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NECESSARY CONDITIONS OF FIRST-ORDER FOR AN OPTIMAL BOUNDARY CONTROL PROBLEM FOR VISCOUS DAMAGE PROCESSES IN 2D

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Abstract. Controlling the growth of material damage is an important engineering task with plenty of real world applications. In this paper we approach this topic from the mathematical point of view by investigating an optimal boundary control problem for a damage phase-field model for viscoelastic media. We consider non-homogeneous Neumann data for the displacement field which describe external boundary forces and act as control variables. The underlying hyberbolic-parabolic PDE system for the state variables exhibit highly nonlinear terms which emerge in context with damage processes. The cost functional is of tracking type, and constraints for the control variable are prescribed. Based on recent results from [M.H. Farshbaf−Shaker and C. Heinemann, *Math. Models Methods Appl. Sci.* **25** (2015) 2749–2793], where global-in-time well-posedness of strong solutions to the lower level problem and existence of optimal controls of the upper level problem have been established, we show in this contribution differentiability of the control-to-state mapping, well-posedness of the linearization and existence of solutions of the adjoint state system. Due to the highly nonlinear nature of the state system which has by our knowledge not been considered for optimal control problems in the literature, we present a very weak formulation and estimation techniques of the associated adjoint system. For mathematical reasons the analysis is restricted here to the two-dimensional case. We conclude our results with first-order necessary optimality conditions in terms of a variational inequality together with PDEs for the state and adjoint state system.

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1. Introduction

Damage processes are usually highly nonlinear phenomena and their mathematical investigation is topic of many recent contributions in applied analysis. One modeling approach uses the phase-field method where a "smooth" variable is introduced. In the simplest case this variable is a scalar function and describes the local accumulation of damage in the body and the transition between the damaged and the undamaged material states. The popularity of phase-field models have increased in the last two decades in various fields of applied mathematics, physics and engineering sciences, see [23]. They are used to predict the micro-structure and morphological evolution of two or more different phases and their mixture. Specifically in damage mechanics

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they promise an accurate modeling and are powerful techniques in prediction of material behavior. For instance physical laws such as Griffith-type criteria for crack propagation could be encoded into a PDE/inclusion system and the crack paths need not to be known *a priori*. For more details on modeling issues and mathematical analysis on rate-independent and rate-dependent (also called viscous) damage models we refer to [9, 19, 20, 26] and [3, 8, 10, 24], respectively and for vanishing viscosity results, *e.g.*, to [15, 16]. Numerical simulations for damage phase-field models can be found, *e.g.*, in [5, 9, 19, 26].

Despite theses advantages the mathematical treatment of the resulting systems is challenging for different reasons. First, the evolution law for the damage process contains a difficult type of nonlinear (coupling) term, a nonlinear operator acting on the time-derivative of the evolution variable and, depending on the type of model, even constraints on the state and/or its time-derivative. Secondly, damage processes are usually coupled with a model for elasticity where the material stiffness depends on the damage variable. An increase of damage usually results in a weakening of the stiffness and may eventually lead to a complete rupture. The analysis of such processes is a challenging endeavor from the mathematical point of view since crucial *a priori* estimates on the displacement field fail to hold in situations where the stiffness tensor degenerates (see, *e.g.*, [4, 11]). In recent works damage evolution systems have also been coupled with further processes such as heat conduction [25], phase separation [10], chemical reactions [18], and plasticity [2].

In engineering problems one is interested in prediction and even more relevant in control or manipulation of damage evolution in order to prevent, for instance, complete failure of a structural component, see [7] for some examples and references.

Our main goal in this work is to provide a mathematical basis for optimal control problems of a time-continuous damage model including a first-order optimality system which has not been accomplished so far to the best knowledge of the authors. To state our problem let us fix an open, bounded and smooth domain $\Omega \subset \mathbb{R}^n$ with $n \in \{1, 2\}$ where the material is located in the reference configuration and let $T > 0$ be a final time. Furthermore let Γ be the boundary to Ω and ν the outward unit normal. We put $Q := \Omega \times (0, T)$ and $\Sigma := \Gamma \times (0, T)$.

We investigate the following optimal control problem:

(CP) *Minimize the cost-functional*

$$\mathcal{J}(\chi, b) := \frac{\lambda_T}{2}\|\chi(T) - \chi_T\|^2_{L^2(\Omega)} + \frac{\lambda_\Sigma}{2}\|b\|^2_{L^2(\Sigma;\mathbb{R}^n)} \tag{1.1}$$

subject to the hyperbolic-parabolic initial-boundary value problem

$$u_{tt} - \operatorname{div}\big(\mathbb{C}(\chi)\varepsilon(u) + \mathbb{D}\varepsilon(u_t)\big) = \ell \qquad \text{a.e. in} \quad Q, \tag{1.2}$$

$$\chi_t + \xi(\chi_t) - \Delta\chi_t - \Delta\chi + \frac{1}{2}\mathbb{C}'(\chi)\varepsilon(u) : \varepsilon(u) + f'(\chi) = 0 \qquad \text{a.e. in} \quad Q, \tag{1.3}$$

$$\big(\mathbb{C}(\chi)\varepsilon(u) + \mathbb{D}\varepsilon(u_t)\big)\cdot\nu = b \qquad \text{a.e. on} \quad \Sigma, \tag{1.4}$$

$$\nabla(\chi + \chi_t)\cdot\nu = 0 \qquad \text{a.e. on} \quad \Sigma, \tag{1.5}$$

$$u(0) = u^0,\ u_t(0) = v^0,\ \chi(0) = \chi^0 \qquad \text{a.e. on} \quad \Omega \tag{1.6}$$

and subject to the control constraint (other types will also be allowed)

$$\mathcal{B}_{\mathrm{adm}} := \big\{b \in \mathcal{B} \mid b_{\min} \le b \le b_{\max} \text{ a.e. in } \Sigma \text{ and } \|b\|_{\mathcal{B}} \le R\big\}. \tag{1.7}$$

In the above problem the Banach space

$$\mathcal{B} := L^2(0, T; H^{1/2}(\Gamma;\mathbb{R}^n)) \cap H^1(0, T; L^2(\Gamma;\mathbb{R}^n)) \tag{1.8}$$

is endowed with its natural norm $\|\cdot\|_{\mathcal{B}} := \|\cdot\|_{L^2(0,T;H^{1/2})} + \|\cdot\|_{H^1(0,T;L^2)}$. The box constraints $b_{\min},\ b_{\max} \in L^\infty(\Sigma)$ satisfy $b_{\min} \le b_{\max}$ a.e. in Σ. Moreover χ_T is a given target function and λ_T, λ_Σ and R are some prescribed positive constants. The damage-rate-dependent function ξ is assumed to be twice differentiable with bounded derivatives, monotonically increasing and it vanishes for negative values (see assumptions (A3) and (B2) below). As we will explain in the following it may arise from a regularization process of the subdifferential of the indicator function $I_{(-\infty,0]}$.

The coupled PDE system (1.2)−(1.3) with its initial-boundary conditions (1.4)−(1.6) models the lower lever problem and consists of the momentum balance equation (1.2) (according to the Kelvin−Voigt rheology) and a parabolic equation (1.3) which governs the evolution of a phase-field variable χ. The displacement field is denoted by u and the variable χ is usually interpreted in relation with the density of micro-defects and therefore influences the material stiffness $\mathbb{C}(\cdot)$ which is considered as a function of χ. Moreover the external volume forces are specified by ℓ, the external surface forces by b, the linearized strain tensor by $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the stress tensor by $\sigma = \mathbb{C}(\chi)\varepsilon(u) + \mathbb{D}\varepsilon(u_t)$. The first summand of σ contains the elastic contribution whereas the second summand models viscous effects. The coefficient $\mathbb{C}$ designates the fourth-order damage-dependent stiffness tensor and $\mathbb{D}$ the (damage independent) viscosity tensor. Please note that we assume the viscosity $\mathbb{D}$ to be independent of the damage variable. Damage-dependent viscosities would lead, among other things, to additional terms in the adjoint system to which the estimation techniques in Lemma 3.10 below cannot be easily adapted (see also the discussion in Sect. 4). We also think that this restriction helps to keep the present contribution better understandable. The limitation of our results to 2D is of technical nature since we make use of enhanced *a priori* estimates proven in [7] which strongly relies on a 2D version of the Gagliardo−Nirenberg inequality.

For a mechanical motivation of system (1.2)−(1.6) by means of balance laws and constitutive relations we refer to [9]. Global-in-time well-posedness of strong solutions of the state system (1.2)−(1.6) in 2D and existence of optimal controls for (CP) have been established in [7]. For further existence, uniqueness and vanishing viscosity results for viscous and rate-independent damage models by making use of higher-order Laplacians we refer to the work [15]. Besides these results necessary optimality conditions for (CP) have been left open and are the topic of the present paper.

Our main result is stated in Theorem 3.11 which contains necessary conditions of first-order for minimizers of (CP). We remark that in this contribution we do not include the sub-differential $\partial I_{(-\infty,0]}(\chi_t)$ in the damage law (1.3) and use $\xi(\chi_t)$ instead. On the one hand the incorporation of $\partial I_{(-\infty,0]}(\chi_t)$ in the lower lever problem seems currently out of reach for necessary optimality conditions to (CP) and, on the other hand, could be approximated by $\xi(\chi_t)$. In particular ξ may stem from a regularization process of the subdifferential of the indicator function $I_{(-\infty,0]}$ (*e.g.* Yosida- or C^∞-approximations, see [21], Chap. 5) which models the so-called irreversibility condition as done in [7]. Nevertheless, to the author's best knowledge, a nonlinearity such as $\xi(\chi_t)$ has not yet been considered in the optimal control literature. The occurrence of the time-derivative of χ in the ξ-nonlinearity leads to the difficult integral term $\iint \xi'(\chi_t)q\psi_t$ (where ψ denotes a test-function, q an adjoint variable) in the adjoint system. We propose a very weak formulation and prove an existence result for the adjoint system. To this end we will consider regularizations and derive *a priori* estimates by testing the system with, roughly speaking, time-integrated versions of the adjoint variables (see (3.32) and Lem. 3.10). A limit passage eventually yields very weak solutions to the adjoint system.

Let us put our work into perspective. In contrast to modeling and analytical aspects of damage models the mathematical literature concerning associated optimal control is rather scarce. Beside the work [7] we refer to [27] as well as to the recent preprint [22] for optimality systems of time-discretized and regularized damage phase-field models which follow the approach "first time-discretize then optimize". A general framework for shape optimization problems in context with semi-linear variational inequalities and, as an application, optimality systems for time-discretized damage models are explored in [12]. We also point to the work [17] for existence of optimal controls for a time-continuous degenerating damage model in terms of an obstacle problem where the irreversibility condition has been dropped. Furthermore and in opposition to a phase-field approach there is a rich literature employing sharp crack models with prescribed paths for optimization problems. We refer to [13, 14] and the references therein.

Structure of the paper.

The paper is organized as follows. In the next section, we list our assumptions and recall the well-posedness result from [7], which will be the starting point for a deeper analysis of the solution operator. In Section 3 we prove differentiability of the control-to-state operator and set up the linearized and adjoint problem. More precisely we establish existence of solutions to both systems and well-posedness to the first one. These intermediate results are summarized in Proposition 3.1, Propositions 3.2 and 3.3. At the end of this section the full optimality system is derived. We conclude with open problems connected to (CP) in the last section.

During the course of the presented analysis we make repeated use of standard inequalities and embedding theorems, in particular exploiting continuous embeddings $H^2(\Omega) \subset L^\infty(\Omega)$ as well as $H^1(\Omega) \subset L^p(\Omega)$ for any $p \in [1, \infty)$ valid in the two-dimensional case.

2. Assumptions and preliminary results

Let us collect the assumptions which are used throughout this work and restate known results obtained in [7] concerning the state system (1.2)−(1.6) which we extensively use for the rest of our paper. For well-posedness of the state system we need the following assumptions.

Assumptions

(A1) $\Omega \subseteq \mathbb{R}^n$ with $n \in \{1, 2\}$ is assumed to be a bounded C^2-domain.

(A2) The damage-dependent stiffness tensor satisfies $\mathbb{C}(\cdot) = \mathsf{c}(\cdot)\mathbf{C}$, where the coefficient function c is assumed to be of the form $\mathsf{c} = \mathsf{c}_1 + \mathsf{c}_2$, where $\mathsf{c}_1 \in C^1_{\text{loc}}(\mathbb{R})$ is convex and $\mathsf{c}_2 \in C^1_{\text{loc}}(\mathbb{R})$ is concave. Moreover, we assume that $\mathsf{c}, \mathsf{c}_1', \mathsf{c}_2'$ (here $'$ denotes its derivative) are bounded, Lipschitz continuous and $\mathsf{c}(x) \geq 0$ for all $x \in \mathbb{R}$. The 4th order stiffness tensor $\mathbf{C} \in \mathcal{L}(\mathbb{R}^{n\times n}_{\text{sym}}; \mathbb{R}^{n\times n}_{\text{sym}})$ is assumed to be symmetric and positive definite, *i.e.*

$$\mathbf{C}_{ijlk} = \mathbf{C}_{jilk} = \mathbf{C}_{lkij} \text{ and } e : \mathbf{C}e \geq \eta |e|^2 \text{ for all } e \in \mathbb{R}^{n\times n}_{\text{sym}} \tag{2.1}$$

with constant $\eta > 0$.

(A3) The function $\xi : \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous, monotonically increasing and $\xi(x) = 0$ for $x \leq 0$.

(A4) The 4th order viscosity tensor $\mathbb{D}$ is given by $\mathbb{D} = \mu\mathbf{C}$ with $\mu \in (0, +\infty)$ and does **not** depend on the damage variable.

(A5) The damage-dependent potential function f is assumed to fulfill $f \in C^1_{\text{loc}}(\mathbb{R})$ and the first derivative f' is Lipschitz continuous.

Regarding existence of solutions of the optimal control problem (CP) we make the following additional assumptions.

Assumptions

(O1) There are given non-negative constants λ_T and λ_Σ.

(O2) The target damage profile is given by $\chi_T \in L^2(\Omega)$.

(O3) The admissible set of boundary controls $\mathcal{B}_{\text{adm}} \subseteq \mathcal{B}$ is assumed to be non-empty, closed and bounded. $\mathcal{B}$ is given by (1.8). Furthermore, let the constant $R > 0$ be such that

$$\|b\|_{\mathcal{B}} \leq R \quad \text{for all } b \in \mathcal{B}_{\text{adm}}.$$

Remark 2.1.

(i) In particular we may choose $\mathcal{B}_{\mathrm{adm}}$ as in (1.7) which is then a bounded subset of $L^\infty(\Sigma)$.
(ii) Note that in [7] the assumptions for c_1 and c_2 are stated as $\mathsf{c}_1 \in C^{1,1}(\mathbb{R})$ convex and $\mathsf{c}_2 \in C^{1,1}(\mathbb{R})$ concave with $\mathsf{c}, \mathsf{c}_1', \mathsf{c}_2'$ bounded and $\mathsf{c} \geq 0$. There, $C^{1,1}(\mathbb{R})$ denotes the space of differentiable functions whose derivatives are Lipschitz continuous. However we do not require c_1 or c_2 to be bounded. In order to avoid confusion we choose the formulation in (A2) above.

For the analytical investigation of (CP) we will employ the following function spaces

$$\mathcal{Q} := \mathcal{U} \times \mathcal{X} \quad \subset \quad \dot{\mathcal{Q}} := \dot{\mathcal{U}} \times \dot{\mathcal{X}} \quad \subset \quad \overline{\mathcal{Q}} := \overline{\mathcal{U}} \times \overline{\mathcal{X}}$$

with the following definitions

- *space for the state system:*
  $\mathcal{U} := H^1(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^n))$,
  $\mathcal{X} := H^1(0,T;H^2(\Omega))$,
- *space for the linearized state system:*
  $\dot{\mathcal{U}} := H^1(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;H^1(\Omega;\mathbb{R}^n)^*)$,
  $\dot{\mathcal{X}} := H^1(0,T;H^1(\Omega))$,
- *space for the adjoint system:*
  $\overline{\mathcal{U}} := L^2(0,T;H^1(\Omega;\mathbb{R}^n)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$,
  $\overline{\mathcal{X}} := L^2(0,T;H^1(\Omega))$.

Observe that the above spaces are Banach spaces when equipped with their natural norms. The following results are taken from ([7], Thm. 2.11–2.12 and Cor. 2.13–2.14):

Theorem 2.2. *Suppose that the general assumptions* (A1)$-$(A5) *are satisfied. Then, we have:*

(i) *The state system* (1.2)$-$(1.6) *has for any $b \in \mathcal{B}$, $\ell \in L^2(0,T;L^2(\Omega))$ and initial values $u^0 \in H^2(\Omega;\mathbb{R}^n)$, $v^0 \in H^1(\Omega;\mathbb{R}^n)$ and $\chi^0 \in H^2(\Omega)$ a unique solution $(u,\chi) \in \mathcal{Q}$ (which we call strong solution).*
(ii) *Suppose that also* (O3) *is fulfilled and that ℓ, u^0, v^0 and χ^0 are fixed. Then there exists a positive constant K_1^* (depending on R) such that for every $b \in \mathcal{B}_{\mathrm{adm}}$ the associated solution $(u,\chi) \in \mathcal{Q}$ satisfies*

$$\|(u,\chi)\|_{\mathcal{Q}} \leq K_1^*. \tag{2.2}$$

(iii) *Under the assumption in (ii) there also exists a positive constant K_2^* (depending on R) such that the following holds: whenever $b_1, b_2 \in \mathcal{B}_{\mathrm{adm}}$ are given and $(u_1,\chi_1), (u_2,\chi_2) \in \mathcal{Q}$ denote the associated solutions of the state system, we then have*

$$\|u_1 - u_2\|_{\dot{\mathcal{U}}} + \|\chi_1 - \chi_2\|_{\dot{\mathcal{X}}} \leq K_2^* \|b_1 - b_2\|_{L^2(\Sigma)}. \tag{2.3}$$

Remark 2.3.

(i) Note that in [7] a weaker stability estimate

$$\|u_1 - u_2\|_{W^{1,\infty}(0,T;L^2) \cap H^1(0,T;H^1)} + \|\chi_1 - \chi_2\|_{H^1(0,T;H^1)} \leq C\|b_1 - b_2\|_{L^2(\Sigma)}. \tag{2.4}$$

has been proven. To obtain (2.3) one also needs to establish (here $C > 0$ depends on R)

$$\|u_1 - u_2\|_{H^2(0,T;(H^1)^*)} \leq C\|b_1 - b_2\|_{L^2(\Sigma)}$$

which follows by (2.4), (2.2) and a comparison of the corresponding terms in (1.2). In fact we find by subtraction for a.e. $t \in (0,T)$ and all $\varphi \in H^1(\Omega)$:

$$\begin{aligned}\langle \partial_{tt}u_1(t) - \partial_{tt}u_2(t), \varphi\rangle_{H^1} = &-\int_\Omega (\mathbb{C}(\chi_1(t)) - \mathbb{C}(\chi_2(t)))\varepsilon(u_1(t)) : \varepsilon(\varphi) + \mathbb{C}(\chi_2(t))\varepsilon(u_1(t) - u_2(t)) : \varepsilon(\varphi)\,\mathrm{d}x\\ &- \int_\Omega \mathbb{D}\varepsilon(\partial_t u_1(t) - \partial_t u_2(t)) : \varepsilon(\varphi)\,\mathrm{d}x + \int_\Gamma (b_1(t) - b_2(t))\cdot\varphi\,\mathrm{d}x\end{aligned}$$

and, consequently,

$$\begin{aligned}&\|\partial_{tt}u_1 - \partial_{tt}u_2\|_{L^2(0,T;(H^1)^*)}\\ &\quad\le C\|\chi_1 - \chi_2\|_{L^2(0,T;H^1)} + C\|u_1 - u_2\|_{H^1(0,T;H^1)} + C\|b_1 - b_2\|_{L^2(\Sigma)} \le \widetilde{C}\|b_1 - b_2\|_{L^2(\Sigma)}.\end{aligned}$$

(ii) It follows from Theorem 2.2, in particular, that the control-to-state mapping $\mathcal{S} : \mathcal{B} \to \mathcal{Q}$ given by $S(b) := (u, \chi)$ is well defined. Moreover, $\mathcal{S}$ is Lipschitz continuous when viewed as a mapping from the subset $\mathcal{B}_{\mathrm{adm}}$ of $\mathcal{B}$ into the space $\dot{\mathcal{Q}}$.

With a proof that resembles ([7], Thm. 3.6) and needs no repetition here, we can show the following existence result for optimal controls:

Theorem 2.4 (*cf.* [7], Thm. 3.6)**.** *Suppose that the Assumptions* (A1)–(A5) *and* (O1)–(O3) *are fulfilled. Then the optimal control problem (CP) admits a solution* $(\chi, b) \in \mathcal{X} \times \mathcal{B}_{\mathrm{adm}}$.

In the present contribution we proceed with a first-order optimality system which will require the following enhanced differentiability assumptions in addition to the Assumptions (A1)–(A5) and (O1)–(O3):
Assumptions.

(B1) $\mathbb{C}(\cdot) = \mathsf{c}(\cdot)\mathbf{C}$ from (A2) is assumed to satisfy $\mathsf{c} \in C^3_{\mathrm{loc}}(\mathbb{R})$;
(B2) ξ from (A3) is assumed to satisfy $\xi \in C^2_{\mathrm{loc}}(\mathbb{R})$ and ξ'' is bounded;
(B3) f from (A5) is assumed to be $f \in C^2_{\mathrm{loc}}(\mathbb{R})$;
(B4) $\mathcal{B}_{\mathrm{adm}}$ from (O3) is assumed to be convex.

3. Analysis of a first-order optimality system to (CP)

In this section our aim is to derive a first-order optimality system to the optimal control problem (CP). We will prove Gâteaux-differentiability of the solution operator and weak solvability of a corresponding adjoint problem. The latter one requires several approximation schemes and carefully designed estimations to handle the term $\int_Q \xi'(\chi_t)q\psi_t$ (weak form) which arises from the difficult non-linearity $\xi(\chi_t)$ in the state system. A priori estimates for the approximated system in the adjoint space $\overline{\mathcal{Q}}$ are derived by testing it with certain modified anti-derivatives with respect to time of the adjoint variables p and q. A challenging part in the calculations is to obtain the *a priori* estimates globally-in-time on the entire interval $[0, T]$. Finally, at the end of this section, we will assemble the pieces and derive a first-order optimality system.

3.1. Differentiability of the control-to-state mapping and the linearized state system

This part is devoted to the proof of Gâteaux-differentiability of the control-to-state mapping $\mathcal{S} : \mathcal{B} \to \dot{\mathcal{Q}}$. This endeavor is splitted into a series of intermediate results which we briefly describe below:

- Proposition 3.1: By considering difference quotients of the state system in combination with a limit passage we prove existence of the linearized state system and a differentiability property of the control-to-state mapping $\mathcal{S} : \mathcal{B} \to \dot{\mathcal{Q}}$ in a weak topology.
- Proposition 3.2: We establish a stability result and, consequently, well-posedness of the linearized problem.

- Proposition 3.3: Based on the previous results we are in the position to show Gâteaux-differentiability of $\mathcal{S}$ by refining the estimates for the difference quotients of the state system and the linearized system.

Proposition 3.1 (Convergence to the linearized problem)**.** *Suppose that the Assumptions* (A1)−(A5) *and* (B1)−(B3) *are fulfilled. Then we have:*

(i) *The control-to-state mapping $\mathcal{S}:\mathcal{B}\to\dot{\mathcal{Q}}$ is differentiable in the following sense:*

$$\frac{\mathcal{S}(b+\lambda h)-\mathcal{S}(b)}{\lambda}\rightharpoonup(\dot u,\dot\chi)\ \textit{weakly-star in}\ \dot{\mathcal{Q}}\ \textit{as}\ \lambda\to 0 \tag{3.1}$$

for all $b,h\in\mathcal{B}$.

(ii) *Furthermore the limit function $(\dot u,\dot\chi)\in\dot{\mathcal{Q}}$ in (3.1) is a weak solution of the linearized state system at $(u,\chi)=\mathcal{S}(b)\in\mathcal{Q}$ in direction $h\in\mathcal{B}$, i.e. $(\dot u,\dot\chi)\in\dot{\mathcal{Q}}$ fulfills*

$$\begin{aligned}&\int_0^T\langle\dot u_{tt},\varphi\rangle_{H^1}\,\mathrm{d}t+\int_Q\mathbb{C}'(\chi)\dot\chi\varepsilon(u):\varepsilon(\varphi)+\mathbb{C}(\chi)\varepsilon(\dot u):\varepsilon(\varphi)+\mathbb{D}\varepsilon(\dot u_t):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t\\&\quad=\int_\Sigma h\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\int_Q\nabla\dot\chi\cdot\nabla\psi+\nabla\dot\chi_t\cdot\nabla\psi+\dot\chi_t\psi+\xi'(\chi_t)\dot\chi_t\psi+\frac12\mathbb{C}''(\chi)\dot\chi\varepsilon(u):\varepsilon(u)\psi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_Q\mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u)\psi+f''(\chi)\dot\chi\psi\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned}\tag{3.3}$$

for all $(\varphi,\psi)\in\overline{\mathcal{Q}}$ and with initial values $u(0)=u_t(0)=\chi(0)=0$.

Proof. Let $\lambda\in\mathbb{R}$ and $b,b+\lambda h\in\mathcal{B}$ and define $(u,\chi):=\mathcal{S}(b)$ and $(u^\lambda,\chi^\lambda):=\mathcal{S}(b+\lambda h)$. By Theorem 2.2 (ii)−(iii) there exist positive constants K_1^* and K_2^* (depending on R) such that

$$\|u^\lambda\|_{\mathcal{U}}+\|\chi^\lambda\|_{\mathcal{X}}\le K_1^*,\qquad\qquad \left\|\frac{u^\lambda-u}{\lambda}\right\|_{\dot{\mathcal{U}}}+\left\|\frac{\chi^\lambda-\chi}{\lambda}\right\|_{\dot{\mathcal{X}}}\le K_2^*\|h\|_{L^2(0,T;L^2)}. \tag{3.4}$$

Therefore the sequence $\{(\frac{u^\lambda-u}{\lambda},\frac{\chi^\lambda-\chi}{\lambda})\}$ is uniformly bounded in $\dot{\mathcal{U}}\times\dot{\mathcal{X}}$ with respect to λ and we may extract a weakly convergent subsequence and obtain by omitting the subscript

$$\left(\frac{u^\lambda-u}{\lambda},\frac{\chi^\lambda-\chi}{\lambda}\right)\to(\dot u,\dot\chi)\ \text{weakly-star in}\ \dot{\mathcal{Q}}\ \text{as}\ \lambda\to 0. \tag{3.5}$$

In the next step we are going to show that $(\dot u,\dot\chi)\in\dot{\mathcal{Q}}$ is a weak solution of the linearized system (3.2)−(3.3) at $(u,\chi)=\mathcal{S}(b)\in\mathcal{Q}$ in direction h. We sketch the passage to the limit for the nonlinear terms. To this end we prove the following convergence statements as $\lambda\to 0$:

(a) $$\int_Q\frac{\mathbb{C}(\chi^\lambda)\varepsilon(u^\lambda)-\mathbb{C}(\chi)\varepsilon(u)}{\lambda}:\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t\to\int_Q\mathbb{C}'(\chi)\dot\chi\varepsilon(u):\varepsilon(\varphi)+\mathbb{C}(\chi)\varepsilon(\dot u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t,$$

(b) $$\begin{aligned}&\int_Q\frac12\frac{\mathbb{C}'(\chi^\lambda)\varepsilon(u^\lambda):\varepsilon(u^\lambda)-\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u)}{\lambda}\psi\,\mathrm{d}x\,\mathrm{d}t\\&\qquad\to\int_Q\frac12\mathbb{C}''(\chi)\dot\chi\varepsilon(u):\varepsilon(u)+\mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u)\psi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

(c) $$\int_Q\frac{f'(\chi^\lambda)-f'(\chi)}{\lambda}\psi\,\mathrm{d}x\,\mathrm{d}t\to\int_Q f''(\chi)\dot\chi\psi\,\mathrm{d}x\,\mathrm{d}t,$$

(d) $$\int_Q\left(\frac{\xi(\chi_t^\lambda)-\xi(\chi_t)}{\lambda}\right)\psi\,\mathrm{d}x\,\mathrm{d}t\to\int_Q\xi'(\chi_t)\dot\chi_t\psi\,\mathrm{d}x\,\mathrm{d}t.$$

and test-functions $(\varphi,\psi)\in\overline{\mathcal{Q}}$.

To (a): We apply the following splitting

$$\int_Q \frac{\mathbb{C}(\chi^\lambda)\varepsilon(u^\lambda)-\mathbb{C}(\chi)\varepsilon(u)}{\lambda}:\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\underbrace{\int_Q \frac{\mathbb{C}(\chi^\lambda)-\mathbb{C}(\chi)}{\lambda}\varepsilon(u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t}_{=:T_1}+\underbrace{\int_Q \mathbb{C}(\chi^\lambda)\varepsilon\Big(\frac{u^\lambda-u}{\lambda}\Big):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t}_{=:T_2}.$$

The first term is treated by the mean value theorem applied to each tensor component (applicable due to Ass. (A2)), *e.g.*

$$\frac{\mathbb{C}_{ijkl}(\chi^\lambda)-\mathbb{C}_{ijkl}(\chi)}{\lambda}=\mathbb{C}'_{ijkl}(\chi+\bar\lambda_{ijkl}(\chi^\lambda-\chi))\left(\frac{\chi^\lambda-\chi}{\lambda}\right)$$

for values $\bar\lambda_{ijkl}\in[0,\lambda]$,

$$T_1=\int_Q\{\mathbb{C}'_{ijkl}(\chi+\bar\lambda_{ijkl}(\chi^\lambda-\chi))\}_{0\le i,j,k,l\le 1}\left(\frac{\chi^\lambda-\chi}{\lambda}\right)\varepsilon(u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t$$

By using the dominated convergence theorem of Lebesgue and the *a priori* estimates in (3.4), we obtain for a subsequence

$$T_1\to\int_Q\mathbb{C}'(\chi)\dot\chi\varepsilon(u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t,\qquad\qquad T_2\to\int_Q\mathbb{C}(\chi)\varepsilon(\dot u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t.$$

To (b): *Via* splitting and similar arguments we obtain for a subsequence $\lambda\to 0$

$$\int_Q\frac{1}{2}\frac{\mathbb{C}'(\chi^\lambda)\varepsilon(u^\lambda):\varepsilon(u^\lambda)-\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u)}{\lambda}\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$=\underbrace{\int_Q\frac{1}{2}\frac{\mathbb{C}'(\chi^\lambda)-\mathbb{C}'(\chi)}{\lambda}\varepsilon(u):\varepsilon(u)\psi\,\mathrm{d}x\,\mathrm{d}t}_{\to\int_Q\frac{1}{2}\mathbb{C}''(\chi)\dot\chi\varepsilon(u):\varepsilon(u)\psi\,\mathrm{d}x\,\mathrm{d}t}+\underbrace{\int_Q\mathbb{C}'(\chi^\lambda)\varepsilon\left(\frac{u^\lambda-u}{\lambda}\right):\varepsilon\left(\frac{u^\lambda+u}{2}\right)\psi\,\mathrm{d}x\,\mathrm{d}t}_{\to\int_Q\mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u)\psi\,\mathrm{d}x\,\mathrm{d}t}.$$

To (c)–(d): These properties follow by similar arguments with less effort. □

Proposition 3.2 (Stability of the linearized problem)**.** *Let $b\in\mathcal{B}$ be given and denote $(u,\chi)=\mathcal{S}(b)$. Furthermore, let $h_1,h_2\in\mathcal{B}$ be two given directions. Then for a weak solution $(\dot{\mathbf{u}},\dot{\boldsymbol\chi})$ of the linearized system (3.2)–(3.3) to the associated direction $\mathbf{h}$ we have the Lipschitz estimate*

$$\|\dot{\mathbf{u}}\|_{H^1(0,T;H^1)\cap W^{1,\infty}(0,T;L^2)}+\|\dot{\boldsymbol\chi}\|_{H^1(0,T;H^1)}\le C\|\mathbf{h}\|_{L^2(0,T;L^2(\Gamma;\mathbb{R}^n))}.$$

Proof. Testing equations (3.2)–(3.3) with $(\dot{\mathbf{u}}_t,\dot{\boldsymbol\chi}_t)$ yield

$$\int_0^t\langle\dot{\mathbf{u}}_{tt},\dot{\mathbf{u}}_t\rangle_{H^1}\,\mathrm{d}s+\int_0^t\int_\Omega\mathbb{C}'(\chi)\dot{\boldsymbol\chi}\varepsilon(u):\varepsilon(\dot{\mathbf{u}}_t)+\mathbb{C}(\chi)\varepsilon(\dot{\mathbf{u}}):\varepsilon(\dot{\mathbf{u}}_t)+\mathbb{D}\varepsilon(\dot{\mathbf{u}}_t):\varepsilon(\dot{\mathbf{u}}_t)\,\mathrm{d}x\,\mathrm{d}s$$
$$=\int_0^t\int_\Gamma\mathbf{h}\cdot\dot{\mathbf{u}}_t\,\mathrm{d}x\,\mathrm{d}s,$$
$$\int_0^t\int_\Omega\nabla\dot{\boldsymbol\chi}\cdot\nabla\dot{\boldsymbol\chi}_t+|\nabla\dot{\boldsymbol\chi}_t|^2+|\dot{\boldsymbol\chi}_t|^2+\xi'(\chi_t)|\dot{\boldsymbol\chi}_t|^2+\frac{1}{2}\mathbb{C}''(\chi)\varepsilon(u):\varepsilon(u)\dot{\boldsymbol\chi}\dot{\boldsymbol\chi}_t\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^t\int_\Omega\mathbb{C}'(\chi)\varepsilon(\dot{\mathbf{u}}):\varepsilon(u)\dot{\boldsymbol\chi}_t+f''(\chi)\dot{\boldsymbol\chi}\dot{\boldsymbol\chi}_t\,\mathrm{d}x\,\mathrm{d}s=0.$$

We estimate the following terms occurring on the left-hand side of these equations by making use of Young's and Hölder's inequalities, standard Sobolev embeddings, the regularity of the state variables (u, χ) and the properties of the β-regularization:

$$\begin{aligned}
\left|\int_0^t \int_\Omega \mathbb{C}'(\chi)\dot{\boldsymbol{\chi}}\varepsilon(u) : \varepsilon(\dot{\mathbf{u}}_t)\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \|\mathbb{C}'(\chi)\|_{L^\infty(L^\infty)}\|\dot{\boldsymbol{\chi}}\|_{L^2(0,t;L^4)}\|\varepsilon(u)\|_{L^\infty(L^4)}\|\varepsilon(\dot{\mathbf{u}}_t)\|_{L^2(0,t;L^2)}\\
&\leq \delta\|\varepsilon(\dot{\mathbf{u}}_t)\|^2_{L^2(0,t;L^2)} + C_\delta\|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)},\\
\left|\int_0^t \int_\Omega \mathbb{C}(\chi)\varepsilon(\dot{\mathbf{u}}) : \varepsilon(\dot{\mathbf{u}}_t)\,\mathrm{d}x\,\mathrm{d}s\right| &\leq \delta\|\varepsilon(\dot{\mathbf{u}}_t)\|^2_{L^2(0,t;L^2)} + C_\delta\|\varepsilon(\dot{\mathbf{u}})\|^2_{L^2(0,t;L^2)},\\
\int_0^t \int_\Omega \xi'(\chi_t)|\dot{\boldsymbol{\chi}}_t|^2\,\mathrm{d}x\,\mathrm{d}s &\geq 0,\\
\left|\int_0^t \int_\Omega \mathbb{C}''(\chi)\varepsilon(u) : \varepsilon(u)\dot{\boldsymbol{\chi}}\dot{\boldsymbol{\chi}}_t\,\mathrm{d}x\,\mathrm{d}s\right| &\leq \|\mathbb{C}''(\chi)\|_{L^\infty(L^\infty)}\|\varepsilon(u)\|^2_{L^\infty(L^4)}\|\dot{\boldsymbol{\chi}}\|_{L^2(0,t;L^4)}\|\dot{\boldsymbol{\chi}}_t\|_{L^2(0,t;L^4)}\\
&\leq \delta\|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;H^1)} + C_\delta\|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)},\\
\left|\int_0^t \int_\Omega \mathbb{C}'(\chi)\varepsilon(\dot{\mathbf{u}}) : \varepsilon(u)\dot{\boldsymbol{\chi}}_t\,\mathrm{d}x\,\mathrm{d}s\right| &\leq \|\mathbb{C}'(\chi)\|_{L^\infty(L^\infty)}\|\varepsilon(u)\|_{L^\infty(L^4)}\|\varepsilon(\dot{\mathbf{u}})\|_{L^2(0,t;L^2)}\|\dot{\boldsymbol{\chi}}_t\|_{L^2(0,t;L^4)}\\
&\leq \delta\|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;H^1)} + C_\delta\|\varepsilon(\dot{\mathbf{u}})\|^2_{L^2(0,t;L^2)},\\
\left|\int_0^t \int_\Omega f''(\chi)\dot{\boldsymbol{\chi}}\dot{\boldsymbol{\chi}}_t\,\mathrm{d}x\,\mathrm{d}s\right| &\leq \|f''(\chi)\|_{L^\infty(L^\infty)}\|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;L^2)}\|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;L^2)}\\
&\leq \delta\|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;L^2)} + C_\delta\|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;L^2)}.
\end{aligned}$$

The right-hand side of the first equation is treated by Young's inequality and the trace theorem *via*

$$\int_0^t \int_\Gamma \mathbf{h}\cdot\dot{\mathbf{u}}_t\,\mathrm{d}x\,\mathrm{d}s \leq \delta\|\dot{\mathbf{u}}_t\|^2_{L^2(0,t;H^1(\Omega;\mathbb{R}^n))} + C_\delta\|\mathbf{h}\|^2_{L^2(0,t;L^2(\Gamma;\mathbb{R}^n))}.$$

Moreover, by making use of the embedding $H^1(\Omega;\mathbb{R}^n) \hookrightarrow H^1(\Omega;\mathbb{R}^n)^*$ given by $u \mapsto (u,\cdot)_{L^2}$, we find

$$\frac{1}{2}\|\dot{\mathbf{u}}_t(t)\|^2_{L^2} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle\dot{\mathbf{u}}_t(s), \dot{\mathbf{u}}_t(s)\rangle_{(H^1)^*\times H^1}\,\mathrm{d}s = \int_0^t \langle\dot{\mathbf{u}}_{tt}(s), \dot{\mathbf{u}}_t(s)\rangle_{(H^1)^*\times H^1}\,\mathrm{d}s.$$

Applying these calculations and adding the equations above, we obtain

$$\begin{aligned}
&\|\dot{\mathbf{u}}_t(t)\|^2_{L^2} + \|\nabla\dot{\boldsymbol{\chi}}(t)\|^2_{L^2} + \|\varepsilon(\dot{\mathbf{u}}_t)\|^2_{L^2(0,t;L^2)} + \|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;H^1)}\\
&\quad\leq \delta\Big(\|\dot{\mathbf{u}}_t\|^2_{L^2(0,t;H^1)} + \|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;H^1)}\Big) + C_\delta\Big(\|\dot{\mathbf{u}}\|^2_{L^2(0,t;L^2)} + \|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)} + \|\mathbf{h}\|^2_{L^2(0,t;L^2(\Gamma;\mathbb{R}^n))}\Big).
\end{aligned}$$

Now, adding $\|\dot{\mathbf{u}}_t\|^2_{L^2(0,t;L^2)} + \|\dot{\mathbf{u}}\|^2_{L^2(0,t;H^1)} + \|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)}$ on both sides, applying Korn's inequality and choosing $\delta > 0$ small enough, we find

$$\begin{aligned}
&\|\dot{\mathbf{u}}_t(t)\|^2_{L^2} + \|\nabla\dot{\boldsymbol{\chi}}(t)\|^2_{L^2} + \|\dot{\mathbf{u}}\|^2_{H^1(0,t;H^1)} + \|\dot{\boldsymbol{\chi}}\|^2_{H^1(0,t;H^1)}\\
&\quad\leq C\Big(\|\dot{\mathbf{u}}_t\|^2_{L^2(0,t;L^2)} + \|\dot{\mathbf{u}}\|^2_{L^2(0,t;H^1)} + \|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)} + \|\mathbf{h}\|^2_{L^2(0,t;L^2(\Gamma;\mathbb{R}^n))}\Big).
\end{aligned}$$

This yields with the help of the estimates

$$\|\dot{\mathbf{u}}\|^2_{L^2(0,t;H^1)} \leq C\int_0^t \|\dot{\mathbf{u}}_t\|^2_{L^2(0,s;H^1)}\,\mathrm{d}s,$$
$$\|\dot{\boldsymbol{\chi}}\|^2_{L^2(0,t;H^1)} \leq C\int_0^t \|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,s;H^1)}\,\mathrm{d}s,$$
$$\|\dot{\boldsymbol{\chi}}(t)\|^2_{L^2} \leq C\|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,t;L^2)}\,\mathrm{d}s,$$

the following inequality

$$\begin{aligned}&\|\dot{\mathbf{u}}_t(t)\|^2_{L^2} + \|\dot{\boldsymbol{\chi}}(t)\|^2_{H^1} + \|\dot{\mathbf{u}}\|^2_{H^1(0,t;H^1)} + \|\dot{\boldsymbol{\chi}}\|^2_{H^1(0,t;H^1)}\\ &\quad\leq C\|\mathbf{h}\|^2_{L^2(0,T;L^2(\Gamma;\mathbb{R}^n))} + C\int_0^t \Big(\|\dot{\mathbf{u}}_t\|^2_{L^2} + \|\dot{\mathbf{u}}_t\|^2_{L^2(0,s;H^1)} + \|\dot{\boldsymbol{\chi}}_t\|^2_{L^2(0,s;H^1)} + \|\dot{\boldsymbol{\chi}}\|^2_{H^1}\Big)\,\mathrm{d}s.\end{aligned}$$

Gronwall's lemma shows the claim. □

Proposition 3.3 (Strong differentiability of the control-to-state mapping)**.** *Under the assumptions of Proposition 3.1 the convergence (3.1) is even strong in $\dot{\mathcal{Q}}$. Moreover, the operator $\mathcal{S}:\mathcal{B}\to\dot{\mathcal{Q}}$ is Gâteaux-differentiable and we have $\langle D\mathcal{S}(b),h\rangle = (\dot{u},\dot{\chi})$.*

Proof. Let $\lambda\in\mathbb{R}$ and $b, b+\lambda h\in\mathcal{B}$ and define

$$(u^\lambda,\chi^\lambda) := \mathcal{S}(b+\lambda h), \qquad (u,\chi) := \mathcal{S}(b).$$

Furthermore, let $(\dot{u},\dot{\chi})$ be the unique solution of the linearized system (3.2)–(3.3) at b in direction h. We consider the following system arising from the calculations

$$\begin{aligned}\Big[\text{PDE system (1.2)–(1.6) for } b+\lambda h\Big] \;-\; &\Big[\text{PDE system (1.2)–(1.6) for } b\Big]\\ &-\lambda\times\Big[\text{PDE system (3.2)–(3.3) for } (b,h)\Big].\end{aligned}$$

By introducing the functions $(y^\lambda,z^\lambda)\in\dot{\mathcal{Q}}$ as

$$y^\lambda := u^\lambda - u - \lambda\dot{u} \qquad z^\lambda := \chi^\lambda - \chi - \lambda\dot{\chi},$$

the resulting system can be written as

$$\left\{\begin{aligned}&\int_0^t \langle y^\lambda_{tt},\varphi\rangle_{H^1}\,\mathrm{d}s + \int_0^t\int_\Omega \big(\mathbb{C}(\chi^\lambda)-\mathbb{C}(\chi)-\lambda\dot{\chi}\mathbb{C}'(\chi)\big)\,\varepsilon(u):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega \mathbb{C}(\chi^\lambda)\varepsilon(y^\lambda):\varepsilon(\varphi) + \mathbb{D}\varepsilon(y^\lambda_t):\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}s = 0,\end{aligned}\right. \tag{3.6}$$

$$\left\{\begin{aligned}&\int_0^t\int_\Omega \nabla z^\lambda\cdot\nabla\psi + \nabla z^\lambda_t\cdot\nabla\psi + z^\lambda_t\psi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \big(\xi(\chi^\lambda_t)-\xi(\chi_t)-\lambda\dot{\chi}_t\xi'(\chi_t)\big)\,\psi\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega \frac{1}{2}\big(\mathbb{C}'(\chi^\lambda)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \mathbb{C}'(\chi)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \lambda\dot{\chi}\mathbb{C}''(\chi)\varepsilon(u):\varepsilon(u)\big)\,\psi\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega \frac{1}{2}\big(\mathbb{C}'(\chi)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u^\lambda) - \lambda\mathbb{C}'(\chi)\varepsilon(\dot{u}):\varepsilon(u)\big)\,\psi\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega \frac{1}{2}\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(y^\lambda)\psi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \big(f'(\chi^\lambda)-f'(\chi)-\lambda\dot{\chi}f''(\chi)\big)\,\psi\,\mathrm{d}x\,\mathrm{d}s\\ &\quad= 0.\end{aligned}\right. \tag{3.7}$$

Now, testing (3.6) with $\varphi = y_t^\lambda$ and (3.7) with $\psi = z_t^\lambda$ and adding both equations, we find

$$
\begin{aligned}
&\int_0^t \langle y_{tt}^\lambda, y_t^\lambda\rangle_{H^1}\,\mathrm{d}s + \int_0^t\int_\Omega \mathbb{D}\varepsilon(y_t^\lambda):\varepsilon(y_t^\lambda) + \nabla z^\lambda\cdot\nabla z_t^\lambda + |\nabla z_t^\lambda|^2 + |z_t^\lambda|^2\,\mathrm{d}x\,\mathrm{d}s\\
&= \underbrace{-\int_0^t\int_\Omega \big(\mathbb{C}(\chi^\lambda) - \mathbb{C}(\chi) - \lambda\dot\chi\mathbb{C}'(\chi)\big)\,\varepsilon(u):\varepsilon(y_t^\lambda)\,\mathrm{d}x\,\mathrm{d}s,}_{=:T_1}\\
&\quad \underbrace{-\int_0^t\int_\Omega \mathbb{C}(\chi^\lambda)\varepsilon(y^\lambda):\varepsilon(y_t^\lambda)\,\mathrm{d}x\,\mathrm{d}s}_{=:T_2}\underbrace{-\int_0^t\int_\Omega \big(\xi(\chi_t^\lambda)-\xi(\chi_t)-\lambda\dot\chi_t\xi'(\chi_t)\big)\,z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{=:T_3}\\
&\quad \underbrace{-\int_0^t\int_\Omega \frac12\big(\mathbb{C}'(\chi^\lambda)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \mathbb{C}'(\chi)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \lambda\dot\chi\mathbb{C}''(\chi)\varepsilon(u):\varepsilon(u)\big)\,z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{=:T_4}\\
&\quad \underbrace{-\int_0^t\int_\Omega \frac12\big(\mathbb{C}'(\chi)\varepsilon(u^\lambda):\varepsilon(u^\lambda) - \mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u^\lambda) - \lambda\mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u)\big)\,z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{T_5}\\
&\quad \underbrace{-\int_0^t\int_\Omega \frac12\big(\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u^\lambda) - \mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u) - \lambda\mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u)\big)\,z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{T_6}\\
&\quad \underbrace{-\int_0^t\int_\Omega \frac12\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(y^\lambda)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{=:T_7}\underbrace{-\int_0^t\int_\Omega \big(f'(\chi^\lambda)-f'(\chi)-\lambda\dot\chi f''(\chi)\big)\,z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s}_{=:T_8}.
\end{aligned}
$$

To proceed, we make use of the following estimates: By using Taylor's theorem and boundedness

$$
\|\chi\|_{L^\infty(Q)} + \|\chi^\lambda\|_{L^\infty(Q)} \le L \tag{3.8}
$$

with respect to λ (applying Thm. 2.2 (ii)) as well as $\|\xi'\|_{L^\infty(Q)} + \|\xi''\|_{L^\infty(Q)} < +\infty$ and Assumptions (B1)–(B3), we obtain the following estimates

$$
\begin{aligned}
\Big|\mathbb{C}(\chi^\lambda) - \mathbb{C}(\chi) - \lambda\dot\chi\mathbb{C}'(\chi)\Big| &\le \sup_{|x|<L}|\mathbb{C}'(x)||z^\lambda| + C\sup_{|x|<L}|\mathbb{C}''(x)||\chi^\lambda-\chi|^2\\
&\le C(|z^\lambda| + |\chi^\lambda-\chi|^2),\\
\Big|\mathbb{C}'(\chi^\lambda) - \mathbb{C}'(\chi) - \lambda\dot\chi\mathbb{C}''(\chi)\Big| &\le \sup_{|x|<L}|\mathbb{C}''(x)||z^\lambda| + C\sup_{|x|<L}|\mathbb{C}'''(x)||\chi^\lambda-\chi|^2\\
&\le C(|z^\lambda| + |\chi^\lambda-\chi|^2),\\
\Big|f'(\chi^\lambda) - f'(\chi) - \lambda\dot\chi f''(\chi)\Big| &\le C(|z^\lambda| + |\chi^\lambda-\chi|^2).
\end{aligned}
$$

The above estimates, the *a priori* estimates

$$
\|(u^\lambda,\chi^\lambda)\|_{\mathcal{Q}} \le C, \qquad \|(u,\chi)\|_{\mathcal{Q}} \le C
$$

(in particular (3.8)) and a special vector-valued version of the Gagliardo−Nirenberg inequality in 2D

$$\|w\|_{L^4} \le C\|w\|_{H^1}^{1/2}\|w\|_{L^2}^{1/2} \quad \text{for all } w \in H^1(\Omega;\mathbb{R}^m)$$

allow us to treat the terms T_1, T_2, T_4, T_5, T_6, T_7 and T_8 as follows

$$\begin{aligned}
T_1 &\le C\int_0^t\int_\Omega |z^\lambda||\varepsilon(u)||\varepsilon(y_t^\lambda)| + |\chi^\lambda-\chi|^2|\varepsilon(u)||\varepsilon(y_t^\lambda)|\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\left(\|z^\lambda\|_{L^2(0,t;L^3)} + \|\chi^\lambda-\chi\|^2_{L^2(0,t;L^6)}\right)\|\varepsilon(u)\|_{L^\infty(L^6)}\|\varepsilon(y_t^\lambda)\|_{L^2(0,t;L^2)}\\
&\le \delta\|\varepsilon(y_t^\lambda)\|^2_{L^2(0,t;L^2)} + C_\delta\|z^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|\chi^\lambda-\chi\|^4_{L^2(0,t;H^1)},\\
T_2 &\le \delta\|\varepsilon(y_t^\lambda)\|^2_{L^2(0,t;L^2)} + C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)},\\
T_4 &= -\int_0^t\int_\Omega \frac12\left(\mathbb{C}'(\chi^\lambda)-\mathbb{C}'(\chi)-\lambda\dot\chi\mathbb{C}''(\chi)\right)\varepsilon(u^\lambda):\varepsilon(u^\lambda)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s\\
&\quad -\int_0^t\int_\Omega \frac12\lambda\dot\chi\mathbb{C}''(\chi)\varepsilon(u^\lambda+u):\varepsilon(u^\lambda-u)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\int_0^t\int_\Omega |z^\lambda||\varepsilon(u^\lambda)|^2|z_t^\lambda|\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_\Omega |\chi^\lambda-\chi|^2|\varepsilon(u^\lambda)|^2|z_t^\lambda|\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + |\lambda|C\int_0^t\int_\Omega |\dot\chi||\varepsilon(u^\lambda+u)||\varepsilon(u^\lambda-u)||z_t^\lambda|\,\mathrm{d}x\,\mathrm{d}s\\
&\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|\varepsilon(u^\lambda)\|^2_{L^\infty(L^4)}\|z^\lambda\|^2_{L^2(0,t;H^1)}\\
&\quad + C\|\varepsilon(u^\lambda)\|^2_{L^\infty(L^5)}\|\chi^\lambda-\chi\|^2_{L^\infty(L^5)}\|z_t^\lambda\|_{L^1(0,t;L^5)}\\
&\quad + |\lambda|C\|\dot\chi\|_{L^\infty(L^6)}\|\varepsilon(u^\lambda+u)\|_{L^\infty(L^6)}\|\varepsilon(u^\lambda-u)\|_{L^2(L^2)}\|z_t^\lambda\|_{L^2(0,t;L^6)}\\
&\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|z^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|\chi^\lambda-\chi\|^4_{H^1(H^1)} + |\lambda|^2C_\delta\|\varepsilon(u^\lambda-u)\|^2_{L^2(L^2)},\\
T_5 &= -\frac12\int_0^t\int_\Omega \mathbb{C}'(\chi)\varepsilon(y^\lambda):\varepsilon(u^\lambda)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s - \frac{\lambda}{2}\int_0^t\int_\Omega \mathbb{C}'(\chi)\varepsilon(\dot u):\varepsilon(u^\lambda-u)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\|\varepsilon(y^\lambda)\|_{L^2(0,t;L^2)}\|\varepsilon(u^\lambda)\|_{L^\infty(L^4)}\|z_t^\lambda\|_{L^2(0,t;L^4)}\\
&\quad + |\lambda|C\int_0^t \|\varepsilon(\dot u)\|_{L^2}\|\varepsilon(u^\lambda-u)\|_{L^4}\|z_t^\lambda\|_{L^4}\,\mathrm{d}s \quad \text{(apply Gagliardo−Nirenberg)}\\
&\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)}\\
&\quad + |\lambda|C\|\varepsilon(\dot u)\|_{L^\infty(L^2)}\int_0^t \|\varepsilon(u^\lambda-u)\|_{H^1}^{1/2}\|\varepsilon(u^\lambda-u)\|_{L^2}^{1/2}\|z_t^\lambda\|_{L^4}\,\mathrm{d}s\\
&\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)} + C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)} + |\lambda|^2C_\delta\|\varepsilon(u^\lambda-u)\|_{L^2(L^2)},
\end{aligned}$$

$$\begin{aligned}
T_6 &= -\frac{1}{2}\int_0^t\int_\Omega \mathbb{C}'(\chi)\varepsilon(y^\lambda):\varepsilon(u)z_t^\lambda\,\mathrm{d}x\,\mathrm{d}s\\
&\le C\|\varepsilon(y^\lambda)\|_{L^2(0,t;L^2)}\|\varepsilon(u^\lambda)\|_{L^\infty(L^4)}\|z_t^\lambda\|_{L^2(0,t;L^4)}\\
&\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)}+C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)}\\
T_7 &\le \delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)}+C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)},\\
T_8 &\le \delta\|z_t^\lambda\|^2_{L^2(0,t;L^2)}+C_\delta\|\chi^\lambda-\chi\|^4_{H^1(H^1)}+C_\delta\|z^\lambda\|^2_{L^2(0,t;L^2)}.
\end{aligned}$$

Due to the low time-regularity of the damage variables, the term T_3 needs to be treated differently. To this end, we find by the mean value theorem with

$$\mu\in\left[\min\{\chi_t^\lambda,\chi_t\},\max\{\chi_t^\lambda,\chi_t\}\right]\ \text{suitably chosen,}$$

Young's inequality, $\xi'\ge 0$ and the monotonicity of ξ' that

$$\begin{aligned}
-\left(\xi(\chi_t^\lambda)-\xi(\chi_t)-\lambda\dot\chi_t\xi'(\chi_t)\right)z_t^\lambda &= -\left(\xi'(\mu)(\chi_t^\lambda-\chi_t)-\lambda\dot\chi_t\xi'(\chi_t)\right)z_t^\lambda\\
&= -\xi'(\mu)(z_t^\lambda+\lambda\dot\chi_t)z_t^\lambda+\lambda\dot\chi_t\xi'(\chi_t)z_t^\lambda\\
&\le -\xi'(\mu)\lambda\dot\chi_t z_t^\lambda+\lambda\dot\chi_t\xi'(\chi_t)z_t^\lambda\\
&= -(\xi'(\mu)-\xi'(\chi_t))\lambda\dot\chi_t z_t^\lambda\\
&\le \delta|z_t^\lambda|^2+|\lambda|^2C_\delta|\xi'(\mu)-\xi'(\chi_t)|^2|\dot\chi_t|^2\\
&\le \delta|z_t^\lambda|^2+|\lambda|^2C_\delta|\xi'(\chi_t^\lambda)-\xi'(\chi_t)|^2|\dot\chi_t|^2.
\end{aligned}$$

For further considerations we define $f_\lambda\in L^\infty(Q)$ by

$$f_\lambda:=|\xi'(\chi_t^\lambda)-\xi'(\chi_t)|^2$$

and thus obtain

$$T_3\le\delta\|z_t^\lambda\|^2_{L^2(0,t;L^2)}+|\lambda|^2C_\delta\int_0^t\int_\Omega f_\lambda|\dot\chi_t|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Note that due to continuity of the solution operator $\mathcal{S}:\mathcal{B}\to\dot{\mathcal{Q}}$ by Theorem 2.2 (iii) we find $\chi_t^\lambda\to\chi_t$ strongly in $L^2(0,T;L^2(\Omega))$ as $\lambda\to 0$. Taking also the boundedness and continuity of ξ' (see (A3) and (B2)) into account we observe that $f_\lambda\overset{\star}{\rightharpoonup}0$ weakly-star in $L^\infty(Q)$ as $\lambda\to 0$ and in particular

$$\int_0^t\int_\Omega f_\lambda|\dot\chi_t|^2\,\mathrm{d}x\,\mathrm{d}s\to 0\qquad\text{as }\lambda\to 0. \tag{3.9}$$

Applying all the estimates for $T_1,\dots,T_8$ we obtain

$$\begin{aligned}
&\|y_t^\lambda(t)\|^2_{L^2}+\|\nabla z^\lambda(t)\|^2_{L^2}+\|\varepsilon(y_t^\lambda)\|^2_{L^2(0,t;L^2)}+\|z_t^\lambda\|^2_{L^2(0,t;H^1)}\\
&\le\delta\|z_t^\lambda\|^2_{L^2(0,t;H^1)}+\delta\|\varepsilon(y_t^\lambda)\|^2_{L^2(0,t;L^2)}+C_\delta\|z^\lambda\|^2_{L^2(0,t;H^1)}+C_\delta\|\varepsilon(y^\lambda)\|^2_{L^2(0,t;L^2)}\\
&\quad+|\lambda|^2C_\delta\int_0^t\int_\Omega f_\lambda|\dot\chi_t|^2\,\mathrm{d}x\,\mathrm{d}s+C_\delta\|\chi^\lambda-\chi\|^4_{H^1(H^1)}+|\lambda|^2C_\delta\|\varepsilon(u^\lambda-u)\|^2_{L^2(L^2)}\\
&\quad+|\lambda|^2C_\delta\|\varepsilon(u^\lambda-u)\|_{L^2(L^2)}.
\end{aligned}$$

Adding $\|z^\lambda\|^2_{L^2(0,t;H^1)} + \|y^\lambda\|^2_{L^2(0,t;H^1)} + \|y^\lambda_t\|^2_{L^2(0,t;L^2)}$ on both sides, using the estimate

$$\frac{1}{2}\|z^\lambda_t\|_{L^2(0,t;L^2)} \geq c\|z^\lambda(t)\|^2_{L^2}$$

on the left-hand side and the estimate

$$\|y^\lambda\|^2_{L^2(0,t;H^1)} \leq C\int_0^t \|y^\lambda_t\|^2_{L^2(0,s;H^1)}\,\mathrm{d}s$$

on the right-hand side, applying Korn's inequality and choosing $\delta > 0$ small, we obtain

$$\begin{aligned}
&\|y^\lambda_t(t)\|^2_{L^2} + \|z^\lambda(t)\|^2_{H^1} + \|y^\lambda\|^2_{H^1(0,t;H^1)} + \|z^\lambda\|^2_{H^1(0,t;H^1)}\\
&\leq C\int_0^t \|z^\lambda\|^2_{H^1} + \|y^\lambda_t\|^2_{L^2(0,s;H^1)} + \|y^\lambda_t\|^2_{L^2}\,\mathrm{d}s\\
&\quad + C\|\chi^\lambda - \chi\|^4_{H^1(H^1)} + C|\lambda|^2\|u^\lambda - u\|^2_{L^2(H^1)} + C|\lambda|^2\|u^\lambda - u\|_{L^2(H^1)}\\
&\quad + C|\lambda|^2\int_Q f_\lambda|\dot{\chi}_t|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}
&\|y^\lambda_t(t)\|^2_{L^2} + \|z^\lambda(t)\|^2_{H^1} + \|y^\lambda\|^2_{H^1(0,t;H^1)} + \|z^\lambda\|^2_{H^1(0,t;H^1)}\\
&\quad\leq C\Big(\|\chi^\lambda - \chi\|^4_{H^1(H^1)} + |\lambda|^2\|u^\lambda - u\|^2_{L^2(H^1)} + |\lambda|^2\|u^\lambda - u\|_{L^2(H^1)} + |\lambda|^2\int_Q f_\lambda|\dot{\chi}_t|^2\,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}$$

Due to Lipschitz continuity of the solution operator on bounded subsets of $\mathcal{B}$ as in Theorem 2.2 (iii) (we consider a ball containing b and $b + \lambda h$), we find

$$\begin{aligned}
&\|y^\lambda_t(t)\|^2_{L^2} + \|z^\lambda(t)\|^2_{H^1} + \|y^\lambda\|^2_{H^1(0,t;H^1)} + \|z^\lambda\|^2_{H^1(0,t;H^1)}\\
&\quad\leq C\Big(|\lambda|^4\|h\|^4_{H^1(H^1)} + |\lambda|^4\|h\|^2_{H^1(H^1)} + |\lambda|^3\|h\|_{H^1(H^1)} + |\lambda|^2\int_Q f_\lambda|\dot{\chi}_t|^2\,\mathrm{d}x\,\mathrm{d}s\Big).
\end{aligned}$$

Taking also (3.9) into account we end up with

$$\frac{\|y^\lambda_t(t)\|_{L^2} + \|z^\lambda(t)\|_{H^1} + \|y^\lambda\|_{H^1(0,t;H^1)} + \|z^\lambda\|_{H^1(0,t;H^1)}}{|\lambda|} \to 0 \qquad \text{as } \lambda \to 0.$$

□

Remark 3.4. The argumentation above is not sufficient to prove Fréchet differentiability because in that case one has to prove that $\int_0^t\int_\Omega |\xi'(\chi^h_t) - \xi'(\chi_t)|^2|\dot{\chi}^h_t|^2\,\mathrm{d}x\,\mathrm{d}s/\|h\|^2_{\mathcal{B}} \to 0$ as $h \to 0$ in $\mathcal{B}$ in the estimate for T_3. However for our purposes Gâteaux-differentiability turns out to be sufficient.

3.2. Adjoint state problem

Let us firstly give a short motivation for the derivation of the adjoint system and then continue with rigorous analysis:

By utilizing the differentiability of the solution operator $\mathcal{S}$ obtained in Proposition 3.3 we find for the derivative of the cost-functional $\mathcal{J}$ composed with the χ-part of the solution operator $\mathcal{S} : b \mapsto (u(b), \chi(b))$ *via* the chain rule

$$\big\langle D_b\mathcal{J}(\chi(b), b), h\big\rangle = \big\langle \partial_\chi\mathcal{J}(\chi(b), b), \dot{\chi}[h]\big\rangle + \big\langle \partial_b\mathcal{J}(\chi(b), b), h\big\rangle \qquad \text{for all } h \in \mathcal{B}, \tag{3.10}$$

where ∂_χ and ∂_b denote the partial derivatives with respect to the corresponding variables. Now, to rewrite the expression in terms of PDEs, the adjoint system is introduced as follows: Our goal is to find a pair of functions $(p,q) \in \overline{\mathcal{Q}}$ such that

$$\big\langle \partial_\chi \mathcal{J}(\chi(b),b), \dot{\chi}[h] \big\rangle = \big\langle (p,q), \mathcal{C}(h) \big\rangle, \tag{3.11}$$

where $\mathcal{C} : \mathcal{B} \to \overline{\mathcal{Q}}^*$ (note that $\overline{\mathcal{Q}}^*$ denotes the topological dual of $\overline{\mathcal{Q}}$) specifies the operator mapping the control variable b to the right-hand side of (3.2)–(3.3). More precisely

$$\big\langle (p,q), \mathcal{C}(h) \big\rangle = \int_\Sigma p \cdot h \,\mathrm{d}x\,\mathrm{d}t, \tag{3.12}$$

i.e. the right-hand side of (3.2)–(3.3) tested with (p,q). We call (p,q) the adjoint variables to the linearized solutions $(\dot{u},\dot{\chi})$ at (u,χ). Even though the adjoint variable q does not appear directly in (3.11) (by taking (3.12) into account) it will be used for intermediate steps as shown below.

In order to derive an explicit PDE system for (p,q) (which will be justified rigorously afterward), we proceed formally and test (3.2)–(3.3) with (p,q). Then, adding both resulting equations yield

$$\begin{aligned}
&\int_0^T \langle \dot{u}_{tt}, p\rangle_{H^1}\,\mathrm{d}t + \int_Q \mathbb{C}'(\chi)\dot{\chi}\varepsilon(u) : \varepsilon(p) + \mathbb{C}(\chi)\varepsilon(\dot{u}) : \varepsilon(p) + \mathbb{D}\varepsilon(\dot{u}_t) : \varepsilon(p)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \nabla\dot{\chi}\cdot\nabla q + \nabla\dot{\chi}_t\cdot\nabla q + \dot{\chi}_t q + \xi'(\chi_t)\dot{\chi}_t q + \frac{1}{2}\mathbb{C}''(\chi)\dot{\chi}\varepsilon(u) : \varepsilon(u) q\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \mathbb{C}'(\chi)\varepsilon(\dot{u}) : \varepsilon(u) q + f''(\chi)\dot{\chi} q\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \big\langle (p,q), \mathcal{C}(h)\big\rangle.
\end{aligned} \tag{3.13}$$

Consequently the adjoint variables should satisfy for all "appropriate" test-functions (φ,ψ):

$$\begin{aligned}
&\int_0^T \langle \varphi_{tt}, p\rangle_{H^1}\,\mathrm{d}t + \int_Q \mathbb{C}'(\chi)\psi\varepsilon(u) : \varepsilon(p) + \mathbb{C}(\chi)\varepsilon(\varphi) : \varepsilon(p) + \mathbb{D}\varepsilon(\varphi_t) : \varepsilon(p)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \nabla\psi\cdot\nabla q + \nabla\psi_t\cdot\nabla q + \psi_t q + \xi'(\chi_t)\psi_t q + \frac{1}{2}\mathbb{C}''(\chi)\psi\varepsilon(u) : \varepsilon(u) q\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_Q \mathbb{C}'(\chi)\varepsilon(\varphi) : \varepsilon(u) q + f''(\chi)\psi q\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \big\langle \partial_\chi \mathcal{J}(\chi(b),b), \psi\big\rangle.
\end{aligned} \tag{3.14}$$

In this case we can recover (3.11) by using (3.14) tested with $(\varphi,\psi) = (\dot{u},\dot{\chi})$ and using (3.13). Note that (3.14) can be equivalently recasted as the system

$$\int_0^T \langle p, \varphi_{tt}\rangle_{H^1}\,\mathrm{d}t + \int_Q \mathbb{C}(\chi)\varepsilon(p) : \varepsilon(\varphi) + \mathbb{C}'(\chi)\varepsilon(u) q : \varepsilon(\varphi) + \mathbb{D}\varepsilon(p) : \varepsilon(\varphi_t)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.15}$$

$$\begin{aligned}
&\int_Q q\psi_t + \xi'(\chi_t) q\psi_t + \nabla q\cdot\nabla\psi + \nabla q\cdot\nabla\psi_t + \frac{1}{2}\mathbb{C}''(\chi)\varepsilon(u) : \varepsilon(u) q\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_Q \mathbb{C}'(\chi)\varepsilon(u) : \varepsilon(p)\psi + f''(\chi) q\psi\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \lambda_T\big(\chi(T) - \chi_T\big)\psi(T)\,\mathrm{d}x.
\end{aligned} \tag{3.16}$$

In the pointwise formulation this reads as follows:

$$\begin{aligned}
&p_{tt} - \operatorname{div}(\mathbb{C}(\chi)\varepsilon(p) + \mathbb{C}'(\chi)\varepsilon(u)q - \mathbb{D}\varepsilon(p_t)) = 0 && \text{in} \quad Q,\\
&- q_t - (\xi'(\chi_t)q)_t + \Delta q_t - \Delta q + \frac{1}{2}\mathbb{C}''(\chi)\varepsilon(u) : \varepsilon(u)q + \mathbb{C}'(\chi)\varepsilon(u) : \varepsilon(p) + f''(\chi)q = 0 && \text{in} \quad Q,\\
&(\mathbb{C}(\chi)\varepsilon(p) + \mathbb{C}'(\chi)\varepsilon(u)q - \mathbb{D}\varepsilon(p_t)) \cdot \nu = 0 && \text{on} \quad \Sigma,\\
&\nabla p \cdot \nu = 0 && \text{on} \quad \Sigma\\
&\text{with the final-time conditions}\\
&p(T) = p_t(T) = 0 && \text{in} \quad \Omega,\\
&- \Delta q(T) + q(T) + \xi'(\chi_t(T))q(T) = \lambda_T(\chi(T) - \chi_T) && \text{in} \quad \Omega,\\
&\nabla q(T) \cdot \nu = 0 && \text{on} \quad \Gamma.
\end{aligned}$$

The PDE system above is a backward in time boundary value problem for (p, q), where q itself fulfills an elliptic PDE at the final-time T. Our task is now to prove existence of solutions in a weak sense.

Proposition 3.5 (Existence of very weak solution to the adjoint problem)**.** *Suppose that the Assumptions* (A1)–(A5), (O1)–(O2) *and* (B1)–(B3) *are fulfilled. Furthermore let* (u, χ) *be a solution of the state system corresponding to* $b \in \mathcal{B}$, *i.e.* $(u, \chi) := \mathcal{S}(b)$. *Then there exists a pair of function* $(p, q) \in \overline{\mathcal{Q}}$ *(weak solution) such that*

(3.15)–(3.16) *holds for all test-functions* $(\varphi, \psi) \in \dot{\mathcal{Q}}$ *with* $\varphi(0) = \varphi_t(0) = \psi(0) = 0$.

We prove Proposition 3.5 in several steps whose intermediate results are highlighted by corresponding lemmas. The idea is the following:

First, we will apply a time transformation and investigate a modified adjoint system where the coefficient functions of the original adjoint system are substituted by smoother versions. More precisely the state variable χ on which some of the coefficient functions depend is replaced by a smooth approximation $\chi_\alpha \to \chi$ gained, *e.g.*, *via* convolution (see step 1 below). This will enable us to time-differentiate the nonlinear term $\xi'(\partial_t\chi_\alpha)q$ and obtain suitable *a priori* estimates. Existence of solutions for this regularized system will be achieved by utilizing a time-discretization scheme with time step size τ and a limit analysis $\tau \downarrow 0$. In the time-discrete setting the regularized adjoint system is an elliptic problem which can be solved by standard methods. We then derive *a priori* estimates (energy estimates) uniformly in τ in order to pass to the limit $\tau \downarrow 0$. After solving the regularized adjoint system, *i.e.* for $\alpha > 0$, we transform it to the very weak formulation as used in (3.15)–(3.16), where time-derivatives only occur on the test-functions. Then, roughly speaking, we test the resulting system with certain modified anti-derivatives with respect to time of p_α and q_α and end up with *a priori* estimates uniformly in α in the large space $\overline{\mathcal{Q}}$. Finally the limit passage $\alpha \downarrow 0$ can be performed in the regularized adjoint system.

Step 1. *setup time-transformation and* α*-regularization*

In the first step we consider a transformation of the adjoint system above to an inital-boundary value problem by using the time transformation $t \mapsto T - t$. We find (we keep the notation (p, q) for the transformed variables)

$$\begin{aligned}
&p_{tt} - \operatorname{div}(\mathbb{C}(\chi)\varepsilon(p) + \mathbb{C}'(\chi)\varepsilon(u)q + \mathbb{D}\varepsilon(p_t)) = 0 && \text{in} \quad Q, && (3.17)\\
&q_t + (\xi'(-\chi_t)q)_t - \Delta q_t - \Delta q + \frac{1}{2}\mathbb{C}''(\chi)\varepsilon(u) : \varepsilon(u)q\\
&\qquad\qquad + \mathbb{C}'(\chi)\varepsilon(u) : \varepsilon(p) + f''(\chi)q = 0 && \text{in} \quad Q, && (3.18)\\
&(\mathbb{C}(\chi)\varepsilon(p) + \mathbb{C}'(\chi)\varepsilon(u)q + \mathbb{D}\varepsilon(p_t)) \cdot \nu = 0 && \text{on} \quad \Sigma, && (3.19)\\
&\nabla p \cdot \nu = 0 && \text{on} \quad \Sigma && (3.20)
\end{aligned}$$

with the initial conditions

$$p(0) = p_t(0) = 0 \qquad \text{in} \quad \Omega, \tag{3.21}$$

$$-\Delta q(0) + q(0) + \xi'(-\chi_t(0))q(0) = \lambda_T(\chi(T) - \chi_T) \qquad \text{in} \quad \Omega, \tag{3.22}$$

$$\nabla q(0) \cdot \nu = 0 \qquad \text{on} \quad \Gamma. \tag{3.23}$$

Secondly, to obtain rigorous existence results, we will firstly work with a regularized version of the state variable (u, χ). To this end, let $\{\chi_\alpha\} \subseteq C^\infty(\overline{Q})$ be a smooth approximation sequence such that

$$\chi_\alpha \to \chi \text{ in } H^1(0,T; H^2(\Omega)) \text{ as } \alpha \downarrow 0.$$

The regularized system is obtained by replacing χ by its regularization χ_α in (3.17)–(3.23). In the next two steps of the proof we will prove existence of solutions *via* a time-discretization argument. In the last step we will perform $\alpha \downarrow 0$.

Step 2. *setup time-discretization of the α-regularized problem*

To keep the notation simple we omit the explicit dependence on $\alpha > 0$ in this step. We consider the following time-discretization scheme: Let $\{0, \tau, 2\tau, \ldots, T\}$ denote an equidistant partition of $[0, T]$ with time step size $\tau := T/M$ and $M \in \mathbb{N}$. Moreover, let denote the first and second difference operators by $D_k(p) := \frac{p^k - p^{k-1}}{\tau}$ and $D_k(D_k(p)) := \frac{p^k - 2p^{k-1} + p^{k-2}}{\tau^2}$. For an arbitrary sequence $\{h^k\}_{k=0,\ldots,M}$ we define the piecewise constant and linear interpolation as

$$\overline{h}_\tau(t) := h^k,\ \underline{h}_\tau(t) := h^{k-1},\ h_\tau(t) := \frac{t-(k-1)\tau}{\tau}h^k + \frac{k\tau - t}{\tau}h^{k-1} \quad \text{for } t \in ((k-1)\tau, k\tau]. \tag{3.24}$$

With these preparations the time-discretized version of the system in step 1 reads in a weak formulation as

find $\{(p^k, q^k)\}_{k=1,\ldots,M} \subseteq H^1(\Omega; \mathbb{R}^n) \times H^1(\Omega)$ such that

$$\int_\Omega D_k(D_k(p)) \cdot \varphi + \big(\mathbb{C}(\chi^k)\varepsilon(p^k) + \mathbb{C}'(\chi^k)\varepsilon(u^k)q^{k-1} + \mathbb{D}\varepsilon(D_k(p))\big) : \varepsilon(\varphi)\,\mathrm{d}x = 0 \tag{3.25}$$

$$\begin{aligned}&\int_\Omega D_k(q)\psi + \mathsf{a}^k D_k(q)\psi - \mathsf{b}^k q^k \psi + D_k(\nabla q^k)\cdot\nabla\psi + \nabla q^k \cdot \nabla\psi \\ &\quad + \frac{1}{2}\mathbb{C}''(\chi^k)\varepsilon(u^k) : \varepsilon(u^k) q^k \psi + \mathbb{C}'(\chi^k)\varepsilon(u^k) : \varepsilon(p^k)\psi + f''(\chi^k) q^k \psi\,\mathrm{d}x = 0,\end{aligned} \tag{3.26}$$

for all $(\varphi, \psi) \in H^1(\Omega; \mathbb{R}^n) \times H^1(\Omega)$, where $\{\mathsf{a}^k\}_{k=0,\ldots,M}$ with $\mathsf{a}^k \geq 0$ and $\{\mathsf{b}^k\}_{k=0,\ldots,M}$ are time-discretizations of $\xi'(-\partial_t\chi_\alpha)$ and $\xi''(-\partial_t\chi_\alpha)\partial_{tt}\chi_\alpha$ respectively (note that ξ'' exists due to assumption (B2)), such that

$$\overline{\mathsf{a}} \to \xi'(-\partial_t\chi_\alpha) \qquad \text{strongly in} \quad L^\infty(Q),$$

$$\overline{\mathsf{b}} \to \xi''(-\partial_t\chi_\alpha)\partial_{tt}\chi_\alpha \qquad \text{strongly in} \quad L^\infty(Q)$$

Remark 3.6. Note that in (3.25)–(3.26) we use a time-discretization in a form which allows for a decoupled system of linear elliptic equations.

We proceed recursively and construct p^k, q^k from $p^{k-1}, p^{k-2}, q^{k-1}$. The initial values are given by $p^0 = p^{-1} = 0$ and q^0 is the weak solution of

$$\int_\Omega \nabla q^0 \cdot \nabla\varphi + q^0\varphi + \xi'(-\partial_t\chi_\alpha(0))q^0\varphi - \lambda_T\big(\chi(T) - \chi_T\big)\varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in H^1(\Omega). \tag{3.27}$$

Lemma 3.7. *There exist* $\{(p^k, q^k)\}_{k=1,\ldots,M} \subseteq H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega)$*, which fulfill* (3.25)–(3.26).

Proof. Employing Lax–Milgram's theorem for given $q^{k-1} \in H^1(\Omega)$ the equation (3.25) admits a solution $p^k \in H^1(\Omega;\mathbb{R}^n)$. Moreover, by standard theory of partial differential equations of second order we obtain for given $p^k \in H^1(\Omega;\mathbb{R}^n)$ a solution $q^k \in H^1(\Omega)$ to the equation (3.26). □

Step 3. *a priori estimates for the α-regularized time-discrete system and limit passage $\tau \downarrow 0$* After setting up the time-discrete scheme and existence, we will establish *a priori* estimates uniformly in τ in order to perform $\tau \downarrow 0$.

Lemma 3.8. *There exists a constant $C > 0$ (possibly depending on α) independent of τ such that*

$$\|p_\tau\|_{H^1(0,T;H^1)\cap W^{1,\infty}(0,T;L^2)} \le C, \qquad \|w_\tau\|_{H^1(0,T;(H^1)^*)} \le C, \qquad \|q_\tau\|_{H^1(0,T;H^1)} \le C,$$

where w_τ denotes the linear interpolation of $\{w^k\}_{k=0,\ldots,M}$ defined by $w^k := D_k(p)$.

Proof. By testing (3.25) with $p^k - p^{k-1}$, testing (3.26) with $q^k - q^{k-1}$ and summing over $k = 1, \ldots, \bar{t}/\tau$, we obtain the estimates

$$\begin{aligned}&\frac{1}{2}\|\partial_t p(t)\|_{L^2}^2 + c\|\varepsilon(\partial_t p)\|_{L^2(0,\bar{t};L^2)}^2 \\ &\qquad \le \underbrace{-\int_0^{\bar{t}}\int_\Omega \mathbb{C}(\overline{\chi})\varepsilon(\overline{p}) : \varepsilon(\partial_t p)\,\mathrm{d}x\,\mathrm{d}s}_{=:T_1} \underbrace{-\int_0^{\bar{t}}\int_\Omega \mathbb{C}'(\overline{\chi})\varepsilon(\overline{u}) : \varepsilon(\partial_t p)\underline{q}\,\mathrm{d}x\,\mathrm{d}s}_{=:T_2}\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\frac{1}{2}\|\nabla\overline{q}(t)\|_{L^2}^2 + \|\partial_t q\|_{L^2(0,\bar{t};H^1)}^2 \\ &\qquad \le \underbrace{-\int_0^{\bar{t}}\int_\Omega \mathsf{a}|\partial_t q|^2\,\mathrm{d}x\,\mathrm{d}s}_{=:T_3} \underbrace{-\int_0^{\bar{t}}\int_\Omega \overline{\mathsf{b}}\overline{q}\,\partial_t q\,\mathrm{d}x\,\mathrm{d}s}_{=:T_4} \underbrace{-\int_0^{\bar{t}}\int_\Omega \frac{1}{2}\mathbb{C}''(\overline{\chi})\varepsilon(\overline{u}) : \varepsilon(\overline{u})\overline{q}\,\partial_t q\,\mathrm{d}x\,\mathrm{d}s}_{=:T_5} \\ &\qquad\quad \underbrace{-\int_0^{\bar{t}}\int_\Omega \mathbb{C}'(\overline{\chi})\varepsilon(\overline{u}) : \varepsilon(\overline{p})\,\partial_t q\,\mathrm{d}x\,\mathrm{d}s}_{=:T_6} \underbrace{-\int_0^{\bar{t}}\int_\Omega f''(\overline{\chi})\overline{q}\,\partial_t q\,\mathrm{d}x\,\mathrm{d}s}_{=:T_7}\end{aligned} \tag{3.29}$$

Hölder's and Young's inequality show

$$
\begin{aligned}
T_1 &\le \|\mathbb{C}(\overline{\chi})\|_{L^\infty(L^\infty)}\|\varepsilon(\overline{p})\|_{L^2(0,\bar{t};L^2)}\|\varepsilon(\partial_t p)\|_{L^2(0,\bar{t};L^2)}\\
&\quad \delta\|\varepsilon(\partial_t p)\|^2_{L^2(0,\bar{t};L^2)} + C_\delta\|\varepsilon(\overline{p})\|^2_{L^2(0,\bar{t};L^2)},\\
T_2 &\le \|\mathbb{C}'(\overline{\chi})\|_{L^\infty(L^\infty)}\|\varepsilon(\overline{u})\|_{L^\infty(L^4)}\|\varepsilon(\partial_t p)\|_{L^2(0,\bar{t};L^2)}\|\underline{q}\|_{L^2(0,\bar{t};L^4)}\\
&\quad \delta\|\varepsilon(\partial_t p)\|^2_{L^2(0,\bar{t};L^2)} + C_\delta\|\underline{q}\|^2_{L^2(0,\bar{t};H^1)},\\
T_3 &\le 0,\\
T_4 &\le \|\overline{\mathsf{b}}\|_{L^\infty(L^\infty)}\|\overline{q}\|_{L^2(0,\bar{t};L^2)}\|\partial_t q\|_{L^2(0,\bar{t};L^2)}\\
&\quad \delta\|\partial_t q\|^2_{L^2(0,\bar{t};L^2)} + C_\delta\|\overline{q}\|^2_{L^2(0,\bar{t};L^2)},\\
T_5 &\le \|\mathbb{C}''(\overline{\chi})\|_{L^\infty(L^\infty)}\|\varepsilon(\overline{u})\|^2_{L^\infty(L^6)}\|\overline{q}\|_{L^2(0,\bar{t};L^6)}\|\partial_t q\|_{L^2(0,\bar{t};L^2)}\\
&\quad \delta\|\partial_t q\|^2_{L^2(0,\bar{t};L^2)} + C_\delta\|\overline{q}\|^2_{L^2(0,\bar{t};H^1)},\\
T_6 &\le \|\mathbb{C}'(\overline{\chi})\|_{L^\infty(L^\infty)}\|\varepsilon(\overline{u})\|_{L^\infty(L^4)}\|\varepsilon(\overline{p})\|_{L^2(0,\bar{t};L^2)}\|\partial_t q\|_{L^2(0,\bar{t};L^4)}\\
&\quad \le \delta\|\partial_t q\|^2_{L^2(0,\bar{t};H^1)} + C_\delta\|\varepsilon(\overline{p})\|^2_{L^2(0,\bar{t};L^2)},\\
T_7 &\le \|f''(\overline{\chi})\|_{L^\infty(L^\infty)}\|\overline{q}\|_{L^2(0,\bar{t};L^2)}\|\partial_t q\|_{L^2(0,\bar{t};L^2)}\\
&\quad \delta\|\partial_t q\|^2_{L^2(0,\bar{t};L^2)} + C_\delta\|\overline{q}\|^2_{L^2(0,\bar{t};L^2)}.
\end{aligned}
$$

All in all we obtain by adding the inequalities in (3.28)−(3.29), applying above estimates and readjusting the constants (δ, C_δ)

$$
\begin{aligned}
&\|\partial_t p(t)\|^2_{L^2} + \|\varepsilon(\partial_t p)\|^2_{L^2(0,\bar{t};L^2)} + \|\nabla\overline{q}(t)\|^2_{L^2} + \|\partial_t q\|^2_{L^2(0,\bar{t};H^1)}\\
&\quad \le \delta\left(\|\varepsilon(\partial_t p)\|^2_{L^2(0,\bar{t};L^2)} + \|\partial_t q\|^2_{L^2(0,\bar{t};H^1)}\right) + C_\delta\left(\|\varepsilon(\overline{p})\|^2_{L^2(0,\bar{t};L^2)} + \|\underline{q}\|^2_{L^2(0,\bar{t};H^1)}\right).
\end{aligned}
$$

Furthermore, observe that

$$
\|\underline{q}\|^2_{L^2(0,\bar{t};H^1)} \le C\left(\|q(0)\|^2_{H^1} + \int_0^{\bar{t}}\|\partial_t q\|^2_{L^2(0,s;H^1)}\,\mathrm{d}s\right),
$$

$$
\|\varepsilon(\overline{p})\|^2_{L^2(0,\bar{t};L^2)} \le C\left(\|\varepsilon(\overline{p}(0))\|^2_{L^2} + \int_0^{\bar{t}}\|\varepsilon(\partial_t p)\|^2_{L^2(0,s;L^2)}\,\mathrm{d}s\right).
$$

By means of Korn's inequality and Gronwall's lemma we find the *a priori* estimates

$$
\|p_\tau\|_{W^{1,\infty}(0,T;L^2)\cap H^1(0,T;H^1)} + \|q_\tau\|_{H^1(0,T;H^1)} \le C.
$$

By using (3.25) and the properties of $\mathbb{C}$ and $\mathbb{D}$ we also get

$$
\begin{aligned}
\|\partial_t w\|_{(H^1)^*} &= \sup_{\phi\in H^1,\|\phi\|_{H^1}=1}\langle\partial_t w,\phi\rangle_{H^1}\\
&= \sup_{\phi\in H^1,\|\phi\|_{H^1}=1}\int_\Omega\Big(\mathbb{C}(\overline{\chi})\varepsilon(\overline{p}) + \mathbb{C}'(\overline{\chi})\varepsilon(\overline{u})\underline{q} + \mathbb{D}\varepsilon(\partial_t p)\Big):\varepsilon(\phi)\,\mathrm{d}x\\
&\le \sup_{\phi\in H^1,\|\phi\|_{H^1}=1}\|\mathbb{C}(\overline{\chi})\varepsilon(\overline{p}) + \mathbb{C}'(\overline{\chi})\varepsilon(\overline{u})\underline{q} + \mathbb{D}\varepsilon(\partial_t p)\|_{L^2}\|\varepsilon(\phi)\|_{L^2}\\
&\le C\big(\|\varepsilon(\overline{p})\|_{L^2} + \|\varepsilon(\overline{u})\|^2_{L^4} + \|\underline{q}\|^2_{L^4} + \|\varepsilon(\partial_t p)\|_{L^2}\big).
\end{aligned}
$$

Integrating over time and using the *a priori* estimates above yields

$$\|\partial_t w_\tau\|_{L^2(0,T;(H^1)^*)} \leq C. \qquad \square$$

Now, extracting weakly convergent subsequences we may pass to the limit as $\tau \downarrow 0$ in (3.25)–(3.26) and obtain $(p,q) \in \dot{\mathcal{Q}}$ fulfilling

$$\langle \partial_{tt} p, \varphi\rangle_{H^1} + \int_\Omega \big(\mathbb{C}(\chi)\varepsilon(p) + \mathbb{C}'(\chi)\varepsilon(u)q + \mathbb{D}\varepsilon(\partial_t p)\big) : \varepsilon(\varphi)\,\mathrm{d}x = 0 \tag{3.30}$$

$$\begin{aligned}\int_\Omega &(\partial_t q)\psi + \xi'(-\partial_t\chi_\alpha)(\partial_t q)\psi - \xi_\beta''(-\partial_t\chi_\alpha)(\partial_{tt}\chi_\alpha)q\psi + \nabla\partial_t q\cdot\nabla\psi + \nabla q\cdot\nabla\psi \\ &+ \frac{1}{2}\mathbb{C}''(\chi)\varepsilon(u):\varepsilon(u)q\psi + \mathbb{C}'(\chi)\varepsilon(u):\varepsilon(p)\psi + f''(\chi)q\psi\,\mathrm{d}x = 0\end{aligned} \tag{3.31}$$

for a.e. $t \in (0,T)$, all $(\varphi,\psi) \in H^1(\Omega;\mathbb{R}^n)\times H^1(\Omega)$ and with the initial conditions $p(0) = p_t(0) = 0$ and $q(0) = q^0$ satisfying (3.27). Here we have make use of $w = \partial_t p$ and thus $\partial_t w = \partial_{tt} p$ in $H^1(\Omega;\mathbb{R}^n)^*$.

Step 4. *a priori* estimates for the α-regularized time-continuous system and limit passage $\alpha \downarrow 0$

In this step we are going to derive certain weak *a priori* estimates uniformly in α. In preparation of the corresponding result we prove a technical Lemma.

Lemma 3.9. *Let $s, t \in [0,T]$ be given and define*

$$\hat{p}^t_\alpha(s) := \begin{cases} 0 & \text{if} \quad s\in[t,T],\\ \int_s^t p_\alpha(\tau)\mathrm{d}\tau & \text{if} \quad s\in[0,t),\end{cases} \qquad \hat{q}^t_\alpha(s) := \begin{cases} 0 & \text{if} \quad s\in[t,T],\\ \int_s^t q_\alpha(\tau)\mathrm{d}\tau & \text{if} \quad s\in[0,t).\end{cases} \tag{3.32}$$

Then, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\hat{q}^t_\alpha(0)\|^2_{H^1}\Big) = 2\int_\Omega \Big(\hat{q}^t_\alpha(0)q_\alpha(t) + \nabla\hat{q}^t_\alpha(0)\cdot\nabla q_\alpha(t)\Big)\,\mathrm{d}x.$$

Similarly for $\hat{p}^t_\alpha$.

Proof. In order to differentiate the parametrized integral

$$t \mapsto \int_\Omega f(x,t)\,\mathrm{d}x \quad \text{with} \quad f(x,t) := |\hat{q}^t_\alpha(x,0)|^2 + |\nabla\hat{q}^t_\alpha(x,0)|^2$$

we apply *e.g.* ([6], 5.7 Satz – Zusatz (Differentiation unter dem Integralzeichen)) and check the following properties by noticing that $q_\alpha \in H^1(0,T;H^1(\Omega))$:

- For every $t\in[0,T]$ the function $f(\cdot,t)$ is in $L^1(\Omega)$.
- For a.e. $x\in\Omega$ and $t\in(0,T)$ the function $\partial_t f(x,t)$ is differentiable with respect to t and for the derivative we obtain

$$\partial_s f(x,t) = 2\hat{q}^t_\alpha(x,0)q_\alpha(x,t) + 2\nabla\hat{q}^t_\alpha(x,0)\cdot\nabla q_\alpha(x,t).$$

- Boundedness of the partial derivative:

$$\begin{aligned}|\partial_t f(x,t)| &\leq 2\|q_\alpha(x,\cdot)\|_{L^1(0,T)}\|q_\alpha(x,\cdot)\|_{L^\infty(0,T)} + 2\|\nabla q_\alpha(x,\cdot)\|_{L^1(0,T)}\|\nabla q_\alpha(x,\cdot)\|_{L^\infty(0,T)}\\ &\leq C\|q_\alpha(x,\cdot)\|_{L^1(0,T)}\|q_\alpha(x,\cdot)\|_{H^1(0,T)} + C\|\nabla q_\alpha(x,\cdot)\|_{L^1(0,T)}\|\nabla q_\alpha(x,\cdot)\|_{H^1(0,T)}. \qquad \square\end{aligned}$$

Lemma 3.10. *There exists a constant $C > 0$ independent of α such that*

$$\|(p_\alpha, q_\alpha)\|_{\overline{\mathcal{Q}}} \leq C.$$

Proof. To this end, let $(p_\alpha, q_\alpha) \in \dot{\mathcal{Q}}$ be a solution of (3.30)–(3.31) for $\alpha > 0$ as proven in step 2. Integrating (3.30)–(3.31) in time, applying integration by part and using the initial conditions yield

$$\int_0^T \int_\Omega -\partial_t p_\alpha \cdot \partial_t \varphi + \big(\mathbb{C}(\chi_\alpha)\varepsilon(p_\alpha) + \mathbb{C}'(\chi_\alpha)\varepsilon(u) q_\alpha + \mathbb{D}\varepsilon(\partial_t p_\alpha)\big) : \varepsilon(\varphi) \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{3.33}$$

$$\begin{aligned} &\int_0^T \int_\Omega -q_\alpha \partial_t \psi - \xi'(-\partial_t \chi_\alpha) q_\alpha \partial_t \psi - \nabla q_\alpha \cdot \nabla \partial_t \psi + \nabla q_\alpha \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \frac{1}{2} \mathbb{C}''(\chi_\alpha)\varepsilon(u) : \varepsilon(u) q_\alpha \psi + \mathbb{C}'(\chi)\varepsilon(u) : \varepsilon(p_\alpha)\psi + f''(\chi) q_\alpha \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_\Omega \lambda_T \big(\chi_\alpha(T) - \chi_T\big)\psi(0) \, \mathrm{d}x \end{aligned} \tag{3.34}$$

for all $(\varphi, \psi) \in \dot{\mathcal{Q}}$ with $\varphi(T) = 0$ and $\psi(T) = 0$.

Testing (3.33) with $\hat{p}_\alpha^t$ and (3.34) with $\hat{q}_\alpha^t$ and noticing $p_\alpha = -\partial_t \hat{p}_\alpha^t$ and $q_\alpha = -\partial_t \hat{q}_\alpha^t$ as well as the initial and final-time conditions $p_\alpha(0) = 0$ and $\hat{p}_\alpha^t(T) = 0$, we obtain after integration by parts

$$\int_0^t \int_\Omega \partial_s p_\alpha \cdot p_\alpha + \mathbb{C}(\chi_\alpha)\varepsilon(p_\alpha) : \varepsilon(\hat{p}_\alpha^t) + \mathbb{C}'(\chi_\alpha)\varepsilon(u) q_\alpha : \varepsilon(\hat{p}_\alpha^t) + \mathbb{D}\varepsilon(p_\alpha) : \varepsilon(p_\alpha) \, \mathrm{d}x \, \mathrm{d}s = 0,$$

$$\begin{aligned} &\int_0^t \int_\Omega |q_\alpha|^2 + \xi'(-\partial_s \chi_\alpha)|q_\alpha|^2 + |\nabla q_\alpha|^2 + \nabla(-\partial_s \hat{q}_\alpha^t) \cdot \nabla \hat{q}_\alpha^t + \frac{1}{2}\mathbb{C}''(\chi_\alpha)\varepsilon(u) : \varepsilon(u) q_\alpha \hat{q}_\alpha^t \, \mathrm{d}x \, \mathrm{d}s \\ &\quad + \int_0^t \int_\Omega \mathbb{C}'(\chi_\alpha)\varepsilon(u) : \varepsilon(p_\alpha)\hat{q}_\alpha^t + f''(\chi_\alpha) q_\alpha \hat{q}_\alpha^t \, \mathrm{d}x \, \mathrm{d}s = \int_\Omega \lambda_T \big(\chi_\alpha(T) - \chi_T\big)\hat{q}_\alpha^t(0) \, \mathrm{d}x. \end{aligned}$$

Adding these equations, using $\xi' \geq 0$ (see (A3)) and $q_\alpha^t(T) = 0$ and applying further standard estimates yield

$$\begin{aligned} &\|p_\alpha(t)\|_{L^2}^2 + c\|\varepsilon(p_\alpha)\|_{L^2(0,t;L^2)}^2 + \|q_\alpha\|_{L^2(0,t;H^1)}^2 + \|\nabla \hat{q}_\alpha^t(0)\|_{L^2}^2 \\ &\quad \leq \underbrace{-\int_0^t \int_\Omega \mathbb{C}(\chi_\alpha)\varepsilon(p_\alpha) : \varepsilon(\hat{p}_\alpha^t) \, \mathrm{d}x \, \mathrm{d}s}_{=:T_1} \underbrace{- \int_0^t \int_\Omega \mathbb{C}'(\chi_\alpha)\varepsilon(u) q_\alpha : \varepsilon(\hat{p}_\alpha^t) \, \mathrm{d}x \, \mathrm{d}s}_{=:T_2} \\ &\quad \quad \underbrace{- \int_0^t \int_\Omega \frac{1}{2}\mathbb{C}''(\chi_\alpha)\varepsilon(u) : \varepsilon(u) q_\alpha \hat{q}_\alpha^t \, \mathrm{d}x \, \mathrm{d}s}_{=:T_3} \underbrace{- \int_0^t \int_\Omega \mathbb{C}'(\chi_\alpha)\varepsilon(u) : \varepsilon(p_\alpha)\hat{q}_\alpha^t \, \mathrm{d}x \, \mathrm{d}s}_{=:T_4} \\ &\quad \quad \underbrace{- \int_0^t \int_\Omega f''(\chi_\alpha) q_\alpha \hat{q}_\alpha^t \, \mathrm{d}x \, \mathrm{d}s}_{=:T_5} + \underbrace{\int_\Omega \lambda_T \big(\chi_\alpha(T) - \chi_T\big)\hat{q}_\alpha^t(0) \, \mathrm{d}x}_{=:T_6}. \end{aligned} \tag{3.35}$$

We obtain by standard calculations

$$
\begin{aligned}
T_1 &\le \|\mathbb{C}(\chi_\alpha)\|_{L^\infty(L^\infty)}\|\varepsilon(p_\alpha)\|_{L^2(0,t;L^2)}\|\varepsilon(\hat{p}_\alpha^t)\|_{L^2(0,t;L^2)}\\
&\le \delta\|\varepsilon(p_\alpha)\|_{L^2(0,t;L^2)} + C_\delta\|\varepsilon(\hat{p}_\alpha^t)\|_{L^2(0,t;L^2)},\\
T_2 &\le \|\mathbb{C}'(\chi_\alpha)\|_{L^\infty(L^\infty)}\|\varepsilon(u)\|_{L^\infty(L^4)}\|q_\alpha\|_{L^2(0,t;L^4)}\|\varepsilon(\hat{p}_\alpha^t)\|_{L^2(0,t;L^2)}\\
&\le \delta\|q_\alpha\|^2_{L^2(0,t;H^1)} + C_\delta\|\varepsilon(\hat{p}_\alpha^t)\|^2_{L^2(0,t;L^2)},\\
T_3 &\le \frac{1}{2}\|\mathbb{C}''(\chi_\alpha)\|_{L^\infty(L^\infty)}\|\varepsilon(u)\|^2_{L^\infty(L^4)}\|q_\alpha\|_{L^2(0,t;L^4)}\|\hat{q}_\alpha^t\|_{L^2(0,t;L^4)}\\
&\le \delta\|q_\alpha\|^2_{L^2(0,t;H^1)} + C_\delta\|\hat{q}_\alpha^t\|^2_{L^2(0,t;H^1)},\\
T_4 &\le \|\mathbb{C}'(\chi_\alpha)\|_{L^\infty(L^\infty)}\|\varepsilon(u)\|_{L^\infty(L^4)}\|\varepsilon(p_\alpha)\|_{L^2(0,t;L^2)}\|\hat{q}_\alpha^t\|_{L^2(0,t;L^4)}\\
&\le \delta\|\varepsilon(p_\alpha)\|^2_{L^2(0,t;L^2)} + C_\delta\|\hat{q}_\alpha^t\|^2_{L^2(0,t;H^1)},\\
T_5 &\le \|f''(\chi_\alpha)\|_{L^\infty(L^\infty)}\|q_\alpha\|_{L^2(0,t;L^2)}\|\hat{q}_\alpha^t\|_{L^2(0,t;L^2)}\\
&\le \delta\|q_\alpha\|^2_{L^2(0,t;L^2)} + C_\delta\|\hat{q}_\alpha^t\|^2_{L^2(0,t;L^2)},\\
T_6 &\le \lambda_T\|\chi_\alpha(T)-\chi_T\|_{L^2}\|\hat{q}_\alpha^t(0)\|_{L^2},\\
&\le \delta\|q_\alpha\|^2_{L^2(0,t;L^2)} + C_\delta\|\chi_\alpha(T)-\chi_T\|^2_{L^2}.
\end{aligned}
$$

We observe that it will be indispensable to absorb the terms $\|\hat{q}_\alpha^t\|^2_{L^2(0,t;H^1)}$ and $\|\varepsilon(\hat{p}_\alpha^t)\|^2_{L^2(0,t;L^2)}$ by terms on the left-hand side in (3.35). To this end, we notice that by definition of $\hat{q}_\alpha^t$ we have

$$
\begin{aligned}
\|\hat{q}_\alpha^t\|^2_{L^2(0,t;H^1)} &= \|\hat{q}_\alpha^t(0) - \int_0^s q_\alpha(\tau)\,\mathrm{d}\tau\|^2_{L^2(0,t;H^1)}\\
&\le C\|\hat{q}_\alpha^t(0)\|^2_{L^2(0,t;H^1)} + C\int_0^t \|q_\alpha\|^2_{L^2(0,s;H^1)}.
\end{aligned}
\tag{3.36}
$$

The first term on the right-hand side of (3.36) is treated by a tricky calculations using Lemma 3.9:

$$
\begin{aligned}
&\|\hat{q}_\alpha^t(0)\|^2_{L^2(0,t;H^1)}\\
&= \int_0^t \|\hat{q}_\alpha^t(0)\|^2_{H^1}\,\mathrm{d}s = t\|\hat{q}_\alpha^t(0)\|^2_{H^1} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Big(s\|\hat{q}_\alpha^s(0)\|^2_{H^1}\Big)\,\mathrm{d}s\\
&= \int_0^t \|\hat{q}_\alpha^s(0)\|^2_{H^1}\,\mathrm{d}s + \int_0^t s\frac{\mathrm{d}}{\mathrm{d}s}\Big(\|\hat{q}_\alpha^s(0)\|^2_{H^1}\Big)\,\mathrm{d}s\\
&= \int_0^t \|\hat{q}_\alpha^s(0)\|^2_{H^1}\,\mathrm{d}s + \int_0^t 2s\int_\Omega\Big(\hat{q}_\alpha^s(0)q_\alpha(s) + \nabla\hat{q}_\alpha^s(0)\cdot\nabla q_\alpha(s)\,\mathrm{d}x\Big)\,\mathrm{d}s\\
&\le \int_0^t \|q_\alpha\|^2_{L^2(0,s;H^1)}\,\mathrm{d}s + 2T\int_0^t\int_\Omega C_\delta|\hat{q}_\alpha^s(0)|^2 + \delta|q_\alpha(s)|^2 + C_\delta|\nabla\hat{q}_\alpha^s(0)|^2 + \delta|\nabla q_\alpha(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&= \delta 2T\|q_\alpha\|^2_{L^2(0,t;H^1)} + (C_\delta 2T+1)\int_0^t\int_0^s \|q_\alpha\|^2_{H^1}\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

This yields with (3.36) the crucial estimate

$$\|\hat{q}_\alpha^t\|^2_{L^2(0,t;H^1)} \le \delta\|q_\alpha\|^2_{L^2(0,t;H^1)} + C_\delta \int_0^t \|q_\alpha\|^2_{L^2(0,s;H^1)}.$$

Analogously,

$$\|\varepsilon(\hat{p}^t)\|^2_{L^2(0,t;L^2)} \le \delta\|\varepsilon(p_\alpha)\|^2_{L^2(0,t;L^2)} + C_\delta \int_0^t \|\varepsilon(p_\alpha)\|^2_{L^2(0,s;L^2)}.$$

By using these estimates and the estimates for $T_1, \ldots, T_6$ we obtain from (3.35)

$$\begin{aligned}&\|p_\alpha(t)\|^2_{L^2} + \|\varepsilon(p_\alpha)\|^2_{L^2(0,t;L^2)} + \|q_\alpha\|^2_{L^2(0,t;H^1)}\\ &\quad\le C\int_0^t \left(\|\varepsilon(p_\alpha)\|^2_{L^2(0,s;L^2)} + \|q_\alpha\|^2_{L^2(0,s;H^1)}\right)\mathrm{d}s + C\|\chi_\alpha(T)-\chi_T\|^2_{L^2}.\end{aligned} \tag{3.37}$$

Thus Lemma 3.10 is proven after using Gronwall's lemma. □

The assertion of Proposition 3.5 can now easily be obtained by exploiting the *a priori* estimate of the α-regularized adjoint system from Lemma 3.10. Due to the linearity of the PDE system (3.30)−(3.31) we can pass to the limit $\alpha \downarrow 0$. Thus Proposition 3.5 is proven.

3.3. Derivation of a first-order optimality system

This last part of the section is devoted to collect the results from below in order to prove our main result, namely a necessary optimality system for minimizers of (CP). For reader's convenience we summarize the approach to solve this problem.

From now on we assume that (A1)−(A5), (O1)−(O3) and (B1)−(B4) hold. Let us introduce the so-called "reduced cost functional" given by

$j : \mathcal{B} \to \mathbb{R}$ defined by $j(b) := \mathcal{J}(\mathcal{S}_2(b), b)$

with the cost functional

$\mathcal{J} : \dot{\mathcal{X}} \times \mathcal{B} \to \mathbb{R}$ defined by (1.1)

and the control-to-state operator

$\mathcal{S} : \mathcal{B} \to \dot{\mathcal{U}} \times \dot{\mathcal{X}}$ defined by $\mathcal{S}(b) = (\mathcal{S}_1(b), \mathcal{S}_2(b)) := (u(b), \chi(b))$ solving PDE system (1.3)−(1.6).

Our optimal control problem (CP) can now be restated as

$$(CP') \quad \text{find a minimizer of } j \text{ over } \mathcal{B}_{\mathrm{adm}}.$$

Theorem 2.4 guarantees existence of minimizers to (CP). Let b such a minimizer. We know that $\mathcal{J}$ is Fréchet differentiable and from Proposition 3.3 that $\mathcal{S}$ is Gâteaux-differentiable. Thus j is also Gâteaux-differentiable. Since $\mathcal{B}_{\mathrm{adm}}$ is a bounded, closed and convex subset of $\mathcal{B}$ (see (B4)), the desired necessary condition for optimality is

$$\langle Dj(b), \widehat{b} - b\rangle_{\mathcal{B}} \ge 0 \quad \text{for every } \widehat{b} \in \mathcal{B}_{\mathrm{adm}}. \tag{3.38}$$

Application of the chain rule yields (see (3.10) with $h = \widehat{b} - b$)

$$\left\langle \partial_\chi \mathcal{J}(\mathcal{S}_2(b), b), D\mathcal{S}_2(b)[\widehat{b} - b]\right\rangle_{\dot{\mathcal{X}}} + \left\langle \partial_b \mathcal{J}(\mathcal{S}_2(b), b), \widehat{b} - b\right\rangle_{\mathcal{B}} \ge 0. \tag{3.39}$$

Let (p,q) be a solution of the adjoint problem at $(u,\chi) := \mathcal{S}(b)$ according to Proposition 3.5. Testing equation (3.14) with the admissible pair of test-functions

$$(\varphi,\psi) = (D\mathcal{S}_1(b)[\widehat{b}-b], D\mathcal{S}_2(b)[\widehat{b}-b]) =: (\dot{u},\dot{\chi}) \in \dot{\mathcal{Q}},$$

we can rewrite the first term in (3.39) as the left-hand side of (3.14) tested with $(\dot{u},\dot{\chi})$. Then testing the PDE system for $(\dot{u},\dot{\chi})$ in Proposition 3.1 with the admissible pair of test-functions $(p,q) \in \overline{\mathcal{Q}}$ and adding the resulting equations we end up with (see (3.11)–(3.12))

$$\left\langle \partial_\chi \mathcal{J}(\mathcal{S}_2(b),b), D\mathcal{S}_2(b)[\widehat{b}-b]\right\rangle_{\dot{\mathcal{X}}} = \int_\Sigma p\cdot(\widehat{b}-b)\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore (3.39) is equivalent to

$$\int_\Sigma (p+\lambda_\Sigma b)\cdot(\widehat{b}-b)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \text{ for every } \widehat{b}\in\mathcal{B}_{\mathrm{adm}}. \tag{3.40}$$

In conclusion we have proven the following result:

Theorem 3.11. *Suppose that* (A1)–(A5), (O1)–(O3) *and* (B1)–(B4) *hold. Let* $b \in \mathcal{B}_{\mathrm{adm}}$ *be an optimal control of (CP) with the associated state* $(u,\chi) = \mathcal{S}(b)$ *and some pair of adjoint variables* $(p,q) \in \overline{\mathcal{Q}}$ *that solves the system* (3.15)–(3.16) *according to Proposition* 3.5*. Then* (3.40) *holds.*

4. Conclusion and perspectives

In our preceding work [7] we have proven well-posedness of strong solutions of the state system (1.2)–(1.6) and existence of optimal controls for (CP). Based on these results we have established first-order optimality conditions in this paper. The main result is stated in Theorem 3.11 and provides a basis for further investigations. We conclude our paper with some open problems that could be addressed in future works.

- *Irreversibility condition.* As pointed out in the introduction damage models usually contains a so-called irreversibility condition which is realized *via* the sub-differential term $\partial I_{(-\infty,0]}(\chi_t)$ in the damage law. Equivalently we may introduce a slack variable ζ and write the damage law as

  $$\chi_t + \zeta - \Delta\chi_t - \Delta\chi + \frac{1}{2}\mathbb{C}'(\chi)\varepsilon(u):\varepsilon(u) + f'(\chi) = 0$$

  together with the complementarity conditions

  $$\chi_t\cdot\zeta = 0, \quad \zeta \geq 0, \quad \chi_t \leq 0.$$

  The corresponding optimal control problem then becomes a difficult and unexplored mathematical program with complementarity constraints (MPCC) and it remains open if stationarity conditions can be obtained *via* a limit passage of the regularized version as considered in this paper. As pointed out in [1] (see [7] for our case) optima of the regularized control problem approximate solutions of the MPCC.
- *Different cost functionals.* We have considered an L^2-tracking type cost functional in (CP) since this work is focused on the treatment of a complex and nonlinear state system. This restricts possible applications because (smooth approximations of) cracks only give rise to a small contribution with respect to the L^2-norm. More realistic choices would be the usage of higher-order or even L^∞-cost functionals.
- *Damage-dependent viscosities.* It would be desirable to let not only the stiffness tensor $\mathbb{C}(\cdot)$ but also the viscosity tensor $\mathbb{D}$ in the force balance equation (1.2) to depend on the damage phase-field χ. Existence of strong solutions has already been proven in [7] and well-posedness is also expected for this case. However optimality conditions for the optimal control problem still need to be shown.

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