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A RISK-SENSITIVE MAXIMUM PRINCIPLE FOR A MARKOV REGIME-SWITCHING JUMP-DIFFUSION SYSTEM AND APPLICATIONS

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Abstract. In this paper, we derive a general stochastic maximum principle for a risk-sensitive type optimal control problem of Markov regime-switching jump-diffusion model. The results are obtained *via* a logarithmic transformation and the relationship between adjoint variables and the value function. We apply the results to study both a linear-quadratic optimal control problem and a risk-sensitive benchmarked asset management problem for Markov regime-switching models. In the latter case, the optimal control is of feedback form and is given in terms of solutions to a Markov regime-switching Riccati equation and an ordinary Markov regime-switching differential equation.

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1. Introduction

Applications of Markov regime-switching models in finance and stochastic optimal control have received significant attention recently. Comparing to the traditional models based on the diffusion processes, Markov regime-switching models perform better from the empirical point of view. The basic idea of regime-switching is to modulate the model with a continuous time, finite-state Markov chain where each state represents a regime of the system or level of economic indicator. For example, in the stock market, the up-trend volatility of a stock tends to be smaller than its down-trend volatility (see [24]). Therefore, it is reasonable to describe the market trends by a two states Markov chain. Risk-sensitive control is a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization. There have been many works on risk-sensitive control and its applications to finance; see, for example, [2, 7, 8, 17] and references therein.

There are two main approaches to solve risk-sensitive control problems: the dynamic programming principle and the stochastic maximum principle. Although dynamic programming principle has been the tool predominantly used to study such problems, several papers have been devoted to the stochastic maximum principle, a set of necessary and sufficient conditions that must be satisfied by any optimal control. In [23], the author uses

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a heuristic approach based on large deviation theory to derive a maximum principle. In [3], a measure-valued decomposition and weak control variations are used to obtain a minimum principle for the partial observation problem. In the above mentioned works, the diffusion coefficients are independent of the control variable. The work [14] generalizes the above results by including the control variable in the diffusion coefficients. The paper [19] extends the previous results assuming that the dynamics of the controlled system is modeled by a jump-diffusion process.

There have been a lot of works on classical stochastic optimal control for regime-switching models. The stochastic maximum principle approach to solve this problem was introduced in [9, 10] for the diffusion case and generalized in [25] to the jump-diffusion case. The work [13] proposes a weak maximum principle when the Hamiltonian is non-convex whereas the works [15, 20, 21] study optimal control for Markov regime-switching forward-backward stochastic differential equations. Let us also mention the paper [16] that deals with maximum principle for Markov regime-switching forward-backward stochastic differential equations. Note, however, that the maximum principle proposed in the above works are only of local form and the authors do not consider the second order adjoint processes and adjoint equations. To our best knowledge so far, it appears that the risk-sensitive maximum principle in global form for Markov regime-switching jump-diffusion processes has not yet been established and this is the main goal of the paper.

In this paper, we derive a general risk-sensitive stochastic maximum principle assuming that the state is governed by a continuous time Markov regime-switching jump-diffusion process and the cost functional is of exponential-of-integral type. Following the idea in [14,19], we first reformulate the risk-sensitive control problem as a risk-neutral one by adding a new variable to the state process. Combining the results in [22, 25], an intermediate stochastic maximum principle in global form is obtained by introducing the second order adjoint variable. This intermediate result is not satisfactory because it is more complex than its risk-neutral counterpart (compare with [25], Thm. 3.1). Using a logarithmic transformation and relationship between adjoint variables and the value function, we simplify the adjoint equations and derive a general risk-sensitive maximum principle. Hence the risk-sensitive maximum principle we obtained is of global form. Note that, the adjoint equations and the maximum condition heavily depend on the risk-sensitive parameter and two jump proportional processes (see (3.4) and (3.5)). This shows the fundamental difference between the two cases of diffusion and the jump-diffusion controlled processes. Contrary to any conventional maximum principle, the first order adjoint equation is no longer a linear backward stochastic differential equation. Under additional convexity condition on the Hamiltonian, we show that the maximum condition is also sufficient.

The other motivation of this paper is to solve a risk-sensitive benchmarked asset management problem of a firm when the stock is modeled by both a Markov regime-switching diffusion process and an external factor. It is also assumed that the benchmark depends on the economic factor. Such problem was considered in [6] for the classical factor model without regime-switching and using a dynamic programming approach. In this paper, we use the risk-sensitive maximum principle obtained to find the optimal portfolio strategy that minimizes the risk sensitivity of an investor in such environment. This portfolio strategy is given in a feedback form and depends on the solution to a regime-switching Riccati equation. We also apply our main results to solve a linear-quadratic optimal control problem.

It is worth mentioning that for the risk-sensitive control problem under Markov regime-switching models, there is an additional risk-sensitive parameter and a set of jump martingales related to the Markov chain that appear in adjoint equations. Thus it is not straightforward to give the variational inequality and the maximum condition. Our results generalize those in [14, 19] by including Markov regime-switching and those in [9, 10, 25] by taking into consideration the risk-sensitive parameter. In addition, the accession of jumps in the model make the problem more difficult to deal with, not only for the non-linearity of the first order adjoint equation, but also for the extra jump proportional processes in the Hamiltonian. To the authors' knowledge, this paper is the first attempt to apply a risk-sensitive maximum principle to solve a risk-sensitive benchmarked asset management problem under a Markov regime-switching model.

The paper is organized as follows. In Section 2, we present the optimal control problem and our assumptions. In Section 3, we give statement of the risk-sensitive maximum principle. For the sake of readability, the proofs of the results are given in Section 4. In Section 5, we use the results obtained to solve a risk-sensitive linear-quadratic problem. A risk-sensitive benchmarked asset management problem with Markov regime-switching is also studied. Finally, Section 6 provides our conclusions and proposes some potential extensions of our work.

In the rest of our paper, we shall adopt the following notations.

$M^\top$: the transpose of any vector or matrix M;
$\langle x, y\rangle$: the inner product of $x, y \in \mathbb{R}^L$, that is $\langle x, y\rangle := x^\top y$;
$|x|$: the norm in the Euclidean space.

2. Formulation of the optimal control problems

Let $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete filtered probability space and $T > 0$ be a finite-time horizon. The filtration $\mathbb{F}$ is a right-continuous, P-completed, filtration to which all of the processes defined below including the Markov chain, the Brownian motions and the Poisson random measures are adapted.

We consider an irreducible homogeneous continuous time Markov chain $\{\alpha(t), t \in [0, T]\}$ with a finite-state space $S := \{e_1, e_2, \ldots, e_D\}$, where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$ and the jth component of e_i is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, D$. Here, we denote by $\mathbb{R}$ the set of real numbers and $\mathbb{N}$ the set of natural numbers. The Markov chain is characterized by its Q-matrix $\Lambda := [\lambda_{ij}]_{i,j=1,2,\ldots,D}$ under P. Here, for each $i, j = 1, 2, \ldots, D$, λ_{ij} is the transition intensity of the chain from state e_i to state e_j at time t. Note that $\lambda_{ij} \geq 0$, for $i \neq j$ and $\sum_{j=1}^D \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows, for each $i, j = 1, 2, \ldots, D$ with $i \neq j$, we suppose that $\lambda_{ij} > 0$, and so $\lambda_{ii} < 0$.

It follows from [11] that the following semimartingale representation of the Markov chain $\{\alpha(t), t \in [0, T]\}$ holds

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^\top \alpha(s)\mathrm{d}s + M(t), \tag{2.1}$$

where $\{M(t), t \in [0, T]\}$ is a $\mathbb{R}^D$-valued, $(\mathbb{F}, P)$-martingale.

Now, let us introduce a set of Markov jump martingales associated with the chain α, which will be used to model the controlled state process. For each $i, j = 1, 2, \ldots, D$, with $i \neq j$, and $t \in [0, T]$, let $J^{ij}(t)$ be the number of jumps from state e_i to state e_j up to time t. One can show, using the results in [11] that

$$J^{ij}(t) = \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle \,\mathrm{d}s + m_{ij}(t), \tag{2.2}$$

where $m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i\rangle \langle \mathrm{d}M(s), e_j\rangle$ is a $(\mathbb{F}, P)$-martingale.

For each fixed $j = 1, 2, \ldots, D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t. Then

$$\begin{aligned}\Phi_j(t) := \sum_{i=1, i\neq j}^{D} J^{ij}(t) &= \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle \,\mathrm{d}s + \widetilde{\Phi}_j(t)\\ &= \lambda_j(t) + \widetilde{\Phi}_j(t),\end{aligned} \tag{2.3}$$

where $\widetilde{\Phi}_j(t) := \sum_{i=1, i\neq j}^{D} m_{ij}(t)$ and $\lambda_j(t) := \sum_{i=1, i\neq j}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-), e_i\rangle \,\mathrm{d}s$. Note that, for each $j = 1, 2, \ldots, D$, $\widetilde{\Phi}_j(t)$ is again a $(\mathbb{F}, P)$-martingale.

In what follows, let $L, M, N, K \in \mathbb{N}$. Suppose that $N^i(\mathrm{d}t, \mathrm{d}z), i = 1, 2, \ldots, M$, are independent Poisson random measures on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}_0))$, with $\mathbb{R}^+ := [0, +\infty)$ and $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. Here $\mathcal{B}(\mathbb{R}^+)$ and $\mathcal{B}(\mathbb{R}_0)$ are the Borel σ-fields generated by open subsets of $\mathbb{R}^+$ and $\mathbb{R}_0$ respectively. Assume that the Poisson random measure $N^i(\mathrm{d}t, \mathrm{d}z)$ has the following compensator

$$n^i_\alpha(\mathrm{d}t, \mathrm{d}z) := \nu^i_\alpha(\mathrm{d}z)\mathrm{d}t = \left\langle \alpha(t), \nu^i(\mathrm{d}z)\right\rangle \mathrm{d}t, \tag{2.4}$$

where $\nu^i(\mathrm{d}z) := (\nu^i_{e_1}(\mathrm{d}z), \nu^i_{e_2}(\mathrm{d}z), \ldots, \nu^i_{e_D}(\mathrm{d}z))^\top \in \mathbb{R}^D$. Here we use the subscript α in $n^i_\alpha, i = 1, 2, \ldots, M$ to indicate the dependence of the probability law of the Poisson random measure on the Markov chain $\alpha(t)$. Indeed, for each $j = 1, 2, \ldots, D$, $\nu^i_{e_j}(\mathrm{d}z)$ is the conditional Lévy density of jump sizes of the random measure $N^i(\mathrm{d}t, \mathrm{d}z)$ when $\alpha(t) = e_j$. Moreover, denote $\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z)$ by

$$\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z) := (N^1(\mathrm{d}t, \mathrm{d}z) - n^1_\alpha(\mathrm{d}t, \mathrm{d}z), \ldots, N^M(\mathrm{d}t, \mathrm{d}z) - n^M_\alpha(\mathrm{d}t, \mathrm{d}z))^\top. \tag{2.5}$$

Let U be a nonempty subset of $\mathbb{R}^K$ and $u : [0, T] \times \Omega \to U$ be a control process. We require that $\{u(t, \omega), t \in [0, T]\}$ to be $\mathbb{F}$-predictable and with right limits. In what follows, we set

$$\lambda(t) := (\lambda_1(t), \ldots, \lambda_D(t))^\top, \quad \nu_\alpha(\mathrm{d}z) := (\nu^1_\alpha(\mathrm{d}z), \ldots, \nu^M_\alpha(\mathrm{d}z))^\top.$$

Let $p \geq 1$, denote by $\mathcal{L}^{2p}(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu_\alpha; \mathbb{R}^M)$ the set of integrable functions $k(\cdot) : \mathbb{R}_0 \to \mathbb{R}^M$ such that

$$||k(\cdot)||^{2p}_{\mathcal{L}^{2p}} = \sum_{j=1}^M \int_{\mathbb{R}_0} |k_j(z)|^{2p} \nu^j_\alpha(\mathrm{d}z) < \infty$$

and $\mathcal{M}^{2p}(\mathbb{R}^+; \mathbb{R}^D)$ the set of functions $a(\cdot) : \mathbb{R}^+ \to \mathbb{R}^D$ such that

$$||a(t)||^{2p}_{\mathcal{M}^{2p}} = \sum_{j=1}^D |a_j(t)|^{2p} \lambda_j(t) < \infty.$$

Let $\mathcal{H}$ be a finite-dimensional vector or matrix space. We define

$$\begin{aligned}
L^2(\Omega, \mathcal{F}_T; \mathcal{H}) &:= \left\{\xi : \mathcal{H}\text{-valued } \mathcal{F}_T\text{-measurable random variables, s.t. } E[|\xi|^2] < \infty\right\};\\
L^2_{\mathcal{F}}([0, T]; \mathcal{H}) &:= \left\{f : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-adapted processes, s.t. } E\left[\int_0^T |f(t)|^2 \mathrm{d}t\right] < \infty\right\};\\
L^2_{\mathcal{F},p}([0, T]; \mathcal{H}) &:= \left\{f : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-predictable processes, s.t. } E\left[\int_0^T |f(t)|^2 \mathrm{d}t\right] < \infty\right\};\\
F^2_p([0, T]; \mathcal{H}) &:= \left\{f : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-predictable processes, s.t. } E\left[\int_0^T ||f(t, \cdot)||^2_{\mathcal{L}^2} \mathrm{d}t\right] < \infty\right\};\\
M^2_p([0, T]; \mathcal{H}) &:= \left\{f : \mathcal{H}\text{-valued } \mathcal{F}_t\text{-predictable processes, s.t. } E\left[\int_0^T ||f(t)||^2_{\mathcal{M}^2} \mathrm{d}t\right] < \infty\right\};
\end{aligned}$$

For any given initial time $s \in [0, T)$ and initial state $x \in \mathbb{R}^L$, we assume that the controlled process is given by the following controlled Markov regime-switching system with random jumps:

$$\begin{cases}
\mathrm{d}x(t) = b(t, x(t), u(t), \alpha(t))\mathrm{d}t + \sigma(t, x(t), u(t), \alpha(t))\mathrm{d}W(t)\\
\qquad\qquad + \displaystyle\int_{\mathbb{R}_0} \eta(t, x(t-), u(t-), \alpha(t-), z)\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z)\\
\qquad\qquad + \gamma(t, x(t-), u(t-), \alpha(t-))\mathrm{d}\widetilde{\Phi}(t),\\
x(s) = x, \qquad t \in [s, T].
\end{cases} \tag{2.6}$$

Here $b : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^L, \sigma : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L \times N}, \eta : [0, T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}_0 \to \mathbb{R}^{L \times M}, \gamma : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}^{L \times D}$ are given deterministic continuous, measurable functions, $W(t) := (W_1(t), \ldots, W_N(t))^\top$

is an N-dimensional standard Brownian motion, and $\widetilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z)$ is an M-dimensional Markov regime-switching random measures defined by (2.5), $\widetilde{\Phi}(t) := (\widetilde{\Phi}_1(t), \dots, \widetilde{\Phi}_D(t))^\top$ with $\widetilde{\Phi}_j(t), j = 1, \dots, D$, defined by (2.3).

The cost functional $J^\theta(s, x, e_i; u(\cdot))$ associated with the initial condition $(s, x, e_i) \in [0, T] \times \mathbb{R}^L \times S$ and control process $u(\cdot) \in U$ is given by

$$J^\theta(s, x, e_i; u(\cdot)) := E\left[\mathrm{e}^{\theta\left[g(x(T),\alpha(T))+\int_s^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\right]}\right], \tag{2.7}$$

where $f : [0, T] \times \mathbb{R}^L \times U \times S \to \mathbb{R},\ g : \mathbb{R}^L \times S \to \mathbb{R}$ are given and $\theta > 0$, the risk-sensitive parameter, is a fixed constant.

We say that u is an admissible control, if it belongs to $L^2_{\mathcal{F},p}([0, T]; \mathbb{R}^K)$ and the stochastic differential equation (2.6) has a unique strong solution. We denote by $\mathcal{U}[s, T]$ the set of all admissible controls. Our risk-sensitive stochastic control problem associated with (2.6)−(2.7) is defined as follows:

$$\begin{cases} \text{Minimize} \quad J^\theta(s, x, e_i; u(\cdot)) \\ \text{subject to:} \begin{cases} u(\cdot) \in \mathcal{U}[s, T] \\ (x(\cdot), u(\cdot)) \text{ satisfies (2.6).} \end{cases} \end{cases} \tag{2.8}$$

The value function $V^\theta : [0, T] \times \mathbb{R}^L \times S \to \mathbb{R}$ associated with (2.8) is defined by:

$$V^\theta(s, x, e_i) := \inf_{u(\cdot)\in\mathcal{U}[s,T]} J^\theta(s, x, e_i, u(\cdot)). \tag{2.9}$$

Since the exponential function is involved, we have that $V^\theta(s, x, e_i) \geq 0$ for $e_i \in S$. We will make further assumptions on the above functions.

($\mathcal{A}1$) b, σ, η, γ are uniformly Lipschitz in (x, u) and $b, \sigma, ||\eta||_{\mathcal{L}^{2p}}, ||\gamma||_{\mathcal{M}^{2p}}, p = 1, 2$ are bounded by $C(1+|x|+|u|)$;
($\mathcal{A}2$) $b, \sigma, \eta, \gamma, f, g$ are twice continuously differentiable with respect to x, they and their partial derivatives in x are continuous in (x, u);
($\mathcal{A}3$) $b_x, b_{xx}, \sigma_x, \sigma_{xx}, ||\eta_x||_{\mathcal{L}^{2p}}, ||\eta_{xx}||_{\mathcal{L}^2}, ||\gamma_x||_{\mathcal{M}^{2p}}, ||\gamma_{xx}||_{\mathcal{M}^2}, f_x, f_{xx}, p = 1, 2$ and g_x, g_{xx} are bounded;
($\mathcal{A}4$) f and g are uniformly bounded;
($\mathcal{A}5$) $V^\theta \in C^{1,3}([0, T] \times \mathbb{R}^L \times S)$;
($\mathcal{A}6$) U is a convex subset of $\mathbb{R}^K$.

In the above, ($\mathcal{A}1$)−($\mathcal{A}3$) are usual conditions for risk-neutral maximum principles. ($\mathcal{A}4$) is standard in the risk-sensitive literature to ensure the cost functional (2.7) to be well defined. Due to the relationship between the maximum principle and the dynamic programming is involved, our results depend on the value function $V^\theta(s, x.e_i)$ being sufficiently smooth (see ($\mathcal{A}5$)). For the sufficient maximum principle, we also require U to be convex (see ($\mathcal{A}6$)).

3. Statement of Risk-Sensitive maximum principle

In this section, we present a general maximum principle for the risk-sensitive control problem (2.8) and also sufficient conditions for optimality. The proofs of these results will be given in Section 4.

In what follows, for $\varphi = b, \sigma, \gamma, f$, we define:

$$\begin{cases} \bar\varphi(t) \triangleq \varphi(t, \bar x(t), \bar u(t), \alpha(t)),\ \ \bar\varphi_x(t) \triangleq \varphi_x(t, \bar x(t), \bar u(t), \alpha(t)), \\ \bar\eta(t, z) \triangleq \eta(t, \bar x(t-), \bar u(t-), \alpha(t-), z),\ \ \bar\eta_x(t, z) \triangleq \eta_x(t, \bar x(t-), \bar u(t-), \alpha(t-), z), \\ \bar\varphi_{xx}(t) \triangleq \varphi_{xx}(t, \bar x(t), \bar u(t), \alpha(t)),\ \ \bar\eta_{xx}(t, z) \triangleq \eta_{xx}(t, \bar x(t-), \bar u(t-), \alpha(t-), z), \\ \delta\varphi(t, u) \triangleq \varphi(t, \bar x(t-), u, \alpha(t-)) - \varphi(t, \bar x(t-), \bar u(t), \alpha(t-)), \\ \delta\eta(t, u, z) \triangleq \eta(t, \bar x(t-), u, \alpha(t-), z) - \eta(t, \bar x(t-), \bar u(t), \alpha(t-), z), \end{cases} \tag{3.1}$$

Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be an admissible pair for the system (2.6). Let us introduce the first order adjoint variable $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot),\bar{s}(\cdot)) \in L^2_{\mathcal{F}}([s,T];\mathbb{R}^L)\times L^2_{\mathcal{F},p}([s,T];\mathbb{R}^{L\times N})\times F^2_p([s,T];\mathbb{R}^{L\times M})\times M^2_p([s,T];\mathbb{R}^{L\times D})$ and the second order adjoint variable $(\bar{P}(\cdot),\bar{Q}(\cdot),\bar{R}(\cdot,\cdot),\bar{S}(\cdot)) \in L^2_{\mathcal{F}}([s,T];\mathbb{R}^{L\times L})\times \left(L^2_{\mathcal{F},p}([s,T];\mathbb{R}^{L\times L})\right)^N \times \left(F^2_p([s,T];\mathbb{R}^{L\times L})\right)^M \times \left(M^2_p([s,T];\mathbb{R}^{L\times L})\right)^D$ corresponding to the admissible pair $(\bar{x}(\cdot),\bar{u}(\cdot))$, which are the solutions of the following first order and second order adjoint equations respectively:

$$\begin{cases} \mathrm{d}\bar{p}(t) = -\Big\{\bar{b}_x(t)^\top\bar{p}(t) + \bar{f}_x(t) + \sum\limits_{j=1}^{N}\Big[\bar{\sigma}^j_x(t)^\top\bar{q}_j(t) + \theta\bar{p}(t)^\top\bar{\sigma}^j(t)\Big(\bar{\sigma}^j_x(t)^\top\bar{p}(t)+\bar{q}_j(t)\Big)\Big] \\ \qquad\quad + \sum\limits_{j=1}^{M}\int_{\mathbb{R}_0}\Big[\bar{\eta}^j_x(t,z)^\top\bar{r}_j(t,z) + \mathit{\Theta}_j(t,z)\Big(\bar{\eta}^j_x(t,z)^\top\bar{p}(t) \\ \qquad\quad +\bar{\eta}^j_x(t,z)^\top\bar{r}_j(t,z)+\bar{r}_j(t,z)\Big)\Big]\nu^j_\alpha(\mathrm{d}z) \\ \qquad\quad + \sum\limits_{j=1}^{D}\Big[\bar{\gamma}^j_x(t)^\top\bar{s}_j(t) + \mathit{\Lambda}_j(t)\Big(\bar{\gamma}^j_x(t)^\top\bar{p}(t)+\bar{\gamma}^j_x(t)^\top\bar{s}_j(t)+\bar{s}_j(t)\Big)\Big]\lambda_j(t)\Big\}\mathrm{d}t \\ \qquad\quad + \sum\limits_{j=1}^{N}\bar{q}_j(t)\mathrm{d}W_j(t) + \sum\limits_{j=1}^{M}\int_{\mathbb{R}_0}\bar{r}_j(t,z)\widetilde{N}^j_\alpha(\mathrm{d}t,\mathrm{d}z) + \sum\limits_{j=1}^{D}\bar{s}_j(t)\mathrm{d}\widetilde{\mathit{\Phi}}_j(t), \\ \bar{p}(T) = g_x(\bar{x}(T),\alpha(T)), \end{cases} \tag{3.2}$$

$$\begin{cases} \mathrm{d}\bar{P}(t) = -\Big\{\bar{b}_x(t)^\top\bar{P}(t) + \bar{P}(t)\bar{b}_x(t) + \sum\limits_{j=1}^{N}\Big[\bar{\sigma}^j_x(t)^\top\Big(\bar{P}(t)+\theta\bar{p}(t)\bar{p}(t)^\top\Big)\bar{\sigma}^j_x(t) \\ \qquad +\bar{\sigma}^j_x(t)^\top\Big(\bar{Q}_j(t)+\theta\bar{p}(t)^\top\bar{\sigma}^j(t)\bar{P}(t)+\theta\bar{p}(t)\bar{q}_j(t)^\top\Big) \\ \qquad +\Big(\bar{Q}_j(t)+\theta\bar{p}(t)^\top\bar{\sigma}^j(t)\bar{P}(t)+\theta\bar{q}_j(t)\bar{p}(t)^\top\Big)\bar{\sigma}^j_x(t) \\ \qquad +\theta\bar{p}(t)^\top\bar{\sigma}^j(t)\bar{Q}_j(t)+\theta\bar{q}_j(t)\bar{q}_j(t)^\top\Big] \\ \qquad + \sum\limits_{j=1}^{M}\int_{\mathbb{R}_0}\Big[\bar{\eta}^j_x(t,z)^\top\Big(1+\mathit{\Theta}_j(t,z)\Big)\Big(\bar{P}(t)+\bar{R}_j(t,z) \\ \qquad +\theta(\bar{p}(t)+\bar{r}_j(t,z))(\bar{p}(t)+\bar{r}_j(t,z))^\top\Big)\bar{\eta}^j_x(t,z) \\ \qquad +\bar{\eta}^j_x(t,z)^\top\Big[\Big(1+\mathit{\Theta}_j(t,z)\Big)\Big(\bar{P}(t)+\bar{R}_j(t,z)+\theta(\bar{p}(t)+\bar{r}_j(t,z))\bar{r}_j(t,z)^\top\Big)-\bar{P}(t)\Big] \\ \qquad +\Big[\Big(1+\mathit{\Theta}_j(t,z)\Big)\Big(\bar{P}(t)+\bar{R}_j(t,z)+\theta\bar{r}_j(t,z)(\bar{p}(t)+\bar{r}_j(t,z))^\top\Big)-\bar{P}(t)\Big]\bar{\eta}^j_x(t,z) \\ \qquad +\mathit{\Theta}_j(t,z)\bar{R}_j(t,z)+\Big(1+\mathit{\Theta}_j(t,z)\Big)\theta\bar{r}_j(t,z)\bar{r}_j(t,z)^\top\Big]\nu^j_\alpha(\mathrm{d}z) \\ \qquad + \sum\limits_{j=1}^{D}\Big[\bar{\gamma}^j_x(t)^\top\Big(1+\mathit{\Lambda}_j(t)\Big)\Big(\bar{P}(t)+\bar{S}_j(t)+\theta(\bar{p}(t)+\bar{s}_j(t))(\bar{p}(t)+\bar{s}_j(t))^\top\Big)\bar{\gamma}^j_x(t) \\ \qquad +\bar{\gamma}^j_x(t)^\top\Big[\Big(1+\mathit{\Lambda}_j(t)\Big)\Big(\bar{P}(t)+\bar{S}_j(t)+\theta(\bar{p}(t)+\bar{s}_j(t))\bar{s}_j(t)^\top\Big)-\bar{P}(t)\Big] \\ \qquad +\Big[\Big(1+\mathit{\Lambda}_j(t)\Big)\Big(\bar{P}(t)+\bar{S}_j(t)+\theta\bar{s}_j(t)(\bar{p}(t)+\bar{s}_j(t))^\top\Big)-\bar{P}(t)\Big]\bar{\gamma}^j_x(t) \\ \qquad +\mathit{\Lambda}_j(t)\bar{S}_j(t)+\Big(1+\mathit{\Lambda}_j(t)\Big)\theta\bar{s}_j(t)\bar{s}_j(t)^\top\Big]\lambda_j(t) \\ \qquad +\bar{H}^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),\alpha(t),\bar{p}(t),\bar{q}(t),\bar{r}(t,\cdot),\bar{s}(t))\Big\}\mathrm{d}t \\ \qquad + \sum\limits_{j=1}^{N}\bar{Q}_j(t)\mathrm{d}W_j(t) + \sum\limits_{j=1}^{M}\int_{\mathbb{R}_0}\bar{R}_j(t,z)\widetilde{N}^j_\alpha(\mathrm{d}t,\mathrm{d}z) + \sum\limits_{j=1}^{D}\bar{S}_j(t)d\widetilde{\mathit{\Phi}}_j(t), \\ \bar{P}(T) = g_{xx}(\bar{x}(T),\alpha(T)), \end{cases} \tag{3.3}$$

where σ^j, η^j and γ^j are the jth columns of the matrices σ, η and γ, respectively. For each $t \in [s,T],\ z \in \mathbb{R}_0$,

$$\mathit{\Theta}_j(t,z) = \frac{V^\theta(t, \bar{x}(t-) + \bar{\eta}^j(t,z), \alpha(t-)) - V^\theta(t, \bar{x}(t-), \alpha(t-))}{V^\theta(t, \bar{x}(t-), \alpha(t-))},\ j = 1, \dots, M, \tag{3.4}$$

$$\mathit{\Lambda}_j(t) = \frac{V^\theta(t, \bar{x}(t-) + \bar{\gamma}^j(t), e_j) - V^\theta(t, \bar{x}(t-), \alpha(t-))}{V^\theta(t, \bar{x}(t-), \alpha(t-))},\ j = 1, \dots D, \tag{3.5}$$

and the Hamiltonian $\bar{H}^\theta : [0,T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}^{L+1} \times \mathbb{R}^{L\times N} \times \mathcal{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu_\alpha; \mathbb{R}^{L\times M}) \times \mathcal{M}^2(\mathbb{R}^+; \mathbb{R}^{L\times D}) \to \mathbb{R}$ is defined as

$$\begin{aligned}\bar{H}^\theta(t,x,u,e_i,p,q,r,s) :=& \langle p, b(t,x,u,e_i)\rangle + f(t,x,u,e_i) + \sum_{j=1}^N \sigma^j(t,x,u,e_i)^\top \big(q_j + \theta p p^\top \bar{\sigma}^j(t)\big) \\ &+ \sum_{j=1}^M \int_{\mathbb{R}_0} \Big\langle r_j(t,z) + \mathit{\Theta}_j(t,z)\big(p + r_j(t,z)\big), \eta^j(t,x,u,e_i,z)\Big\rangle \nu_{e_i}^j(\mathrm{d}z) \\ &+ \sum_{j=1}^D \Big\langle s_j(t) + \mathit{\Lambda}_j(t)\big(p + s_j(t)\big), \gamma^j(t,x,u,e_i)\Big\rangle \lambda_j(t). \end{aligned} \tag{3.6}$$

Note that (3.2) is a nonlinear backward stochastic differential equation (BSDE), which is different from the risk-neutral case. In addition, its generator does not satisfy the classical Lipschitz condition for the existence and uniqueness of solution to nonlinear BSDEs. As shown later, our assumptions are sufficient to guaranty the existence of unique solution to (3.2) and (3.3), respectively.

Remark 3.1. A key feature of (3.2) and (3.3) is that it relies on the value function, which involves the function $\mathit{\Theta}_j(\cdot,\cdot)$ and $\mathit{\Lambda}_j(\cdot)$ defined by (3.4) and (3.5), respectively. We call $\mathit{\Theta}_j(\cdot,\cdot)$ and $\mathit{\Lambda}_j(\cdot)$ the jump proportion processes associated with the value function along with the state trajectory $\bar{x}(\cdot)$.

Define $\bar{\mathcal{H}}^\theta : [0,T] \times \mathbb{R}^L \times U \times S \to \mathbb{R}$, associated with the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, as:

$$\begin{aligned}\bar{\mathcal{H}}^\theta(t,x,u,e_i) :=& \langle \bar{p}(t-), b(t,x,u,e_i)\rangle + f(t,x,u,e_i) + \sum_{j=1}^N \Big[\sigma^j(t,x,u,e_i)^\top \big(\bar{q}_j(t) - \bar{P}(t-)\bar{\sigma}^j(t)\big) \\ &+ \frac{1}{2}\sigma^j(t,x,u,e_i)^\top \big(\bar{P}(t-) + \theta \bar{p}(t-)\bar{p}(t-)^\top\big)\sigma^j(t,x,u,e_i)\Big] \\ &+ \sum_{j=1}^M \int_{\mathbb{R}_0} \eta^j(t,x,u,e_i,z)^\top \Big[\bar{r}_j(t,z) + \mathit{\Theta}_j(t,z)\big(\bar{p}(t-) + \bar{r}_j(t,z)\big) \\ &+ \frac{1}{2}\big(1 + \mathit{\Theta}_j(t,z)\big)\big(\bar{P}(t-) + \bar{R}_j(t,z) + \theta(\bar{p}(t-) + \bar{r}_j(t,z))(\bar{p}(t-) + \bar{r}_j(t,z))^\top\big) \\ &\times \big(\eta^j(t,x,u,e_i,z) - 2\bar{\eta}^j(t,z)\big)\Big]\nu_\alpha^j(\mathrm{d}z) \\ &+ \sum_{j=1}^D \gamma^j(t,x,u,e_i)^\top \Big[\bar{s}_j(t) + \mathit{\Lambda}_j(t)\big(\bar{p}(t-) + \bar{s}_j(t)\big) \\ &+ \frac{1}{2}\big(1 + \mathit{\Lambda}_j(t)\big)\big(\bar{P}(t-) + \bar{S}_j(t) + \theta(\bar{p}(t-) + \bar{s}_j(t))(\bar{p}(t-) + \bar{s}_j(t))^\top\big) \\ &\times \big(\gamma^j(t,x,u,e_i) - 2\bar{\gamma}^j(t)\big)\Big]\lambda_j(t). \end{aligned} \tag{3.7}$$

The risk-sensitive maximum principle can be stated as follows:

Theorem 3.2 (Risk-sensitive maximum principle)**.** *Suppose that assumptions* $(\mathcal{A}1)$–$(\mathcal{A}5)$ *hold. Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be an optimal pair for the risk-sensitive control problem* (2.8)*. Then there exist processes* $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot,\cdot), \bar{s}(\cdot))$ *and* $(\bar{P}(\cdot), \bar{Q}(\cdot), \bar{R}(\cdot,\cdot), \bar{S}(\cdot))$ *satisfying the first and second order adjoint equations* (3.2) *and* (3.3)*, respectively, such that the following variational inequality holds,*

$$\begin{aligned}
&\bar{H}^\theta(t, \bar{x}(t-), u, \alpha(t-), \bar{p}(t-), \bar{q}(t), \bar{r}(t,\cdot), \bar{s}(t)) - \bar{H}^\theta(t, \bar{x}(t-), \bar{u}(t), \alpha(t-), \bar{p}(t-), \bar{q}(t), \bar{r}(t,\cdot), \bar{s}(t)) \\
&+ \frac{1}{2}\sum_{j=1}^{N} \delta\sigma^j(t,u)^\top \Big(\bar{P}(t-) + \theta\bar{p}(t-)\bar{p}(t-)^\top\Big)\delta\sigma^j(t,u) \\
&+ \frac{1}{2}\sum_{j=1}^{M} \int_{\mathbb{R}_0} \Big[\delta\eta^j(t,u,z)^\top \Big(1 + \Theta_j(t,z)\Big)\Big(\bar{P}(t-) + \bar{R}_j(t,z) \\
&+ \theta(\bar{p}(t-) + \bar{r}_j(t,z))(\bar{p}(t-) + \bar{r}_j(t,z))^\top\Big)\delta\eta^j(t,u,z)\Big]\nu_\alpha^j(\mathrm{d}z) \\
&+ \frac{1}{2}\sum_{j=1}^{D} \Big[\delta\gamma^j(t,u)^\top \Big(1 + \Lambda_j(t)\Big)\Big(\bar{P}(t-) + \bar{S}_j(t) \\
&+ \theta(\bar{p}(t-) + \bar{s}_j(t))(\bar{p}(t-) + \bar{s}_j(t))^\top\Big)\delta\gamma^j(t,u)\Big]\lambda_j(t) \geq 0 \quad \forall u \in U, \ \textit{a.e. } t \in [s,T], \ P\textit{-a.s.},
\end{aligned} \tag{3.8}$$

or equivalently

$$\bar{\mathcal{H}}^\theta(t, \bar{x}(t-), \bar{u}(t), \alpha(t-)) = \inf_{u \in U} \bar{\mathcal{H}}^\theta(t, \bar{x}(t-), u, \alpha(t-)), \quad \textit{a.e. } t \in [s,T], \ P\textit{-a.s.} \tag{3.9}$$

Remark 3.3. The above equation (3.9) expressed by the minimum over U is still called the maximum condition for the control problem just as in the classical optimal control problem.

Sufficient conditions for the optimality of the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ are given as follows:

Theorem 3.4 (Sufficient conditions for optimality)**.** *Suppose that assumptions* $(\mathcal{A}1)$–$(\mathcal{A}6)$ *hold. Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be an admissible pair, and* $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot,\cdot), \bar{s}(\cdot))$, $(\bar{P}(\cdot), \bar{Q}(\cdot), \bar{R}(\cdot,\cdot), \bar{S}(\cdot))$ *be the associated first and second order adjoint variables, respectively. Suppose that for each* $e_i \in S$, $x \mapsto g(x, e_i)$ *is convex,* $(x,u) \mapsto \bar{H}^\theta(t, x, u, e_i, \bar{p}(t-), \bar{q}(t), \bar{r}(t,\cdot), \bar{s}(t))$ *is convex for all* $t \in [0,T]$, P*-a.s., and* (3.9) *holds. Then* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is an optimal pair for the problem* (2.8)*.*

Remark 3.5.

(1) Let us mention that since the jump proportion processes $\Theta(\cdot,\cdot)$ and $\Lambda(\cdot)$ given by (3.4) and (3.5) respectively do not depend on the control variable u, they will not bring any difficulty when applying the maximum condition (3.9) to look for the optimal control.

(2) Assuming that $D = 1$ and letting $\eta = \gamma \equiv 0$ in (2.6), Theorems 3.2 and 3.4 are reduced to ([14], Thms. 3.1 and 3.2) and in this case, the first and second order adjoint equation together with the $\bar{H}^\theta$-function and $\bar{\mathcal{H}}^\theta$-function (3.2), (3.3), (3.6), (3.7) coincide with (5), (6), (7) and (8) in[14], respectively.

Proofs of the preceding two theorems are deferred to Section 4.

4. Proofs of the main results

This section is devoted to proofs of the two main theorems of the paper (Thms. 3.2 and 3.4). The proofs will be done in several steps. In the first step, we reformulate the risk-sensitive problem (2.8) as a risk-neutral one. Then the existing risk-neutral maximum principle can be used directly. In Step 2, we transform the first order adjoint equation to a relative simple form. The transformation of the second order adjoint equation is done in Step 3. In Step 4, we complete the proofs by transforming the variational inequality and maximum condition into the one stated in the previous section.

Step 1. Applying risk-neutral maximum principle

The classical risk-neutral maximum principle for regime-switching jump-diffusion systems (see [25]) cannot directly be applied to problem (2.8). In this step, we first reformulate problem (2.8) as a risk-neutral one (see (4.1)). Then we define the Hamiltonian and the first order adjoint equation from which we obtain a maximum principle in local form (compare with [25]). In order to obtain a risk-sensitive maximum principle of the risk-neutral problem in global form, we define the Hamiltonian and associated second order adjoint equation. Combining the first and second order adjoint equations, a general stochastic maximum principle is obtained in terms of the variational inequality.

Let us consider the following risk-neutral control problem:

$$\begin{cases} \text{Minimize} \quad J^\theta(s,x,y,e_i;u(\cdot)) = E\Big[\mathrm{e}^{\theta[g(x(T),\alpha(T))+y(T)]}\Big] \\ \text{subject to:} \begin{cases} \mathrm{d}x(t) = b(t,x(t),u(t),\alpha(t))\mathrm{d}t + \sigma(t,x(t),u(t),\alpha(t))\mathrm{d}W(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \eta(t,x(t-),u(t-),\alpha(t-),z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z) \\ \qquad\qquad + \gamma(t,x(t-),u(t-),\alpha(t-))\mathrm{d}\widetilde{\Phi}(t), \\ \mathrm{d}y(t) = f(t,x(t),u(t),\alpha(t))\mathrm{d}t, \quad t\in[s,T], \\ x(s)=x, \quad y(s)=y, \quad u(\cdot)\in\mathcal{U}[s,T]. \end{cases} \end{cases} \tag{4.1}$$

Problem (2.8) corresponds to the case when $y = 0$ in (4.1). The value function $V^\theta : [0,T]\times\mathbb{R}^L\times\mathbb{R}\times S \to \mathbb{R}$ associated with (4.1) is:

$$V^\theta(s,x,y,e_i) := \inf_{u(\cdot)\in\mathcal{U}[s,T]} J^\theta(s,x,y,e_i;u(\cdot)). \tag{4.2}$$

Note that $V^\theta(s,x,y,e_i) = \mathrm{e}^{\theta y}V^\theta(s,x,e_i)$. In particular, assumption $(\mathcal{A}5)$ implies that $V^\theta(s,x,y,e_i) \in C^{1,3,\infty}([0,T]\times\mathbb{R}^L\times\mathbb{R}\times S)$.

Assume that $(\mathcal{A}1)-(\mathcal{A}4)$ hold, and let $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ be an optimal triple for the risk-neutral problem (4.1). We introduce the following first and second order adjoint equations:

$$\begin{cases} dp(t) = -\Bigg\{\begin{pmatrix}\bar{b}_x(t) & 0\\ \bar{f}_x(t) & 0\end{pmatrix}^\top p(t) + \sum_{j=1}^N \begin{pmatrix}\bar{\sigma}^j_x(t) & 0\\ 0 & 0\end{pmatrix}^\top q_j(t) \\ \qquad + \sum_{j=1}^M \int_{\mathbb{R}_0}\begin{pmatrix}\bar{\eta}^j_x(t,z) & 0\\ 0 & 0\end{pmatrix}^\top r_j(t,z)\nu^j_\alpha(\mathrm{d}z) + \sum_{j=1}^D \begin{pmatrix}\bar{\gamma}^j_x(t) & 0\\ 0 & 0\end{pmatrix}^\top s_j(t)\lambda_j(t)\Bigg\}\mathrm{d}t \\ \qquad + \sum_{j=1}^N q_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^M\int_{\mathbb{R}_0} r_j(t,z)\widetilde{N}^j_\alpha(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^D s_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ p(T) = \theta\mathrm{e}^{\theta[g(\bar{x}(T),\alpha(T))+\bar{y}(T)]}\begin{pmatrix} g_x(\bar{x}(T),\alpha(T))\\ 1\end{pmatrix}, \end{cases} \tag{4.3}$$

$$
\begin{cases}
dP(t) = -\Bigg\{ \begin{pmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{pmatrix}^\top P(t) + P(t) \begin{pmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{pmatrix} + \sum_{j=1}^{N} \Bigg[\begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top Q_j(t) \\
\qquad + Q_j(t) \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top P(t) \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} \Bigg] \\
\qquad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \Bigg[\begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top P(t) \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top R_j(t,z) \\
\qquad + R_j(t,z) \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top R_j(t,z) \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} \Bigg] \nu_\alpha^j(\mathrm{d}z) \\
\qquad + \sum_{j=1}^{D} \Bigg[\begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top P(t) \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top S_j(t) \\
\qquad + S_j(t) \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top S_j(t) \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} \Bigg] \lambda_j(t) \\
\qquad + \begin{pmatrix} H_{xx}^\theta(t, \bar{x}(t), \bar{u}(t), \alpha(t), p(t), q(t), r(t,\cdot), s(t)) & 0 \\ 0 & 0 \end{pmatrix} \Bigg\} \mathrm{d}t \\
\qquad + \sum_{j=1}^{N} Q_j(t) \mathrm{d}W_j(t) + \sum_{j=1}^{M} \int_{\mathbb{R}_0} R_j(t,z) \widetilde{N}_\alpha^j(\mathrm{d}t, \mathrm{d}z) + \sum_{j=1}^{D} S_j(t) \mathrm{d}\widetilde{\Phi}_j(t), \\
P(T) = \begin{pmatrix} \theta g_x(\bar{x}(T), \alpha(T)) g_x(\bar{x}(T), \alpha(T))^\top + g_{xx}(\bar{x}(T), \alpha(T)) & \theta g_x(\bar{x}(T), \alpha(T)) \\ \theta g_x(\bar{x}(T), \alpha(T))^\top & \theta \end{pmatrix} \\
\qquad \times \theta \mathrm{e}^{\theta[g(\bar{x}(T), \alpha(T)) + \bar{y}(T)]},
\end{cases}
\tag{4.4}
$$

Here the Hamiltonian function $H^\theta : [0,T] \times \mathbb{R}^L \times U \times S \times \mathbb{R}^{L+1} \times \mathbb{R}^{(L+1)\times N} \times \mathcal{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu_\alpha; \mathbb{R}^{(L+1)\times M}) \times \mathcal{M}^2(\mathbb{R}^+; \mathbb{R}^{(L+1)\times D}) \to \mathbb{R}$ is given by

$$
\begin{aligned}
H^\theta(t,x,u,e_i,p,q,r,s) := {}& \left\langle p, \begin{pmatrix} b(t,x,u,e_i) \\ f(t,x,u,e_i) \end{pmatrix} \right\rangle + \sum_{j=1}^{N} \left\langle q_j, \begin{pmatrix} \sigma^j(t,x,u,e_i) \\ 0 \end{pmatrix} \right\rangle \\
& + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \left\langle r_j(t,z), \begin{pmatrix} \eta^j(t,x,u,e_i,z) \\ 0 \end{pmatrix} \right\rangle \nu_{e_i}^j(\mathrm{d}z) \\
& + \sum_{j=1}^{D} \left\langle s_j(t), \begin{pmatrix} \gamma^j(t,x,u,e_i) \\ 0 \end{pmatrix} \right\rangle \lambda_j(t).
\end{aligned}
\tag{4.5}
$$

The adjoint equations (4.3)−(4.4) are linear BSDEs with jumps. Assumptions $(\mathcal{A}1)-(\mathcal{A}3)$ guaranty the existence and uniqueness of solution to the BSDEs (4.3) and (4.4), respectively. For the existence and uniqueness of solution to BSDEs driven by Poisson jumps and Markov chains; see, for example, [5].

Combining ([22], Thm. 2.1) with ([25], Thm. 3.1), the following stochastic maximum principle for the risk-neutral problem (4.1) can be proved.

Proposition 4.1. *Assume that assumptions $(\mathcal{A}1)-(\mathcal{A}4)$ hold and let $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal triple for risk-neutral problem (4.1). Then there exists a unique solution $(p(\cdot), q(\cdot), r(\cdot,\cdot), s(\cdot))$ and $(P(\cdot), Q(\cdot), R(\cdot,\cdot), S(\cdot))$*

of (4.3) *and* (4.4)*, respectively, such that*

$$
\begin{aligned}
&H^\theta(t,\bar{x}(t-),u,\alpha(t-),p(t-),q(t),r(t,\cdot),s(t)) - H^\theta(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),p(t-),q(t),r(t,\cdot),s(t)) \\
&+\frac{1}{2}\sum_{j=1}^{N}\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix}^\top P(t-)\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix} \\
&+\frac{1}{2}\sum_{j=1}^{M}\int_{\mathbb{R}_0}\left[\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix}^\top \Big(P(t-)+R_j(t,z)\Big)\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix}\right]\nu_\alpha^j(\mathrm{d}z) \\
&+\frac{1}{2}\sum_{j=1}^{D}\left[\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}^\top \Big(P(t-)+S_j(t)\Big)\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}\right]\lambda_j(t)\geq 0, \quad \forall u\in U,\ \text{a.e. } t\in[s,T],\ P\text{-a.s.},
\end{aligned}
\tag{4.6}
$$

or equivalently,

$$
\mathcal{H}^\theta(t,\bar{x}(t-),\bar{u}(t),\alpha(t-)) = \inf_{u\in U}\mathcal{H}^\theta(t,\bar{x}(t-),u,\alpha(t-)), \quad \text{a.e. } t\in[s,T],\ P\text{-a.s.}, \tag{4.7}
$$

where the $\mathcal{H}^\theta$*-function for problem* (4.1) *associated with* $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ *is defined by:*

$$
\begin{aligned}
\mathcal{H}^\theta(t,x,u,e_i) = {}& H^\theta(t,x,u,e_i,p(t-),q(t),r(t,\cdot),s(t)) \\
&-\frac{1}{2}\sum_{j=1}^{N}\begin{pmatrix}\bar{\sigma}^j(t)\\0\end{pmatrix}^\top P(t-)\begin{pmatrix}\bar{\sigma}^j(t)\\0\end{pmatrix} + \frac{1}{2}\sum_{j=1}^{N}\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix}^\top P(t-)\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix} \\
&-\frac{1}{2}\sum_{j=1}^{M}\int_{\mathbb{R}_0}\left[\begin{pmatrix}\bar{\eta}^j(t,z)\\0\end{pmatrix}^\top \Big(P(t-)+R_j(t,z)\Big)\begin{pmatrix}\bar{\eta}^j(t,z)\\0\end{pmatrix}\right]\nu_{e_i}^j(\mathrm{d}z) \\
&+\frac{1}{2}\sum_{j=1}^{M}\int_{\mathbb{R}_0}\left[\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix}^\top \Big(P(t-)+R_j(t,z)\Big)\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix}\right]\nu_{e_i}^j(\mathrm{d}z) \\
&-\frac{1}{2}\sum_{j=1}^{D}\left[\begin{pmatrix}\bar{\gamma}^j(t)\\0\end{pmatrix}^\top \Big(P(t-)+S_j(t)\Big)\begin{pmatrix}\bar{\gamma}^j(t)\\0\end{pmatrix}\right]\lambda_j(t) \\
&+\frac{1}{2}\sum_{j=1}^{D}\left[\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}^\top \Big(P(t-)+S_j(t)\Big)\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}\right]\lambda_j(t).
\end{aligned}
\tag{4.8}
$$

Proof. The proof easily follows by combining ([22], Thm. 2.1) and ([25], Thm. 3.1). We omit the details. ☐

The following sufficient conditions for the optimality of $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ can also be derived.

Proposition 4.2. *Suppose that assumptions* $(\mathcal{A}1)$–$(\mathcal{A}4)$ *and* $(\mathcal{A}6)$ *hold. Let* $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ *be an admissible triple,* $(p(\cdot),q(\cdot),r(\cdot,\cdot),s(\cdot))$ *and* $(P(\cdot),Q(\cdot),R(\cdot,\cdot),S(\cdot))$ *satisfy* (4.3) *and* (4.4)*, respectively. Suppose that for each* $e_i\in S$, $x\mapsto g(x,e_i)$ *is a convex function,* $(x,u)\mapsto H^\theta(t,x,u,e_i,p(t),q(t),r(t,\cdot),s(t))$ *is convex for all* $t\in[0,T]$, P*-a.s., and* (4.7) *holds. Then* $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ *is an optimal triple for problem* (4.1).

Proof. The proof follows in a similar way as in ([25], Thm. 3.1); we omit the details. ☐

Remark 4.3. Note that Propositions 4.1 and 4.2 do not depend on assumption $(\mathcal{A}5)$.

Next, we will transform the first order adjoint equation (4.3) to a more familiar form.

Step 2. Transformation of first order adjoint equation

Proposition 4.1 can be regarded as a maximum principle for the underlying risk-sensitive control problem (2.8). However, it is not a desirable one since the adjoint equations involve additional and "unnecessary" components. To overcome this problem, we need to transform the adjoint variables $(p(\cdot), q(\cdot), r(\cdot,\cdot), s(\cdot))$ and $(P(\cdot), Q(\cdot), R(\cdot,\cdot), S(\cdot))$. These changes are inspired by both logarithmic transformations (see, for example, [14]) which are used to derive a partial differential equation for the value function and the relationship between the maximum principle and the dynamic programming principle (see, for example, [25]). We extend the result in [14] to a continuous time Markov regime-switching jump-diffusion model (Lem. 4.4).

Lemma 4.4. *Under assumptions* $(\mathcal{A}1)-(\mathcal{A}5)$*, the first order adjoint equation* (4.3) *reduces to* (3.2)*.*

Proof. Let $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal triple for problem (4.1) and

$$\Big(p(\cdot), q(\cdot), r(\cdot,\cdot), s(\cdot)\Big) \equiv \left(\begin{bmatrix} p_1(\cdot) \\ p_2(\cdot) \end{bmatrix}, \begin{bmatrix} q_1(\cdot) \\ q_2(\cdot) \end{bmatrix}, \begin{bmatrix} r_1(\cdot,\cdot) \\ r_2(\cdot,\cdot) \end{bmatrix}, \begin{bmatrix} s_1(\cdot) \\ s_2(\cdot) \end{bmatrix}\right)$$

$$\in L^2_{\mathcal{F}}([s,T];\mathbb{R}^{L+1}) \times L^2_{\mathcal{F},p}([s,T];\mathbb{R}^{(L+1)\times N}) \times F^2_p([s,T];\mathbb{R}^{(L+1)\times M}) \times M^2_p([s,T];\mathbb{R}^{(L+1)\times D}),$$

be the first order adjoint variables satisfying equation (4.3), here $(p_1(\cdot), q_1(\cdot), r_1(\cdot,\cdot), s_1(\cdot)) \in L^2_{\mathcal{F}}([s,T];\mathbb{R}^{L}) \times L^2_{\mathcal{F},p}([s,T];\mathbb{R}^{L\times N}) \times F^2_p([s,T];\mathbb{R}^{L\times M}) \times M^2_p([s,T];\mathbb{R}^{L\times D})$ and $(p_2(\cdot), q_2(\cdot), r_2(\cdot,\cdot), s_2(\cdot)) \in L^2_{\mathcal{F}}([s,T];\mathbb{R}) \times L^2_{\mathcal{F},p}([s,T];\mathbb{R}^{1\times N}) \times F^2_p([s,T];\mathbb{R}^{1\times M}) \times M^2_p([s,T];\mathbb{R}^{1\times D})$. By the relationship between the maximum principle and the dynamic programming principle (see, for example, [25], Thm. 4.2), we have

$$p(t) = V^{\theta}_{(x,y)}(t, \bar{x}(t), \bar{y}(t), \alpha(t)), \tag{4.9}$$

where $V^{\theta}_{(x,y)}$ denotes the gradient of V^{θ} in (x,y). Let us introduce the following logarithmic transformation of the value function

$$v^{\theta}(t,x,y,e_i) = \frac{1}{\theta}\ln V^{\theta}(t,x,y,e_i), \tag{4.10}$$

Taking gradient on the right hand side of (4.10) and noting (4.9), we have the following transformation of the first order adjoint variable:

$$\tilde{p}(t) = \frac{1}{\theta}\frac{p(t)}{V(t)}, \tag{4.11}$$

where $V(t) := V^{\theta}(t, \bar{x}(t), \bar{y}(t), \alpha(t)) > 0$.

Next, we derive the equation for $\tilde{p}(\cdot) \equiv \begin{pmatrix} \bar{p}(\cdot) \\ \tilde{p}^*(\cdot) \end{pmatrix}$, where $\bar{p}(\cdot)$ is $\mathbb{R}^L$-valued. First notice that $V^{\theta}(s,x,y,e_i)$ is the value function of the risk-neutral problem (4.1). Then it follows from ([25], Eqs. (4.1) and (4.2)) that V must satisfy

$$\begin{cases} dV(t) = \sum\limits_{j=1}^{N} p_1(t)^{\top}\bar{\sigma}^j(t)\mathrm{d}W_j(t) + \sum\limits_{j=1}^{M}\int_{\mathbb{R}_0}[V^{\theta}(t, \bar{x}(t-) + \bar{\eta}^j(t,z), \bar{y}(t), \alpha(t-), z) \\ \qquad -V^{\theta}(t, \bar{x}(t-), \bar{y}(t), \alpha(t-), z)]\widetilde{N}^j_{\alpha}(\mathrm{d}t, \mathrm{d}z) \\ \qquad + \sum\limits_{j=1}^{D}[V^{\theta}(t, \bar{x}(t-) + \bar{\gamma}^j(t), \bar{y}(t), e_j) - V^{\theta}(t, \bar{x}(t-), \bar{y}(t), \alpha(t-))]\mathrm{d}\widetilde{\Phi}_j(t), \\ V(T) = \mathrm{e}^{\theta(g(\bar{x}(T), \alpha(T)) + \bar{y}(T))}. \end{cases} \tag{4.12}$$

Assume that $\tilde{p}(t)$ satisfies an equation of the following form:

$$d\tilde{p}(t) = \alpha(t)dt + \sum_{j=1}^{N} \tilde{q}_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{r}_j(t,z)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^{D} \tilde{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t). \tag{4.13}$$

Using Itô's formula for Markov regime switching jump-diffusion process (see, for example, [25], Thm. 4.1), we obtain:

$$\begin{aligned}
\mathrm{d}p(t) &= \mathrm{d}(\theta V(t)\tilde{p}(t)) \\
&= \theta V(t-)\mathrm{d}\tilde{p}(t) + \sum_{j=1}^{N} \theta p_1(t)^\top \bar{\sigma}^j(t)\tilde{q}_j(t)\mathrm{d}t + \sum_{j=1}^{N} \theta \tilde{p}(t)p_1(t)^\top \bar{\sigma}^j(t)\mathrm{d}W_j(t) \\
&\quad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \theta\Big[V^\theta(t,\bar{x}(t-)+\bar{\eta}^j(t,z),\bar{y}(t),\alpha(t-)) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))\Big]\tilde{r}_j(t,z)\nu_\alpha^j(\mathrm{d}z)\mathrm{d}t \\
&\quad + \sum_{j=1}^{D} \theta\Big[V^\theta(t,\bar{x}(t-)+\bar{\gamma}^j(t),\bar{y}(t),e_j) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))\Big]\tilde{s}_j(t)\lambda_j(t)\mathrm{d}t \\
&\quad + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \theta\Big[V^\theta(t,\bar{x}(t-)+\bar{\eta}^j(t,z),\bar{y}(t),\alpha(t-)) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))\Big]\Big(\tilde{p}(t-)+\tilde{r}_j(t,z)\Big)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}z) \\
&\quad + \sum_{j=1}^{D} \theta\Big[V^\theta(t,\bar{x}(t-)+\bar{\gamma}^j(t),\bar{y}(t),e_j) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))\Big]\Big(\tilde{p}(t-)+\tilde{s}_j(t)\Big)\mathrm{d}\widetilde{\Phi}_j(t).
\end{aligned} \tag{4.14}$$

Dividing both sides of (4.14) by $\theta V(t-)$ and nothing that $p_1(t-) = \theta V(t-)\bar{p}(t-)$, we obtain:

$$\begin{aligned}
\mathrm{d}\tilde{p}(t) = \frac{1}{\theta V(t-)}\mathrm{d}p(t) &- \sum_{j=1}^{N} \theta\bar{p}(t)^\top \bar{\sigma}^j(t)\tilde{q}_j(t)\mathrm{d}t - \sum_{j=1}^{N} \theta\tilde{p}(t)\bar{p}(t)^\top \bar{\sigma}^j(t)\mathrm{d}W_j(t) \\
&- \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{\Theta}_j(t,z)\tilde{r}_j(t,z)\nu_{\alpha(t-)}^j(\mathrm{d}z)\mathrm{d}t - \sum_{j=1}^{D} \tilde{\Lambda}_j(t)\tilde{s}_j(t)\lambda_j(t)\mathrm{d}t \\
&- \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{\Theta}_j(t,z)\Big(\tilde{p}(t-)+\tilde{r}_j(t,z)\Big)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}z) - \sum_{j=1}^{D} \tilde{\Lambda}_j(t)\Big(\tilde{p}(t-)+\tilde{s}_j(t)\Big)\mathrm{d}\widetilde{\Phi}_j(t),
\end{aligned} \tag{4.15}$$

where (for each $t \in [s,T],\ z \in \mathbb{R}_0$)

$$\begin{aligned}
\tilde{\Theta}_j(t,z) &= \frac{V^\theta(t,\bar{x}(t-)+\bar{\eta}^j(t,z),\bar{y}(t),\alpha(t-)) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))}{V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))},\ j=1,\dots,M, \\
\tilde{\Lambda}_j(t) &= \frac{V^\theta(t,\bar{x}(t-)+\bar{\gamma}^j(t),\bar{y}(t),e_j) - V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))}{V^\theta(t,\bar{x}(t-),\bar{y}(t),\alpha(t-))},\ j=1,\dots D.
\end{aligned}$$

Substituting (4.3) into (4.15) leads to

$$\begin{aligned}
\tilde{q}_j(t) &\equiv \begin{pmatrix} \bar{q}_j(t) \\ \tilde{q}_j^*(t) \end{pmatrix} = \frac{q_j(t)}{\theta V(t-)} - \theta\tilde{p}(t)\bar{p}(t)^\top \bar{\sigma}^j(t),\ j=1,\dots,N, \\
\tilde{r}_j(t,\cdot) &\equiv \begin{pmatrix} \bar{r}_j(t,\cdot) \\ \tilde{r}_j^*(t,\cdot) \end{pmatrix} = \frac{r_j(t,\cdot)}{\theta V(t-)} - \tilde{\Theta}_j(t,z)\Big(\tilde{p}(t-)+\tilde{r}_j(t,z)\Big),\ j=1,\dots,M, \\
\tilde{s}_j(t) &\equiv \begin{pmatrix} \bar{s}_j(t) \\ \tilde{s}_j^*(t) \end{pmatrix} = \frac{s_j(t)}{\theta V(t-)} - \tilde{\Lambda}_j(t)\Big(\tilde{p}(t-)+\tilde{s}_j(t)\Big),\ j=1,\dots,D,
\end{aligned} \tag{4.16}$$

where $\bar{q}(\cdot) = (\bar{q}_1(\cdot), \ldots, \bar{q}_N(\cdot))$ is $\mathbb{R}^{L\times N}$-valued, $\bar{r}(\cdot,\cdot) = (\bar{r}_1(\cdot,\cdot), \ldots, \bar{r}_M(\cdot,\cdot))$ is $\mathbb{R}^{L\times M}$-valued, and $\bar{s}(\cdot) = (\bar{s}_1(\cdot), \ldots, \bar{s}_D(\cdot))$ is $\mathbb{R}^{L\times D}$-valued. Substituting (4.16) into (4.15) and by virtue of (4.3), it follows that the transformed first order adjoint variable $\tilde{p}(\cdot)$ satisfies the following equation:

$$\begin{cases} \mathrm{d}\tilde{p}(t) = -\Bigg\{ \begin{pmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{pmatrix}^\top \tilde{p}(t) + \sum_{j=1}^N \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(\tilde{q}_j(t) + \theta \tilde{p}(t)\bar{p}(t)^\top \bar{\sigma}^j(t) \Big) \\ \quad + \sum_{j=1}^M \int_{\mathbb{R}_0} \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(\tilde{r}_j(t,z) + \tilde{\Theta}_j(t,z)\big(\tilde{p}(t-) + \tilde{r}_j(t,z)\big)\Big) \nu_\alpha^j(\mathrm{d}z) \\ \quad + \sum_{j=1}^D \begin{pmatrix} \bar{\gamma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(\tilde{s}_j(t) + \tilde{\Lambda}_j(t)\big(\tilde{p}(t-) + \tilde{s}_j(t)\big)\Big)\lambda_j(t) \\ \quad + \sum_{j=1}^N \theta \bar{p}(t)^\top \bar{\sigma}^j(t)\tilde{q}_j(t)\mathrm{d}t + \sum_{j=1}^M \int_{\mathbb{R}_0} \tilde{\Theta}_j(t,z)\tilde{r}_j(t,z)\big)\nu_\alpha^j(\mathrm{d}z) + \sum_{j=1}^D \tilde{\Lambda}_j(t)\tilde{s}_j(t)\lambda_j(t) \Bigg\} \mathrm{d}t \\ \quad + \sum_{j=1}^N \tilde{q}_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^M \int_{\mathbb{R}_0} \tilde{r}_j(t,z)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^D \tilde{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ \tilde{p}(T) = \begin{pmatrix} g_x(\bar{x}(T),\alpha(T)) \\ 1 \end{pmatrix}, \end{cases} \tag{4.17}$$

where $\tilde{p}(T)$ is easily determined from (4.11) and $p(T)$.

Letting $y = 0$ in(4.1) and expanding (4.17), it can be seen that

$$\tilde{p}^*(t) = 1, \quad \tilde{q}_j^*(t) = \tilde{r}_j^*(t,\cdot) = \tilde{s}_j^*(t) = 0, \quad \forall t \in [s,T], \tag{4.18}$$

and $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot,\cdot), \bar{s}(\cdot))$ is a solution of (3.2). This explains how equation (3.2) is derived. Also, since our derivation can be reversed, it follows from the uniqueness of solution to (4.3) that this solution is unique. □

In the next step, we will transform the second order adjoint equation (4.4) to a relative simple form.

Step 3. Transformation of second order adjoint equation

The transformation of second order adjoint equation is given in the following lemma.

Lemma 4.5. *Assume that assumptions* $(\mathcal{A}1)$–$(\mathcal{A}5)$ *hold. Then the second order adjoint equation* (4.4) *reduces to* (3.3).

Proof. Let $(P(\cdot), Q(\cdot), R(\cdot,\cdot), S(\cdot))$ be the second order adjoint variable satisfying (4.4) and consider the following transformation:

$$\tilde{P}(t) = \frac{1}{\theta}\frac{P(t)}{V(t)} - \theta\tilde{p}(t)\tilde{p}(t)^\top \triangleq \Gamma(t) - \theta\tilde{p}(t)\tilde{p}(t)^\top, \tag{4.19}$$

Assume that

$$\mathrm{d}\Gamma(t) = X(t)\mathrm{d}t + \sum_{j=1}^N Y_j(t)\mathrm{d}W_j(t) + \sum_{j=1}^M \int_{\mathbb{R}_0} Z_j(t,z)\widetilde{N}_\alpha^j(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^D A_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \tag{4.20}$$

Using Itô's formula, (4.12) and (4.19), we get

$$\begin{aligned}
\mathrm{d}P(t) =& \mathrm{d}(\theta V(t)\mathit{\Gamma}(t)) \\
=& \theta V(t-)\mathrm{d}\mathit{\Gamma}(t) + \sum_{j=1}^{N} \theta p_1(t)^\top \bar{\sigma}^j(t) Y_j(t)\mathrm{d}t + \sum_{j=1}^{N} \theta \mathit{\Gamma}(t) p_1(t)^\top \bar{\sigma}^j(t)\mathrm{d}W_j(t) \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \theta \Big[V^\theta(t, \bar{x}(t-) + \bar{\eta}^j(t,z), \bar{y}(t), \alpha(t-)) - V^\theta(t, \bar{x}(t-), \bar{y}(t), \alpha(t-)) \Big] Z_j(t,z)\nu_\alpha^j(\mathrm{d}z)\mathrm{d}t \\
&+ \sum_{j=1}^{D} \theta \Big[V^\theta(t, \bar{x}(t-) + \bar{\gamma}^j(t), \bar{y}(t), e_j) - V^\theta(t, \bar{x}(t-), \bar{y}(t), \alpha(t-)) \Big] A_j(t)\lambda_j(t)\mathrm{d}t \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \theta \Big[V^\theta(t, \bar{x}(t-) + \bar{\eta}^j(t,z), \bar{y}(t), \alpha(t-)) - V^\theta(t, \bar{x}(t-), \bar{y}(t), \alpha(t-)) \Big] \Big(\mathit{\Gamma}(t-) + Z_j(t,z) \Big) \widetilde{N}_\alpha^j(\mathrm{d}t, \mathrm{d}z) \\
&+ \sum_{j=1}^{D} \theta \Big[V^\theta(t, \bar{x}(t-) + \bar{\gamma}^j(t), \bar{y}(t), e_j) - V^\theta(t, \bar{x}(t-), \bar{y}(t), \alpha(t-)) \Big] \Big(\mathit{\Gamma}(t-) + A_j(t) \Big) \mathrm{d}\widetilde{\mathit{\Phi}}_j(t). \qquad (4.21)
\end{aligned}$$

Dividing once more $\theta V(t-)$ and noting that $p_1(t-) = \theta V(t-)\bar{p}(t-)$, we get:

$$\begin{aligned}
\mathrm{d}\mathit{\Gamma}(t) =& \frac{1}{\theta V(t-)}\mathrm{d}P(t) - \sum_{j=1}^{N} \theta \bar{p}(t)^\top \bar{\sigma}^j(t) Y_j(t)\mathrm{d}t - \sum_{j=1}^{N} \theta \mathit{\Gamma}(t)\bar{p}(t)^\top \bar{\sigma}^j(t)\mathrm{d}W_j(t) \\
&- \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{\mathit{\Theta}}_j(t,z) Z_j(t,z)\nu_\alpha^j(\mathrm{d}z)\mathrm{d}t - \sum_{j=1}^{D} \tilde{\mathit{\Lambda}}_j(t) A_j(t)\lambda_j(t)\mathrm{d}t \\
&- \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{\mathit{\Theta}}_j(t,z) \Big(\mathit{\Gamma}(t-) + Z_j(t,z) \Big) \widetilde{N}_\alpha^j(\mathrm{d}t, \mathrm{d}z) \\
&- \sum_{j=1}^{D} \tilde{\mathit{\Lambda}}_j(t) \Big(\mathit{\Gamma}(t-) + A_j(t) \Big) \mathrm{d}\widetilde{\mathit{\Phi}}_j(t). \qquad (4.22)
\end{aligned}$$

Substituting the expression (4.4) into (4.22), we have

$$\begin{aligned}
Y_j(t) \equiv& \frac{Q_j(t)}{\theta V(t-)} - \theta \bar{p}(t)^\top \bar{\sigma}^j(t) \Big(\tilde{P}(t) + \theta \tilde{p}(t)\tilde{p}(t)^\top \Big), \ j = 1, \dots, N, \\
Z_j(t,\cdot) \equiv& \frac{R_j(t,\cdot)}{\theta V(t-)} - \tilde{\mathit{\Theta}}_j(t,z) \Big(\tilde{P}(t-) + \theta \tilde{p}(t-)\tilde{p}(t-)^\top + Z_j(t,z) \Big), \ j = 1, \dots, M, \\
A_j(t) \equiv& \frac{S_j(t)}{\theta V(t-)} - \tilde{\mathit{\Lambda}}_j(t) \Big(\tilde{P}(t-) + \theta \tilde{p}(t-)\tilde{p}(t-)^\top + A_j(t) \Big), \ j = 1, \dots, D. \qquad (4.23)
\end{aligned}$$

Combining (4.5) with (3.6), we can easily obtain:

$$\begin{aligned}
&\frac{1}{\theta V(t-)} \begin{pmatrix} H_{xx}^\theta(t, \bar{x}(t-), \bar{u}(t), \alpha(t-), p(t-), q(t), r(t,\cdot), s(t)) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \bar{H}_{xx}^\theta(t, \bar{x}(t-), \bar{u}(t), \alpha(t-), \bar{p}(t-), \bar{q}(t), \bar{r}(t,\cdot), \bar{s}(t)) & 0 \\ 0 & 0 \end{pmatrix}. \qquad (4.24)
\end{aligned}$$

Applying Itô's formula to (4.19) and using (4.17) and (4.22), we have

$$
\begin{aligned}
\mathrm{d}\tilde{P}(t) = &- \Bigg\{ \begin{pmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{pmatrix}^\top \tilde{P}(t) + \tilde{P}(t) \begin{pmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{pmatrix} \\
&+ \sum_{j=1}^{N} \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(\tilde{P}(t) + \theta \tilde{p}(t) \bar{p}(t)^\top \Big) \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \sum_{j=1}^{N} \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(\tilde{Q}_j(t) + \theta \bar{p}(t)^\top \bar{\sigma}^j(t) \tilde{P}(t) + \theta \tilde{p}(t) \tilde{q}_j(t)^\top \Big) \\
&+ \sum_{j=1}^{N} \Big(\tilde{Q}_j(t) + \theta \bar{p}(t)^\top \bar{\sigma}^j(t) \tilde{P}(t) + \theta \tilde{q}_j(t) \tilde{p}(t)^\top \Big) \begin{pmatrix} \bar{\sigma}_x^j(t) & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \sum_{j=1}^{N} \Big[\theta \bar{p}(t)^\top \bar{\sigma}^j(t) \tilde{Q}_j(t) + \theta \bar{q}_j(t) \bar{q}_j(t)^\top \Big] \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(1 + \tilde{\Theta}_j(t,z) \Big) \Big(\tilde{P}(t) + \tilde{R}_j(t,z) \\
&+ \theta (\tilde{p}(t) + \tilde{r}_j(t,z)) (\tilde{p}(t) + \tilde{r}_j(t,z))^\top \Big) \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} \nu_\alpha^j(\mathrm{d}z) \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big[\Big(1 + \tilde{\Theta}_j(t,z) \Big) \Big(\tilde{P}(t) + \tilde{R}_j(t,z) + \theta (\tilde{p}(t) + \tilde{r}_j(t,z)) \tilde{r}_j(t,z)^\top \Big) - \tilde{P}(t) \Big] \nu_\alpha^j(\mathrm{d}z) \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \Big[\Big(1 + \tilde{\Theta}_j(t,z) \Big) \Big(\tilde{P}(t) + \tilde{R}_j(t,z) + \theta \tilde{r}_j(t,z) (\tilde{p}(t) + \tilde{r}_j(t,z))^\top \Big) - \tilde{P}(t) \Big] \begin{pmatrix} \bar{\eta}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} \nu_\alpha^j(\mathrm{d}z) \\
&+ \sum_{j=1}^{M} \int_{\mathbb{R}_0} \Big[\tilde{\Theta}_j(t,z) \tilde{R}_j(t,z) + \Big(1 + \tilde{\Theta}_j(t,z) \Big) \theta \tilde{r}_j(t,z) \tilde{r}_j(t,z)^\top \Big] \nu_\alpha^j(\mathrm{d}z) \\
&+ \sum_{j=1}^{D} \begin{pmatrix} \bar{\gamma}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big(1 + \tilde{\Lambda}_j(t) \Big) \Big(\tilde{P}(t) + \tilde{S}_j(t) + \theta (\tilde{p}(t) + \tilde{s}_j(t)) (\tilde{p}(t) + \tilde{s}_j(t))^\top \Big) \begin{pmatrix} \bar{\gamma}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} \lambda_j(t) \\
&+ \sum_{j=1}^{D} \begin{pmatrix} \bar{\gamma}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix}^\top \Big[\Big(1 + \tilde{\Lambda}_j(t) \Big) \Big(\tilde{P}(t) + \tilde{S}_j(t) + \theta (\tilde{p}(t) + \tilde{s}_j(t)) \tilde{s}_j(t)^\top \Big) - \tilde{P}(t) \Big] \lambda_j(t) \\
&+ \sum_{j=1}^{D} \Big[\Big(1 + \tilde{\Lambda}_j(t) \Big) \Big(\tilde{P}(t) + \tilde{S}_j(t) + \theta \tilde{s}_j(t) (\tilde{p}(t) + \tilde{s}_j(t))^\top \Big) - \tilde{P}(t) \Big] \begin{pmatrix} \bar{\gamma}_x^j(t,z) & 0 \\ 0 & 0 \end{pmatrix} \lambda_j(t) \\
&+ \sum_{j=1}^{D} \Big[\tilde{\Lambda}_j(t) \tilde{S}_j(t) + \Big(1 + \tilde{\Lambda}_j(t) \Big) \theta \tilde{s}_j(t) \tilde{s}_j(t)^\top \Big] \lambda_j(t) \\
&+ \begin{pmatrix} \bar{H}_{xx}^\theta(t, \bar{x}(t), \bar{u}(t), \alpha(t), \bar{p}(t), \bar{q}(t), \bar{r}(t,\cdot), \bar{s}(t)) & 0 \\ 0 & 0 \end{pmatrix} \Bigg\} \mathrm{d}t \\
&+ \sum_{j=1}^{N} \tilde{Q}_j(t) \mathrm{d}W_j(t) + \sum_{j=1}^{M} \int_{\mathbb{R}_0} \tilde{R}_j(t,z) \widetilde{N}_\alpha^j(\mathrm{d}t, \mathrm{d}z) + \sum_{j=1}^{D} \tilde{S}_j(t) \mathrm{d}\widetilde{\Phi}_j(t),
\end{aligned}
$$

with terminal condition

$$\tilde{P}(T) = \begin{pmatrix} g_{xx}(\bar{x}(T), \alpha(T)) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
\tilde{Q}_j(t) =& \frac{Q_j(t)}{\theta V(t-)} - \theta \bar{p}(t)^\top \bar{\sigma}^j(t) \Big(\tilde{P}(t) + \theta \tilde{p}(t)\tilde{p}(t)^\top \Big) - \theta \tilde{p}(t)\tilde{q}_j(t)^\top - \theta \tilde{q}_j(t)\tilde{p}(t)^\top, \ j = 1, \ldots, N, \\
\tilde{R}_j(t, \cdot) =& \frac{R_j(t, \cdot)}{\theta V(t-)} - \tilde{\varTheta}_j(t, z) \Big(\tilde{P}(t-) + \theta \tilde{p}(t-)\tilde{p}(t-)^\top + Z_j(t, z) \Big) \\
& - \theta \tilde{p}(t-)\tilde{r}_j(t, z)^\top - \theta \tilde{r}_j(t, z)\tilde{p}(t-)^\top - \theta \tilde{r}_j(t, z)\tilde{r}_j(t, z)^\top, \ j = 1, \ldots, M, \\
\tilde{S}_j(t) =& \frac{S_j(t)}{\theta V(t-)} - \tilde{\varLambda}_j(t) \Big(\tilde{P}(t-) + \theta \tilde{p}(t-)\tilde{p}(t-)^\top + A_j(t) \Big) \\
& - \theta \tilde{p}(t-)\tilde{s}_j(t)^\top - \theta \tilde{s}_j(t)\tilde{p}(t-)^\top - \theta \tilde{s}_j(t)\tilde{s}_j(t)^\top, \ j = 1, \ldots, D.
\end{aligned} \tag{4.25}$$

Therefore, it follows that

$$\begin{aligned}
\tilde{P}(t) &= \begin{pmatrix} \bar{P}(t) & 0 \\ 0 & 0 \end{pmatrix}, \\
\tilde{Q}_j(t) &= \begin{pmatrix} \bar{Q}_j(t) & 0 \\ 0 & 0 \end{pmatrix}, \ j = 1, \ldots, N, \\
\tilde{R}_j(t, \cdot) &= \begin{pmatrix} \bar{R}_j(t, \cdot) & 0 \\ 0 & 0 \end{pmatrix}, \ j = 1, \cdots, M, \\
\tilde{S}_j(t) &= \begin{pmatrix} \bar{S}(t) & 0 \\ 0 & 0 \end{pmatrix}, \ j = 1, \ldots, D,
\end{aligned} \tag{4.26}$$

where $(\bar{P}(\cdot), \bar{Q}(\cdot), \bar{R}(\cdot, \cdot), \bar{S}(\cdot))$ is the solution of (3.3). As in the first order case, this solution is unique. □

In the next step, we will transform the variational inequality (4.6) and the maximum condition (4.7) to a more familiar form as stated in Theorem 3.2.

Step 4. Maximum condition

The transformation of the variational inequality and the maximum condition are given in the following lemma.

Lemma 4.6. *Under ssumptions ($\mathcal{A}1$)–($\mathcal{A}5$), the variational inequality (4.6) and the maximum condition (4.7) reduce to (3.8) and (3.9), respectively.*

Proof. To start, consider the variational inequality (4.6). It can be shown that, in view of (4.11), (4.16) and (4.18), for each $e_i \in S$,

$$H^\theta(t, x, u, e_i, p(t-), q(t), r(t, \cdot), s(t)) = \theta V(t-) \bar{H}^\theta(t, x, u, e_i, \bar{p}(t-), \bar{q}(t), \bar{r}(t, \cdot), \bar{s}(t)), \tag{4.27}$$

where H^θ and $\bar{H}^\theta$ are defined by (4.5) and (3.6), respectively. By (4.19), (4.25) and (4.26), we can easily obtain:

$$\frac{1}{2}\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix}^\top P(t-)\begin{pmatrix}\delta\sigma^j(t,u)\\0\end{pmatrix}=\frac{\theta V(t-)}{2}\delta\sigma^j(t,u)^\top\Big(\bar{P}(t-)+\theta\bar{p}(t-)\bar{p}(t-)^\top\Big)\delta\sigma^j(t,u), \tag{4.28}$$

$$\begin{aligned}&\frac{1}{2}\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix}^\top\Big(P(t-)+R_j(t,z)\Big)\begin{pmatrix}\delta\eta^j(t,u,z)\\0\end{pmatrix} \qquad (4.29)\\ =&\frac{\theta V(t-)}{2}\delta\eta^j(t,u,z)^\top\Big(1+\Theta_j(t,z)\Big)\Big(\bar{P}(t-)+\bar{R}_j(t,z)+\theta(\bar{p}(t-)+\bar{r}_j(t,z))(\bar{p}(t-)+\bar{r}_j(t,z))^\top\Big)\delta\eta^j(t,u,z)\\ &\frac{1}{2}\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}^\top\Big(P(t-)+S_j(t)\Big)\begin{pmatrix}\delta\gamma^j(t,u)\\0\end{pmatrix}\\ =&\frac{\theta V(t-)}{2}\delta\gamma^j(t,u)^\top\Big(1+\Lambda_j(t)\Big)\Big(\bar{P}(t-)+\bar{S}_j(t)+\theta(\bar{p}(t-)+\bar{s}_j(t))(\bar{p}(t-)+\bar{s}_j(t))^\top\Big)\delta\gamma^j(t,u). \qquad (4.30)\end{aligned}$$

Since $V(t-)>0$, it follows that the variational inequality (4.6) is equivalent to

$$\begin{aligned}&\bar{H}^\theta(t,\bar{x}(t-),u,\alpha(t-),\bar{p}(t-),\bar{q}(t),\bar{r}(t,\cdot),\bar{s}(t))-\bar{H}^\theta(t,\bar{x}(t-),\bar{u}(t),\alpha(t-),\bar{p}(t-),\bar{q}(t),\bar{r}(t,\cdot),\bar{s}(t))\\ &+\frac{1}{2}\sum_{j=1}^N\delta\sigma^j(t,u)^\top\Big(\bar{P}(t-)+\theta\bar{p}(t-)\bar{p}(t-)^\top\Big)\delta\sigma^j(t,u)\\ &+\frac{1}{2}\sum_{j=1}^M\int_{\mathbb{R}_0}\Big[\delta\eta^j(t,u,z)^\top\Big(1+\Theta_j(t,z)\Big)\Big(\bar{P}(t-)+\bar{R}_j(t,z)\\ &+\theta(\bar{p}(t-)+\bar{r}_j(t,z))(\bar{p}(t-)+\bar{r}_j(t,z))^\top\Big)\delta\eta^j(t,u,z)\Big]\nu_\alpha^j(\mathrm{d}z)\\ &+\frac{1}{2}\sum_{j=1}^D\Big[\delta\gamma^j(t,u)^\top\Big(1+\Lambda_j(t)\Big)\Big(\bar{P}(t-)+\bar{S}_j(t)\\ &+\theta(\bar{p}(t-)+\bar{s}_j(t))(\bar{p}(t-)+\bar{s}_j(t))^\top\Big)\delta\gamma^j(t,u)\Big]\lambda_j(t)\geq 0\quad\forall u\in U,\ \text{a.e. } t\in[s,T],\ P\text{-a.s.}\end{aligned} \tag{4.31}$$

which gives (3.8). The equivalent condition (3.9) is obtained *via* direct manipulation. □

This completes proofs for both Theorems 3.2 and 3.4.

5. Applications

In this section, we apply our results to solve first a linear-quadratic (LQ) optimal control problem and second, a risk-sensitive benchmarked asset management for Markov regime-switching models. For simplicity, we suppose that all the controled processes are 1-dimensional.

5.1. Application to LQ risk-sensitive control under regime-switching

Consider the following LQ risk-sensitive control problem:

$$\begin{cases}\text{Minimize}\quad J^\theta(x,e_i;u(\cdot))=E\left[\mathrm{e}^{\theta\left[\int_0^T\left(M(t,\alpha(t))x(t)+\frac{1}{2}N(t,\alpha(t))u(t)^2\right)\mathrm{d}t+Gx(T)\right]}\right],\\ \text{subject to:}\begin{cases}\mathrm{d}x(t)=[A(t,\alpha(t))x(t)+B(t,\alpha(t))u(t)+b(t,\alpha(t))]\mathrm{d}t\\ \qquad\qquad+[C(t,\alpha(t))u(t)+\sigma(t,\alpha(t))]\mathrm{d}W(t)\\ \qquad\qquad+\displaystyle\int_{\mathbb{R}_0}\eta(t,\alpha(t-),z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z)+\gamma(t,\alpha(t-))\mathrm{d}\widetilde{\Phi}(t),\\ x(0)=x,\qquad t\in[0,T].\end{cases}\end{cases} \tag{5.1}$$

We assume that $U = \mathbb{R}$ and for each $e_i \in S$, $A(\cdot, e_i), B(\cdot, e_i), b(\cdot, e_i), C(\cdot, e_i), \sigma(\cdot, e_i), N(\cdot, e_i) > 0, M(\cdot, e_i) \in L^\infty_{\mathcal{F}}([0, T]; \mathbb{R})$, $\eta(\cdot, e_i, \cdot) \in \mathcal{L}^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu_\alpha; \mathbb{R})$, $\gamma(\cdot, e_i) \in \mathcal{M}^2(\mathbb{R}^+; \mathbb{R}^D)$, $G, x \in \mathbb{R}$ and $\theta > 0$ is given. Here $L^\infty_{\mathcal{F}}([0, T]; \mathbb{R})$ denotes the set of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted essentially bounded processes. We also need the following assumption: for any admissible pair $(x(\cdot), u(\cdot))$,

$$E\left[\mathrm{e}^{\theta\left[\int_0^T |M(t,\alpha(t))x(t)+\frac{1}{2}N(t,\alpha(t))u(t)^2|\mathrm{d}t+|Gx(T)|\right]}\right] < \infty. \tag{5.2}$$

This is a special case of our risk-sensitive control problem (2.6)−(2.8) with the initial time equals 0 and $x(t)$ satisfies a linear SDE. Obviously, the solution of $x(t)$ is given in the following:

$$\begin{aligned}
x(T) =& x\mathrm{e}^{\int_o^T A(s,\alpha(s))\mathrm{d}s} + \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\Big[B(t,\alpha(t))u(t) + b(t,\alpha(t))\Big]\mathrm{d}t \\
&+ \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\Big[C(t,\alpha(t))u(t) + \sigma(t,\alpha(t))\Big]\mathrm{d}W(t) \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\eta(t,\alpha(t-),z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z) \\
&+ \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\gamma(t,\alpha(t-))\mathrm{d}\widetilde{\Phi}(t) \\
\triangleq& x\Gamma(T) + \Sigma(T),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma(T) =& \mathrm{e}^{\int_o^T A(s,\alpha(s))\mathrm{d}s}, \\
\Sigma(T) =& \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\Big[B(t,\alpha(t))u(t) + b(t,\alpha(t))\Big]\mathrm{d}t \\
&+ \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\Big[C(t,\alpha(t))u(t) + \sigma(t,\alpha(t))\Big]\mathrm{d}W(t) \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\eta(t,\alpha(t-),z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z) \\
&+ \int_0^T \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\gamma(t,\alpha(t-))\mathrm{d}\widetilde{\Phi}(t).
\end{aligned}$$

Substituting the solution for $x(t)$ into $J^\theta(x, e_i; u(\cdot))$, we have

$$\begin{aligned}
J^\theta(x, e_i; u(\cdot)) =& E\bigg[\exp\bigg(\theta x\bigg[\int_0^T M(t,\alpha(t))\Gamma(t)\mathrm{d}t + G\Gamma(T)\bigg]\bigg) \\
&\times \exp\bigg(\theta\int_0^T \Big[M(t,\alpha(t))\Sigma(t) + \frac{1}{2}N(t,\alpha(t))u(t)^2\Big]\mathrm{d}t + \theta G\Sigma(T)\bigg)\bigg].
\end{aligned}$$

Under the above assumptions on the coefficients, it is easy to see that $J^\theta(x, e_i; u(\cdot))$ is infinitely differentiable with respect to x. So assumptions $(\mathcal{A}1)-(\mathcal{A}6)$ made in Section 2 hold, we can use our maximum principle (Thms. 3.2 and 3.4) to solve the above risk-sensitive optimal control problem. First of all, associated with an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, the adjoint equations (3.2) and (3.3) become

$$\begin{cases} d\bar{p}(t) = -\big\{A(t,\alpha(t))\bar{p}(t) + M(t,\alpha(t)) + \theta\bar{p}(t)\big(C(t,\alpha(t))\bar{u}(t) + \sigma(t,\alpha(t))\big)\bar{q}(t) \\ \qquad + \int_{\mathbb{R}_0} \Theta(t,z)\bar{r}(t,z)\nu_\alpha(\mathrm{d}z) + \sum_{j=1}^{D} \Lambda_j(t)\bar{s}_j(t)\lambda_j(t)\big\}\mathrm{d}t \\ \qquad +\bar{q}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0} \bar{r}(t,z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^{D} \bar{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ \bar{p}(T) = G, \end{cases} \tag{5.3}$$

$$\begin{cases} d\bar{P}(t) = -\big\{2A(t,\alpha(t))\bar{P}(t) + \theta\bar{p}(t)(C(t,\alpha(t))\bar{u}(t) + \sigma(t,\alpha(t)))\bar{Q}(t) \\ \qquad + \theta\bar{q}(t)^2 + \int_{\mathbb{R}_0} \big[(1+\Theta(t,z))\theta\bar{r}(t,z)^2 + \Theta(t,z)\bar{R}(t,z)\big]\nu_\alpha(\mathrm{d}z) \\ \qquad + \sum_{j=1}^{D} \big[(1+\Lambda_j(t))\theta\bar{s}_j(t)^2 + \Lambda_j(t)\bar{S}_j(t)\big]\lambda_j(t)\big\}\mathrm{d}t \\ \qquad + \bar{Q}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0} \bar{R}(t,z)\widetilde{N}_\alpha(\mathrm{d}t,\mathrm{d}z) + \sum_{j=1}^{D} \bar{S}_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ \bar{P}(T) = 0, \end{cases} \tag{5.4}$$

It follows from its terminal condition that the BSDE (5.3) is deterministic, that is $\bar{q}(t) = \bar{r}(t,\cdot) = \bar{s}(t) = 0$. Then $\bar{p}(\cdot)$ is the solution of the following equation:

$$\begin{cases} d\bar{p}(t) = -[A(t,\alpha(t))\bar{p}(t) + M(t,\alpha(t))]\mathrm{d}t, \\ \bar{p}(T) = G. \end{cases} \tag{5.5}$$

The unique solution of (5.5) is given by

$$\bar{p}(t) = \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s} \left(\int_t^T M(s,\alpha(s))\mathrm{e}^{-\int_s^T A(r,\alpha(r))\mathrm{d}r}\mathrm{d}s + G \right).$$

Similarly, $(\bar{P}(t), \bar{Q}(t), \bar{R}(t,\cdot), \bar{S}(t)) = (0,0,0,0)$ is the unique solution of (5.4). The associated $\mathcal{H}^\theta$-function is

$$\begin{aligned} \bar{\mathcal{H}}^\theta(t,x,u,e_i) :=& \bar{p}(t-)[A(t,e_i)x + B(t,e_i)u + b(t,e_i)] + M(t,e_i)x \\ &+ \frac{1}{2}N(t,e_i)u^2 + \frac{1}{2}\theta\bar{p}(t-)^2\big(C(t,e_i)u + \sigma(t,e_i)\big)^2 \\ &+ \int_{\mathbb{R}_0} \eta(t,e_i,z)\Big[\Theta(t,z)\bar{p}(t-) - \frac{1}{2}\big(1+\Theta(t,z)\big)\theta\bar{p}(t-)^2\eta(t,e_i,z)\Big]\nu_\alpha(\mathrm{d}z) \\ &+ \sum_{j=1}^{D} \gamma^j(t,e_i)\Big[\Lambda_j(t)\bar{p}(t-) - \frac{1}{2}\big(1+\Lambda_j(t)\big)\theta\bar{p}(t-)^2\gamma^j(t,e_i)\Big]\lambda_j(t). \end{aligned}$$

By the maximum condition (3.9), we obtain a candidate optimal control

$$\bar{u}(t) = -\frac{B(t,\alpha(t-))\bar{p}(t-) + \theta C(t,\alpha(t-))\sigma(t,\alpha(t-))\bar{p}(t-)^2}{N(t,\alpha(t-)) + \theta C(t,\alpha(t-))^2\bar{p}(t-)^2}, \tag{5.6}$$

where $\bar{p}(\cdot)$ is determined by (5.5). By (5.1), (5.5) and (5.6), we can easily check assumption (5.2) holds for the candidate optimal pair $(\bar{x}(\cdot), u(\cdot))$.

Finally, we verify that, in the present case, the necessary conditions of optimality (Thm. 3.2) are also sufficient. In view of the sufficient conditions given in Theorem 3.4, it suffices to show the convexity of $g(x, e_i) = Gx$ and the joint convexity of the Hamiltonian $\bar{H}^\theta(t, x, u, e_i) = \bar{H}^\theta(t, x, u, e_i, \bar{p}(t-), \bar{q}(t), \bar{r}(t, \cdot), \bar{s}(t))$ in (x, u). Since g is linear in x, it is obviously convex. On the other hand,

$$\begin{aligned}\bar{H}^\theta(t, x, u, e_i) =& \bar{p}(t-)[A(t, e_i)x + B(t, e_i)u + b(t, e_i)] + M(t, e_i)x \\ &+ \frac{1}{2}N(t, e_i)u^2 + \frac{1}{2}\theta\bar{p}(t-)^2\big(C(t, e_i)u + \sigma(t, e_i)\big)^2 \\ &+ \int_{\mathbb{R}_0} \Theta(t, z)\bar{p}(t-)\eta(t, e_i, z)\nu_{e_i}(\mathrm{d}z) + \sum_{j=1}^{D} \Lambda_j(t)\bar{p}(t-)\gamma^j(t, e_i)\lambda_j(t)\end{aligned} \tag{5.7}$$

is quadratic in u and linear in x. Since $N(t, e_i)$ is positive, $\bar{H}^\theta(t, x, u, e_i)$ is convex with respect to (x, u). The above analysis yields the following theorem:

Theorem 5.1. *The optimal control for the risk-sensitive control problem* (5.1) *is given by*

$$\bar{u}(t) = -\frac{B(t, \alpha(t-))\bar{p}(t-) + \theta C(t, \alpha(t-))\sigma(t, \alpha(t-))\bar{p}(t-)^2}{N(t, \alpha(t-)) + \theta C(t, \alpha(t-))^2\bar{p}(t-)^2}, \tag{5.8}$$

with

$$\bar{p}(t) = \mathrm{e}^{\int_t^T A(s,\alpha(s))\mathrm{d}s}\left(\int_t^T M(s, \alpha(s))\mathrm{e}^{-\int_s^T A(r,\alpha(r))\mathrm{d}r}\mathrm{d}s + G\right).$$

5.2. Application to risk-sensitive benchmarked and asset management under regime-switching

In this section, we apply our stochastic maximum principle to solve a risk-sensitive benchmarked asset management problem in a Markov regime-switching financial market. We assume that $\gamma = \eta = 0$, that is we consider a Markov regime-switching diffusion factor model. Our model is inspired by that of [2] (see also [6]). More specifically, we assume that there is one economic factor x that determines the performance of the market and which evolves according to:

$$\mathrm{d}x(t) = \Big(b(t, \alpha(t)) + C(t, \alpha(t))x(t)\Big)\mathrm{d}t + \sigma(t, \alpha(t))\mathrm{d}W(t), \quad x(0) = x \in \mathbb{R}. \tag{5.9}$$

The dynamics of the risky asset S is governed by:

$$\frac{\mathrm{d}S(t)}{S(t)} = \Big(a(t, \alpha(t)) + A(t, \alpha(t))x(t)\Big)\mathrm{d}t + \xi(t, \alpha(t))\mathrm{d}Z(t), \quad S(0) = s \in \mathbb{R}, \tag{5.10}$$

where Z is a one-dimensional standard Brownian motion, correlated with W: $\mathrm{Cov}\big(Z(t), W(t)\big) = \rho_0 t$ in which ρ_0 is a constant valued in $[-1, 1]$. In addition, by standard Gaussian linear regression, $W(t)$ can be rewritten as $\mathrm{d}W(t) = \rho_0\mathrm{d}Z(t) + \rho\mathrm{d}B(t)$, where $\rho = \sqrt{1 - \rho_0^2}$ and $B(t)$ is a standard Brownian motion which is independent of $Z(t)$.

The dynamics of the riskless asset S_0 is given by:

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = \Big(a_0(t, \alpha(t)) + A_0(t, \alpha(t))x(t)\Big)\mathrm{d}t, \quad S_0(0) = s_0 \in \mathbb{R}. \tag{5.11}$$

Here we assume that all the functions $b(t, e_i)$, $C(t, e_i)$, $\sigma(t, e_i) > 0$, $a(t, e_i)$, $A(t, e_i)$, $\xi(t, e_i) > 0$, $a_0(t, e_i)$, $A_0(t, e_i)$ are deterministic continuous functions and uniformly bounded in t.

Let $\mathcal{G}_t = \sigma((S(s), L(s), x(s)), 0 \leq s \leq t)$ be the σ-field generated by the security, benchmark (defined below) and factor process up to time t. The proportion of wealth invested in the risky asset is defined by an $\mathbb{R}$-valued stochastic process $u(t)$.

Definition 5.2. An investment process $u(\cdot)$ is said to be admissible if the following conditions are satisfied:

(1) $u(\cdot)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t\geq 0}$;
(2) $P(\int_0^T |u(t)|^2 \mathrm{d}t < +\infty) = 1$;
(3) $E\chi_T^u = 1$, with χ_T^u given by (5.20).

The set of all admissible strategies on $[0,T]$ is denoted by $\mathcal{U}[0,T]$.

Let $X^u(t)$ be the investor's wealth at time t corresponding to the portfolio strategy u. Then, by the self financing condition, $X^u(t)$ evolves according to:

$$\begin{aligned}\frac{\mathrm{d}X^u(t)}{X^u(t)} =& u(t)\frac{\mathrm{d}S(t)}{S(t)} + (1-u(t))\frac{\mathrm{d}S_0(t)}{S_0(t)}\\ =&\Big[a_0(t,\alpha(t)) + A_0(t,\alpha(t))x(t) + \big(\tilde{a}(t,\alpha(t)) + \tilde{A}(t,\alpha(t))x(t)\big)u(t)\Big]\mathrm{d}t\\ &+ \xi(t,\alpha(t))u(t)\mathrm{d}Z(t),\end{aligned} \tag{5.12}$$

with $X^u(0) = x_0$, $\tilde{a}(t,\alpha(t)) = a(t,\alpha(t)) - a_0(t,\alpha(t))$ and $\tilde{A}(t,\alpha(t)) = A(t,\alpha(t)) - A_0(t,\alpha(t))$.

Here, we are also interested in portfolio management relative to the performance of a benchmark. The benchmark also depends on the economic factor x. It can for example represent the wealth process of some investors or the value process of a non-traded asset. We assume that the dynamics of $L(t)$ satisfies the following SDE

$$\frac{\mathrm{d}L(t)}{L(t)} = \Big(a_1(t,\alpha(t)) + A_1(t,\alpha(t))x(t)\Big)\mathrm{d}t + \beta(t,\alpha(t))\mathrm{d}Z(t), \quad L(0) = l \in \mathbb{R}, \tag{5.13}$$

where $a_1(t,e_i)$, $A_1(t,e_i)$, $\beta(t,e_i) > 0$ are deterministic continuous functions and uniformly bounded in t.

In risk-sensitive control, the investor's objective is to select a control strategy $u(t)$ to maximize the criterion

$$J(x_0,u,e_i;\theta) := -\frac{2}{\theta}\ln E\Big[\mathrm{e}^{-\frac{\theta}{2}F(T,u)}\Big], \tag{5.14}$$

where F is a given reward function, and the risk-sensitive parameter $\theta > 0$ is an exogenous parameter representing the investor's degree of risk aversion. A Taylor expansion of (5.14) around $\theta = 0$ evidences the vital role played by the risk-sensitive parameter:

$$J(x_0,u,e_i;\theta) := E[F(T,u)] - \frac{\theta}{4}Var[F(T,u)] + o(\theta^2). \tag{5.15}$$

This criterion amounts to maximizing $E[F(T,u)]$ subject to a penalty for variance; for a general overview; see, for example, [23].

Similar to [6], since we look at the performance of the portfolio relative to the benchmark, we consider the log of excess return of the asset portfolio over its benchmark, that is, $F(T,u)$ is defined as

$$F(T,u) := \ln\frac{X^u(T)}{L(T)} = \ln X^u(T) - \ln L(T). \tag{5.16}$$

By Itô's formula, the log of excess return in response to a strategy u is

$$\begin{aligned}F(T,u) =& \ln\frac{x_0}{l} + \int_0^T \Big[a_0(t,\alpha(t)) + A_0(t,\alpha(t))x(t) + \big(\tilde{a}(t,\alpha(t)) + \tilde{A}(t,\alpha(t))x(t)\big)u(t)\\ &- \frac{1}{2}\xi(t,\alpha(t))^2u(t)^2 - a_1(t,\alpha(t)) - A_1(t,\alpha(t))x(t) + \frac{1}{2}\beta(t,\alpha(t))^2\Big]\mathrm{d}t\\ &+ \int_0^T \Big(\xi(t,\alpha(t))u(t) - \beta(t,\alpha(t))\Big)\mathrm{d}Z(t).\end{aligned} \tag{5.17}$$

Hence

$$\mathrm{e}^{-\frac{\theta}{2}F(T,u)} = \left(\frac{x_0}{l}\right)^{-\frac{\theta}{2}} \exp\left\{\theta\int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\right\}\chi_T^u, \tag{5.18}$$

where

$$\begin{aligned} f(t,x,u,e_i) =& \frac{1}{4}\left(\frac{\theta}{2}+1\right)\xi(t,e_i)^2u^2 - \frac{1}{2}a_0(t,e_i) - \frac{1}{2}A_0(t,e_i)x - \frac{1}{2}\big(\tilde{a}(t,e_i)+\tilde{A}(t,e_i)x\big)u \\ &+ \frac{1}{2}a_1(t,e_i) + \frac{1}{2}A_1(t,e_i)x - \frac{\theta}{4}\xi(t,e_i)\beta(t,e_i)u + \frac{1}{4}\left(\frac{\theta}{2}-1\right)\beta(t,e_i)^2, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \chi_T^u =& \exp\bigg\{-\frac{\theta}{2}\int_0^T \Big(\xi(t,\alpha(t))u(t)-\beta(t,\alpha(t))\Big)\mathrm{d}Z(t) \\ &- \frac{1}{2}\left(\frac{\theta}{2}\right)^2\int_0^T \Big(\xi(t,\alpha(t))u(t)-\beta(t,\alpha(t))\Big)^2\mathrm{d}t\bigg\}. \end{aligned} \tag{5.20}$$

For each $u(\cdot)\in\mathcal{U}[0,T]$, define a new probability measure P^u on $\mathcal{G}_T$ via

$$\frac{\mathrm{d}P^u}{\mathrm{d}P} := \chi_T^u. \tag{5.21}$$

From Girsanov's theorem, it follows that

$$Z^u(t) = Z(t) + \frac{\theta}{2}\int_0^t \Big(\xi(s,\alpha(s))u(s)-\beta(s,\alpha(s))\Big)\mathrm{d}s \tag{5.22}$$

is a standard Brownian motion under the measure P^u. Thus, the dynamics of $x(t)$ under P^u is

$$\begin{aligned} \mathrm{d}x(t) =& \left(b(t,\alpha(t)) + C(t,\alpha(t))x(t) - \frac{\theta}{2}\rho_0\sigma(t,\alpha(t))\big(\xi(t,\alpha(t))u(t)-\beta(t,\alpha(t))\big)\right)\mathrm{d}t \\ &+ \rho_0\sigma(t,\alpha(t))\mathrm{d}Z^u(t) + \rho\sigma(t,\alpha(t))\mathrm{d}B(t). \end{aligned} \tag{5.23}$$

We can now introduce the auxiliary criterion functional under the measure P^u:

$$\tilde{J}(x_0,x,u,e_i;\theta) = \mathrm{E}^u\left[\exp\left\{\theta\int_0^T f(t,x(t),u(t),\alpha(t))\mathrm{d}t\right\}\right], \tag{5.24}$$

where $\mathrm{E}^u[\cdot]$ denotes the expectation taken with respect to the measure P^u. Clearly, our original problem has the same optimal strategy as minimizing (5.24) subject to (5.23).

Similar to the previous subsection, it is not hard to check that assumptions $(\mathcal{A}1)$–$(\mathcal{A}6)$ made in Section 2 are satisfied. Hence, we can apply Theorem 3.2 and 3.4 to solve problem (5.23)–(5.24). Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be an optimal pair. First of all, the first order adjoint equation (3.2) reduces to

$$\begin{cases} d\bar{p}(t) = -\big[C(t,\alpha(t))\bar{p}(t) + \theta\sigma(t,\alpha(t))\bar{p}(t)(\rho_0\bar{q}_1(t)+\rho\bar{q}_2(t)) \\ \qquad\qquad + \frac{1}{2}A_1(t,\alpha(t)) - \frac{1}{2}A_0(t,\alpha(t)) - \frac{1}{2}\tilde{A}(t,\alpha(t))\bar{u}(t)\big]\mathrm{d}t \\ \qquad\qquad + \bar{q}_1(t)\mathrm{d}Z^u(t) + \bar{q}_2(t)\mathrm{d}B(t) + \sum_{j=1}^{D}\bar{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ \bar{p}(T) = 0. \end{cases} \tag{5.25}$$

Since the diffusion coefficients in the dynamics of the factor (5.23) is control independent, the second order adjoint variables disappear automatically and the Hamiltonian function (3.7) takes the form

$$\begin{aligned}\bar{\mathcal{H}}^{\theta}(t,x,u,e_i) = &\left(b(t,e_i)+C(t,e_i)x-\frac{\theta}{2}\rho_0\sigma(t,e_i)\big(\xi(t,e_i)u-\beta(t,e_i)\big)\right)\bar{p}(t)\\ &+\frac{1}{4}\left(\frac{\theta}{2}+1\right)\xi(t,e_i)^2u^2-\frac{1}{2}a_0(t,e_i)-\frac{1}{2}A_0(t,e_i)x-\frac{1}{2}\big(\tilde{a}(t,e_i)+\tilde{A}(t,e_i)x\big)u\\ &+\frac{1}{2}a_1(t,e_i)+\frac{1}{2}A_1(t,e_i)x-\frac{\theta}{4}\xi(t,e_i)\beta(t,e_i)u+\frac{1}{4}\left(\frac{\theta}{2}-1\right)\beta(t,e_i)^2\\ &+\sigma(t,e_i)(\rho_0\bar{q}_1(t)+\rho\bar{q}_2(t))+\frac{1}{2}\theta\sigma(t,e_i)^2\bar{p}(t)^2.\end{aligned}\tag{5.26}$$

By the maximum condition (3.9), we obtain

$$\bar{u}(t)=\frac{2\theta\rho_0\sigma(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))}\bar{p}(t)+\frac{2\tilde{A}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))^2}\bar{x}(t)+\frac{2\tilde{a}(t,\alpha(t))+\theta\xi(t,\alpha(t))\beta(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))^2}.\tag{5.27}$$

Substituting (5.27) into the SDE (5.23) and BSDE (5.25) yield

$$\begin{cases} d\bar{x}(t)=\Big(b(t,\alpha(t))+C(t,\alpha(t))\bar{x}(t)-\dfrac{\rho_0^2\theta^2\sigma(t,\alpha(t))^2}{\theta+2}\bar{p}(t)-\dfrac{\rho_0\theta\sigma(t,\alpha(t))\tilde{A}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))}\bar{x}(t)\\ \qquad -\dfrac{\rho_0\theta\sigma(t,\alpha(t))\tilde{a}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))}+\dfrac{\rho_0^2\theta\sigma(t,\alpha(t))\beta(t,\alpha(t))}{\theta+2}\Big)\mathrm{d}t\\ \qquad +\rho_0\sigma(t,\alpha(t))\mathrm{d}Z^u(t)+\rho\sigma(t,\alpha(t))\mathrm{d}B(t),\\ d\bar{p}(t)=-\Big(C(t,\alpha(t))\bar{p}(t)+\theta\sigma(t,\alpha(t))\bar{p}(t)(\rho_0\bar{q}_1(t)+\rho\bar{q}_2(t))+\dfrac{1}{2}A_1(t,\alpha(t))-\dfrac{1}{2}A_0(t,\alpha(t))\\ \qquad -\dfrac{\rho_0\theta\sigma(t,\alpha(t))\tilde{A}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))}\bar{p}(t)-\dfrac{\tilde{A}(t,\alpha(t))^2}{(\theta+2)\xi(t,\alpha(t))^2}\bar{x}(t)-\dfrac{\tilde{a}(t,\alpha(t))\tilde{A}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))^2}\\ \qquad -\dfrac{\theta\beta(t,\alpha(t))\tilde{A}(t,\alpha(t))}{2(\theta+2)\xi(t,\alpha(t))}\Big)\mathrm{d}t+\bar{q}_1(t)\mathrm{d}Z^u(t)+\bar{q}_2(t)\mathrm{d}B(t)+\sum\limits_{j=1}^{D}\bar{s}_j(t)\mathrm{d}\widetilde{\Phi}_j(t),\\ \bar{x}(0)=x,\quad \bar{p}(T)=0. \end{cases}\tag{5.28}$$

Therefore, an optimal pair $(\bar{x}(\cdot),\bar{u}(\cdot))$ can be obtained by solving the above FBSDE (forward-backward stochastic differential equation). We conjecture the solution of (5.28) to be in the following form:

$$\bar{p}(t)=\phi(t,\alpha(t))\bar{x}(t)+\varphi(t,\alpha(t)),\tag{5.29}$$

where $\phi(t,e_i),\varphi(t,e_i),i=1,\ldots,D$, are deterministic and differentiable functions which are to be determined. From (5.28), ϕ,φ must satisfy the following terminal boundary condition:

$$\phi(T,e_i)=0\qquad\text{and}\qquad\varphi(T,e_i)=0,\quad i=1,\ldots,D.$$

Applying Itô's formula to the right-hand side of (5.29) yields

$$
\begin{aligned}
\mathrm{d}\bar{p}(t) = \Bigg\{ & \phi'(t,\alpha(t))\bar{x}(t) + \varphi'(t,\alpha(t)) + \phi(t,\alpha(t)) \Bigg[b(t,\alpha(t)) + C(t,\alpha(t))\bar{x}(t) \\
& - \frac{\rho_0^2\theta^2\sigma(t,\alpha(t))^2}{\theta+2}\bar{p}(t) - \frac{\rho_0\theta\sigma(t,\alpha(t))\tilde{A}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))}\bar{x}(t) - \frac{\rho_0\theta\sigma(t,\alpha(t))\tilde{a}(t,\alpha(t))}{(\theta+2)\xi(t,\alpha(t))} \\
& + \frac{\rho_0^2\theta\sigma(t,\alpha(t))\beta(t,\alpha(t))}{\theta+2} \Bigg] + \sum_{j=1}^{D} \Big[\big(\phi(t,e_j) - \phi(t,\alpha(t))\big)\bar{x}(t) + \varphi(t,e_j) - \varphi(t,\alpha(t)) \Big] \lambda_j(t) \Bigg\} \mathrm{d}t \\
& + \rho_0\phi(t,\alpha(t))\sigma(t,\alpha(t))dZ^u(t) + \rho\phi(t,\alpha(t))\sigma(t,\alpha(t))\mathrm{d}B(t) \\
& + \sum_{j=1}^{D} \Big[\big(\phi(t,e_j) - \phi(t,\alpha(t))\big)\bar{x}(t) + \varphi(t,e_j) - \varphi(t,\alpha(t)) \Big] \mathrm{d}\widetilde{\Phi}_j(t).
\end{aligned}
\tag{5.30}
$$

Comparing the coefficients with (5.28), we get

$$\bar{q}_1(t) = \rho_0\phi(t,\alpha(t))\sigma(t,\alpha(t)), \tag{5.31}$$

$$\bar{q}_2(t) = \rho\phi(t,\alpha(t))\sigma(t,\alpha(t)), \tag{5.32}$$

$$\bar{s}_j(t) = \big(\phi(t,e_j) - \phi(t,\alpha(t))\big)\bar{x}(t) + \varphi(t,e_j) - \varphi(t,\alpha(t)). \tag{5.33}$$

and

$$
\begin{cases}
\phi'(t,e_i) + \dfrac{\rho^2\theta^2+2\theta}{\theta+2}\sigma(t,e_i)^2\phi(t,e_i)^2 + 2\left(C(t,e_i) - \dfrac{\rho_0\theta\sigma(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)} \right)\phi(t,e_i) \\
\qquad - \dfrac{\tilde{A}(t,e_i)^2}{(\theta+2)\xi(t,e_i)^2} + \sum_{j=1}^{D} \lambda_{ij}\big(\phi(t,e_j) - \phi(t,e_i)\big) = 0, \\
\phi(T,e_i) = 0,
\end{cases}
\tag{5.34}
$$

$$
\begin{cases}
\varphi'(t,e_i) + \left(C(t,e_i) - \dfrac{\rho_0\theta\sigma(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)} + \dfrac{\rho^2\theta^2+2\theta}{\theta+2}\sigma(t,e_i)^2\phi(t,e_i) \right)\varphi(t,e_i) \\
\qquad + \left(b(t,e_i) - \dfrac{\rho_0\theta\sigma(t,e_i)\tilde{a}(t,e_i)}{(\theta+2)\xi(t,e_i)} + \dfrac{\rho_0^2\theta\sigma(t,e_i)\beta(t,e_i)}{\theta+2} \right)\phi(t,e_i) \\
\qquad + \dfrac{1}{2}A_1(t,e_i) - \dfrac{1}{2}A_0(t,e_i) - \dfrac{\tilde{a}(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)^2} \\
\qquad - \dfrac{\theta\beta(t,e_i)\tilde{A}(t,e_i)}{2(\theta+2)\xi(t,e_i)} + \sum_{j=1}^{D} \lambda_{ij}\big(\varphi(t,e_j) - \varphi(t,e_i)\big) = 0, \\
\varphi(T,e_i) = 0.
\end{cases}
\tag{5.35}
$$

Equation (5.34) is a Markov regime-switching Riccati ordinary differential equation (ODE) and is difficult to solve in general. However, we can obtain a representation of the solution of (5.34) using Radon's lemma (see Lem. A.1), which provides a general relation between linear systems of differential equations and Riccati differential equations. Radon's lemma has been used in the literature (see [4,18]) to turn a matrix-valued Riccati ODE into a solvable linear system of ODEs.

Using the same argument as in ([12], Eq. 4.13) to deal with a system of coupled ODEs, we define for each $i = 1, 2, \ldots, D$,

$$\begin{aligned}\mathit{\Delta}(t,e_i) =& C(t,e_i) - \frac{\rho_0\theta\sigma(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)} + \frac{1}{2}\lambda_{ii},\\ \mathit{\Upsilon}(t,e_i) =& \frac{\rho^2\theta^2+2\theta}{\theta+2}\sigma(t,e_i)^2,\\ \mathit{\Pi}(t,e_i) =& -\frac{\tilde{A}(t,e_i)^2}{(\theta+2)\xi(t,e_i)^2} + \sum_{j=1,j\neq i}^{D}\lambda_{ij}\phi(t,e_j).\end{aligned}$$

It follows from equation (5.34) that $\phi(t, e_i)$ satisfies the following Riccati ODE:

$$\begin{cases}\phi'(t,e_i) + \mathit{\Upsilon}(t,e_i)\phi(t,e_i)^2 + 2\mathit{\Delta}(t,e_i)\phi(t,e_i) + \mathit{\Pi}(t,e_i) = 0,\\ \phi(T,e_i) = 0.\end{cases} \tag{5.36}$$

Then by Lemma A.1, we obtain (set $\tau := T - t$)

$$\phi(t,e_i) = \frac{\tilde{R}_2(\tau,e_i)}{\tilde{R}_1(\tau,e_i)}, \tag{5.37}$$

where $\tilde{R}_1(0,e_i) = 1$, $\tilde{R}_2(0,e_i) = 0$ and $\tilde{R}(\tau,e_i) = (\tilde{R}_1(\tau,e_i), \tilde{R}_2(\tau,e_i))^\top$ satisfies the following linear system of ODEs:

$$\frac{\mathrm{d}\tilde{R}(\tau,e_i)}{\mathrm{d}\tau} = \begin{pmatrix}-\mathit{\Delta}(T-\tau,e_i) & -\mathit{\Upsilon}(T-\tau,e_i)\\ \mathit{\Pi}(T-\tau,e_i) & \mathit{\Delta}(T-\tau,e_i)\end{pmatrix}\tilde{R}(\tau,e_i). \tag{5.38}$$

Once the solution of $\phi(t, e_i)$ is derived, we obtain by Itô's formula that

$$\mathrm{e}^{\int_t^T \mathit{\Gamma}(s,\alpha(s))\mathrm{d}s}\varphi(T,\alpha(T)) = \varphi(t,\alpha(t)) + \int_t^T \mathrm{e}^{\int_t^s \mathit{\Gamma}(r,\alpha(r))\mathrm{d}r} M(s,\alpha(s))\mathrm{d}s + N(t), \tag{5.39}$$

where

$$\begin{aligned}M(t,e_i) =& \ \varphi'(t,e_i) + \mathit{\Gamma}(t,e_i)\varphi(s,e_i) + \sum_{j=1}^{D}\lambda_{ij}\big(\varphi(t,e_j)-\varphi(t,e_i)\big),\\ N(t) =& \sum_{j=1}^{D}\int_t^T \mathrm{e}^{\int_t^s \mathit{\Gamma}(r,\alpha(r))\mathrm{d}r}\big(\varphi(s,e_j)-\varphi(s,\alpha(s-))\big)\mathrm{d}\widetilde{\mathit{\Phi}}_j(s),\\ \mathit{\Gamma}(t,e_i) =& \ C(t,e_i) - \frac{\rho_0\theta\sigma(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)} + \frac{\rho^2\theta^2+2\theta}{\theta+2}\sigma(t,e_i)^2\phi(t,e_i),\\ \mathit{\Sigma}(t,e_i) =& \left(b(t,e_i) - \frac{\rho_0\theta\sigma(t,e_i)\tilde{a}(t,e_i)}{(\theta+2)\xi(t,e_i)} + \frac{\rho_0^2\theta\sigma(t,e_i)\beta(t,e_i)}{\theta+2}\right)\phi(t,e_i)\\ &+\frac{1}{2}A_1(t,e_i) - \frac{1}{2}A_0(t,e_i) - \frac{\tilde{a}(t,e_i)\tilde{A}(t,e_i)}{(\theta+2)\xi(t,e_i)^2} - \frac{\theta\beta(t,e_i)\tilde{A}(t,e_i)}{2(\theta+2)\xi(t,e_i)}.\end{aligned}$$

It is easy to check that $N(t)$ is a martingale under our assumptions. From (5.35), we know that $M(t,e_i) = -\mathit{\Sigma}(t,e_i)$ and $\varphi(T,\alpha(T)) = 0$. Then by taking expectation on both sides of (5.39), $\varphi(t,e_i)$ admits the following Feynman-Kac representation:

$$\varphi(t,e_i) = E\left[\int_t^T \mathrm{e}^{\int_t^s \mathit{\Gamma}(r,\alpha(r))\mathrm{d}r}\mathit{\Sigma}(s,\alpha(s))\mathrm{d}s\,\Big|\,\alpha(t)=e_i\right]. \tag{5.40}$$

With the above choice of $\phi(t,e_i)$ and $\varphi(t,e_i)$ (defined by (5.37) and (5.40), respectively), a candidate for the optimal control is

$$\begin{aligned}\bar{u}(t)=\frac{2}{(\theta+2)\xi(t,\alpha(t))^2}\Big[&\tilde{a}(t,\alpha(t))+\Big(\tilde{A}(t,\alpha(t))+\theta\xi(t,\alpha(t))\sigma(t,\alpha(t))\phi(t,\alpha(t))\Big)\bar{x}(t)\\&+\frac{\theta}{2}\xi(t,\alpha(t))\beta(t,\alpha(t))+\theta\xi(t,\alpha(t))\sigma(t,\alpha(t))\varphi(t,\alpha(t))\Big].\end{aligned}\tag{5.41}$$

Thus, from Theorem 3.4, to ensure that (5.41) is indeed optimal, we only need to check that $\bar{H}^\theta(t,x,u,e_i,\bar{p}(t),\bar{q}_1(t),\bar{q}_2(t),\bar{s}(t))$ is convex in (x,u). Obviously,

$$\begin{aligned}\bar{H}^\theta(t,x,u,e_i,\bar{p}(t),\bar{q}_1(t),\bar{q}_2(t),\bar{s}(t))&=\left(b(t,e_i)+C(t,e_i)x-\frac{\theta}{2}\rho_0\sigma(t,e_i)\big(\xi(t,e_i)u-\beta(t,e_i)\big)\right)\bar{p}(t)\\&+\frac{1}{4}\left(\frac{\theta}{2}+1\right)\xi(t,e_i)^2u^2-\frac{1}{2}a_0(t,e_i)-\frac{1}{2}A_0(t,e_i)x-\frac{1}{2}\big(\tilde{a}(t,e_i)+\tilde{A}(t,e_i)x\big)u\\&+\frac{1}{2}a_1(t,e_i)+\frac{1}{2}A_1(t,e_i)x-\frac{\theta}{4}\xi(t,e_i)\beta(t,e_i)u+\frac{1}{4}\left(\frac{\theta}{2}-1\right)\beta(t,e_i)^2\\&+\sigma(t,e_i)(\rho_0\bar{q}_1(t)+\rho\bar{q}_2(t))+\theta\sigma(t,e_i)^2\bar{p}(t)^2\end{aligned}\tag{5.42}$$

is quadratic in u and linear in x. Therefore, it is convex with respect to (x,u). The above analysis yields the following theorem:

Theorem 5.3. *The optimal portfolio strategy for the risk-sensitive benchmarked asset management problem (5.14) subject to (5.9), (5.12) and (5.13) is given in a feedback form by (5.41), where $\phi(t,\alpha(t))$ and $\varphi(t,\alpha(t))$ in (5.41) are defined by (5.37) and (5.40), respectively.*

From Theorem 5.3, we get easily the following corollary which corresponds to ([6], Thm. 1).

Corollary 5.4. *Assume that $\rho_0=D=1$, $b(t,\alpha(t))=b(t),C(t,\alpha(t))=C(t),a_1(t,\alpha(t))=a_1(t),A_1(t,\alpha(t))=A_1(t),\xi(t,\alpha(t))=\xi(t),\tilde{a}(t,\alpha(t))=\tilde{a}(t),\tilde{A}(t,\alpha(t))=\tilde{A}(t),\sigma(t,\alpha(t))=\sigma(t)$ and $\beta(t,\alpha(t))=\beta(t)$. Then the optimal portfolio strategy $\bar{u}(\cdot)$ reduces to*

$$\bar{u}(t)=\frac{2}{(\theta+2)\xi(t)^2}\left[\tilde{a}(t)+\Big(\tilde{A}(t)+\theta\xi(t)\sigma(t)\psi(t)\Big)\bar{x}(t)+\frac{\theta}{2}\xi(t)\beta(t)+\theta\xi(t)\sigma(t)\kappa(t)\right],\tag{5.43}$$

where $\psi(t)$ and $\kappa(t)$ are solutions of the following ordinary differential equations:

$$\begin{cases}\psi'(t)+\dfrac{2\theta}{\theta+2}\sigma(t)^2\psi(t)^2+2\left(C(t)-\dfrac{\theta\sigma(t)\tilde{A}(t)}{(\theta+2)\xi(t)}\right)\psi(t)-\dfrac{\tilde{A}(t)^2}{(\theta+2)\xi(t)^2}=0,\\ \psi(T)=0,\end{cases}\tag{5.44}$$

$$\begin{cases}\kappa'(t)+\left(C(t)-\dfrac{\theta\sigma(t)\tilde{A}(t)}{(\theta+2)\xi(t)}+\dfrac{2\theta}{\theta+2}\sigma(t)^2\psi(t)\right)\kappa(t)+\left(b(t)-\dfrac{\theta\sigma(t)\tilde{a}(t)}{(\theta+2)\xi(t)}+\dfrac{\theta\sigma(t)\beta(t)}{\theta+2}\right)\psi(t)\\ \qquad+\dfrac{1}{2}A_1(t)-\dfrac{1}{2}A_0(t)-\dfrac{\tilde{a}(t)\tilde{A}(t)}{(\theta+2)\xi(t)^2}-\dfrac{\theta\beta(t)\tilde{A}(t)}{2(\theta+2)\xi(t)}=0,\\ \kappa(T)=0.\end{cases}\tag{5.45}$$

Remark 5.5. The solution to equation (5.44) can also be obtained by Lemma A.1 and hence the solution to (5.45) will follow. Our result can be seen as an extension of ([6], Thm. 1) to the Markov regime-switching case. Note that the authors in [6] use the dynamic programming approach, which differs from ours.

6. Conclusion

In this paper, we have proved a sufficient and necessary risk-sensitive maximum principle in a general Markov regime-switching jump-diffusion setting by using a logarithmic transformation approach. As applications of the obtained results, we have discussed both a linear-quadratic optimal control problem and a risk-sensitive benchmarked asset management problem in a Markov regime-switching financial market. Note that in this work, the terminal condition and the Hamiltonian are all assumed to be convex. The latter is not always satisfied in some interesting applications (see [13, 15] for the classical maximum principle) and hence the risk-sensitive maximum principle for Markov regime-switching model with non-convex Hamiltonian is needed. It will also be interesting to solve the infinite horizon risk-sensitive maximum principle for Markov regime-switching system. We hope to address these problems in the future research.

Appendix A. Radon's lemma

The proposed solution to the Markov regime-switching Riccati equation (5.34) is based on the following Radon's lemma from ([1], Thm. 3.1.1).

Lemma A.1. *If $K(\cdot)$ is a differentiable function and satisfies the following Riccati ODE:*

$$\begin{cases} \dfrac{\mathrm{d}K(t)}{\mathrm{d}t} + \mu(t)K(t)^2 + 2\beta(t)K(t) + \omega(t) = 0, \\ K(T) = 0. \end{cases} \tag{A.1}$$

Here $\mu(t), \beta(t), \omega(t)$ are deterministic continuous functions, then the explicit solution of $K(\cdot)$ is given by $K(t) = \frac{R_2(\tau)}{R_1(\tau)}$, where $\tau := T - t$, $R_1(\tau), R_2(\tau)$ are differentiable and $R(\tau) = (R_1(\tau), R_2(\tau))^\top$ is the fundamental matrix solution of the following linear system of ODEs:

$$\frac{\mathrm{d}R(\tau)}{\mathrm{d}\tau} = \begin{pmatrix} -\beta(T-\tau) & -\mu(T-\tau) \\ \omega(T-\tau) & \beta(T-\tau) \end{pmatrix} R(\tau), \tag{A.2}$$

with initial conditions $R_1(0) = 1$ and $R_2(0) = 0$. Particularly, if all the coefficients $\mu(t), \beta(t), \omega(t)$ are constants, i.e. $\mu(t) = \mu, \beta(t) = \beta$ and $\omega(t) = \omega$, the solution of $R(\tau)$ is given by

$$R(\tau) = \exp\left[\begin{pmatrix} -\beta & -\mu \\ \omega & \beta \end{pmatrix}\tau\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{A.3}$$

Remark A.2. Since the matrix exponential in (A.2) is easy to compute given modern computing power, the Radon's lemma greatly simplifies the procedure of calculation.

Proof. From (A.2), we have

$$\begin{cases} R_1'(\tau) = \dfrac{\mathrm{d}R_1(\tau)}{\mathrm{d}\tau} = -\beta(T-\tau)R_1(\tau) - \mu(T-\tau)R_2(\tau), \\ R_2'(\tau) = \dfrac{\mathrm{d}R_2(\tau)}{\mathrm{d}\tau} = \omega(T-\tau)R_1(\tau) + \beta(T-\tau)R_2(\tau). \end{cases} \tag{A.4}$$

Then, by the above definition of $K(t)$, we get

$$\begin{aligned} \frac{\mathrm{d}K(t)}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{R_2(\tau)}{R_1(\tau)}\right)\cdot\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\left(\frac{R_2(\tau)}{R_1(\tau)}\right)' = -\frac{R_2'(\tau)R_1(\tau) - R_1'(\tau)R_2(\tau)}{R_1(\tau)^2} \\ =& -\omega(T-\tau) - 2\beta(T-\tau)\frac{R_2(\tau)}{R_1(\tau)} - \mu(T-\tau)\left(\frac{R_2(\tau)}{R_1(\tau)}\right)^2 \\ =& -\omega(t) - 2\beta(t)K(t) - \mu(t)K(t)^2. \end{aligned} \tag{A.5}$$

Hence, $K(t) = \frac{R_2(\tau)}{R_1(\tau)}$ is a solution of the Riccati ODE (A.1). □

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References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu and G. Jank, Matrix Riccati Equations in Control and Systems Theory. Springer, Basel-Boston-Berlin (2003).

[2] T.R. Bielecki and S.R. Pliska, Risk-sensitive dynamic asset management. *Appl. Math. Optimiz.* **39** (1999) 337–360.

[3] D. Charalambous and J.L. Hibey, Minimum principle for partially observable nonlinear risk-sensitive control problems using measure-valued decompositions. *Stochastics.* **57** (1996) 247–288.

[4] M.C. Chiu and H.Y. Wong, Mean-variance principle of managing cointegrated risky assets and random liabilities. *Oper. Res. Lett.* **41** (2013) 98–106.

[5] S. Crepey, About the Pricing Equations in Finance. Springer, Berlin (2010).

[6] M. Davis and S. Lleo, Risk-sensitive benchmarked asset management. *Quant. Financ.* **8** (2008) 415–426.

[7] M. Davis and S. Lleo, Jump-diffusion risk-sensitive asset management I: diffusion factor model. *SIAM J. Financ. Math.* **2** (2011) 22–54.

[8] M. Davis and S. Lleo, Jump-diffusion risk-sensitive asset management II: jump-diffusion factor model. *SIAM J. Control. Optimiz.* **51** (2013) 1441–1480.

[9] C. Donnelly, Sufficient stochastic maximum principle in a regime-switching diffusion model. *Appl. Math. Optimiz.* **64** (2011) 155–169.

[10] C. Donnelly and A.J. Heunis, Quadratic risk minimization in a regime-switching model with portfolio constraints. *SIAM J. Control. Optimiz.* **50** (2012) 2431–2461.

[11] R.J. Elliott, L. Aggoun and J.B. Moore, Hidden Markov Models: Estimation and Control. Springer, New York (1994).

[12] R.J. Elliott and T.K. Siu, A stochastic differential game for optimal investment of an insurer with regime switching. *Quant. Financ.* **11** (2011) 365–380.

[13] Y. Li and H. Zheng, Weak necessary and sufficient stochastic maximum principle for markovian regime-switching diffusion models. *Appl. Math. Optimiz.* **71** (2013) 1–39.

[14] A.E.B. Lim and X.Y. Zhou, A new risk-sensitive maximum principle. *IEEE T. Automat. Control.* **50** (2005) 958–966.

[15] O. Menoukeu−Pamen, Maximum principles of Markov regime-switching forward backward stochastic differential equations with jumps and partial information. arXiv:1403.2901 (2014).

[16] O. Menoukeu−Pamen and R. Momeya, A maximum principle for Markov regime-switching forward-backward stochastic differential games and applications. *Math. Meth. Oper. Res.* **85** (2017) 349–388.

[17] H. Nagai and S. Peng, Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon. *Ann. Appl. Probab.* **12** (2002) 173–195.

[18] Y. Shen, X. Zhang and T.K. Siu, Mean-variance portfolio selection under a constant elasticity of variance model. *Oper. Res. Lett.* **42** (2014) 337–342.

[19] J. Shi and Z. Wu, A risk-sensitive stochastic maximum principle for optimal control of jump diffusions and its applications. *Acta. Math. Sci.* **31** (2011) 419–433.

[20] Z. Sun, Maximum principle for forward-backward stochastic control system under G-expectation and relation to dynamic programming. *J. Comput. Appl. Math.* **296** (2016) 753–775.

[21] Z. Sun, J. Guo and X. Zhang, Maximum principle for Markov regime-switching forward-backward stochastic control system with jumps and relation to dynamic programming. *J. Optimiz. Theory. Appl.* DOI:10.1007/s10957-017-1068-5 (2017).

[22] S. Tang and X. Li, Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J. Control. Optimiz.* **32** (1994) 1447–1475.

[23] P. Whittle, Risk-Sensitive Optimal Control. Wiley, New York (1990).

[24] Q. Zhang, Stock trading: an optimal selling rule. *SIAM J. Control. Optimiz.* **40** (2001) 64–87.

[25] X. Zhang, R.J. Elliott and T.K. Siu, A stochastic maximum principle for a markov regime-switching jump-diffusion model and its application to finance. *SIAM J. Control. Optimiz.* **50** (2012) 964–990.