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UPPER SEMICONTINUITY OF THE LAMINATION HULL *

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Abstract. Let $K \subseteq \mathbb{R}^{2\times 2}$ be a compact set, let K^{rc} be its rank-one convex hull, and let $L(K)$ be its lamination convex hull. It is shown that the mapping $K \mapsto \overline{L(K)}$ is not upper semicontinuous on the diagonal matrices in $\mathbb{R}^{2\times 2}$, which was a problem left by Kolář. This is followed by an example of a 5-point set of 2×2 symmetric matrices with non-compact lamination hull. Finally, another 5-point set K is constructed, which has $L(K)$ connected, compact and strictly smaller than K^{rc}.

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1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the space of $m \times n$ matrices with real entries. Two matrices $X, Y \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(X - Y) = 1$ are called *rank-one connected*. A set $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ is *lamination convex* if

$$\lambda X + (1-\lambda)Y \in \mathcal{S} \quad \text{for all} \quad \lambda \in [0,1],$$

whenever $X, Y \in \mathcal{S}$ are rank-one connected. For a set $K \subseteq \mathbb{R}^{m\times n}$, the smallest lamination convex set containing K is denoted by $L(K)$.

This work contains a counterexample to a question posed in [5], concerning the continuity of the mapping $K \mapsto \overline{L(K)}$ on $\mathbb{R}^{2\times 2}$. The example is similar to Example 2.2 in [1]. This is followed by a 5-point set K of symmetric 2×2 matrices with non-compact $L(K)$, similar to Example 2.4 in [5]. Then, another 5-point set K is constructed which has $L(K)$ connected, compact and strictly smaller than K^{rc}. This is contrasted with Proposition 2.5 in [8], which says that $K^{pc} = L(K) = K$ if K is connected, compact and has no rank-one connections. Finally, a weaker version of this result is given for sets with rank-one connections.

2. Main results

Define the Hausdorff distance between two compact sets K_1, K_2 in $\mathbb{R}^{m\times n}$ by

$$\rho(K_1, K_2) = \inf\{\epsilon \geq 0 : K_1 \subseteq U_\epsilon(K_2) \text{ and } K_2 \subseteq U_\epsilon(K_1)\},$$

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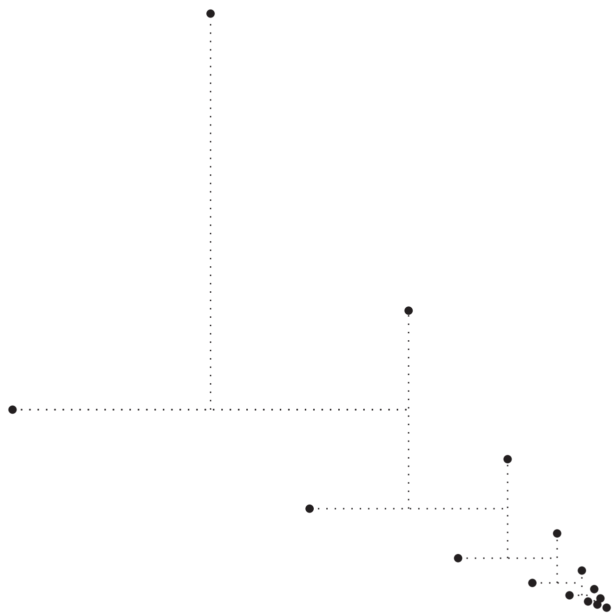

Figure 1: The set K_0 from Theorem 2.1. The dotted lines are rank-one lines in $L(K)$, where K is a small perturbation of K_0.

where $U_\epsilon(K)$ is the open ϵ-neighbourhood of K, corresponding to the Euclidean distance. Let $\mathcal{K}$ be the set of compact subsets of $\mathbb{R}^{m\times n}$. A function $f : \mathcal{K} \to \mathcal{K}$ is upper semicontinuous if for every $\epsilon > 0$ and for every $K_0 \in \mathcal{K}$, there exists a $\delta > 0$ such that $f(K) \subseteq U_\epsilon(f(K_0))$ whenever $\rho(K, K_0) < \delta$. It is known that the function $K \mapsto K^{rc}$ is upper semicontinuous on the compact subsets of $\mathbb{R}^{m\times n}$ (see for example the proof of Thm. 1 in [7], Example 4.18 in [4], or Thm. 3.2 in [9]). The following example (pictured in Fig. 1) shows that this fails on diagonal matrices in $\mathbb{R}^{2\times 2}$, for the lamination convex hull.

Theorem 2.1. *There exists a compact set K_0 of diagonal matrices in $\mathbb{R}^{2\times 2}$ such that the mapping $K \mapsto \overline{L(K)}$ is not upper semicontinuous at K_0.*

Proof. Identify the space of 2×2 diagonal matrices with $\mathbb{R}^2$ in the natural way. Let

$$K_0 = \{(1,0)\} \cup \bigcup_{n=0}^{\infty} \left\{ \left(1 - \frac{3}{2^{n+1}}, \frac{1}{2^{n+1}}\right), \quad \left(1 - \frac{1}{2^n}, \frac{3}{2^{n+1}}\right) \right\}.$$

The set K_0 is compact and has no rank-one connections, thus $L(K_0) = K_0$. For each integer $n \geq -1$ let

$$P_n = \left(1 - \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right).$$

Given $\delta > 0$, choose a positive integer N large enough to ensure that $\frac{1}{2^N} < \delta$, and let $K = K_0 \cup \{P_N\}$, so that $\rho(K, K_0) < \delta$. Then

$$\left(1 - \frac{1}{2^N}, \frac{1}{2^{N+1}}\right) = \frac{1}{2}\left(1 - \frac{3}{2^{N+1}}, \frac{1}{2^{N+1}}\right) + \frac{1}{2} P_N \in L(K),$$

and hence

$$P_{N-1} = \frac{1}{2}\left(1 - \frac{1}{2^N}, \frac{1}{2^{N+1}}\right) + \frac{1}{2}\left(1 - \frac{1}{2^N}, \frac{3}{2^{N+1}}\right) \in L(K).$$

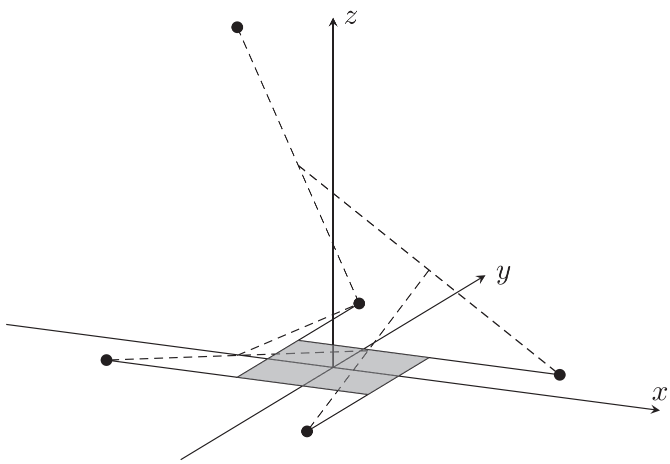

Figure 2: A 5-point set $K \subseteq \mathbb{R}^{2\times 2}_{\text{tri}}$ together with 5 rank-one lines in $L(K)$. The dashed lines indicate rank-one lines in $L(K)$, which spiral toward the diagonal plane and make $L(K)$ non-compact.

It follows by induction that $(0,1) = P_{-1} \in L(K)$. Since $\rho(P_{-1}, L(K_0)) \geq \frac{1}{2}$, this shows that the function $K \mapsto \overline{L(K)}$ is not upper semicontinuous at K_0. $\square$

The next result gives two examples of 5-point subsets of $\mathbb{R}^{2\times 2}$, each with a non-compact lamination hull. The upper-triangular example is pictured in Figure 2. It consists of 4 points in the diagonal plane arranged in a T_4 configuration, together with a point whose projection onto the diagonal plane is a corner of the inner rectangle of the T_4 configuration.

Throughout, the upper triangular matrix $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ will be identified with the point $(x, y, z) \in \mathbb{R}^3$. The symmetric example uses essentially the same idea as in Figure 2, so the matrix $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ will also be denoted by the point $(x, y, z) \in \mathbb{R}^3$. Since the cases are treated separately, the notations do not conflict. The symmetric notation also differs from the usual identification, used for example in [5]. The space of 2×2 upper triangular matrices is denoted by $\mathbb{R}^{2\times 2}_{\text{tri}}$, and the space of 2×2 symmetric matrices by $\mathbb{R}^{2\times 2}_{\text{sym}}$. Up to linear isomorphisms preserving rank-one directions, these are the only two 3-dimensional subspaces of $\mathbb{R}^{2\times 2}$ (see [2], Cor. 6 or [6], Lem. 3.1)

Theorem 2.2.

(i) *There exists a 5-point set* $K \subseteq \mathbb{R}^{2\times 2}_{\text{tri}}$ *such that* $L(K)$ *is not compact.*
(ii) *There exists a 5-point set* $K \subseteq \mathbb{R}^{2\times 2}_{\text{sym}}$ *such that* $L(K)$ *is not compact.*

Proof. For part (i) let $x_1 < x_2$, $y_2 < y_1$, $z_0 > 0$ and $\alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0$. Let

$$P_0 = (x_1, y_2, 0), \quad P_1 = (x_1, y_1, 0), \quad P_2 = (x_2, y_1, 0), \quad P_3 = (x_2, y_2, 0),$$

and set

$$A_0 = (x_1, y_1 + \alpha_0, 0), \quad A_1 = (x_2 + \alpha_1, y_1, 0),$$
$$A_2 = (x_2, y_2 - \alpha_2, 0), \quad A_3 = (x_1 - \alpha_3, y_2, 0).$$

For $i \in \{0, 1, 2, 3\}$ let $A_4 = A_0$ and

$$\lambda_i = \frac{\det(A_i - A_{i+1})}{\det(A_i - A_{i+1}) - \det(P_i - A_{i+1})} \in (0, 1),$$

let $X_0 = P_0 + (0,0,z_0)$ and $K = \{A_0, A_1, A_2, A_3, X_0\}$. For $i \geq 0$ let

$$X_{i+1} = (1 - \lambda_{i \bmod 4})A_{i \bmod 4} + \lambda_{i \bmod 4} X_i,$$

so that for $i \geq 0$ and $k \in \{0,1,2,3\}$, induction gives

$$X_{4i+k} = P_k + (\lambda_0\lambda_1\lambda_2\lambda_3)^i \left(\prod_{j=0}^{k-1} \lambda_j\right)(0,0,z_0), \quad \det(X_i - A_{i \bmod 4}) = 0,$$

which implies that $X_i \in L(K)$ for every $i \geq 0$. Hence $P_0 \in \overline{L(K)}$, and it remains to show that $P_0 \notin L(K)$. This follows from the fact that

$$\{(x,y,z) \in \mathbb{R}^{2\times 2}_{\mathrm{tri}} : z > 0\} \cup \{A_0, A_1, A_2, A_3\}$$

is a lamination convex set containing K, which does not contain P_0.

For part (ii), let all the scalars and diagonal points be the same as in part (i). Using the symmetric notation let $Y_0 = P_0 + (\xi_1, \xi_2, \xi_3)$ where $\xi_3 > 0$ and

$$\xi_1 = \frac{1}{2}\left(-\alpha_3 + \sqrt{\alpha_3^2 - \frac{4\alpha_3\xi_3^2}{y_1 + \alpha_0 - y_2}}\right), \quad \xi_2 = \frac{-\xi_1(y_1 + \alpha_0 - y_2)}{\alpha_3}.$$

so that $\det(Y_0 - A_0) = \det(Y_0 - A_3) = 0$, and $Y_0 \to P_0$ as $\xi_3 \to 0$. The fact that $\det(P_0 - A_1) > 0 > \det(A_0 - A_1)$ means that

$$\det(Y_0 - A_1) > 0 > \det(A_0 - A_1),$$

whenever $\xi_3 \in (0, \epsilon_1)$, for some $\epsilon_1 > 0$. Set $B_0 = Y_0$. For $i \in \{0,1,2,3\}$ and B_i with

$$\det(B_i - A_{i+1}) \neq 0 \text{ and } \operatorname{sgn}\det(B_i - A_{i+1}) \neq \operatorname{sgn}\det(A_i - A_{i+1}),$$

let

$$B_{i+1} = (1 - t_i)A_i + t_i B_i, \text{ where } t_i = \frac{\det(A_i - A_{i+1})}{\det(A_i - A_{i+1}) - \det(B_i - A_{i+1})} \in (0,1),$$

so that $\det(B_{i+1} - A_{i+1}) = 0$. By induction $t_i \to \lambda_i$ as $\xi_3 \to 0$ for $i \in \{0,1,2,3\}$, $B_i \to P_{i \bmod 4}$ as $\xi_3 \to 0$ for each $i \in \{0,1,2,3,4\}$, and B_1, B_2, B_3, B_4 all exist if ξ_3 is sufficiently small. Hence there exists $\epsilon_2 > 0$ such that $(t_0t_1t_2t_3) < \frac{1}{2}(1 + \lambda_0\lambda_1\lambda_2\lambda_3)$ and B_1, B_2, B_3, B_4 all exist whenever $\xi_3 \in (0, \epsilon_2)$. Put $(\eta_1, \eta_2, \eta_3) = B_4 - P_0$. Then since $\det(B_4 - A_0) = \det(B_4 - A_3) = 0$,

$$\eta_1 = \frac{1}{2}\left(-\alpha_3 \pm \sqrt{\alpha_3^2 - \frac{4\alpha_3\eta_3^2}{y_1 + \alpha_0 - y_2}}\right), \quad \eta_2 = \frac{-\eta_1(y_1 + \alpha_0 - y_2)}{\alpha_3}. \tag{2.1}$$

But since $B_4 \to P_0$ as $\xi_3 \to 0$, there exists $\epsilon_3 > 0$ such that the sign in (2.1) is positive whenever $\xi_3 \in (0, \epsilon_3)$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. If $\xi_3 \in (0, \epsilon)$. then

$$\eta_3 = (t_0t_1t_2t_3)\xi_3 < \frac{1}{2}(1 + \lambda_0\lambda_1\lambda_2\lambda_3)\xi_3. \tag{2.2}$$

Therefore let $K = \{A_0, A_1, A_2, A_3, Y_0\}$, and set $Y_1 = B_4$. Then $Y_1 \in \underline{L(K)}$ by the preceding working. By (2.2), iterating this process gives a sequence $Y_n \in L(K)$ with $Y_n \to P_0 \in \overline{L(K)}$. Again the point P_0 is not in $L(K)$ since

$$\{(x,y,z) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}} : z > 0\} \cup \{A_0, A_1, A_2, A_3\}$$

is a lamination convex set separating P_0 from K. Hence $L(K)$ is not compact. $\square$

A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *rank-one convex* if

$$f(\lambda X + (1-\lambda)Y) \le \lambda f(X) + (1-\lambda) f(Y) \text{ for all } \lambda \in [0,1],$$

whenever $\operatorname{rank}(X-Y) \le 1$. The rank-one convex hull of a compact set $K \subseteq \mathbb{R}^{m\times n}$ is defined by

$$K^{rc} = \{X \in \mathbb{R}^{m\times n} : f(X) \le 0 \quad \forall \text{ rank-one convex } f \text{ with } f|_K \le 0\}.$$

The polyconvex hull is defined similarly via polyconvex functions; a function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is polyconvex if there exists a convex function $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $f(X) = g(X, \det X)$ for all $X \in \mathbb{R}^{2\times 2}$. For compact K, the following characterisation of K^{pc} will be used (see Theorem 1.9 in [4]):

$$K^{pc} = \{\overline{\mu} : \mu \in \mathscr{M}_{pc}(K)\}, \tag{2.3}$$

where $\mathscr{M}_{pc}(K)$ is the class of probability measures supported in K which satisfy Jensen's inequality for all polyconvex f;

$$f(\overline{\mu}) \le \int_{\mathbb{R}^{2\times 2}} f(X)\,\mathrm{d}\mu(X) \quad \text{ where } \quad \overline{\mu} = \int_{\mathbb{R}^{2\times 2}} X\,\mathrm{d}\mu(X).$$

Definition 2.3. An ordered set $\{X_i\}_{i=1}^4 \subseteq \mathbb{R}^{m\times n}$ without rank-one connections is called a T_4 configuration if there exist matrices $P, C_1, C_2, C_3, C_4 \in \mathbb{R}^{m\times n}$ and real numbers $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4 > 1$ satisfying

$$\operatorname{rank} C_i = 1 \text{ for } 1 \le i \le 4, \quad \sum_{i=1}^4 C_i = 0,$$

and

$$\begin{aligned} X_1 &= P + \mu_1 C_1 \\ X_2 &= P + C_1 + \mu_2 C_2 \\ X_3 &= P + C_1 + C_2 + \mu_3 C_3 \\ X_4 &= P + C_1 + C_2 + C_3 + \mu_4 C_4. \end{aligned} \tag{2.4}$$

An unordered set $\{X_i\}_{i=1}^4$ is a T_4 configuration if it has at least one ordering which is a T_4 configuration.

The following result is a slight generalisation of Theorem 1 in [7] (see also Cor. 3 in [3]). The proof is similar to the one in [7], with minor technical changes.

Theorem 2.4. *If $K \subseteq \mathbb{R}^{2\times 2}$ is compact, and does not have a T_4 configuration $\{X_i\}_{i=1}^4$ with at least two X_i, X_j in distinct connected components of $L(K)$, then*

$$K^{rc} = \bigcup_i (U_i \cap K)^{rc} \quad \text{and} \quad K^{qc} = \bigcup_i (U_i \cap K)^{qc},$$

where the U_i are the connected components of $L(K)$.

On diagonal matrices the conclusion reduces to $K^{rc} = L(K)$. The following proposition shows that this fails in the full space $\mathbb{R}^{2\times 2}$.

Proposition 2.5. *There exists a 5-point set $K \subseteq \mathbb{R}^{2\times 2}$ with $L(K)$ connected, compact and strictly smaller than K^{rc}.*

Proof. Fix $\epsilon \in (0,1)$, let

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -\epsilon & -1 \\ -\epsilon^2 & -\epsilon \end{pmatrix}, \quad X_4 = \begin{pmatrix} -\epsilon & \epsilon^2 \\ 1 & -\epsilon \end{pmatrix},$$

and let

$$\mu_1 = \frac{1+2\epsilon}{\epsilon(1-\epsilon^2)}, \quad \mu_2 = 1 + \epsilon^2 \mu_1, \quad \mu_3 = 1 + \left(\frac{1+\epsilon^2}{\epsilon} \right) \mu_2, \quad \mu_4 = 1 + \epsilon^2 \mu_3, \tag{2.5}$$

so that

$$\mu_1 = 1 + \frac{\mu_4}{\epsilon(1+\epsilon^2)}\cdot \tag{2.6}$$

Set

$$P_1 = \frac{1}{\epsilon(\mu_1 - 1)} \begin{pmatrix} -\epsilon & 0 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{\mu_1 \epsilon} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$P_3 = \frac{1}{\mu_2} \begin{pmatrix} 0 & 0 \\ \epsilon & 1 \end{pmatrix}, \quad P_4 = \frac{1}{\mu_3 \epsilon} \begin{pmatrix} -\epsilon^2 & -\epsilon \\ \epsilon & 1 \end{pmatrix},$$

and let $C_i = P_{i+1} - P_i$, where $P_5 := P_1$. Then clearly $\operatorname{rank} C_i = 1$ for all i, whilst (2.5) and (2.6) imply that this is a solution of (2.4). Let

$$K = \{0, X_1, X_2, X_3, X_4\}, \qquad \text{so that} \qquad L(K) = \bigcup_{i=1}^{4} [0, X_i].$$

To prove the second formula for $L(K)$, it suffices to show that the set $\mathcal{S} = \bigcup_{i=1}^{4} [0, X_i]$ is lamination convex. For $i \neq j$, the fact that $\det X_i = \det X_j = 0$ and $\det(X_i - X_j) \neq 0$ implies that $\det(X_i - tX_j) \neq 0$ whenever $t \in (0,1]$, since the determinant is linear along rank-one lines. It follows similarly that $\det(sX_i - tX_j) \neq 0$ for $s, t \in (0,1]$, and so the only rank-one connected pairs in $\mathcal{S}$ are 0 and tX_i for any i. Hence $\mathcal{S}$ is lamination convex. By Lemma 2 in [7], the point P_1 is in $K^{rc} \setminus L(K)$, so this proves the proposition. □

The preceding example contrasts with Lemma 3 in [8], which states (in a weakened form) that $K^{pc} = K$ if K is a connected compact subset of $\mathbb{R}^{2\times 2}$ without rank-one connections. The example shows that the assumption that K has no rank-one connections cannot be weakened to $L(K) = K$. The reason is that $\det(X - Y)$ cannot change sign on connected subsets of $\mathbb{R}^{2\times 2}$ without rank-one connections, whilst it can on lamination convex sets. If the assumption that $\det(X - Y)$ does not change sign is added, K^{pc} is equal to the lamination hull of order 2: given a set $K \subseteq \mathbb{R}^{m\times n}$, let $L^{(0)}(K) = K$ and define $L^{(k)}(K)$ inductively by

$$L^{(k+1)}(K) = \bigcup_{\substack{X,Y \in L^{(k)}(K) \\ \operatorname{rank}(X-Y) \leq 1}} [X, Y].$$

Proposition 2.6. *If $K \subseteq \mathbb{R}^{2\times 2}$ is a compact set such that $\det(X - Y) \geq 0$ for every $X, Y \in K$, then $K^{pc} = L^{(2)}(K)$.*

Proof. If μ is a probability measure supported in K with $\det \overline{\mu} = \int_{\mathbb{R}^{2\times 2}} \det X \, d\mu$, then as in [8],

$$\int_{\mathbb{R}^{2\times 2}} \int_{\mathbb{R}^{2\times 2}} \det(X - Y) \, \mathrm{d}\mu(X) \, \mathrm{d}\mu(Y) = 0,$$

and therefore $\det(X - Y) = 0$ whenever X and Y are in the support of μ. This implies (see the following Lem. 2.7) that the support of μ is contained in a 2-dimensional affine plane P consisting only of rank-one directions. Therefore $\overline{\mu} \in (K \cap P)^{co}$, and so Carathéodory's Theorem gives 3 points $X_i \in K \cap P$ such that $\overline{\mu}$ is a convex combination $\overline{\mu} = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, and without loss of generality $\lambda_1 \neq 0$. Then $\frac{\lambda_1}{\lambda_1+\lambda_2} \cdot X_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot X_2 \in P \cap L^{(1)}(K)$ since P is a plane consisting of rank-one directions, and similarly

$$\overline{\mu} = (\lambda_1 + \lambda_2) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot X_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot X_2 \right) + \lambda_3 X_3 \in L^{(2)}(K).$$

It follows from (2.3) that $K^{pc} = L^{(2)}(K)$. □

Lemma 2.7. *Let* $X_0, Y_0 \in \mathbb{R}^{m \times n}$ *satisfy* $\operatorname{rank}(X_0 - Y_0) = 1$*, and let*

$$\mathcal{S} = \{X \in \mathbb{R}^{m \times n} : \operatorname{rank}(X - X_0) \leq 1 \textit{ and } \operatorname{rank}(X - Y_0) \leq 1\}.$$

Then:

(i) $\mathcal{S} = P_1 \cup P_2$*, where* P_1 *is an* m*-dimensional affine plane and* P_2 *is an* n*-dimensional affine plane, and for each fixed* i, $\operatorname{rank}(X - Y) \leq 1$ *for* $X, Y \in P_i$.
(ii) *The planes* P_1 *and* P_2 *satisfy*

$$\operatorname{rank}(X - Y) > 1 \quad \textit{for} \quad X \in P_1 \setminus P_2 \quad \textit{and} \quad Y \in P_2 \setminus P_1.$$

Proof. By translation invariance it may be assumed that $Y_0 = 0$, so that $\operatorname{rank} X_0 = 1$ and $X_0 = v_0 w_0^T$ for some nonzero $v_0 \in \mathbb{R}^m$, $w_0 \in \mathbb{R}^n$. Let

$$P_1 = \{x w_0^T : x \in \mathbb{R}^m\}, \quad P_2 = \{v_0 y^T : y \in \mathbb{R}^n\}.$$

If $X \in \mathcal{S}$ then $X = vw^T$ for some $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, and

$$X - X_0 = vw^T - v_0 w_0^T = ab^T, \tag{2.7}$$

for some $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. Suppose for a contradiction that $X \notin P_1 \cup P_2$. Then since $X \notin P_1$ there exists a vector $w_0^\perp$ such that $\langle w_0, w_0^\perp \rangle = 0$ and $\langle w, w_0^\perp \rangle \neq 0$. Right multiplying both sides of (2.7) with $w_0^\perp$ gives

$$v = \frac{\langle b, w_0^\perp \rangle a}{\langle w, w_0^\perp \rangle}, \qquad \text{and similarly} \qquad w = \frac{\langle a, v_0^\perp \rangle b}{\langle v, v_0^\perp \rangle}.$$

Let $\lambda = \frac{\langle a, v_0^\perp \rangle \langle b, w_0^\perp \rangle}{\langle v, v_0^\perp \rangle \langle w, w_0^\perp \rangle}$. Then $\lambda \neq 1$ by (2.7) since $v_0 w_0^T \neq 0$, and therefore

$$X = vw^T = \left(\frac{\lambda}{\lambda - 1} \right) v_0 w_0^T \in P_1 \cap P_2,$$

which is a contradiction. This proves part (i).

For part (ii), let $X = x w_0^T \in P_1 \setminus P_2$, let $Y = v_0 y^T \in P_2 \setminus P_1$ and suppose for a contradiction that $\operatorname{rank}(X - Y) = 1$. Then by part (i), $Y = xz^T$ for some nonzero $z \in \mathbb{R}^n$, and therefore $x = \frac{v_0 \langle y, z \rangle}{\|z\|^2}$, which contradicts the fact that $X \notin P_2$. □

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