^t(v)(w) = I_{K(v(t))}(w), \qquad w \in \mathbb{R},\ t \in [0,1],\ v \in \Lambda, \tag{2.6}$$

where $I_{K(v)}$ is the indicator function of the set $K(v)$, the latter being defined as above. Furthermore, let

$$\Phi(v)(w) = \int_0^1 \varphi^t(v)(w(t))\,\mathrm{d}t, \quad w \in L^2([0,1], \mathbb{R}),$$

and

$$S_{K(v)} = \{w \in L^2([0,1], \mathbb{R});\ w(t) \in K(v(t)) \quad \text{for a.e. } t \in [0,1]\},$$

$v \in L^2([0,1], \mathbb{R})$. Then, it is easy to see that

$$\Phi(v) = I_{S_{K(v)}}, \quad v \in \Lambda, \tag{2.7}$$

where $I_{S_{K(v)}}$ is the indicator function of the set $S_{K(v)}$. Moreover, we have

Proposition 2.2 ([12], Lem. 5.1)**.** *If* $f \in L^2([0,1], \mathbb{R})$, *then*

$$f \in \partial\Phi(v)(w) \quad \text{if and only if} \quad f(t) \in \partial\varphi^t(v)(w(t)) \quad \text{a.e. on } [0,1]$$

for any $v \in \Lambda$, $w \in L^2([0,1], \mathbb{R})$.

For a measurable multivalued mapping $F : [0,1] \to \mathbb{R}$ denote by

$$S(F) = \{f \in L^2([0,1], \mathbb{R});\ f(t) \in F(t) \quad \text{for a.e. } t \in [0,1]\},$$

Then, we have

Proposition 2.3 ([13], Prop. 4.2)**.** *Let* $F_1, F_2 : [0,1] \to \mathbb{R}$ *be two measurable multivalued mappings. Then,*

$$\mathrm{haus}_L(S(F_1), S(F_2)) \leq \left(\int_0^1 \left[\mathrm{haus}(F_1(t), F_2(t)) \right]^2 \mathrm{d}t \right)^{\frac{1}{2}},$$

where $\mathrm{haus}(\cdot,\cdot)$ *and* $\mathrm{haus}_L(\cdot,\cdot)$ *are the Hausdorff metrics on the spaces of nonempty closed bounded subsets from* $\mathbb{R}$ *and* $L^2([0,1], \mathbb{R})$, *respectively.*

The following statement is easily obtained from ([1], Def.-Prop. 3.21 and [4], Prop. 4.7.15):

Proposition 2.4. *Let X be a reflexive Banach space and $\{K_n\}_{n\geq 1}$, K be a sequence of nonempty closed bounded convex subsets of X such that* $\mathrm{haus}_X(K_n, K) \to 0$ *as $n \to \infty$. Then, $I_{K_n} \overset{M}{\to} I_K$.*

Here, haus_X denotes the Hausdorff metric on the space of nonempty closed bounded subsets of X.

By a *solution* of control system (1.1)–(1.4) we mean a pair (x, u), $x = (v, w) \in C([-r, 1], \mathbb{R}^2)$, $u \in L^2(T, \mathbb{R})$ such that the restriction $x|_{[0,1]}$ is absolutely continuous, $v(\tau) = v_0(\tau)$, $w(\tau) = w_0(\tau)$, $\tau \in [-r, 0]$, $w_0(0) \in K(v_0(0))$, and (1.1), (1.2), (1.4) hold a.e. on $[0, 1]$.

Note that from (1.2) and (2.2) it follows that for a solution (x, u), $x = (v, w)$, we necessarily have $w|_{[0,1]} \in K(v|_{[0,1]})$.

3. Existence

In this section we show that our system (1.1)–(1.4) has solutions.

First, we consider the undelayed system:

$$a_1\dot{v}(t) + a_2\dot{w}(t) = \varphi(t) \qquad \text{a.e. on} \quad [0, 1], \tag{3.1}$$

$$-\dot{w}(t) \in \partial I_{K(v(t))}(w(t)) \qquad \text{a.e. on} \quad [0, 1], \tag{3.2}$$

$$v(0) = v_0(0), \qquad w(0) = w_0(0), \tag{3.3}$$

for some $\varphi \in L^2([0, 1], \mathbb{R})$, where $K(v)$ is given by either $H(K)a$ or $H(K)b$. Its *solution* is a pair $(v, w) \in W^{1,2}([0, 1], \mathbb{R}^2)$ such that (3.1)–(3.3) hold.

Lemma 3.1. *For any $\varphi \in L^2([0, 1], \mathbb{R})$, $\|\varphi\|_{L^2([0,1],\mathbb{R})} \leq M$, $M > 0$, there exists a unique solution $x(\varphi) = (v(\varphi), w(\varphi))$ of system (3.1)–(3.3) such that*

$$\|x(\varphi)(t)\| \leq N, \quad t \in [0, 1], \tag{3.4}$$

$$\|\dot{x}(\varphi)\|_{L^2(T,\mathbb{R}^2)} \leq N, \tag{3.5}$$

for some $N > 0$ depending on M, $(v_0(0), w_0(0))$ and the constants in Hypotheses $H(a)$, $H(K)$ only. Moreover, for any two solutions $x(\varphi_i)$, $\varphi_i \in L^2(T, \mathbb{R})$, $i = 1, 2$, we have

$$\|x(\varphi_1)(t) - x(\varphi_2)(t)\| \leq R_0 \int_0^t |\varphi_1(s) - \varphi_2(s)|\mathrm{d}s, \quad t \in [0, 1], \tag{3.6}$$

for some $R_0 > 0$ independent of M.

Proof. A proof in the case of generalized play operator (corresponding to $H(K)b$) can be derived from its non-constant coefficients counterpart considered in [11] (see Lem. 4.1, Thm. 4.1 and inequality (4.17) of this reference).

Here, we give an elementary proof of the lemma for the case of stop operator, *i.e.* when $H(K)a$ holds. From this hypothesis and (2.2), (3.2) we see that $\dot{w}(t) = 0$ if $h_L < w(t) < h_R$, $t \in [0, 1]$. Obviously, $\dot{w}(t) = 0$ when $w(t) = h_L$ or $w(t) = h_R$, so that always $\dot{w}(t) = 0$ and w is thus a piecewise constant. The estimates for v and $\dot{v}$ then follow from equation (3.1), while inequality (3.6), implying the uniqueness, is an easy consequence of the latter equation and the Gronwall−Bellman lemma. The existence of solutions is given by Carathéodory's existence theorem for ODEs. □

Denote by $\mathcal{T} : L^2([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}^2)$ the operator which with each $\varphi \in L^2([0,1],\mathbb{R})$ associates the unique solution $x(\varphi) = (v(\varphi), w(\varphi))$ of system (3.1)–(3.3), *i.e.*

$$x(\varphi) = (v(\varphi), w(\varphi)) = \mathcal{T}\varphi. \tag{3.7}$$

Now, we prove that the operator $\mathcal{T} : \omega\text{-}L^2([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}^2)$ is continuous. Here, the notation $\omega\text{-}L^2([0,1],\mathbb{R})$ means that the space $L^2([0,1],\mathbb{R})$ is considered to be equipped with the weak topology. To this end, let $\varphi_n \to \varphi_*$, $n \geq 1$, in $\omega\text{-}L^2([0,1],\mathbb{R})$ and $x(\varphi_n) = (v(\varphi_n), w(\varphi_n))$, $n \geq 1$, be the corresponding solutions of (3.1)–(3.3). Then, the sequence φ_n, $n \geq 1$, is bounded in $L^2([0,1],\mathbb{R})$ and, thus, there exists some $M > 0$ such that

$$\|\varphi_n\|_{L^2([0,1],\mathbb{R})} \leq M.$$

In view of estimates (3.4), (3.5) we then infer that up to subsequences:

$$(v_n, w_n) \to (v_*, w_*) \qquad \text{in} \quad C([0,1],\mathbb{R}^2),$$

$$(\dot{v}_n, \dot{w}_n) \to (\dot{v}_*, \dot{w}_*) \qquad \text{in} \quad \omega\text{-}L^2([0,1],\mathbb{R}),$$

for some $(v_*, w_*) \in W^{1,2}([0,1],\mathbb{R}^2)$. Obviously, (v_*, w_*) satisfy equation (3.1) with $\varphi(t) = \varphi_*(t)$, $t \in [0,1]$.

To show that (v_*, w_*) satisfy inclusion (3.2) as well, let Λ be the collection of functions $v : [0,1] \to \mathbb{R}$ such that

$$|v(t)| \leq N, \;\; t \in [0,1], \quad \|\dot{v}\|_{L^2([0,1],\mathbb{R})} \leq N,$$

where N is the constant from (3.4), (3.5). Then, $\Lambda \subset C([0,1],\mathbb{R})$ is compact and $v_n, v_* \in \Lambda$, $n \geq 1$. From Proposition 2.1 in view of (2.6), (2.7) it follows that if $f \in L^2([0,1],\mathbb{R})$, then

$$f \in \partial I_{S_{K(v)}}(w) \quad \text{if and only if} \quad f(t) \in \partial I_{K(v(t))}(w(t)) \quad \text{a.e. on } [0,1] \tag{3.8}$$

for $v \in \Lambda$, $w \in L^2([0,1],\mathbb{R})$. Applying Proposition 2.2 and $H(K)$ we further infer that

$$\begin{aligned}\text{haus}_L(S_{K(v_1)}, S_{K(v_2)}) &\leq \left(\int_0^1 \left[\text{haus}(K(v_1(t)), K(v_2(t)))\right]^2 \text{d}t\right)^{\frac{1}{2}} \\ &\leq L_0 \left(\int_0^1 |v_1(t)) - v_2(t)|^2 \text{d}t\right)^{\frac{1}{2}},\end{aligned}$$

for any $v_1, v_2 \in L^2([0,1],\mathbb{R})$ and some $L_0 > 0$. Hence, Proposition 2.3 implies that

$$I_{S_{K(v_n)}} \xrightarrow{M} I_{S_{K(v)}}. \tag{3.9}$$

From (3.2) and (3.8) we infer that

$$-\dot{w}_n \in \partial I_{S_{K(v_n)}}(w_n), \quad n \geq 1, \tag{3.10}$$

in particular, $w_n \in S_{K(v_n)}$ and, thus, $I_{S_{K(v_n)}}(w_n) = 0$. From (3.9) it follows that

$$0 \leq I_{S_{K(v_*)}}(w_*) \leq \liminf_{n\to\infty} I_{S_{K(v_n)}}(w_n) = 0,$$

which implies that $w_* \in S_{K(v_*)}$. Furthermore, for any $f \in S_{K(v_*)}$ there exists a sequence $f_n \in S_{K(v_n)}$ converging to f in $L^2(T,\mathbb{R})$ such that $I_{S_{K(v_n)}}(f_n) \to I_{S_{K(v_*)}}(f)$. Then, from the definition of the subdifferential in view of (3.10) we obtain

$$-\dot{w}_n(f_n - w_n) \leq I_{S_{K(v_n)}}(f_n) - I_{S_{K(v_n)}}(w_n) = 0.$$

Passing to the limit as $n \to \infty$ in this inequality gives

$$-\dot{w}_*(f - w_*) \leq 0 = I_{S_{K(v_*)}}(f) - I_{S_{K(v_*)}}(w_*).$$

From the arbitrariness of $f \in S_{K(v_*)}$ we deduce that

$$-\dot{w}_* \in \partial I_{S_{K(v_*)}}(w_*),$$

which in turn implies that

$$-\dot{w}_*(t) \in \partial I_{K(v_*(t))}(w_*(t)) \quad \text{a.e. on} \qquad [0,1],$$

i.e. (v_*, w_*) satisfy (3.2). Since for a fixed $\varphi \in L^2([0,1],\mathbb{R})$ system (3.1)–(3.3) has a unique solution, we have $(v_*, w_*) = \mathcal{T}\varphi_*$. Therefore, the operator $\mathcal{T} : \omega\text{-}L^2([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}^2)$ is continuous.

Theorem 3.2. *Control system* (1.1)–(1.4) *has a solution, which is unique for any given measurable* $u(t) \in [u_\alpha, u_\beta]$, $t \in [0,1]$.

Proof. Let (x,u), $x = (v,w)$ be a possible solution of (1.1)–(1.4). Defining

$$\varphi_*(t) = g(v_t, w_t)u(t), \quad t \in [0,1], \tag{3.11}$$

we see from (1.4) and (2.3) that

$$|\varphi_*(t)| \leq (K_0 + k(|w_t|_\infty)|v_t|_\infty)\, m, \quad t \in [0,1], \tag{3.12}$$

where

$$m = \max\{|u_\alpha|, |u_\beta|\}. \tag{3.13}$$

Since $w_0 \in \mathcal{C}_0$, from (1.2), (2.2) and (2.5) or $H(K)a$ it follows that there exists a constant $C > 0$ such that

$$|w_t|_\infty \leq C \qquad \text{for any } t \in [0,1].$$

Therefore, from (3.12) and the properties of the function k (*cf.* $H(h)(1)$) we infer that

$$|\varphi_*(t)| \leq C_1 + C_2|v_t|_\infty, \quad t \in [0,1], \tag{3.14}$$

for some positive constants C_1 and C_2. In particular, $\varphi_* \in L^2([0,1],\mathbb{R})$. The definition of a solution and the existence of a unique solution to system (3.1)–(3.3) implies that $x|_{[0,1]}$ is the unique solution of (3.1)–(3.3) with $\varphi(t) = \varphi_*(t)$ defined by (3.11). Let $y(t)$ be the solution of (3.1)–(3.3) corresponding to $\varphi(t) \equiv 0$. Then, from (3.6) and (3.14) we obtain

$$\|x(t) - y(t)\| \leq R_0 \int_0^t (C_1 + C_2\|x_s\|_\infty)\, \mathrm{d}s, \quad t \in [0,1]. \tag{3.15}$$

Denoting $x_0(t) = (v_0(t), w_0(t))$, $t \in [-r, 0]$, and defining

$$\tilde{y}(t) = \begin{cases} x_0(t),\ t \in [-r,0], \\ y(t), \quad t \in [0,1], \end{cases}$$

from (3.4) and the fact that $x_0 \in \mathcal{C}_0 \times \mathcal{C}_0$ we deduce that

$$\|\tilde{y}\|_{C([-r,1],\mathbb{R}^2)} \leq N_0$$

for some $N_0 > 0$. Hence, from (3.15) we see that

$$\|x_t\|_\infty \leq N_0 + R_0 \int_0^t (C_1 + C_2\|x_s\|_\infty)\, \mathrm{d}s. \tag{3.16}$$

From the Gronwall−Bellman lemma we conclude that

$$\|x_t\|_\infty \le M_0, \quad t \in [0,1], \tag{3.17}$$

for some $M_0 > 0$. Therefore, (1.4) and (2.3) imply that the right-hand side of (1.1) is bounded in $L^2([0,1],\mathbb{R})$ and Lemma 3.1 tells then that

$$\|\dot{x}\|_{L^2([0,1],\mathbb{R}^2)} \le N, \tag{3.18}$$

for some $N > 0$ depending on the constants of the problem only.

Given the a priori estimates (3.17) and (3.18) for solutions of our system (1.1)–(1.4), we now prove the existence of solutions. To this end, consider the operator $\mathrm{pr} : \mathbb{R} \to [-M_0, M_0]$ of projection onto the interval $[-M_0, M_0]$, where M_0 is the constant from (3.17), defined as follows

$$\mathrm{pr}(x) = \begin{cases} x, & x \in [-M_0, M_0], \\ M_0 \dfrac{x}{|x|}, & |x| > M_0. \end{cases}$$

If (x,u), $x = (v,w)$ is a solution of (1.1)–(1.4) then, as shown above, $\|x_t\|_\infty \le M_0$ and, thus, $(\mathrm{pr}\, v)_t = v_t$, $(\mathrm{pr}\, w)_t = w_t$, $t \in [0,1]$. Hence, (x,u) is a solution of system (1.1)–(1.4) with $g(v_t, w_t)$ in (1.1) changed by $g((\mathrm{pr}\, v)_t, (\mathrm{pr}\, w)_t)$. Conversely, if (x,u) is a solution of this latter system, similarly to (3.16) we obtain

$$\|(v,w)_t\|_\infty \le N_0 + R_0 \int_0^t (C_1 + C_2 \|(\mathrm{pr}\, v)_s, (\mathrm{pr}\, w)_s)\|_\infty)\, \mathrm{d}s. \tag{3.19}$$

Since $|\,\mathrm{pr}\, x| \le |x|$, $x \in \mathbb{R}$, (3.19) implies (3.17). Consequently, (x,u) is a solution to (1.1)–(1.4). Therefore, the replacement of $g(v_t, w_t)$ with $g((\mathrm{pr}\, v)_t, (\mathrm{pr}\, w)_t)$ in (1.1) does not affect the solutions of system (1.1)–(1.4).

Let now a measurable $u_*(t) \in [u_\alpha, u_\beta]$ be fixed and define

$$\mathcal{G}(x)(t) = g((\mathrm{pr}\, v)_t, (\mathrm{pr}\, w)_t)\, u_*(t) \tag{3.20}$$

for $x = (v,w) \in C([-r,1],\mathbb{R}^2)$, $t \in [0,1]$. From (2.3) it follows that $\mathcal{G}(x) \in L^2([0,1],\mathbb{R})$ and from (2.4) we see that the operator $\mathcal{G} : C([-r,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$ is continuous. Next, let

$$S_R = \{\varphi \in L^2([0,1],\mathbb{R});\ |\varphi(t)| \le R \qquad \text{for a.e.} \quad t \in [0,1]\}, \tag{3.21}$$

where $R = m(K_0 + c_0 M_0)$, $c_0 = \max\limits_{|w| \le M_0} k(w)$, m, K_0 and M_0 are the constants from (3.13), (2.3) and (3.17), respectively. Define the operator $\widetilde{\mathcal{T}} : L^2([0,1],\mathbb{R}) \to C([-r,1],\mathbb{R}^2)$ by the rule:

$$\widetilde{\mathcal{T}}\varphi(t) = \begin{cases} x_0(t), & t \in [-r,0], \\ \mathcal{T}\varphi(t), & t \in [0,1], \end{cases} \tag{3.22}$$

where $\mathcal{T}$ is the operator (3.7). Since $\mathcal{T} : \omega\text{-}L^2([0,1],\mathbb{R}^2) \to C([0,1],\mathbb{R}^2)$ is continuous and $x_0 \in \mathcal{C}_0 \times \mathcal{C}_0$, we see that the operator $\widetilde{\mathcal{T}}$ is continuous from $\omega\text{-}S_R$ to $C([-r,1],\mathbb{R}^2)$. Consider now the operator $\widetilde{\mathcal{G}} : S_R \to L^2([0,1],\mathbb{R})$:

$$\widetilde{\mathcal{G}}(\varphi) = \mathcal{G}(\widetilde{\mathcal{T}}\varphi),$$

where $\mathcal{G}$ is the operator (3.20). From (3.20), (3.13), (2.3) and (3.21) we see that

$$|\widetilde{\mathcal{G}}(\varphi)| \le R, \qquad \varphi \in S_R.$$

Moreover, the continuity of the operators $\mathcal{G} : C([-r,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R})$ and $\widetilde{\mathcal{T}} : \omega\text{-}S_R \to C([-r,1],\mathbb{R}^2)$ implies that the operator $\widetilde{\mathcal{G}} : \omega\text{-}S_R \to \omega\text{-}S_R$ is continuous. Since S_R is obviously convex, from the Schauder fixed point theorem we conclude that there exists $\varphi_* \in S_R$ such that

$$\varphi_* = \widetilde{\mathcal{G}}(\varphi_*) = \mathcal{G}(\widetilde{\mathcal{T}}\varphi_*).$$

Define $x_* = (v_*, w_*) := \widetilde{\mathcal{T}}\varphi_*$. Then, from (3.20) it follows that

$$\varphi_*(t) = g((\mathrm{pr}\, v_*)_t, (\mathrm{pr}\, w_*)_t)\, u_*(t).$$

From (3.22) we see that $x_*(t) = \mathcal{T}\varphi_*(t)$, $t \in [0,1]$. Therefore, $x_*|_{[0,1]}$ is a solution of (3.1)–(3.3) with $\varphi(t) = \varphi_*(t)$. Hence, (x_*, u_*) is a solution of (1.1)–(1.4).

To prove the uniqueness, let $u_1(t), u_2(t) \in [u_\alpha, u_\beta]$, $t \in [0,1]$, be two different admissible controls and $x_1 = (v_1, w_1)$, $x_2 = (v_2, w_2)$ be the corresponding solutions to (1.1)–(1.3). Setting

$$\bar v := v_1 - v_2, \quad \bar w := w_1 - w_2,$$

let $s_{\bar v}$ be a measurable selection of

$$\mathrm{sign}(\bar v) = \begin{cases} 1 & \text{if} \quad \bar v > 0, \\ [-1, 1] & \text{if} \quad \bar v = 0, \\ -1 & \text{if} \quad \bar v < 0. \end{cases} \tag{3.23}$$

Testing (1.1) with $s_{\bar v}$, from (3.13) and the fact that $\dot{\bar v} s_{\bar v} = \frac{\mathrm{d}}{\mathrm{d}t}|\bar v|$ we obtain

$$a_1 \frac{\mathrm{d}}{\mathrm{d}t}|\bar v| + a_2 \dot{\bar w} s_{\bar v} \le m|g((v_1)_t, (w_1)_t) - g((v_2)_t, (w_2)_t)| \\ + |g((v_1)_t, (w_1)_t)|\,|u_1(t) - u_2(t)|, \quad t \in [0,1].$$

This, in view of $H(a)$, (2.3), (2.4) and (3.17), further gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar v| + \dot{\bar w} s_{\bar v} \le M_1(|\bar v_t|_\infty + |\bar w_t|_\infty) + M_2|u_1(t) - u_2(t)|, \quad t \in [0,1], \tag{3.24}$$

for some constants $M_1, M_2 > 0$ depending on M_0, m and a_1, a_2.

Next, we show that

$$\dot{\bar w} s_{\bar v} \ge \frac{\mathrm{d}}{\mathrm{d}t}|\bar w|. \tag{3.25}$$

In fact, if either $\bar v, \bar w \ge 0$ or $\bar v, \bar w \le 0$, by definition (3.23) of the function sign we can choose $s_{\bar w} = s_{\bar v}$, so that (3.25) trivially holds. In case when $\bar v \ge 0$, $\bar w < 0$ we necessarily have

$$\begin{array}{ll} & w_1 < f_R(v_1),\ w_2 > f_L(v_2) \qquad \text{in case of} \quad H(K)b, \\ \text{or} & \\ & w_1 < h_R, \qquad w_2 > h_L \qquad \text{in case of} \quad H(K)a. \end{array} \tag{3.26}$$

Indeed, if not, then, in view of $H(K)b(1)$, $w_1 = f_R(v_1) \ge f_R(v_2) \ge w_2$ ($w_1 = h_R \ge w_2$) or $w_2 = f_L(v_2) \le f_L(v_1) \le w_1$ ($w_2 = h_L \le w_1$) contradicting $\bar w < 0$. Rewriting (1.2) for (v_1, w_1) as the variational inequality:

$$-\dot w_1(z - w_1) \le 0 \qquad \text{for all} \qquad z \in K(v_1)$$

and testing the latter with $z = f_R(v_1)$ ($z = h_R$) we see, in view of (3.26), that $\dot w_1 \ge 0$. Similarly, we obtain $\dot w_2 \le 0$, so that

$$\dot{\bar w} \ge 0.$$

Since, in our case, $s_{\bar w} = -1 \le s_{\bar v}$, (3.25) is obtained by multiplication of the above inequality by $s_{\bar v} - s_{\bar w}$. The case when $\bar v \le 0$, $\bar w > 0$ is treated likewise.

Now, from (3.24), (3.25) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\bar v_t|_\infty + |\bar w_t|_\infty\right) \le M_1(|\bar v_t|_\infty + |\bar w_t|_\infty) + M_2|u_1(t) - u_2(t)|, \quad t \in [0,1],$$

and the uniqueness follows by applying the Gronwall−Bellman lemma. □

4. Bang-bang principle

In optimal linear control theory the celebrated bang-bang principle states, roughly, that any attainable state of a control system can be reached by a bang-bang control, *i.e.* a control function valued in the set of extreme points of the constraint set. For example, in case the constraint set is an interval $[u_\alpha, u_\beta]$ such a bang-bang control u is provided by a function with only values u_α or u_β, so as if to represent the situation when u "bangs" from u_α to u_β back and forth, hence the name of the principle. Below we give a bang-bang type result for our control system (1.1)–(1.4).

Theorem 4.1. *For any solution (x, u) of control system (1.1)–(1.4) there exists a sequence u_n, $n \geq 1$, of bang-bang controls: $u_n(t) \in \{u_\alpha, u_\beta\}$, $t \in [0,1]$, such that the corresponding solutions x_n of (1.1)–(1.3) converge to x in $C([-r,1], \mathbb{R}^2)$.*

Proof. Let $(x, u) \in C([-r,1], \mathbb{R}^2) \times L^2([0,1], \mathbb{R})$, $x = (v, w)$, a solution to (1.1)–(1.4), be given. Then, there exists a sequence $u_n(t) \in \{u_\alpha, u_\beta\}$, $t \in [0,1]$, $n \geq 1$, such that

$$u_n \to u \qquad \text{in} \qquad \omega\text{-}L^2([0,1], \mathbb{R}).$$

Indeed, since the convex hull of the set $\{u_\alpha, u_\beta\}$ consisting of the end points of the control interval $[u_\alpha, u_\beta]$ clearly coincides with $[u_\alpha, u_\beta]$, from ([2], Thm. 3) we conclude that for any finite interval $T \subset [0,1]$ there exists a measurable function $u_T(t) \in \{u_\alpha, u_\beta\}$, $t \in [0,1]$, such that

$$\int_T u(t)\, \mathrm{d}t = \int_T u_T(t)\, \mathrm{d}t.$$

Defining now the sequence $u_n : [0,1] \to \{u_\alpha, u_\beta\}$ by the rule

$$u_n(t) = \sum_{i=1}^{n} u_{i,n}(t) \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(t), \quad t \in [0,1],$$

where χ is the characteristic function of a set and

$$\int_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} u(t)\, \mathrm{d}t = \int_{\left[\frac{i-1}{n}, \frac{i}{n}\right]} u_{i,n}(t)\, \mathrm{d}t, \quad n \in \mathbb{N},\ i = 1, \ldots, n,$$

we see that

$$\sup_{0 \leq t \leq 1} \left| \int_0^t (u(s) - u_n(s))\, \mathrm{d}s \right| \leq 2\, m \frac{1}{n},$$

where, recall that, $m = \max\{|u_\alpha|, |u_\beta|\}$. The results of [7] now imply the claim.

Let $x_n = (v_n, w_n)$, $n \geq 1$, be the states, *i.e.* solutions to (1.1)–(1.3), corresponding to the controls u_n. Setting $\varphi_n(t) = g((v_n)_t, (w_n)_t))u_n(t)$, $t \in [0,1]$, $n \geq 1$, from (2.3) and (3.17) we see that

$$\|\varphi_n\|_{L^2([0,1],\mathbb{R})} \leq M$$

for some $M > 0$. The continuity of the operator $\widetilde{\mathcal{T}} : \omega\text{-}L^2([0,1], \mathbb{R}) \to C([-r,1], \mathbb{R}^2)$ defined by (3.22) and the properties of the function g imply then that x_n converges in $C([-r,1], \mathbb{R}^2)$ as $n \to \infty$ to some $y = (y_1, y_2) \in C([-r,1], \mathbb{R}^2)$ and $y = \widetilde{\mathcal{T}}\varphi$, where $\varphi(t) = g((y_1)_t, (y_2)_t))\, u(t)$, $t \in [0,1]$. Now, we employ inequality (3.6) to show that

$$\|x(t) - y(t)\| \leq R_0 \int_0^t |g(v_s, w_s)\, u(s) - g((y_1)_s, (y_2)_s)\, u(s)|\, \mathrm{d}s, \quad t \in [0,1].$$

Hypothesis $H(h)(2)$ and (1.4), (3.13) further imply that

$$|v_t - (y_1)_t|_\infty + |w_t - (y_2)_t|_\infty \leq R_0 m L \int_0^t (|v_s - (y_1)_s|_\infty + |w_s - (y_2)_s|_\infty)\, \mathrm{d}s, \qquad t \in [0,1],$$

where L depends on M_0. From the Gronwall−Bellman lemma we finally conclude that $x = y$ and the theorem follows. □

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