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ANISOTROPIC FREE-DISCONTINUITY FUNCTIONALS AS THE Γ-LIMIT OF SECOND-ORDER ELLIPTIC FUNCTIONALS

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Abstract. We provide an approximation result for free-discontinuity functionals of the form

$$\mathcal{F}(u)=\int_\Omega f(x,u,\nabla u)\mathrm{d}x+\int_{S_u\cap\Omega}\theta(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1},\quad u\in SBV^2(\Omega),$$

where f is quadratic in the gradient-variable and θ is an arbitrary smooth Finsler metric. The approximating functionals are of Ambrosio-Tortorelli type and depend on the Hessian of the edge variable through a suitable nonhomogeneous metric ϕ.

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1. Introduction

The partitioning process known as image segmentation is a major task in image analysis. This process deals with the detection of objects and object contours in possibly distorted digital images and aims, among others, to distintuish between meaningful objects and the so-called noise. Mathematically, a grey-scale image can be represented by a scalar function g defined on the image domain Ω (*e.g.*, a rectangle) such that, at each point $x\in\Omega$, $g(x)$ measures the brightness of the image. Since the function g is by nature discontinuous, an image-segmentation problem consists in finding a function u which is still discontinuous, but whose discontinuities are now located only along the relevant object-contours. Further, the function u must provide a smooth approximation of g far from the image contours. Thus the discontinuity set of u decomposes the picture into regions with relatively uniform brightness and the function u is interpreted as a restored version of the input image g. In mathematical terms an image-segmentation problem can be recast into a variational framework by minimizing a so-called free-discontinuity functional, *i.e.*, a functional consisting of competing volume and surface terms. A prototipical free-discontinuity functional used in image segmentation is the celebrated Mumford-Shah functional [28], *i.e.*,

$$\mathcal{MS}(u)=\int_\Omega|\nabla u|^2\mathrm{d}x+\mathcal{H}^{n-1}(S_u)+\int_\Omega|u-g|^2\mathrm{d}x,\tag{1.1}$$

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where the last term in (1.1) represents a so-called "fidelity term", since it measures the fidelity of the restored image u to the input image g. Then the natural functional space where to set the problem is $SBV(\Omega)$: the space of special functions of bounded variation introduced by De Giorgi and Ambrosio [23]. When one looks for minimizers of (1.1), the first term in (1.1) penalizes strong variations of u outside its discontinuity set S_u, while due to the surface term in (1.1) this discontinuity set has to be as "small" as possible. Clearly, the surface term in (1.1) only allows to control the "length" of S_u and it provides no information on its shape. For this reason, functionals as in (1.1) are in general not suited to analyze, *e.g.*, certain biomedical images, where the detection of small structure with a specific geometry may play a crucial role (this is the case, *e.g.*, of thin tubular structures such as blood vessels in MRI). In particular, to remove noise from such images without deleting the important small structures, one needs to consider a more general free-discontinuity functional of the form

$$\mathcal{E}(u) = \int_\Omega f(x,u,\nabla u)\mathrm{d}x + \int_{S_u\cap\Omega} \theta(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{1.2}$$

where $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to [0,+\infty)$ is a Borel function, continuous in (u,ξ), convex and quadratic in ξ, while $\theta : \Omega\times\mathbb{R}^n \to [0,+\infty)$ is a convex smooth Finsler metric. (In (1.2) and in what follows we drop the fidelity term as it represents a "lower-order term" in our analysis.) A functional of type (1.2) which, in particular, may distinguish between noise and sets having a relevant geometry, is the following anisotropic variant of the Mumford-Shah functional:

$$\mathcal{MS}^{anis}(u) = \int_\Omega |\nabla u|^2\mathrm{d}x + \int_{S_u\cap\Omega} \sqrt{\langle M(x)\nu_u,\nu_u\rangle}\mathrm{d}\mathcal{H}^{n-1}, \tag{1.3}$$

where $M : \Omega \to \mathbb{R}^{n\times n}_{sym}$ is a Riemannian metric associated to the input image g. In [30] the author studies existence of minimizers of (1.3) and provides an elliptic approximation of (1.3) *via* Γ-convergence. Indeed, despite the fact that the existence theory for minimizers of functionals of the form (1.2) is by now quite well understood [2,3,4], it remains difficult to compute these minimizers numerically in a robust and efficient way. For this reason, one wishes to approximate functionals as in (1.2) with sequences of functionals defined on spaces of smoother functions. In the case where the function θ does not depend explicitly on the space variable x, an elliptic approximation of (1.2) *via* Γ-convergence is obtained in [20]. Both the approximations in [30] and [20] rely on the original Ambrosio and Tortorelli approximation [6,7] of the Mumford-Shah functional in the spirit of [26,27]. In [7] the authors introduced the family of elliptic functionals

$$\mathcal{AT}_\varepsilon(u,v) := \int_\Omega (v^2+\eta_\varepsilon)|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int \frac{(v-1)^2}{\varepsilon} + \varepsilon|\nabla v|^2\mathrm{d}x, \quad u,v\in W^{1,2}(\Omega) \tag{1.4}$$

(here $\varepsilon > 0$ and $0 < \eta_\varepsilon \ll \varepsilon$) and study its limit as ε tends to zero. Intuitively, in this model the factor $1/\varepsilon$ in the second integral requires that $v \approx 1$ in large regions of Ω, as $\varepsilon \to 0$, while the first integral forces the so-called edge variable v to approach zero in those regions where the gradient of u is steep. Hence, one expects that v approaches $1-\chi_{S_u}$, thus detecting the discontinuity set S_u. In [20] this approach has been extended to approximate functionals as in (1.2) when $\theta = \theta(\nu)$, defining the approximating functionals by

$$\mathcal{E}_\varepsilon(u,v) = \int_\Omega (v^2+\eta_\varepsilon)f(x,u,\nabla u)\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon\theta^2(\nabla v)\mathrm{d}x, \quad u,v\in W^{1,2}(\Omega).$$

In the present paper we want to investigate a second-order approximation of (1.2), that is, an approximation where the approximating functionals depend on the Hessian matrix of the edge variable v. In the last years, second-order models have been used in the setting of Cahn-Hilliard phase transitions to approximate the perimeter functionals [8,16,17,21] and recently this approach has been also extended to the Ambrosio-Tortorelli approximation of the Mumford-Shah functional [15] (see also [19]). More precisely, in [15] the authors prove that the functionals defined in (1.4) Γ-converge to $\mathcal{MS}$ also when $\varepsilon|\nabla v|^2$ is replaced by $\varepsilon^3|\nabla^2 v|^2$ or $\varepsilon^3|\Delta v|^2$.

Moreover, the computational results for these second-order models indicate several advantages if compared with the Ambrosio-Tortorelli approximation. Then, in view of the good numerical results obtained in [15] we propose here a second-order model to approximate functionals of the general form (1.2) with θ now explicitly depending on x. It is worth pointing out that considering a singular perturbation depending on $\nabla^2 v$ instead of ∇v makes the problem intrinsically different from the ones studied in [20] and [30]. Indeed, now it is less clear how to relate the $n \times n$ matrix $\nabla^2 v$ to the vector $\nu_u \in S^{n-1}$ appearing in the anisotropic limit model (1.2), since θ is defined on vectors. Therefore, we cannot use the same strategy as in [20], but we have to find a suitable convex metric ϕ defined on $n \times n$-matrices (for more details see Sects. 2 and 3) to define our approximating functionals. To give an intuitive idea of the construction of ϕ, let us assume for a moment that ϕ is given. Then we can apply ϕ to the tensor product $\nu \otimes \nu \in \mathbb{R}^{n \times n}$, for $\nu \in \mathbb{R}^n$. Since $\phi(x, \nu \otimes \nu)$ is 2-homogeneous and quadratic in ν, $\sqrt{\phi(x, \nu \otimes \nu)}$ defines a Finsler metric on Ω, which though may fail to be convex in ν (notice that the convexity of θ is a necessary condition for the lower semicontinuity of $\mathcal{E}$, which otherwise could not be a Γ-limit). The above consideration suggests, nevertheless, that given θ in (1.2) we should look for a metric ϕ such that

$$\sqrt{\phi(x, \nu \otimes \nu)} = \theta(x, \nu) \tag{1.5}$$

is satisfied for all x and ν. Suppose now that we have found such a metric ϕ. Then, in this mathematical framework a common procedure to reduce the n-dimensional problem to the one-dimensional problem is the so-called slicing-procedure. In the anisotropic setting, usually this procedure heavily relies on the properties of the dual (and the bidual) of the metric θ (see, *e.g.*, [13], Prop. 4.3). Indeed, the main property one uses is the fact that a convex Finsler metric can be written as

$$\theta(x, \nu) = \sup_{|\xi|=1} \frac{|\langle \xi, \nu \rangle|}{\theta_\circ(x, \xi)},$$

where $\theta_\circ : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is the dual of θ. This suggests that, in addition to (1.5), our metric ϕ should satisfy also

$$\sqrt{\phi_\circ(x, \xi \otimes \xi)} = \theta_\circ(x, \xi)$$

for all x and ξ, where $\phi_\circ$ denotes the dual of ϕ with respect to the matrix scalar product. Combining these considerations then motivates the following definition

$$\phi(x, A) = \sup_{\xi, \nu \in S^{n-1}} \frac{|\langle A\nu, \xi \rangle|}{\theta_\circ(x, \xi)\theta_\circ(x, \nu)},$$

and the choice of the approximating functionals

$$\mathcal{F}_\varepsilon(u, v) = \int_\Omega (v^2 + \eta_\varepsilon) f(x, u, \nabla u) \mathrm{d}x + \frac{1}{2\sqrt{2}} \int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v) \mathrm{d}x,$$

where $u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$. Then the main result of the paper is a Γ-convergence result which asserts that, when $\varepsilon \to 0$, the family $(\mathcal{F}_\varepsilon)$ Γ-converges in the strong $(L^1(\Omega) \times L^1(\Omega))$-topology to the functional $\mathcal{E}$ defined in (1.2). To prove this result, we first investigate the one-dimensional problem, which already contains the main information needed for the n-dimensional analysis. We then deduce the n-dimensional result from the one-dimensional result appealing to the above-mentioned slicing-procedure in combination with the blow-up method of Fonseca and Müller [22]. The latter will allow us to deal with the inhomogeneity in the surface part of the functionals $\mathcal{F}_\varepsilon$, which otherwise could not be treated by means of a slicing-procedure.

The structure of the paper is as follows. In Section 2 we fix the notation needed throughout the paper and recall some preliminary results. In Section 3 we state the main result of the paper and we also discuss some examples of Finsler metrics θ for which an explicit formula for the corresponding metric ϕ can be given. Those examples include the case $\theta(x, \nu) = \sqrt{\langle M(x)\nu, \nu \rangle}$ thus providing a second-order approximation of (1.3). Section 4 is devoted to the study of the so-called optimal profile problem, a minimization problem which describes the minimal cost of a transition from the value 0 to the value 1 in terms of the one-dimensional unscaled "Modica-Mortola part" of the energy. The analysis of this optimal profile problem then enables us to investigate the one-dimensional case in Section 5. Eventually, using the tools mentioned above, in Section 6 we prove the Γ-convergence result in dimension n. In Section 7 we finally study the existence of minimizers of $\mathcal{F}_\varepsilon$ and their asymptotic behavior.

2. Notation and preliminaries

In this section we fix the notation and recall some preliminary results that we will use in the following. We start with some basic notation. Let $n \geq 1$. Throughout this paper Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary and $\mathcal{A}(\Omega)$ the family of open subsets of Ω. If $A', A \in \mathcal{A}(\Omega)$ are such that $A' \subset\subset A$, we say that φ is a cut-off function between A' and A if $\varphi \in C_c^\infty(A)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on A'. By $\mathcal{L}^n$ and $\mathcal{H}^k$ we denote the Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. For $p \in [1, +\infty]$ we use standard notation for the Lebesgue spaces $L^p(\Omega)$ and the Sobolev spaces $W^{1,p}(\Omega)$. Let $\nu, \xi \in \mathbb{R}^n$; we use the notation $\langle \nu, \xi \rangle$ for the scalar product, $|\nu|$ for the euclidian norm and $\nu \otimes \xi = (\nu_i \xi_j)_{ij} \in \mathbb{R}^{n\times n}$ denotes the matrix consisting of the entrywise products of ν and ξ. If $A, B \in \mathbb{R}^{n\times n}$, then $AB \in \mathbb{R}^{n\times n}$ denotes the matrix product, $A : B = \mathrm{tr}(B^T A)$ the scalar product, and $\|A\| = \sqrt{A : A}$ the euclidian norm. We set $S^{n-1} := \{\nu \in \mathbb{R}^n : \; |\nu| = 1\}$ and for every $\nu \in S^{n-1}, x_0 \in \mathbb{R}^n$, and $\rho > 0$ we denote by $Q_\rho^\nu(x_0) \subset \mathbb{R}^n$ the cube centered at x_0 with side length ρ and with one face orthogonal to ν. If $x_0 = 0$ and $\rho = 1$ we simply write Q^ν.

We also recall some notation and basic results concerning measure theory and special functions of bounded variation. For the general theory we refer to [5]. Let $\mathcal{M}_b(\Omega)$ be the set of all bounded Radon measures on Ω. Then we say that a sequence (μ_k) in $\mathcal{M}_b(\Omega)$ converges weakly* in $\mathcal{M}_b(\Omega)$ to $\mu \in \mathcal{M}_b(\Omega)$ ($\mu_k \overset{*}{\rightharpoonup} \mu$) if

$$\int_\Omega \varphi \mathrm{d}\mu_k \to \int_\Omega \varphi \mathrm{d}\mu \quad \forall\, \varphi \in C_0(\Omega).$$

For every $u \in SBV(\Omega)$ we write ∇u for the approximate gradient of u, S_u for the approximate discontinuity set of u, and ν_u for the generalized outer normal to S_u. Finally, u^+ and u^- are the traces of u on S_u. We enlarge the space $SBV(\Omega)$ to the space $GSBV(\Omega)$ which consists of all functions $u \in L^1(\Omega)$ such that for each $M \in \mathbb{N}$ the truncated function $u^M := -M \vee (u \wedge M)$ belongs to $SBV(\Omega)$. Moreover, we set

$$SBV^2(\Omega) := \{u \in SBV(\Omega) : \nabla u \in L^2(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}$$

and

$$GSBV^2(\Omega) := \{u \in GSBV(\Omega) : \nabla u \in L^2(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Clearly, we have $SBV^2(\Omega) \cap L^\infty(\Omega) = GSBV^2(\Omega) \cap L^\infty(\Omega)$.

We say that an integrand $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ belongs to the class $\mathcal{I} = \mathcal{I}(m_1, m_2)$ if f satisfies the following conditions.

(i) f is a Borel function;
(ii) for a.e. $x \in \Omega$, $f(x, \cdot, \cdot)$ is continuous;
(iii) for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$, $f(x, u, \cdot)$ is convex;
(iv) there exist constants $0 < m_1 \leq m_2 < +\infty$ such that

$$m_1|\xi|^2 \leq f(x, u, \xi) \leq m_2|\xi|^2 \quad \forall\, (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

The following two results are consequences of ([2], Thm. 2.1 and [25], Thm. 1.2), respectively.

Theorem 2.1. *Let (u_k) be a sequence in $GSBV(\Omega)$ such that $\|u_k\|_{L^1}$, $\|\nabla u_k\|_{L^2}$ and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then there exist $u \in GSBV(\Omega)$ and a subsequence (u_{k_j}) such that $u_{k_j} \to u$ a.e. in Ω. In addition, $\nabla u_{k_j} \rightharpoonup \nabla u$ weakly in $L^1(\Omega; \mathbb{R}^n)$.*

Theorem 2.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ belong to the class $\mathcal{I}$ and let (u_k) be a sequence in $GSBV(\Omega)$, $u \in GSBV(\Omega)$ such that $u_k \to u$ a.e. in Ω, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega, \mathbb{R}^n)$ and such that $\|\nabla u_k\|_{L^2}$ and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then*

$$\int_\Omega f(x, u, \nabla u)\mathrm{d}x \le \liminf_{k\to+\infty} \int_\Omega f(x, u_k, \nabla u_k)\mathrm{d}x.$$

Moreover, we recall the following interpolation inequality that can be found in ([1], Thm. 4.14).

Proposition 2.3. *Let $U \subset \mathbb{R}^n$ be open, bounded and with Lipschitz boundary and let $\varepsilon_0 > 0$. Then there exists a constant $c_0 = c_0(\varepsilon_0, U) > 0$ such that*

$$c_0\varepsilon \int_U |\nabla v|^2\mathrm{d}x \le \frac{1}{\varepsilon}\int_U v^2\mathrm{d}x + \varepsilon^3 \int_U \|\nabla^2 v\|^2\mathrm{d}x$$

for every $0 < \varepsilon \le \varepsilon_0$ and for every $v \in W^{2,2}(U)$.

Furthermore, we briefly recall some basic properties of Finsler metrics, for more details see, *e.g.*, [9, 10, 11]. We call a function $\theta : \Omega \times \mathbb{R}^n \to [0, +\infty)$ a Finsler metric on Ω, if θ is continuous and satisfies the following properties:

$$\theta(x, t\nu) = |t|\theta(x, \nu) \quad \forall\ (x, \nu) \in \Omega \times \mathbb{R}^n, \quad \forall\ t \in \mathbb{R}, \tag{2.2}$$

$$\lambda|\nu| \le \theta(x, \nu) \le \Lambda|\nu| \quad \forall\ (x, \nu) \in \Omega \times \mathbb{R}^n, \tag{2.3}$$

for some $0 < \lambda \le \Lambda < +\infty$. If, in addition, for every $x \in \Omega$ the map $\nu \mapsto \theta(x, \nu)$ is convex, we say that θ is a convex Finsler metric. We denote by $\theta \in \mathscr{M}^n(\Omega)$ the class of all convex Finsler metrics. For every $\theta \in \mathscr{M}^n(\Omega)$ the dual $\theta_\circ : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is defined as

$$\theta_\circ(x, \xi) := \sup\{\langle \xi, \nu\rangle :\ \nu \in \mathbb{R}^n,\ \theta(x, \nu) \le 1\}.$$

It can be shown that $\theta_\circ \in \mathscr{M}^n(\Omega)$, $\theta_{\circ\circ} = \theta$, and that $\theta_\circ$ can be equivalently written as

$$\theta_\circ(x, \xi) := \sup_{|\nu|=1} \frac{|\langle \xi, \nu\rangle|}{\theta(x, \nu)}.$$

(Notice that in the literature usually the dual $\theta_\circ$ is defined for any Finsler metric θ. In that case, $\theta_\circ \in \mathscr{M}^n(\Omega)$ and $\theta_{\circ\circ}$ coincides with the convex envelope of θ.) Analogously, we denote by $\mathscr{M}^{n\times n}(\Omega)$ the class of all continuous functions $\phi : \Omega \times \mathbb{R}^{n\times n} \to [0, +\infty)$ satisfying

$$\phi(x, tA) = |t|\phi(x, A) \quad \forall\ (x, A) \in \Omega \times \mathbb{R}^{n\times n}, \quad \forall\ t \in \mathbb{R}, \tag{2.4}$$

$$m\|A\| \le \phi(x, A) \le M\|A\| \quad \forall\ (x, A) \in \Omega \times \mathbb{R}^{n\times n}, \tag{2.5}$$

for some $0 < m \le M < +\infty$ and such that for every $x \in \Omega$ the map $A \mapsto \phi(x, A)$ is convex on $\mathbb{R}^{n\times n}$. Finally, we define the dual of ϕ by

$$\phi_\circ(x, B) := \sup\{B : A :\ A \in \mathbb{R}^{n\times n},\ \phi(x, A) \le 1\} = \sup_{\|A\|=1} \frac{|B : A|}{\phi(x, A)}.$$

Again, it holds that $\phi_\circ \in \mathscr{M}^{n\times n}(\Omega)$ and $\phi_{\circ\circ} = \phi$.

Remark 2.4 (Continuity Properties of θ and ϕ)**.** Since we will heavily use them in what follows, we state here the following continuity properties that are satisfied by θ and ϕ, respectively. Let $\theta \in \mathscr{M}^n(\Omega)$. Then for every $\eta > 0$ and every $x_0 \in \Omega$, there exists $\delta = \delta(x_0, \eta) > 0$ depending only on x_0, η such that

$$|\theta(x_0, \nu) - \theta(x, \nu)| \le \eta \quad \forall\ \nu \in S^{n-1}, \quad \forall\ x \in B_\delta(x_0). \tag{2.6}$$

Further, if $\phi \in \mathscr{M}^{n\times n}(\Omega)$, using (2.4) and (2.5) it can be shown that ϕ^2 satisfies the following continuity hypotheses. For every $\eta > 0$ and every $x_0 \in \Omega$, there exists $\delta = \delta(x_0, \eta) > 0$ depending only on x_0, η such that

$$|\phi^2(x_0, A) - \phi^2(x, A)| \leq \eta\phi^2(x, A) \quad \forall\, A \in \mathbb{R}^{n\times n} \quad \forall\, x \in B_\delta(x_0). \tag{2.7}$$

We also state here a density result that can be found in [18] and that we will use to prove the limsup-inequality. We start by fixing some notation that has been introduced in [18]. We denote by $\mathcal{W}(\Omega)$ the set of all functions $u \in SBV^2(\Omega)$ satisfying the following properties

(a) S_u is essentially closed, *i.e.*, $\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0$;
(b) $\overline{S_u}$ is the intersection of Ω with the union of a finite number of pairwise disjoint closed and convex sets each contained in an $(n-1)$-dimensional hyperplane, and whose (relative) boundaries are C^∞;
(c) $u \in W^{k,\infty}(\Omega \setminus \overline{S_u})$ for all $k \in \mathbb{N}$.

Then the following is a consequence of ([18], Thm. 3.1, Rems. 3.2 and 3.3).

Theorem 2.5. *Let $\theta \in \mathscr{M}^n(\Omega)$ and let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (u_j) in $\mathcal{W}$ such that*

$$\begin{aligned}
&u_j \to u \text{ strongly in } L^1(\Omega),\\
&\nabla u_j \to \nabla u \text{ strongly in } L^2(\Omega;\mathbb{R}^n),\\
&\limsup_{j\to+\infty} \|u_j\|_{L^\infty} \leq \|u\|_{L^\infty},\\
&\lim_{j\to+\infty} \int_{\Omega\cap S_{u_j}} \theta(x, \nu_{u_j})\mathrm{d}\mathcal{H}^{n-1} = \int_{\Omega\cap S_u} \theta(x, \nu_u)\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

We finally introduce the functionals which will be considered in what follows. We set

$$\mathcal{MS}(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\mathrm{d}x + \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV^2(\Omega),\ v = 1 \text{ a.e.},\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{2.8}$$

and

$$\mathcal{F}(u,v) := \begin{cases} \displaystyle\int_\Omega f(x,u,\nabla u)\mathrm{d}x + \int_{S_u\cap\Omega} \theta(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in GSBV^2(\Omega),\ v = 1 \text{ a.e.},\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{2.9}$$

where we assume that f belongs to the class $\mathcal{I}$ and that $\theta \in \mathscr{M}^n(\Omega)$. Moreover, we consider the following functionals depending on a parameter $\varepsilon > 0$. Throughout this paper, the parameter $\varepsilon > 0$ varies in a strictly decreasing sequence converging to zero,

$$\mathcal{AT}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega v^2|\nabla u|^2\mathrm{d}x & \displaystyle +\frac{1}{2}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon|\nabla v|^2\mathrm{d}x\\ & \text{if } (u,v) \in W^{1,2}(\Omega)\times W^{1,2}(\Omega),\ 0 \leq v \leq 1,\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{2.10}$$

$$\mathcal{E}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega v^2|\nabla u|^2\mathrm{d}x & \displaystyle +\frac{1}{2\sqrt{2}}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\|\nabla^2 v\|^2\mathrm{d}x\\ & \text{if } (u,v) \in W^{1,2}(\Omega)\times W^{2,2}(\Omega),\ v\nabla u \in L^2(\Omega;\mathbb{R}^n),\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{2.11}$$

and

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega v^2 f(x,u,\nabla u)\mathrm{d}x & \displaystyle + \frac{1}{2\sqrt{2}}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2 v)\mathrm{d}x \\ & \text{if } (u,v)\in W^{1,2}(\Omega)\times W^{2,2}(\Omega),\ v\nabla u\in L^2(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases} \tag{2.12}$$

where $\phi \in \mathscr{M}^{n\times n}(\Omega)$.

Remark 2.6 (Domain of the Γ-limit)**.** In ([15], Thm. 4.2) it has been proved that the functionals $\mathcal{E}_\varepsilon$ $\Gamma(L^1)$-converge to $\mathcal{MS}$. Moreover, for any pair (u,v) in the domain of $\mathcal{F}_\varepsilon$, conditions (2.1) and (2.5) directly give that

$$\begin{aligned} m_2\int_\Omega v^2|\nabla u|^2\mathrm{d}x + \frac{M}{2\sqrt{2}}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\|\nabla^2 v\|^2\mathrm{d}x \\ \geq \mathcal{F}_\varepsilon(u,v) \geq m_1\int_\Omega v^2|\nabla u|^2\mathrm{d}x + \frac{m}{2\sqrt{2}}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\|\nabla^2 v\|^2\mathrm{d}x. \end{aligned}$$

Hence, if we set $\mathcal{F}'(u,v) := \Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u,v)$, $\mathcal{F}''(u,v) := \Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u,v)$, then ([15], Thm. 4.2) yields that $\mathcal{F}'(u,v) = \mathcal{F}''(u,v) = +\infty$, whenever (u,v) is not contained in the domain of $\mathcal{MS}$.

3. Statement of the main result

The main result of this paper is the following theorem, which ensures that under suitable conditions on ϕ the functionals $\mathcal{F}_\varepsilon$ defined in (2.12) Γ-converge with respect to the strong $(L^1(\Omega)\times L^1(\Omega))$-topology to the functional $\mathcal{F}$ defined as in (2.9).

Theorem 3.1. *Let $f\in\mathcal{I}$ and $\phi\in\mathscr{M}^{n\times n}(\Omega)$. Suppose that there exists $\theta\in\mathscr{M}^n(\Omega)$ such that*

$$\sqrt{\phi(x,\nu\otimes\nu)} = \theta(x,\nu), \tag{3.1}$$

$$\sqrt{\phi_\circ(x,\xi\otimes\xi)} = \theta_\circ(x,\xi), \tag{3.2}$$

for all $x\in\Omega$ and for all $\xi,\nu\in\mathbb{R}^n$. Then the functionals $\mathcal{F}_\varepsilon$ defined in (2.12) Γ-converge in the strong $\big(L^1(\Omega)\times L^1(\Omega)\big)$-topology to the functional $\mathcal{F}$ defined in (2.9).

We will prove Theorem 3.1 in Section 6 gathering Propositions 6.1 and 6.2, which give the liminf-inequality and the limsup-inequality, respectively. In particular the liminf-inequality heavily relies on conditions (3.1) and (3.2), which allow to obtain the n-dimesional result from the one-dimensional result *via* slicing. Since at a first glance (3.1) and (3.2) might appear quite restrictive we will briefly discuss them in the following remark. In particular, we will show that given an arbitrary $\theta\in\mathscr{M}^n(\Omega)$ we can always find a $\phi\in\mathscr{M}^{n\times n}(\Omega)$ such that (3.1) and (3.2) are fulfilled. Moreover, we discuss two explicit examples of metrics ϕ and θ that satisfy (3.1) and (3.2).

Remark 3.2. Let $\phi\in\mathscr{M}^{n\times n}(\Omega)$. Then we can associate a Finsler metric $\overline{\phi}$ to ϕ by setting

$$\overline{\phi}(x,\nu) := \sqrt{\phi(x,\nu\otimes\nu)} \quad \forall\ (x,\nu)\in\Omega\times\mathbb{R}^n. \tag{3.3}$$

Indeed, $\phi(x,\nu\otimes\nu)$ is 2-homogeneous and quadratic in ν. Thus $\overline{\phi}$ satisfies (2.2) and (2.3). It may, however, fail to be convex in ν. The convexity of the surface integrand being a necessary condition for the lower semicontinuity of the functional $\mathcal{F}$ defined as in (2.9), it seems quite natural to assume that $\overline{\phi}\in\mathscr{M}^n(\Omega)$. This is precisely what (3.1) requires. Then condition (3.2) reads as

$$(\overline{\phi})_\circ(x,\xi\otimes\xi) = \overline{\phi_\circ}(x,\xi) \quad \forall\ (x,\xi)\in\Omega\times\mathbb{R}^n, \tag{3.4}$$

where $\overline{\phi_\circ}$ is defined according to (3.3) replacing ϕ with $\phi_\circ$. In this way, we ensure that it does not matter if we first take the dual of ϕ with respect to the matrix scalar product and then take the square root or if we first consider the square root of ϕ and afterwards take the dual with respect to the scalar product on $\mathbb{R}^n \times \mathbb{R}^n$. Hence, for every $\phi \in \mathscr{M}^{n\times n}(\Omega)$ such that $\overline{\phi}$ belongs to $\mathscr{M}^n(\Omega)$ and satisfies (3.4) Theorem 3.1 implies that the functionals $\mathcal{F}_\varepsilon$ in (2.12) Γ-converge to $\mathcal{F}$ in (2.9) with $\theta = \overline{\phi}$.

On the other hand, from the viewpoint of approximation the most interesting question is wether for any $\theta \in \mathscr{M}^n(\Omega)$ one can find a metric $\phi \in \mathscr{M}^{n\times n}(\Omega)$ satisfying (3.1) and (3.2). In fact, such a ϕ can always be obtained by setting

$$\phi(x,A) := \sup_{\xi,\nu\in S^{n-1}} \frac{|\langle A\nu,\xi\rangle|}{\theta_\circ(x,\xi)\theta_\circ(x,\nu)} \quad \forall\,(x,A)\in\Omega\times\mathbb{R}^{n\times n}. \tag{3.5}$$

It can be easily checked that $\phi \in \mathscr{M}^{n\times n}(\Omega)$, while a direct computation shows that (3.1) and (3.2) are satisfied. Thus, using the definition in (3.5) Theorem 3.1 allows us to approximate the anisotropic functional $\mathcal{F}$ for any $\theta \in \mathscr{M}^n(\Omega)$.

Notice that for $\theta \in \mathscr{M}^n(\Omega)$ given, the metric ϕ obtained by (3.5) is not the unique metric in $\mathscr{M}^{n\times n}(\Omega)$ satisfying (3.1) and (3.2). Indeed, we complete this section with two examples of specific functions $\theta \in \mathscr{M}^n(\Omega)$ for which a corresponding $\phi \in \mathscr{M}^{n\times n}(\Omega)$ satisfying (3.1) and (3.2) can be defined in a quite intuitive way.

Example 3.3 (Quadratic forms)**.** We consider here the anisotropic variant of the Mumford and Shah model analyzed in [30], *i.e.*,

$$\mathcal{MS}^{anis}(u) := \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{S_u\cap\Omega} \sqrt{\langle M(x)\nu_u,\nu_u\rangle}\,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.6}$$

where for every $x \in \Omega$ the matrix $M(x) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and the mapping $x \mapsto M(x)$ satisfies the following hypotheses:

(i) There exist two constants $0 < \lambda \le \Lambda < +\infty$ such that

$$\lambda|\nu|^2 \le \langle M(x)\nu,\nu\rangle \le \Lambda|\nu|^2 \quad \forall\,(x,\nu)\in\Omega\times\mathbb{R}^n \quad \text{(ellipticity)}. \tag{3.7}$$

(ii) There exist $\alpha > 0$ and $L \ge 0$ such that

$$\|M(x)-M(y)\| \le L|x-y|^\alpha \quad \forall\, x,y\in\Omega \quad \text{(Hölder-continuity)}. \tag{3.8}$$

Then in [30] it is shown that (3.6) can be obtained as limit of first-order Ambrosio-Tortorelli approximations. We now briefly describe how to approximate (3.6) *via* second-order functionals. We start by setting a few notation. Let $\theta(x,\nu) := \sqrt{\langle M(x)\nu,\nu\rangle}$. For every $A \in \mathbb{R}^{n\times n}$ we set

$$\operatorname{vec} A := (a_{11},\dots,a_{n1},a_{12},\dots,a_{n2},\dots,a_{1n},\dots,a_{nn})^{\mathrm{T}},$$

and for $A, B \in \mathbb{R}^{n\times n}$ we define the Kronecker product $A\otimes B \in \mathbb{R}^{n^2\times n^2}$ as

$$A\otimes B := \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}.$$

Finally, we set

$$\phi(x,A) := \sqrt{\langle (M(x)\otimes M(x))\operatorname{vec} A, \operatorname{vec} A\rangle} \quad \forall\,(x,A)\in\Omega\times\mathbb{R}^{n\times n}. \tag{3.9}$$

Then ϕ clearly satisfies (2.4), while conditions (3.7) and (3.8) ensure that (2.3) holds true and that ϕ is continuous in (x,A). Moreover, using basic properties of the Kronecker product, for which we refer to [24], it can be easily checked that

$$\phi(x,\nu\otimes\nu) = \langle M(x)\nu,\nu\rangle = \theta^2(x,\nu) \tag{3.10}$$

and

$$\phi_\circ(x,\xi\otimes\xi) = \left\langle M(x)^{-1}\xi,\xi\right\rangle = \theta_\circ^2(x,\xi) \tag{3.11}$$

for all $x \in \Omega$ and for all $\xi, \nu \in S^{n-1}$, where $M(x)^{-1}$ denotes the inverse of $M(x)$. Thus, by Theorem 3.1 the functionals

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega v^2|\nabla u|^2 \mathrm{d}x & \displaystyle +\frac{1}{2\sqrt{2}}\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3 \left\langle (M(x)\otimes M(x))\,\mathrm{vec}\,\nabla^2 v, \mathrm{vec}\,\nabla^2 v\right\rangle^2 \mathrm{d}x \\ & \text{if } (u,v) \in W^{1,2}(\Omega)\times W^{2,2}(\Omega),\ v\nabla u \in L^2(\Omega;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega), \end{cases}$$

provide a second-order approximation of (3.6). We notice that in general the metric ϕ defined in (3.9) does not coincide with the formula in (3.5). Nevertheless, (3.10) implies that they coincide on symmetric rank-one matrices of the type $\nu\otimes\nu$ with $\nu \in S^{n-1}$. Finally, we remark that for our purpose we can replace the Hölder-continuity assumption in (3.8) with the weaker continuity assumption as follows. For every $\eta > 0$ and for every $x_0 \in \Omega$ there exists $\delta = \delta(x_0,\eta) > 0$ such that

$$\|M(x_0) - M(x)\| \leq \eta \quad \forall\, x \in B_\delta(x_0). \tag{3.12}$$

Namely, it can be checked that, if (3.12) is satisfied, then (2.7) still holds true, which will be sufficient for our analysis.

Example 3.4 (p-norms)**.** We discuss here an example of an homogeneous θ of the form

$$\theta(x,\nu) = \theta(\nu) := \left(\sum_{i=1}^n |\nu_i|^p\right)^{\frac{1}{p}} \quad \forall\, (x,\nu) \in \Omega\times\mathbb{R}^n, \tag{3.13}$$

where $p \in [1,\infty)$ and $\nu = (\nu_1,\dots,\nu_n)$. For every $(x,A) \in \Omega\times\mathbb{R}^{n\times n}$ we consider

$$\phi(x,A) = \phi(A) := \left(\sum_{i,j=1}^n |a_{ij}|^p\right)^{\frac{1}{p}}. \tag{3.14}$$

By construction, $\theta \in \mathscr{M}^n(\Omega)$ and $\phi \in \mathscr{M}^{n\times n}(\Omega)$ and for $p \in (1,\infty)$ the duals of θ, ϕ are given by

$$\theta_\circ(\xi) = \left(\sum_{i=1}^n |\xi_i|^{p'}\right)^{\frac{1}{p'}} \quad \text{and} \quad \phi_\circ(B) = \left(\sum_{i,j=1}^n |b_{ij}|^{p'}\right)^{\frac{1}{p'}},$$

respectively, where $p' = \frac{p}{p-1}$ denotes the dual exponent. Then, a direct computation yields that

$$\sqrt{\phi(\nu\otimes\nu)} = \theta(\nu) \quad \text{and} \quad \sqrt{\phi_\circ(\xi\otimes\xi)} = \theta_\circ(\xi) \tag{3.15}$$

holds for all $\nu, \xi \in \mathbb{R}^n$. Let $p = 1$. Then $\theta_\circ$ and $\phi_\circ$ are given by

$$\theta_\circ(\xi) = \sup_{1\leq i\leq n} |\nu_i| \quad \text{and} \quad \phi_\circ(B) = \sup_{1\leq i,j\leq n} |b_{ij}|,$$

respectively. Again, it can be checked that (3.15) holds true. Hence, for $p \in [1,\infty)$ Theorem 3.1 ensures that the functional $\mathcal{F}$ defined as in (2.9) with θ given by (3.13) can be approximated *via* the second-order functionals $\mathcal{F}_\varepsilon$ in (2.12) with ϕ given by (3.14).

4. The optimal profile problem

In this section we study the minimization problem

$$\mathbf{m} := \inf\left\{ \int_0^{+\infty} (h-1)^2 + a^2(h'')^2 \mathrm{d}t :\ h \in W^{2,2}_{loc}(0,+\infty),\ h(0) = h'(0) = 0,\right.$$
$$\left. h(t) = 1 \text{ if } t > M \text{ for some } M > 0 \right\}, \tag{4.1}$$

where $a > 0$ is a fixed constant. To motivate the presence of the constant a, assume for a moment that the metric ϕ does not depend explicitly on x. Then, due to the one-homogeneity, in dimension $n = 1$ we can always write $\phi^2(h'') = \phi^2(1)(h'')^2$ and thus, writing $a = \phi(1)$, the value $\mathbf{m}$ in (4.1) represents the minimal cost of a transition from the value 0 to the value 1 in terms of the one-dimensional unscaled Modica-Mortola part of the energy on the positive real half line. In what follows we refer to this problem as the optimal profile problem. The main result of this section is Lemma 4.1, which shows that the infimum in (4.1) is actually a minimum and the corresponding minimum value is given by $\mathbf{m} = \sqrt{2a}$. The proof follows the argument in ([15], Thm. 3.1). We first introduce the following auxiliary problems. Let $G : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be given by

$$G(w, z) := \inf\left\{ \int_0^1 (g-1)^2 + a^2(g'')^2 \mathrm{d}t :\ g \in C^2([0,1]),\ g(0) = w,\ g(1) = 1, g'(0) = z,\ g'(1) = 0 \right\}.$$

Testing G with a third-order polynomial, one can check that

$$\lim_{(w,z)\to(1,0)} G(w, z) = 0.$$

Finally, we set

$$\tilde{\mathbf{m}} := \inf\left\{ \int_0^{+\infty} (h-1)^2 + a^2(h'')^2 \mathrm{d}t :\ h \in W^{2,2}_{loc}(0,+\infty),\ h(0) = h'(0) = 0,\ \lim_{t\to+\infty} h(t) = 1 \right\}. \tag{4.2}$$

Now we are ready to prove the following lemma.

Lemma 4.1. *Let $\mathbf{m}$, $\tilde{\mathbf{m}}$ be given by (4.1), (4.2) respectively. Then $\mathbf{m} = \tilde{\mathbf{m}} = \sqrt{2a}$ and the infimum in (4.2) is a minimum, i.e.,*

$$\tilde{\mathbf{m}} = \min\left\{ \int_0^{+\infty} (h-1)^2 + a^2(h'')^2 \mathrm{d}t :\ h \in W^{2,2}_{loc}(0,+\infty),\ h(0) = h'(0) = 0,\right.$$
$$\left. \lim_{t\to+\infty} h(t) = 1 \right\} = \sqrt{2a}.$$

Proof. The standard Euler-Lagrange equation associated to (4.2) is given by the fourth order ordinary differential equation

$$h(t) - 1 + a^2 h^{(4)}(t) = 0 \quad \text{in} \quad (0, +\infty) \tag{4.3}$$

subjected to the boundary conditions $h(0) = h'(0) = 0$. A solution to (4.3) is then given by

$$h_a(t) = 1 - \sqrt{2}e^{-\frac{t}{\sqrt{2a}}} \sin\left(\frac{t}{\sqrt{2a}} + \frac{\pi}{4} \right) \tag{4.4}$$

and the corresponding minimum value is given by

$$\int_0^{+\infty} (h_a - 1)^2 + (h_a'')^2 \mathrm{d}t = \sqrt{2a}.$$

Clearly, we have that $\tilde{\mathbf{m}} \leq \mathbf{m}$. Hence, to deduce the thesis it remains to show that $\mathbf{m} \leq \tilde{\mathbf{m}}$. Let $\eta > 0$. We choose a sequence (x_i) such that $x_i \to +\infty$ as $i \to +\infty$ and for fixed $i \in \mathbb{N}$ we choose $g \in C^2([0,1])$ such that $g(0) = h_a(x_i)$, $g(1) = 1$, $g'(0) = h'(x_i)$, $g'(1) = 0$ and

$$\int_0^1 (g-1)^2 + a^2 (g'')^2 \mathrm{d}t \leq G(h_a(x_i), h_a'(x_i)) + \eta,$$

where h_a is the function given in (4.4). Now, we define $g_i(t) := g(t - x_i)$ and

$$h_a^i(t) := \begin{cases} h_a(t) & \text{if } 0 \leq t \leq x_i \\ g_i(t) & \text{if } x_i \leq t \leq x_i + 1 \\ 1 & \text{if } t \geq x_i + 1. \end{cases}$$

In this way, h_a^i belongs to $W^{2,2}_{loc}(0,+\infty)$ and satisfies the boundary conditions in the definition of $\mathbf{m}$. Thus we achieve

$$\begin{aligned} \tilde{\mathbf{m}} &= \int_0^{+\infty} (h_a - 1)^2 + a^2 (h_a'')^2 \mathrm{d}t \\ &\geq \int_0^{+\infty} (h_a^i - 1)^2 + a^2 ((h_a^i)'')^2 \mathrm{d}t - \int_{x_i}^{x_i+1} (g_i - 1)^2 + a^2 (g_i'')^2 \mathrm{d}t \\ &\geq \int_0^{+\infty} (h_a^i - 1)^2 + a^2 ((h_a^i)'')^2 \mathrm{d}t - G(h_a(x_i), h_a'(x_i)) - \eta \\ &\geq \mathbf{m} - G(h_a(x_i), h_a'(x_i)) - \eta. \end{aligned}$$

Since $\lim_{(w,z)\to(1,0)} G(w,z) = 0$, we may conclude letting first i go to infinity and then η go to zero. $\square$

Remark 4.2. Notice that if $\mathbf{m}_d$ is given by

$$\mathbf{m}_d := \min \left\{ \int_0^{+\infty} (h-1)^2 + a^2 (h'')^2 \mathrm{d}t : \; h \in W^{2,2}_{loc}(0,+\infty), \; h(0) = d, \; h'(0) = 0, \; \lim_{t\to+\infty} h(t) = 1 \right\}$$

for some $d \in \mathbb{R}$, then $\mathbf{m}_d = (d-1)^2 \sqrt{2a}$ and hence

$$\lim_{d\to 0} \mathbf{m}_d = \sqrt{2a} = \mathbf{m}.$$

5. The one-dimensional case

In this section we study the asymptotic behavior of the functionals $\mathcal{F}_\varepsilon$ when the space dimension is one. Throughout this section we write $\Omega = I = (a,b) \subset \mathbb{R}$. We will see that the one-dimensional problem already contains the main information needed to analyze the n-dimensional problem, which will be studied combining the slicing procedure with the blow-up method introduced by Fonseca and Müller in [22] (see also [12]). Thanks to the continuity properties (2.6) and (2.7) satisfied by ϕ and θ, for the analysis it would be enough to study the one-dimensional problem in the homogeneous case, *i.e.*, for $\phi = \phi(A)$. Nevertheless, for the sake of completeness, we prefer to prove the result in its full generality also in dimension one. Note that due to the 1-homogeneity of ϕ for every $(t,z) \in I \times \mathbb{R}$ we have $\phi(t,z) = |z|\phi(t,1)$. Then we rewrite the functionals $\mathcal{F}_\varepsilon$ as

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_a^b v^2 f(t,u,u')\mathrm{d}t + \frac{1}{2\sqrt{2}}\int_a^b \frac{(v-1)^2}{\varepsilon} & +\varepsilon^3\phi^2(t,1)(v'')^2\mathrm{d}t \\ & \text{if } (u,v) \in W^{1,2}(I) \times W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I) \times L^1(I). \end{cases} \tag{5.1}$$

Note that in dimension one we do not need the requirement $vu' \in L^2(I)$ since automatically $v \in L^\infty(I)$. The following result will be crucial to deduce the lower bound inequality in the proof of Theorem 3.1.

Theorem 5.1. *Let $\phi \in \mathscr{M}^1(I)$ and let $f \in \mathcal{I}$. Then the functionals $\mathcal{F}_\varepsilon$ given by (5.1) Γ-converge in the strong $(L^1(I) \times L^1(I))$-topology to the functional $\mathcal{E}$ defined on $L^1(I) \times L^1(I)$ as*

$$\mathcal{E}(u,v) := \begin{cases} \displaystyle\int_a^b f(t,u,u')\mathrm{d}t + \sum_{t \in S_u} \sqrt{\phi(t,1)} & \textit{if } u \in SBV^2(I),\ v = 1 \textit{ a.e.}, \\ +\infty & \textit{otherwise in } L^1(I) \times L^1(I). \end{cases}$$

In all that follows C denotes a generic positive constant that may vary from line to line.

Proof. We divide the proof into two main steps proving first the liminf-inequality and then the limsup-inequality

Step 1: Liminf-inequality. Let (u_ε), (v_ε) be sequences in $L^1(I)$ and $u, v \in L^1(I)$ be such that $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(I)$. We claim that

$$\mathcal{E}(u,v) \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon). \tag{5.2}$$

To prove (5.2), we can clearly restrict to the case where

$$\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty. \tag{5.3}$$

From (5.3) we directly deduce that

$$\frac{1}{\varepsilon}\int_a^b (v_\varepsilon - 1)^2\mathrm{d}t \le C,$$

for some constant C, uniformly in ε and thus $\|v_\varepsilon - 1\|_{L^2} \to 0$, which then yields $v = 1$ a.e. Moreover, Proposition 2.3 implies that there exists a constant $c_0 > 0$ such that for ε sufficiently small we have

$$c_0\varepsilon\int_a^b (v_\varepsilon')^2\mathrm{d}t \le \int_a^b \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3(v_\varepsilon'')^2\mathrm{d}t,$$

which in view of (2.5) and (5.3) gives

$$\int_a^b \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon(v_\varepsilon')^2\mathrm{d}t \le C \quad \text{for } \varepsilon \text{ sufficiently small.} \tag{5.4}$$

Then we can apply ([14], Lem. 6.2 and Rem. 6.3) to deduce that for every $\gamma \in (0,1)$ there exists $S_\gamma \subset I$ finite such that for every $I' \subset\subset I \setminus S_\gamma$ fixed and for ε sufficiently small we have

$$1 - \gamma \le v_\varepsilon \le 1 + \gamma \quad \text{on } I'. \tag{5.5}$$

Appealing to (2.1) and (5.5), for every $I' \subset\subset I \setminus S_\gamma$ we thus achieve

$$m_1(1-\gamma)^2\int_{I'} (u_\varepsilon')^2\mathrm{d}t \le (1-\gamma)^2\int_{I'} f(t,u_\varepsilon,u_\varepsilon')\mathrm{d}t \le \int_{I'} v_\varepsilon^2 f(t,u_\varepsilon,u_\varepsilon')\mathrm{d}t \le C, \tag{5.6}$$

for some $0 < C < +\infty$, uniformly with respect to ε, from which we deduce both $u \in W^{1,2}(I')$ and $u'_\varepsilon \rightharpoonup u'$ weakly in $L^2(I')$. Hence, by the arbitrariness of I', we obtain that $u \in SBV^2(I)$ and $S_u \subset S_\gamma$. Moreover, using the convexity of $\xi \mapsto f(t, u, \xi)$, we deduce that

$$\liminf_{\varepsilon\to 0} \int_a^b v_\varepsilon^2 f(t, u_\varepsilon, u'_\varepsilon)\mathrm{d}t \geq (1-\gamma)^2 \liminf_{\varepsilon\to 0} \int_{I'} f(t, u_\varepsilon, u'_\varepsilon)\mathrm{d}t \geq (1-\gamma)^2 \int_{I'} f(t, u, u')\mathrm{d}t. \tag{5.7}$$

Assume now that $S_u = \{t_1, \dots, t_N\} \subset S_\gamma$ for some $N \in \mathbb{N}$. Then, to prove (5.2), we have to show that

$$\liminf_{\varepsilon\to 0} \frac{1}{2\sqrt{2}} \int_a^b \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v''_\varepsilon)^2\mathrm{d}t \geq \sum_{i=1}^N \sqrt{\phi(t_i, 1)}.$$

To this end we use the same argument as in ([15], Thm. 4.1), now also choosing carefully the intervals around each discontinuity point t_i in order to overcome the difficulties due to the inhomogeneity of θ. We fix $\eta > 0$. Then, thanks to (2.7) and the fact that there exist only finitely many discontinuity points $t_i, \dots t_N$, we may deduce that there exists $\delta > 0$ such that for all $t \in (a, b)$ satisfying $|t - t_i| \leq \delta$ for some $t_i \in S_u$ we have

$$\phi^2(t, 1) \geq \frac{1}{1+\eta}\phi^2(t_i, 1). \tag{5.8}$$

Now, for each $t_i \in S_u$ up to choosing δ small enough the intervals $I_i := (t_i - \delta, t_i + \delta) \subset (a, b)$ are pairwise disjoint. Hence, appealing to (5.8) we get

$$\begin{aligned}\int_a^b \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v''_\varepsilon)^2\mathrm{d}t &\geq \sum_{i=1}^N \int_{t_i-\delta}^{t_i+\delta} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v''_\varepsilon)^2\mathrm{d}t \\ &\geq \frac{1}{1+\eta}\sum_{i=1}^N \int_{t_i-\delta}^{t_i+\delta} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v''_\varepsilon)^2\mathrm{d}t. \end{aligned} \tag{5.9}$$

Then, to prove the liminf-inequality it suffices to show that for every $i \in \{1, \dots, N\}$ it holds that

$$\liminf_{\varepsilon\to 0} \frac{1}{2\sqrt{2}} \int_{I_i} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v''_\varepsilon)^2\mathrm{d}t \geq \sqrt{\phi(t_i, 1)}. \tag{5.10}$$

In order to prove (5.10), we consider the intervals $I'_i := (t_i - \frac{\delta}{2}, t_i + \frac{\delta}{2}) \subset\subset I_i$ and we set $n_i := \liminf_{\varepsilon\to 0} \inf_{I'_i}(v_\varepsilon(t))^2$. Then $n_i = 0$, since otherwise by (2.1) we would get

$$\int_{I'_i} (u'_\varepsilon)^2\mathrm{d}t \leq \frac{1}{n_i m_1}\int_{I'_i} v_\varepsilon^2 f(t, u_\varepsilon, u'_\varepsilon)\mathrm{d}t < +\infty,$$

and hence $u'_\varepsilon \rightharpoonup u'$ weakly in $W^{1,2}(I'_i)$, which is a contradiction since $t_i \in I'_i$. Thus $n_i = 0$ and we may choose a sequence (s_ε^i) in I'_i such that $v_\varepsilon(s_\varepsilon^i) \to 0$. On the other hand, since (up to subsequences) $v_\varepsilon \to 1$ a.e., we can also find $r_\varepsilon^i, \tilde{r}_\varepsilon^i \in I_i$ such that $\tilde{r}_\varepsilon^i < s_\varepsilon^i < r_\varepsilon^i$ and $v_\varepsilon(r_\varepsilon^i), v_\varepsilon(\tilde{r}_\varepsilon^i) \to 1$. Moreover, appealing to the Hölder inequality we achieve that

$$\int_a^b \varepsilon|v'_\varepsilon|\mathrm{d}t \leq \sqrt{\varepsilon(b-a)}\left(\int_a^b \varepsilon(v'_\varepsilon)^2\mathrm{d}t\right)^{1/2},$$

which in view of the interpolation inequality and of (5.3) implies that $\varepsilon v'_\varepsilon$ converges to zero in $L^1(a, b)$. Thus, without loss of generality, we may assume that $\varepsilon v'_\varepsilon(r_\varepsilon^i), \varepsilon v'_\varepsilon(\tilde{r}_\varepsilon^i) \to 0$. Finally, we choose $\tilde{s}_\varepsilon^i \in (\tilde{r}_\varepsilon^i, r_\varepsilon^i)$ as a minimum point for v_ε on $[\tilde{r}_\varepsilon^i, r_\varepsilon^i]$. Since by hypotheses $v_\varepsilon \in W^{2,2}(I) \hookrightarrow C^1(I)$, we know that $v'_\varepsilon(\tilde{s}_\varepsilon^i) = 0$ and moreover, for ε sufficiently small we can assume that

$$v_\varepsilon(\tilde{s}_\varepsilon^i) \leq v_\varepsilon(s_\varepsilon^i) < v_\varepsilon(r_\varepsilon^i),$$

and hence $v_\varepsilon(\tilde{s}^i_\varepsilon) < 1$. We have

$$\int_{I_i} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v_\varepsilon'')^2\mathrm{d}t \geq \int_{\tilde{r}^i_\varepsilon}^{\tilde{s}^i_\varepsilon} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v_\varepsilon'')^2\mathrm{d}t + \int_{\tilde{s}^i_\varepsilon}^{r^i_\varepsilon} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v_\varepsilon'')^2\mathrm{d}t. \tag{5.11}$$

We now estimate the second term in (5.11) the computations for the first one being analogous. By the change of variables $z = (t - \tilde{s}^i_\varepsilon)/\varepsilon$, setting $w_\varepsilon(z) := v_\varepsilon(\varepsilon z + \tilde{s}^i_\varepsilon)$ we get

$$\int_{\tilde{s}^i_\varepsilon}^{r^i_\varepsilon} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t_i,1)(v_\varepsilon'')^2\mathrm{d}t = \int_0^{\frac{r_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon}} (w_\varepsilon - 1)^2 + \phi^2(t_i,1)(w_\varepsilon'')^2\mathrm{d}z.$$

Now we use the infimum problem defining G, introduced in Section 4, with $a = \phi(t_i,1)$ and we choose $g^i_\varepsilon \in C^2([0,1])$ such that

$$g^i_\varepsilon(0) = v_\varepsilon(r^i_\varepsilon), \quad (g^i_\varepsilon)'(0) = \varepsilon v_\varepsilon'(r^i_\varepsilon), \quad g^i_\varepsilon(1) = 1, \quad (g^i_\varepsilon)'(1) = 0,$$

and such that

$$\int_0^1 (g^i_\varepsilon - 1)^2 + \phi^2(t_i,1)((g^i_\varepsilon)'')^2\mathrm{d}t \leq G\left(v_\varepsilon(r^i_\varepsilon), \varepsilon v_\varepsilon'(r^i_\varepsilon)\right) + \eta.$$

By construction, $G\left(v_\varepsilon(r^i_\varepsilon), \varepsilon v_\varepsilon'(r^i_\varepsilon)\right)$ tends to 0 as $\varepsilon \to 0$. Finally, we define the functions $\tilde{v}^i_\varepsilon$ as

$$\tilde{v}^i_\varepsilon(z) := \begin{cases} w_\varepsilon(z) & \text{if } 0 \leq \frac{r^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon}, \\ g^i_\varepsilon(z - \frac{r^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon}) & \text{if } \frac{r^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon} \leq z \leq \frac{r^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon} + 1, \\ 1 & \text{if } z \geq \frac{r^i_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon} + 1. \end{cases}$$

In this way, we clearly have that $\tilde{v}^i_\varepsilon \in W^{2,2}_{loc}(0,+\infty)$ and that it satisfies the boundary conditions in the definition of $\mathbf{m}_d$ in Remark 4.2 with $d = v_\varepsilon(\tilde{s}^i_\varepsilon)$. Hence, if we choose again $a = \phi(t_i,1)$ in the definition of $\mathbf{m}_d$, due to the fact that $v_\varepsilon(\tilde{s}^i_\varepsilon) \leq v_\varepsilon(s^i_\varepsilon) < 1$ for ε sufficiently small, we get

$$\begin{aligned}\int_0^{\frac{r_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon}} (w_\varepsilon - 1)^2 + \phi^2(t_i,1)(w_\varepsilon'')^2\mathrm{d}z &= \int_0^{+\infty} (\tilde{v}_\varepsilon - 1)^2 + \phi^2(t_i,1)(\tilde{v}_\varepsilon'')^2\mathrm{d}z - \int_0^1 (g^i_\varepsilon - 1)^2 + \phi^2(t_i,1)((g^i_\varepsilon)'')^2\mathrm{d}t \\ &\geq \mathbf{m}_{v_\varepsilon(\tilde{s}^i_\varepsilon)} - G\left(v_\varepsilon(r^i_\varepsilon), \varepsilon v_\varepsilon'(r^i_\varepsilon)\right) - \eta \\ &\geq \min_{d \leq v_\varepsilon(s^i_\varepsilon)} \mathbf{m}_d - G\left(v_\varepsilon(r^i_\varepsilon), \varepsilon v_\varepsilon'(r^i_\varepsilon)\right) - \eta \\ &= \mathbf{m}_{v_\varepsilon(s^i_\varepsilon)} - G\left(v_\varepsilon(r^i_\varepsilon), \varepsilon v_\varepsilon'(r^i_\varepsilon)\right) - \eta.\end{aligned}$$

Appealing to Remark 4.2 we deduce that

$$\liminf_{\varepsilon \to 0} \int_0^{\frac{r_\varepsilon - \tilde{s}^i_\varepsilon}{\varepsilon}} (w_\varepsilon - 1)^2 + \phi^2(t_i,1)(w_\varepsilon'')^2\mathrm{d}z \geq \sqrt{2\phi(t_i,1)} - \eta. \tag{5.12}$$

Since we get the same estimate on the first integral in (5.11), from (5.7), (5.9), and (5.12) we may deduce that

$$\begin{aligned}\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) &\geq \liminf_{\varepsilon \to 0} \int_a^b v_\varepsilon^2 f(t, u_\varepsilon, u_\varepsilon')\mathrm{d}t + \liminf_{\varepsilon \to 0} \frac{1}{2\sqrt{2}} \int_a^b \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v_\varepsilon'')^2\mathrm{d}t \\ &\geq (1-\gamma)^2 \int_{I'} f(t,u,u')\mathrm{d}t + \frac{1}{1+\eta}\left(\sum_{i=1}^N \sqrt{\phi(t_i,1)} - \frac{\eta}{2\sqrt{2}}\right).\end{aligned}$$

Letting $I' \nearrow (I \setminus S_\gamma)$ we then achieve

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \geq (1-\gamma)^2 \int_I f(t,u,u')\mathrm{d}t + \frac{1}{1+\eta}\left(\sum_{i=1}^N \sqrt{\phi(t_i,1)} - \frac{\eta}{2\sqrt{2}}\right),$$

and we conclude letting $\eta \to 0$ and $\gamma \to 0$.

Step 2: Limsup-inequality. Let $u, v \in L^1(I)$. We claim that there exist sequences (u_ε) in $W^{1,2}(I)$ and (v_ε) in $W^{2,2}(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$, $v_\varepsilon \to v$ in $L^1(I)$ and such that

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \mathcal{E}(u,v). \tag{5.13}$$

To prove (5.13), it is sufficient to consider the case, where $v = 1$ a.e. in I and $u \in SBV^2(I)$. Otherwise the inequality is trivially satisfied. Thus, we may write $S_u = \{t_1, \dots, t_N\}$ such that $t_i < t_{i+1}$ for every $1 \leq i \leq N-1$. We set $t_0 := a$ and $t_{N+1} := b$ and $\tau := \min_{0\leq i\leq N}(t_{i+1} - t_i)$. Moreover, for $1 \leq i \leq N$ we consider the intervals $I_i := \left[\frac{t_{i-1}+t_i}{2}, \frac{t_i+t_{i+1}}{2}\right]$. We start by defining the sequence u_ε converging to u. To this end, we choose a sequence $0 < \xi_\varepsilon \to 0$ such that $\xi_\varepsilon \ll \varepsilon$ and smooth cut-off functions φ_ε^i between $(t_i - \frac{\xi_\varepsilon}{2}, t_i + \frac{\xi_\varepsilon}{2})$ and $(t_i - \xi_\varepsilon, t_i + \xi_\varepsilon)$. We may assume that for ε sufficiently small $\xi_\varepsilon < \frac{\tau}{2}$. Define the sequence

$$u_\varepsilon(t) := u(t)\left(1 - \sum_{i=1}^N \varphi_\varepsilon^i(t)\chi_{I_i}(t)\right);$$

then $u_\varepsilon \in W^{1,2}(I)$ and it can be easily checked that $u_\varepsilon \to u$ in $L^1(I)$. Indeed, we directly achieve that $u_\varepsilon \to u$ a.e. in I and by the dominated convergence Theorem we obtain the $L^1(I)$-convergence. In order to define the sequence v_ε converging to v, we fix $\eta > 0$ and according to Lemma 4.1 we choose functions $g_\eta^i \in W^{2,2}_{loc}(0,+\infty)$ such that $g_\eta^i(0) = (g_\eta^i)'(0) = 0$ and $g_\eta^i(t) = 1$ if $t \geq M_\eta^i$ for some $M_\eta^i > 0$ and such that

$$\int_0^{+\infty} (g_\eta^i - 1)^2 + \phi(t_i,1)^2((g_\eta^i)'')^2\mathrm{d}t \leq \sqrt{2\phi(t_i,1)} + \eta. \tag{5.14}$$

Now we fix $T > \max\{M_\eta^1, \dots, M_\eta^N\}$ and assume that ε is sufficiently small such that $\frac{\tau - 2\xi_\varepsilon}{2\varepsilon} > T$. Then we can define v_ε as

$$v_\varepsilon(t) := \begin{cases} 0 & \text{if there exists } i \in \{1,\dots,N\} \text{ s.t. } |t - t_i| < \xi_\varepsilon, \\ g_\eta^i\left(\frac{|t-t_i| - \xi_\varepsilon}{\varepsilon}\right) & \text{if there exists } i \in \{1,\dots,N\} \text{ s.t. } \xi_\varepsilon \leq |t-t_i| < \xi_\varepsilon + \varepsilon T, \\ 1 & \text{if for all } i \in \{1,\dots,N\}\ |t-t_i| \geq \xi_\varepsilon + \varepsilon T \\ & \text{or } t \in \left[a, \frac{a+t_1}{2}\right] \cup \left[\frac{t_N+b}{2}, b\right]. \end{cases}$$

By construction, $v_\varepsilon \in W^{2,2}(I)$ and $v_\varepsilon \to 1$ in $L^1(I)$. Using again (2.7), we can assume that there exists $\delta \in (0, \tau/2)$ such that for all $t \in (a,b)$ satisfying $|t - t_i| \leq \delta$ for some $t_i \in S_u$ we have

$$\phi^2(t_i,1) \leq (1+\eta)\phi^2(t,1).$$

We can also assume that ε is sufficiently small such that $\xi_\varepsilon + \varepsilon T \leq \delta$. Then, by a change of variables we get

$$\begin{aligned}
\int_{I_i} \frac{(v_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v_\varepsilon'')^2\mathrm{d}t &= \int_{t_i-\xi_\varepsilon-\varepsilon T}^{t_i-\xi_\varepsilon} \frac{1}{\varepsilon}\left(g_\eta^i\left(\frac{t_i-t-\xi_\varepsilon}{\varepsilon}\right)-1\right)^2 + \varepsilon^3\phi^2(t,1)((g_\eta^i)'')^2\mathrm{d}t \\
&\quad + \int_{t_i+\xi_\varepsilon}^{t_i+\xi_\varepsilon+\varepsilon T} \frac{1}{\varepsilon}\left(g_\eta^i\left(\frac{t-t_i-\xi_\varepsilon}{\varepsilon}\right)-1\right)^2 + \varepsilon^3\phi^2(t,1)((g_\eta^i)'')^2\mathrm{d}t + 2\frac{\xi_\varepsilon}{\varepsilon} \\
&\leq (1+\eta)\bigg(\int_{t_i-\xi_\varepsilon-\varepsilon T}^{t_i-\xi_\varepsilon} \frac{1}{\varepsilon}\left(g_\eta^i\left(\frac{t_i-t-\xi_\varepsilon}{\varepsilon}\right)-1\right)^2 + \varepsilon^3\phi^2(t_i,1)((g_\eta^i)'')^2\mathrm{d}t \\
&\quad + \int_{t_i+\xi_\varepsilon}^{t_i+\xi_\varepsilon+\varepsilon T} \frac{1}{\varepsilon}\left(g_\eta^i\left(\frac{t-t_i-\xi_\varepsilon}{\varepsilon}\right)-1\right)^2 + \varepsilon^3\phi^2(t_i,1)((g_\eta^i)'')^2\mathrm{d}t\bigg) + 2\frac{\xi_\varepsilon}{\varepsilon} \\
&= (1+\eta)2\int_0^T (g_\eta^i-1)^2 + \phi^2(t_i,1)((g_\eta^i)'')^2\mathrm{d}z + 2\frac{\xi_\varepsilon}{\varepsilon} \\
&\leq (1+\eta)2\left(\sqrt{2\phi(t_i,1)}+\eta\right) + 2\frac{\xi_\varepsilon}{\varepsilon}\cdot
\end{aligned} \tag{5.15}$$

Moreover, we have

$$\int_{I_i} v_\varepsilon^2 f(t,u_\varepsilon,u_\varepsilon')\mathrm{d}t = \int_{I_i\setminus[t_i-\xi_\varepsilon,t_i+\xi_\varepsilon]} v_\varepsilon^2 f(t,u_\varepsilon,u_\varepsilon')\mathrm{d}t = \int_{I_i} v_\varepsilon^2 f(t,u,u')\mathrm{d}t. \tag{5.16}$$

Gathering (5.15) and (5.16), summing over all i, due to the dominated convergence theorem we finally achieve

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon) &\leq \sum_{i=1}^N \left(\limsup_{\varepsilon\to 0}\int_{I_i} v_\varepsilon^2 f(t,u,u')\mathrm{d}t + \limsup_{\varepsilon\to 0}\frac{1}{2\sqrt{2}}\int_{I_i}\frac{(v_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(t,1)(v_\varepsilon'')^2\mathrm{d}t\right) \\
&\leq \int_a^b f(t,u,u')\mathrm{d}t + (1+\eta)\sum_{i=1}^N\sqrt{\phi(t_i,1)} + \frac{(1+\eta)}{\sqrt{2}}\eta.
\end{aligned}$$

Hence, we deduce the limsup-inequality by the arbitrariness of η. □

6. The n-dimensional case

In this section we study the Γ-convergence of $\mathcal{F}_\varepsilon$ in the n-dimensional setting. The result established in the one-dimensional case will be crucial for our purpose. We will prove Theorem 3.1 gathering Propositions 6.1 and 6.2 below. Thus, from now on we assume that we are within the hypotheses of Theorem 3.1, *i.e.*, $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ are given by (2.9) and (2.12), respectively, $f \in \mathcal{I}$, and $\phi \in \mathscr{M}^{n\times n}(\Omega)$ satisfies conditions (3.1) and (3.2). For any $A \in \mathcal{A}(\Omega)$ we introduce the localized functionals

$$\mathcal{F}_\varepsilon(u,v,A) := \begin{cases} \displaystyle\int_A v^2 f(x,u,\nabla u)\mathrm{d}x + \frac{1}{2\sqrt{2}}\int_A \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2 v)\mathrm{d}x \\ \qquad\qquad \text{if } (u,v)\in W^{1,2}(A)\times W^{2,2}(A),\ v\nabla u\in L^2(A;\mathbb{R}^n), \\ +\infty \qquad\qquad \text{otherwise in } L^1(A)\times L^1(A), \end{cases} \tag{6.1}$$

$$G_\varepsilon(u,v,A) := \begin{cases} \displaystyle\frac{1}{2\sqrt{2}}\int_A \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2 v)\mathrm{d}x & \text{if } (u,v)\in W^{1,2}(A)\times W^{2,2}(A), \\ & v\nabla u \in L^2(A;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^1(A)\times L^1(A). \end{cases} \tag{6.2}$$

Moreover, for $x_0 \in \Omega$ fixed, we also consider

$$G_\varepsilon(x_0,u,v,A) := \begin{cases} \dfrac{1}{2\sqrt{2}}\displaystyle\int_A \frac{(v-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x_0,\nabla^2 v)\mathrm{d}x & \text{if } (u,v)\in W^{1,2}(A)\times W^{2,2}(A), \\ & v\nabla u \in L^2(A;\mathbb{R}^n), \\ +\infty & \text{otherwise in } L^1(A)\times L^1(A). \end{cases} \tag{6.3}$$

We are ready to prove the following proposition.

Proposition 6.1 (Lower bound)**.** *Let $u,\ v\in L^1(\Omega)$ and let (u_ε), (v_ε) be sequences in $L^1(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $v_\varepsilon \to v$ in $L^1(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon) \geq \mathcal{F}(u,v).$$

Proof. We may restrict our analysis to the case

$$\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon) < +\infty, \tag{6.4}$$

which immediately gives that $v=1$ a.e. in Ω. Moreover, invoking Remark 2.6, we know that the domain of the Γ-limit is $GSBV^2(\Omega)\times\{v=1 \text{ a.e.}\}$. Thus, we have $u\in GSBV^2(\Omega)$ and it remains to prove that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon) \geq \int_\Omega f(x,u,\nabla u) + \int_{S_u\cap\Omega} \theta(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{6.5}$$

To this end, we use the blow-up method due to Fonseca and Müller and we start by defining the measures

$$\mu_\varepsilon := \left[v_\varepsilon^2(\cdot) f\left(\cdot,\mu_\varepsilon(\cdot),\nabla\mu_\varepsilon(\cdot)\right) + \frac{1}{2\sqrt{2}}\left(\frac{(v_\varepsilon(\cdot)-1)^2}{\varepsilon} + \varepsilon^3\phi\left(\cdot,\nabla^2 v_\varepsilon(\cdot)\right)\right)\right]\mathcal{L}^n \llcorner \Omega.$$

In view of (6.4), (μ_ε) is an equi-bounded sequence of positive finite Radon measures. Hence, there exists a positive Radon measure μ such that (up to subsequences) $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures. Appealing to the Radon-Nikodỳm Theorem we can write μ as the sum of three mutually orthogonal measures

$$\mu = \mu_a\mathcal{L}^n + \mu_J\mathcal{H}^{n-1}\llcorner S_\mu + \mu_s.$$

We claim that

$$\mu_a(x_0) \geq f(x_0,\mu(x_0),\nabla\mu(x_0)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x_0\in\Omega \tag{6.6}$$

and

$$\mu_J(x_0) \geq \theta(x_0,\nu_\mu(x_0)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0\in S_\mu. \tag{6.7}$$

Indeed, if (6.6) and (6.7) hold true, choosing an increasing sequence of cutoff functions (φ_k) such that $0\leq\varphi_k\leq 1$ and $\sup_k\varphi_k(x)=1$ in Ω, we get the following estimate for every $k\in\mathbb{N}$

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon) &\geq \liminf_{\varepsilon\to 0}\int_\Omega \left(v_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon) + \frac{1}{2\sqrt{2}}\left(\frac{(v_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2 v_\varepsilon)\right)\right)\varphi_k\mathrm{d}x \\ &= \int_\Omega \varphi_k\mathrm{d}\mu \geq \int_\Omega\varphi_k\mathrm{d}\mu_a + \int_{S_\mu\cap\Omega}\varphi_k\mathrm{d}\mu_J \\ &\geq \int_\Omega f(x,u,\nabla u)\varphi_k\mathrm{d}x + \int_{S_u\cap\Omega}\theta(x,\nu_u)\varphi_k\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

and hence we may conclude using the monotone convergence Theorem.

We start proving (6.6). We claim that for each $A \in \mathcal{A}(\Omega)$ it holds that

$$\liminf_{\varepsilon\to 0}\int_A v_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \geq \int_A f(x,u,\nabla u)\mathrm{d}x. \tag{6.8}$$

Let us assume for a moment that we have proved (6.8). Then, using the Besicovitch derivation Theorem, for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$ we can write

$$\mu_a(x_0) = \lim_{\rho\to 0}\frac{1}{\omega_n\rho^n}\mu(B_\rho(x_0)).$$

Moreover, we can assume that x_0 is a Lebesgue point for μ and invoking the Calderòn-Zygmnund Lemma we may also suppose that μ is approximately differentiable at x_0. Since μ is finite, we also know that $\mu(\partial B_\rho) = 0$ except for countably many ρ and for ρ sufficiently small the upper semicontinuous function $x \mapsto \chi_{\overline{B_\rho(x_0)}}(x)$ has compact support in Ω. Hence, appealing to ([5], Prop. 1.62 (a)), (6.8) applied to the sets $B_\rho(x_0)$ gives

$$\begin{aligned}
\mu_a(x_0) &= \lim_{\rho\to 0}\frac{1}{\omega_n\rho^n}\int_\Omega \chi_{\overline{B_\rho(x_0)}}\mathrm{d}\mu \\
&\geq \lim_{\rho\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\omega_n\rho^n}\int_\Omega \chi_{\overline{B_\rho(x_0)}}\mathrm{d}\mu_\varepsilon \\
&\geq \lim_{\rho\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\omega_n\rho^n}\int_{B_\rho(x_0)} v_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \\
&\geq \lim_{\rho\to 0}\frac{1}{\omega_n\rho^n}\int_{B_\rho(x_0)} f(x,u,\nabla u)\mathrm{d}x \\
&= f\left(x_0,u(x_0),\nabla u(x_0)\right).
\end{aligned}$$

So let us now prove (6.8). First suppose that up to extracting a subsequence (not relabelled) the liminf in (6.8) is a limit. Then, to prove (6.8) we would like to invoke a suitable lower semicontinuity theorem like, *e.g.*, Theorem 2.2. But since we have no uniform bound on $\|\nabla u_\varepsilon\|_{L^2}$, we have to replace the sequence (u_ε) with a suitable sequence (w_ε) still converging to u a.e. and satisfying all requirements of Theorems 2.1 and 2.2. To do so we use the same strategy as in [20] now adapted to the second-order setting. We start by using the interpolation inequality to achieve an upper bound on the L^2-norm of the gradient of v_ε. Indeed, for ε sufficiently small we have

$$c_0\int_\Omega \varepsilon|\nabla v_\varepsilon|^2\mathrm{d}x \leq \int_\Omega \frac{(v_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\|\nabla^2 v_\varepsilon\|^2\mathrm{d}x,$$

which is bounded in view of (2.1) and (2.5). To overcome the difficulty that our sequence (v_ε) is in general not bounded between 0 and 1 (like in the case of [20]), we define the truncated functions $\tilde{v}_\varepsilon := 0 \vee (v_\varepsilon \wedge 1)$. Notice that in this way the functions $\tilde{v}_\varepsilon$ do not belong to $W^{2,2}(\Omega)$ any more, but we still have

$$\nabla\tilde{v}_\varepsilon = \nabla v_\varepsilon \chi_{\{0\leq v_\varepsilon\leq 1\}}$$

and hence, for every $A \in \mathcal{A}(\Omega)$ we may deduce that

$$\varepsilon\int_A|\nabla\tilde{v}_\varepsilon|^2\mathrm{d}x \leq \varepsilon\int_A|\nabla v_\varepsilon|^2\mathrm{d}x \leq C \quad \text{and} \quad \int_A\frac{(\tilde{v}_\varepsilon-1)^2}{\varepsilon}\mathrm{d}x \leq \int_A\frac{(v_\varepsilon-1)^2}{\varepsilon}\mathrm{d}x \leq C, \tag{6.9}$$

where the latter estimate implies that $\|\tilde{v}_\varepsilon - 1\|_{L^2(A)} \to 0$ as $\varepsilon \to 0$ and (up to subsequences) $\tilde{v}_\varepsilon \to 1$ a.e. in A. Now define $\Phi : [0,1] \to [0,1]$ as

$$\Phi(t) := \int_0^t (1-s)\mathrm{d}s = \frac{t(2-t)}{2}\cdot$$

Note that Φ is strictly increasing. Moreover, using the classical "Modica-Mortola-trick" we may deduce that

$$|D\Phi(\tilde{v}_\varepsilon)|(A) = \int_A |\nabla\Phi(\tilde{v}_\varepsilon)|\mathrm{d}x = \int_A (1-\tilde{v}_\varepsilon)|\nabla\tilde{v}_\varepsilon|\mathrm{d}x \le \int_A \frac{(1-\tilde{v}_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla\tilde{v}_\varepsilon|^2\mathrm{d}x \le C$$

uniformly in ε. Hence, if for every $t \in \mathbb{R}$ we set $U_{\varepsilon,t} := \{x \in A : \ \Phi(\tilde{v}_\varepsilon(x)) \ge t\}$, applying the coarea-formula, we may deduce that

$$C \ge |D\Phi(\tilde{v}_\varepsilon)|(A) = \int_{-\infty}^{+\infty} |D\chi_{U_{\varepsilon,t}}|(A)\mathrm{d}t,$$

and thus $\sup_\varepsilon |D\chi_{U_{\varepsilon,t}}| < +\infty$ for almost every $t \in \mathbb{R}$, *i.e.*, $U_{\varepsilon,t}$ has finite perimeter in A independently of ε for almost every $t \in \mathbb{R}$. Now let $0 < \gamma < \gamma' < 1/2 = \Phi(1)$ and denote by $P_\varepsilon(t)$ the perimeter of $U_{\varepsilon,t}$ in Ω. Then, for every $\varepsilon > 0$, by the mean-value theorem, there exists $t_\varepsilon \in (\gamma, \gamma')$ such that

$$(\gamma' - \gamma)P_\varepsilon(t_\varepsilon) \le \int_\gamma^{\gamma'} P_\varepsilon(t)\mathrm{d}t = |D\Phi(\tilde{v}_\varepsilon)|(A) \le C, \tag{6.10}$$

uniformly in ε. We write $U_\varepsilon := U_{\varepsilon,t_\varepsilon}$ and we set $w_\varepsilon := u_\varepsilon\chi_{U_\varepsilon}$. Then $w_\varepsilon \in GSBV(A)$ and

$$\nabla w_\varepsilon = \nabla u_\varepsilon \chi_{U_\varepsilon} \tag{6.11}$$

(see, *e.g.*, [5], Thm. 3.84 and Ex. 4.5). Since $\tilde{v}_\varepsilon$ converges to 1 a.e. and $u_\varepsilon \to u$ in $L^1(\Omega)$, w_ε converges to u a.e. and $\|w_\varepsilon\|_{L^1} \le \|u_\varepsilon\|_{L^1}$ is bounded uniformly in ε. Moreover, considering (6.10) we may deduce that

$$\mathcal{H}^{n-1}\left(S\left(w_\varepsilon\right)\right) \le \mathcal{H}^{n-1}\left(S\left(\chi_{U_\varepsilon}\right)\right) = P_\varepsilon(t_\varepsilon) \le C, \tag{6.12}$$

uniformly with respect to ε. Finally, due to (2.1), by (6.11) and the definition of U_ε, we achieve that

$$\begin{aligned} m_1 \int_A \left(\Phi^{-1}(\gamma)\right)^2 |\nabla w_\varepsilon|^2\mathrm{d}x &\le m_1 \int_{U_\varepsilon} \left(\Phi^{-1}(t_\varepsilon)\right)^2 |\nabla u_\varepsilon|^2\mathrm{d}x \\ &\le \int_{U_\varepsilon} m_1\tilde{v}_\varepsilon^2|\nabla u_\varepsilon|^2\mathrm{d}x \\ &\le \int_A \tilde{v}_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \\ &\le \int_A v_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \le C, \end{aligned} \tag{6.13}$$

uniformly in ε. Here $\Phi^{-1} : [0,1/2] \to [0,1]$ is the inverse of Φ given by $\Phi^{-1}(t) = 1 - \sqrt{1-2t}$ and we have used the fact that Φ and Φ^{-1} are strictly increasing functions. Thus by (6.12), (6.13), the boundedness of $\|w_\varepsilon\|_{L^1(A)}$, and the fact that w_ε converges to u a.e. in A, appealing to the $GSBV$-compactness Theorem 2.1 we may deduce that $\nabla w_\varepsilon \rightharpoonup \nabla u$ weakly in $L^1(A;\mathbb{R}^n)$. Hence, we can use the lower semicontinuity Theorem 2.2 to finally achieve

$$\begin{aligned} \liminf_{\varepsilon\to0} \int_A v_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x &\ge \liminf_{\varepsilon\to0} \int_A \tilde{v}_\varepsilon^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \\ &\ge \liminf_{\varepsilon\to0} \left(\Phi^{-1}(\gamma)\right)^2 \int_{U_\varepsilon} f(x,w_\varepsilon,\nabla w_\varepsilon)\mathrm{d}x \\ &= \liminf_{\varepsilon\to0} \left(\Phi^{-1}(\gamma)\right)^2 \int_A f(x,w_\varepsilon,\nabla w_\varepsilon)\mathrm{d}x \\ &\ge \left(\Phi^{-1}(\gamma)\right)^2 \int_A f(x,u,\nabla u)\mathrm{d}x. \end{aligned}$$

Then, letting γ go to 1/2 we deduce (6.8).

We now prove (6.7). Let $x_0 \in S_u$, $\nu := \nu_u(x_0) \in S^{n-1}$. Again using the Besicovitch derivation theorem, for $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_\mu$ we may write

$$\mu_J(x_0) = \lim_{\rho \to 0} \frac{\mu\left(Q^\nu_\rho(x_0)\right)}{\mathcal{H}^{n-1}\left(Q^\nu_\rho(x_0) \cap S_u\right)}\cdot \tag{6.14}$$

Moreover, by the definition of approximate discontinuity point, for a.e. $x_0 \in S_u$ we have that

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{(Q^\nu_\rho(x_0))^\pm} |u(x) - u^\pm(x_0)| \mathrm{d}x = 0, \tag{6.15}$$

where $(Q^\nu_\rho(x_0))^\pm := \{x \in Q^\nu_\rho(x_0) :\ \pm\langle x - x_0, \nu\rangle > 0\}$. Fix $x_0 \in S_u$ such that (6.14) and (6.15) hold true. Then, arguing as for μ_a, we deduce that

$$\begin{aligned}
\mu_J(x_0) &= \lim_{\rho \to 0} \frac{\mu\left(Q^\nu_\rho(x_0)\right)}{\mathcal{H}^{n-1}\left(Q^\nu_\rho(x_0) \cap S_\mu\right)} = \lim_{\rho \to 0} \int_{\overline{Q^\nu_\rho(x_0)}} 1 \mathrm{d}\mu \geq \lim_{\rho \to 0} \limsup_{\varepsilon \to 0} \frac{1}{\rho^{n-1}} \int_{Q^\nu_\rho(x_0)} 1 \mathrm{d}\mu_\varepsilon \\
&\geq \lim_{\rho \to 0} \limsup_{\varepsilon \to 0} \frac{1}{\rho^{n-1}} \bigg(\int_{Q^\nu_\rho(x_0)} v_\varepsilon^2 f(x, u_\varepsilon, \nabla u_\varepsilon) \mathrm{d}x \\
&+ \frac{1}{2\sqrt{2}} \int_{Q^\nu_\rho(x_0)} \frac{(v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v_\varepsilon) \mathrm{d}x \bigg) \\
&= \lim_{\rho \to 0} \limsup_{\varepsilon \to 0} \rho \bigg(\int_{Q^\nu} v_\varepsilon^2(x_0 + \rho y) f\left(x_0 + \rho y, u_\varepsilon(x_0 + \rho y), \nabla u_\varepsilon(x_0 + \rho y)\right) \mathrm{d}y \\
&+ \frac{1}{2\sqrt{2}} \int_{Q^\nu} \frac{\left(v_\varepsilon(x_0 + \rho y) - 1\right)^2}{\varepsilon} + \varepsilon^3 \phi^2\left(x_0 + \rho y, \nabla^2 v_\varepsilon(x_0 + \rho y)\right) \mathrm{d}y \bigg).
\end{aligned}$$

Now fix $\eta > 0$, $\delta > 0$ as in (2.7). Using conditions (2.1) and (2.7), setting $v^\rho_\varepsilon(y) := v_\varepsilon(x_0 + \rho y)$, $u^\rho_\varepsilon(y) := u_\varepsilon(x_0 + \rho y)$, for $0 < \rho < \frac{2\delta}{\sqrt{n}}$ we then get

$$\mu_J(x_0) \geq \lim_{\rho \to 0} \limsup_{\varepsilon \to 0} \int_{Q^\nu} \frac{m_1}{\rho} (v^\rho_\varepsilon)^2 |\nabla \mu^\rho_\varepsilon|^2 \mathrm{d}y + \frac{1}{1+\eta} \frac{1}{2\sqrt{2}} \int_{Q^\nu} \frac{(v^\rho_\varepsilon - 1)^2}{\left(\frac{\varepsilon}{\rho}\right)} + \left(\frac{\varepsilon}{\rho}\right)^3 \phi^2(x_0, \nabla^2 v^\rho_\varepsilon) \mathrm{d}y.$$

Letting first $\varepsilon \to 0$ and then $\rho \to 0$, in view of (6.15) we find that $v^\rho_\varepsilon \to 1$ in $L^1(Q^\nu)$ and $u^\rho_\varepsilon \to u_0$ in $L^1(Q^\nu)$, where u_0 is given by

$$u_0(x) := \begin{cases} u^+(x_0) & \text{if } \langle x, \nu\rangle \geq 0, \\ u^-(x_0) & \text{if } \langle x, \nu\rangle < 0. \end{cases}$$

By a diagonal argument, we find positive vanishing sequences (ε_h), (ρ_h) such that $\sigma_h := \varepsilon_h/\rho_h \to 0$ as $h \to +\infty$ and such that

$$\mu_J(x_0) \geq \lim_{h \to +\infty} \int_{Q^\nu} m_1 v_h^2 |\nabla \mu_h|^2 \mathrm{d}x + \frac{1}{1+\eta} \frac{1}{2\sqrt{2}} \int_{Q^\nu} \frac{(v_h - 1)^2}{\sigma_h} + \sigma_h^3 \phi^2(x_0, \nabla^2 v_h) \mathrm{d}x, \tag{6.16}$$

where $u_h = u^{\rho_h}_{\varepsilon_h}$, $v_h = v^{\rho_h}_{\varepsilon_h}$. Since now ϕ is fixed in x_0, we can use the slicing procedure to deduce the lower bound. To do so, we first need to introduce some notation. For $\xi \in S^{n-1}$, we denote by $\Pi_\xi := \{y \in \mathbb{R}^n :\ \langle y, \xi\rangle = 0\}$ the hyperplane orthogonal to ξ and for every $A \subset \Omega$, $y \in \Pi_\xi$ we set $A_{\xi,y} := \{t \in \mathbb{R} :\ y + t\xi \in A\}$ and $u_h^{\xi,y} := u_h(y + t\xi)$, $v_h^{\xi,y} := v_h(y + t\xi)$. Finally, we define $G_{\sigma_h}(x_0, u_h, v_h, Q^\nu)$ according to (6.3) and for simplicity we set $G_h := G_{\sigma_h}$. Now let $\xi \in S^{n-1}$ be fixed. Using the previous notation, thanks to the properties of the dual metric and to the fact that $\|\xi \otimes \xi\| = 1$, we may deduce the following estimate:

$$
\begin{aligned}
G_h(x_0, u_h, v_h, Q^\nu) &\geq \frac{1}{2\sqrt{2}} \int_{Q^\nu} \frac{(v_h - 1)^2}{\sigma_h} + \sigma_h^3 \frac{\big((\nabla^2 v_h) : (\xi \otimes \xi)\big)^2}{\phi_\circ^2(x_0, \xi \otimes \xi)} \mathrm{d}x \\
&= \frac{1}{2\sqrt{2}} \int_{\Pi_\xi} \left(\int_{Q^\nu_{\xi,y}} \frac{(v_h(y + t\xi) - 1)^2}{\sigma_h} + \frac{\sigma_h^3}{\phi_\circ^2(x_0, \xi \otimes \xi)} \left\langle \nabla^2 v_h(y + t\xi)\xi, \xi \right\rangle^2 \mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \frac{1}{2\sqrt{2}} \int_{\Pi_\xi} \left(\int_{Q^\nu_{\xi,y}} \frac{(v_h^{\xi,y} - 1)^2}{\sigma_h} + \frac{\sigma_h^3}{\phi_\circ^2(x_0, \xi \otimes \xi)} \left((v_h^{\xi,y})'' \right)^2 \mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \int_{\Pi_\xi} G_h^{\xi,y}(x_0, u_h^{\xi,y}, v_h^{\xi,y}, Q^\nu_{\xi,y}) \mathrm{d}\mathcal{H}^{n-1}(y),
\end{aligned} \tag{6.17}
$$

where $G_h^{\xi,y}$ is defined as

$$
G_h^{\xi,y}(x_0, u, v, I) := \begin{cases} \dfrac{1}{2\sqrt{2}} \displaystyle\int_I \frac{(v-1)^2}{\sigma_h} + \frac{\sigma_h^3}{\phi_\circ^2(x_0, \xi \otimes \xi)} (v'')^2 \mathrm{d}t & \text{if } (u, v) \in W^{1,2}(I) \times W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I) \times L^1(I). \end{cases}
$$

Clearly, in view of (6.16) this gives

$$
\sup_h \int_{\Pi_\xi} \left(\int_{Q^\nu_{\xi,y}} m_1 (v_h^{\xi,y})^2 ((u_h^{\xi,y})')^2 \mathrm{d}t + G_h^{\xi,y}(x_0, u_h^{\xi,y}, v_h^{\xi,y}, Q^\nu_{\xi,y}) \right) \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty,
$$

and hence we may appeal to the liminf-inequality for Theorem 5.1. Thus, passing to the limit in (6.17), in view of Fatou's Lemma, we achieve

$$
\begin{aligned}
\lim_{h \to +\infty} G_h(x_0, u_h, v_h, Q^\nu) &\geq \lim_{h \to +\infty} \int_{\Pi_\xi} G_h^{\xi,y}(x_0, u_h^{\xi,y}, v_h^{\xi,y}, Q^\nu_{\xi,y}) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\geq \int_{\Pi_\xi} \liminf_{h \to +\infty} G_h^{\xi,y}(x_0, u_h^{\xi,y}, v_h^{\xi,y}, Q^\nu_{\xi,y}) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\geq \int_{\Pi_\xi} \frac{1}{\sqrt{\phi_\circ(x_0, \xi \otimes \xi)}} \mathcal{H}^0(S(u^{\xi,y}) \cap Q^\nu_{\xi,y}) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \frac{|\left\langle \nu_u(x_0), \xi \right\rangle|}{\sqrt{\phi_\circ(x_0, \xi \otimes \xi)}}.
\end{aligned} \tag{6.18}
$$

Taking into account the positivity of the first term in (6.16), by (6.18) and the arbitrariness of $\xi \in S^{n-1}$ we deduce that

$$
\mu_J(x_0) \geq \frac{1}{1+\eta} \frac{\langle \nu_\mu(x_0), \xi \rangle}{\sqrt{\phi_\circ(x_0, \xi \otimes \xi)}} \quad \forall\, \xi \in S^{n-1}.
$$

Then, appealing to (3.2), passing to the supremum over $\xi \in S^{n-1}$ we finally get the estimate

$$
\mu_J(x_0) \geq \frac{1}{1+\eta} \theta(x_0, \nu_\mu(x_0)),
$$

and we conclude by letting $\eta \to 0$. □

Proposition 6.2 (Upper bound)**.** *Let $(u, v) \in L^1(\Omega) \times L^1(\Omega)$. Then there exist sequences (u_ε) in $W^{1,2}(\Omega)$, (v_ε) in $W^{2,2}(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $v_\varepsilon \to v$ in $L^1(\Omega)$ and such that*

$$
\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \mathcal{F}(u, v). \tag{6.19}
$$

Proof. We may assume that $u \in GSBV^2(\Omega)$ and $v = 1$ a.e. in Ω, since otherwise there is nothing to prove. Moreover, it suffices to prove (6.19) for all $u \in \mathcal{W}$. Indeed, if we showed (6.19) for all $u \in \mathcal{W}$, for $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ we might take the sequence (u_j) converging to u provided by Theorem 2.5. Thus we would have

$$\int_{S_u \cap \Omega} \theta(x, \nu_u) \mathrm{d}\mathcal{H}^{n-1} = \lim_{j \to +\infty} \int_{S_{u_j} \cap \Omega} \theta(x, \nu_{u_j}) \mathrm{d}\mathcal{H}^{n-1},$$

and invoking Theorem 2.2 and Fatou's Lemma, in view of the continuity hypotheses on f, we would achieve

$$\int_\Omega f(x, u, \nabla u) \mathrm{d}x = \lim_{j \to +\infty} \int_\Omega f(x, u_j, \nabla u_j) \mathrm{d}x,$$

and hence

$$\mathcal{F}(u, 1) = \lim_{j \to +\infty} \mathcal{F}(u_j, 1) \geq \liminf_{j \to +\infty} \mathcal{F}''(u_j, 1) \geq \mathcal{F}''(u, 1). \tag{6.20}$$

Finally, if $u \in GSBV^2(\Omega)$, for $j \in \mathbb{N}$ we might define the truncated function $u^j := -j \vee (u \wedge j)$, satisfying

$$u^j \to u \text{ in } L^1(\Omega), \quad \nabla u^j = \nabla u \chi_{\{|u| \leq j\}} \to \nabla u \text{ in } L^2(\Omega; \mathbb{R}^n) \quad \text{and } \mathcal{H}^{n-1}(S_{u^j}) \leq \mathcal{H}^{n-1}(S_u) < +\infty.$$

Thus, again appealing to Theorem 2.2 and to Fatou's Lemma we could deduce that

$$\int_\Omega f(x, u, \nabla u) \mathrm{d}x = \lim_{j \to +\infty} \int_\Omega f(x, u^j, \nabla u^j) \mathrm{d}x.$$

Moreover, the monotone convergence Theorem yields

$$\lim_{j \to +\infty} \int_{S_{u^j} \cap \Omega} \theta(x, \nu_{u^j}) \mathrm{d}\mathcal{H}^{n-1} = \lim_{j \to +\infty} \int_{S_u \cap \Omega} \theta(x, \nu_u) \chi_{\{|u| \leq j\}} \mathrm{d}\mathcal{H}^{n-1} = \int_{S_u \cap \Omega} \theta(x, \nu_u) \mathrm{d}\mathcal{H}^{n-1},$$

and we might conclude as in (6.20). Hence, it remains to prove (6.19) for all $u \in \mathcal{W}$. To do so, we divide the proof into two steps.

Step 1: We suppose that there exists an $(n-1)$-dimensional hyperplane Π and $K \subset \Pi$ closed and convex such that $\overline{S_u} = \Omega \cap K$.

The main idea of the proof is to divide the interior of the jump set S_u into sufficiently small cubes such that on each cube we can use (2.6) and (2.7) thus reducing to an homogeneous ϕ. Then, on each cube we define two sequences (u_ε^i), (v_ε^i) (here i labels the cubes partitioning S_u) which are intrinsically 1-dimensional so that we can use the results of Section 5. In the above construction the main difficulty will be to glue the sequences (u_ε^i), (v_ε^i) in order to obtain a sequence $(u_\varepsilon, v_\varepsilon)$ which satisfies on Ω the required regularity properties. Specifically to construct a recovery sequence (v_ε) which belongs to $W^{2,2}(\Omega)$ we suitably modify every v_ε^i in a neighborhood of the boundary of each cube of the partition. To this end, we start by introducing some notation. We denote the normal to Π, the distance from Π, and the orthogonal projection onto Π respectively by ν, $d(x)$, $p(x)$. Moreover, for every $h > 0$ we define the sets

$$K_h := \{x \in \Pi : \operatorname{dist}(x, K) \leq h\}.$$

Finally, in what follows, if $Q' = Q'_\rho(x_0) \subset K$ is a generic cube with side length $\rho > 0$, we write

$$Q = Q^\nu_\rho(x_0) = Q' \times \left(x_0 - \frac{\rho\nu}{2}, x_0 + \frac{\rho\nu}{2}\right) \tag{6.21}$$

for the unique cube in Ω with side length $\rho > 0$ whose projection on Π is Q'. Applying Whitney's Covering Theorem (see, *e.g.*, [29], Chap. 1, Thm. 3) to $\Pi \setminus \mathring{S}_u$, we can find a countable family of mutually disjoint open cubes Q'_i such that $\mathring{S}_u = \cup_{i=1}^{+\infty} \overline{Q'_i}$ and such that

$$\operatorname{diam}(Q'_i) \leq \operatorname{dist}(Q'_i, \partial \overline{S_u}) \leq 4 \operatorname{diam}(Q'_i). \tag{6.22}$$

We fix $\rho > 0$ and define the families

$$U_\rho := \left\{Q_i :\ \operatorname{dist}(Q_i', \partial\overline{S_u}) \geq \rho\right\}, \qquad \overline{U_\rho} := \left\{\overline{Q_i} :\ \operatorname{dist}(Q_i', \partial\overline{S_u}) \geq \rho\right\},$$
$$U_\rho' := \left\{Q_i' :\ \operatorname{dist}(Q_i', \partial\overline{S_u}) \geq \rho\right\}, \qquad \overline{U_\rho'} := \left\{\overline{Q_i'} :\ \operatorname{dist}(Q_i', \partial\overline{S_u}) \geq \rho\right\}.$$

In view of condition (6.22) the family U_ρ consists of only finitely many cubes. Indeed, there exists a constant $C = C(n)$ such that $\mathcal{H}^{n-1}(Q_i') \geq C\operatorname{diam}(Q_i')^{n-1}$ for all i and thus

$$\begin{aligned}+\infty > \mathcal{H}^{n-1}(S_u) &\geq \sum_{Q_i\in U_\rho} \mathcal{H}^{n-1}(Q_i') \geq C\sum_{Q_i\in U_\rho}\operatorname{diam}(Q_i')^{n-1}\\ &\geq \frac{C}{4}\sum_{Q_i\in U_\rho}\operatorname{dist}(Q_i',\partial\overline{S_u})^{n-1} \geq \frac{C}{4}\rho^{n-1}\#U_\rho.\end{aligned}$$

Without loss of generality we then write $U_\rho = \{Q_1, \cdots, Q_N\}$, where Q_i are the n-dimensional cubes corresponding to Q_i' defined as in (6.21). Let $\eta > 0$. Since $\overline{Q_1} \cup \cdots \cup \overline{Q_N}$ is compact, due to (2.7) and (2.6) we may find $\delta = \delta(\eta) > 0$ depending only on η such that for all $x, y \in \overline{Q_1} \cup \cdots \cup \overline{Q_N}$ satisfying $|x - y| \leq \delta$ we have

$$|\phi^2(x, A) - \phi^2(y, A)| \leq \eta\phi^2(x, A) \quad \forall\ A \in \mathbb{R}^{n\times n} \tag{6.23}$$

and

$$|\theta(x, \xi) - \theta(y, \xi)| \leq \eta \quad \forall\ \xi \in S^{n-1}. \tag{6.24}$$

For fixed $i \in \{1, \cdots, N\}$ we divide $Q_i' \in U_\rho'$ into $N(i)$ pairwise disjoint open subcubes Q_{ij}' centered at $a_{ij} \in S_u$ such that $\overline{Q_i'} = \bigcup_{j=1}^{N(i)} \overline{Q_{ij}'}$ and such that $\operatorname{diam}(Q_{ij}) \leq 2\delta$. This enables us to define sequences on each subcube Q_{ij} as follows. Appealing to Lemma 4.1, for each $1 \leq j \leq N(i)$ we choose $g_\eta^{ij} \in W^{2,2}_{loc}(0, +\infty)$ such that $g_\eta^{ij}(0) = (g_\eta^{ij})'(0) = 0$ and $g_\eta^{ij}(t) = 1$ for all $t \geq M_\eta^{ij}$ for some constant M_η^{ij} and such that

$$\int_0^{+\infty} (g_\eta^{ij} - 1)^2 + \phi^2(a_{ij}, \nu\otimes\nu)((g_\eta^{ij})'')^2 \mathrm{d}t \leq \sqrt{2\phi(a_{ij}, \nu\otimes\nu)} + \eta = \sqrt{2}\theta(a_{ij}, \nu) + \eta.$$

Moreover, we can find $f_\eta \in W^{2,2}_{loc}(0, +\infty)$ such that $f_\eta(0) = f_\eta'(0) = 0$ and $f_\eta(t) = 1$ for all $t \geq M_\eta$ for some constant M_η and such that

$$\int_0^{+\infty} (f_\eta - 1)^2 + (f_\eta'')^2 \mathrm{d}t \leq \sqrt{2} + \eta.$$

We set $T > \max\left\{M_\eta, \max_{i,j} M_\eta^{ij}\right\}$ and choose $\xi_\varepsilon = o(\varepsilon)$. We finally define the functions

$$h_\varepsilon^{ij}(t) := \begin{cases} 0 & \text{if } |t| \leq \xi_\varepsilon \\ g_\eta^{ij}\left(\frac{|t|-\xi_\varepsilon}{\varepsilon}\right) & \text{if } \xi_\varepsilon \leq |t| \leq \xi_\varepsilon + \varepsilon T \\ 1 & \text{if } |t| \geq \xi_\varepsilon + \varepsilon T, \end{cases}$$

$$\tilde{h}_\varepsilon(t) := \begin{cases} 0 & \text{if } |t| \leq \xi_\varepsilon \\ f_\eta\left(\frac{|t|-\xi_\varepsilon}{\varepsilon}\right) & \text{if } \xi_\varepsilon \leq |t| \leq \xi_\varepsilon + \varepsilon T \\ 1 & \text{if } |t| \geq \xi_\varepsilon + \varepsilon T. \end{cases}$$

We set

$$\begin{aligned} A_\varepsilon &:= \{x \in \mathbb{R}^n :\ p(x) \in K_\varepsilon,\ d(x) \leq \xi_\varepsilon\}, \\ B_\varepsilon &:= \{x \in \mathbb{R}^n :\ p(x) \in K_{2\varepsilon},\ d(x) \leq \xi_\varepsilon + \varepsilon T\}, \\ C_\varepsilon &:= \left\{x \in \mathbb{R}^n :\ p(x) \in K_{\varepsilon/2},\ d(x) \leq \xi_\varepsilon/2\right\} \end{aligned}$$

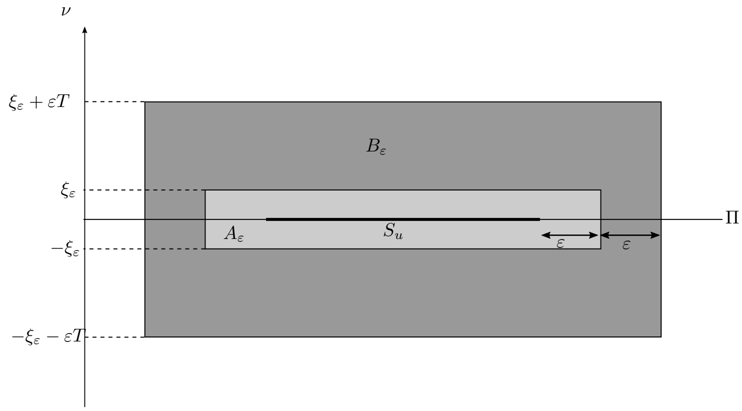

FIGURE 1. The sets A_ε and B_ε.

(see Fig. 1).

Then, choosing a cut-off function ψ_ε between C_ε and A_ε, we define the functions u_ε as

$$u_\varepsilon := (1-\psi_\varepsilon)u.$$

Clearly, $u_\varepsilon \in W^{1,2}(\Omega)$ and thanks to the dominated convergence Theorem $u_\varepsilon \to u$ in $L^1(\Omega)$. Then it remains to construct the sequence (v_ε). To this end, we focus first on a single cube Q_{ij}. In order to not to overburden notation, we drop the indices ij and just write $Q = Q_{ij}$, $a = a_{ij}$, $h_\varepsilon = h_\varepsilon^{ij}$, $g_\eta = g_\eta^{ij}$. Using this notation we define the functions $\hat v_\varepsilon$ and $\tilde v_\varepsilon$ on Q as

$$\hat v_\varepsilon(x) := h_\varepsilon(d(x)), \quad \tilde v_\varepsilon(x) := \tilde h_\varepsilon(d(x)).$$

We will obtain the desired recovery sequence by suitably combining the two sequences $(\hat v_\varepsilon)$ and $(\tilde v_\varepsilon)$ as above. To this end, we first consider the two contributions $G_\varepsilon(u_\varepsilon, \hat v_\varepsilon, Q)$ and $G_\varepsilon(u_\varepsilon, \tilde v_\varepsilon, Q)$, where G_ε is as in (6.2). Since $\hat v_\varepsilon(x) = \tilde v_\varepsilon(x) = 1$ if $d(x) \geq \xi_\varepsilon + \varepsilon T$, we have

$$\begin{aligned} G_\varepsilon(u_\varepsilon, \hat v_\varepsilon, Q) &= G_\varepsilon(u_\varepsilon, \hat v_\varepsilon, Q\cap B_\varepsilon) \\ &= \frac{1}{2\sqrt{2}}\int_{Q\cap A_\varepsilon} \frac{1}{\varepsilon}\mathrm{d}x + \frac{1}{2\sqrt{2}}\int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{(\hat v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(x, \nabla^2 \hat v_\varepsilon)\mathrm{d}x \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} G_\varepsilon(u_\varepsilon, \tilde v_\varepsilon, Q) &= G_\varepsilon(u_\varepsilon, \tilde v_\varepsilon, Q\cap B_\varepsilon) \\ &= \frac{1}{2\sqrt{2}}\int_{Q\cap A_\varepsilon} \frac{1}{\varepsilon}\mathrm{d}x + \frac{1}{2\sqrt{2}}\int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{(\tilde v_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3\phi^2(x, \nabla^2 \tilde v_\varepsilon)\mathrm{d}x. \end{aligned} \tag{6.26}$$

A direct calculation gives

$$\int_{Q\cap A_\varepsilon} \frac{1}{\varepsilon}\mathrm{d}x = \int_{Q'}\left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \frac{1}{\varepsilon}\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(Q')2\frac{\xi_\varepsilon}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.27}$$

To compute the contribution of the second term in (6.25), note first that $\hat{v}_\varepsilon(x) = g_\eta\left(\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right)$ on $Q \cap (B_\varepsilon \setminus A_\varepsilon)$ and $\nabla d(x) = \pm\nu$. Moreover, since by construction $|x-a| \le \delta$ for all $x \in Q$, (6.23) and (6.24) give

$$
\begin{aligned}
&\frac{1}{2\sqrt{2}} \int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{(\hat{v}_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2\hat{v}_\varepsilon)\mathrm{d}x \\
&\le \frac{(1+\eta)}{2\sqrt{2}} \int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{(\hat{v}_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(a,\nabla^2\hat{v}_\varepsilon)\mathrm{d}x \\
&= \frac{(1+\eta)}{2\sqrt{2}} \int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{1}{\varepsilon}\left(g_\eta\left(\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right)-1\right)^2 + \varepsilon^3\phi^2\left(a, g_\eta''\left(\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right)\frac{\nu\otimes\nu}{\varepsilon^2}\right)\mathrm{d}x \\
&= \frac{(1+\eta)}{\sqrt{2}} \int_{Q'} \left(\int_0^T (g_\eta-1)^2 + \phi^2(a,\nu\otimes\nu)(g_\eta'')^2\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\le (1+\eta)\int_{Q'} \theta(a,\nu) + \eta\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\le (1+\eta)\int_{S_u\cap Q} \theta(y,\nu)\mathrm{d}\mathcal{H}^{n-1}(y) + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap Q).
\end{aligned} \tag{6.28}
$$

Using (2.5), the same computations give for the second term in (6.26)

$$
\frac{1}{2\sqrt{2}} \int_{Q\cap(B_\varepsilon\setminus A_\varepsilon)} \frac{(\tilde{v}_\varepsilon-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2\tilde{v}_\varepsilon)\mathrm{d}x \le C(1+\eta)\mathcal{H}^{n-1}(S_u\cap Q). \tag{6.29}
$$

In particular, combining (6.25), (6.26), (6.28) and (6.29), we deduce that $\|\hat{v}_\varepsilon - 1\|^2_{L^2(Q)} \le C\varepsilon$ and $\|\tilde{v}_\varepsilon - 1\|^2_{L^2(Q)} \le C\varepsilon$, which by means of the triangular inequality also gives

$$
\|\hat{v}_\varepsilon - \tilde{v}_\varepsilon\|^2_{L^2(Q)} \le C\varepsilon. \tag{6.30}
$$

Moreover, appealing to the interpolation inequality we get that for ε sufficiently small $\varepsilon\|\nabla\hat{v}_\varepsilon\|^2_{L^2(Q)}$ and $\varepsilon\|\nabla\tilde{v}_\varepsilon\|^2_{L^2(Q)}$ are equibounded. We now modify $\hat{v}_\varepsilon$ in a neighborhood of the boundary using an averaging-slicing procedure. We start by choosing a positive sequence (a_ε) such that $a_\varepsilon \to 0$, $\frac{\varepsilon}{a_\varepsilon^4} \to 0$ as $\varepsilon \to 0$ and we set $b_\varepsilon := [\varepsilon^{-1}]$. For every $0 \le i \le b_\varepsilon$ we define the sets

$$
Q_{\varepsilon,i} := \left\{x \in Q : \ \mathrm{dist}(x,\partial Q) \ge a_\varepsilon - i\frac{a_\varepsilon}{b_\varepsilon}\right\},
$$

and for $1 \le i \le b_\varepsilon$ we consider cut-off functions $\gamma_{\varepsilon,i}$ between $Q_{\varepsilon,i-1}$ and $Q_{\varepsilon,i}$. Finally, we define

$$
v_{\varepsilon,i} := \gamma_{\varepsilon,i}\hat{v}_\varepsilon + (1-\gamma_{\varepsilon,i})\tilde{v}_\varepsilon.
$$

By construction, we have that $\|v_{\varepsilon,i} - 1\|^2_{L^2(Q)} \le C\varepsilon$ for every $1 \le i \le b_\varepsilon$ and $v_{\varepsilon,i} \equiv 0$ on $Q \cap A_\varepsilon$. Since $u_\varepsilon = u$ on $Q \setminus A_\varepsilon$, we easily get

$$
\limsup_{\varepsilon\to 0} \int_Q (v_{\varepsilon,i})^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x = \limsup_{\varepsilon\to 0} \int_{Q\setminus A_\varepsilon} (v_{\varepsilon,i})^2 f(x,u,\nabla u)\mathrm{d}x = \int_Q f(x,u,\nabla u)\mathrm{d}x, \tag{6.31}
$$

independently of i. Moreover, we may write

$$
G_\varepsilon(u_\varepsilon,v_{\varepsilon,i},Q) = G_\varepsilon(u_\varepsilon,\hat{v}_\varepsilon,Q_{\varepsilon,i-1}) + G_\varepsilon(u_\varepsilon,v_{\varepsilon,i},Q_{\varepsilon,i}\setminus Q_{\varepsilon,i-1}) + G_\varepsilon(u_\varepsilon,\tilde{v}_\varepsilon,Q\setminus Q_{\varepsilon,i}). \tag{6.32}
$$

To estimate the third term on the right hand side of (6.32) note that $G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,i}) \leq G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,0})$. Then, by the definition of $\tilde{v}_\varepsilon$, in view of the growth condition on ϕ we get

$$\begin{aligned} G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,0}) &\leq \frac{C}{2\sqrt{2}} \int_{Q\setminus Q_{\varepsilon,0}} \frac{(\tilde{v}_\varepsilon - 1)^2}{\varepsilon} + \varepsilon^3 \|\nabla^2 \tilde{v}_\varepsilon\|^2 \mathrm{d}x \\ &\leq \frac{C}{\sqrt{2}} \int_{(Q\setminus Q_{\varepsilon,0})\cap S_u} \left(\int_0^T (f_\eta - 1)^2 + (f''_\eta)^2 \mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\leq C(1+\eta)\mathcal{H}^{n-1}\left((Q \setminus Q_{\varepsilon,0}) \cap S_u\right) \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{6.33}$$

since

$$\mathcal{H}^{n-1}\left((Q \setminus Q_{\varepsilon,0}) \cap S_u\right) = \mathcal{H}^{n-1}\left(\{x \in Q \cap S_u : \ \operatorname{dist}(x, \partial Q) < a_\varepsilon\}\right) \to 0 \quad \text{as } \varepsilon \to 0,$$

by the choice of the sequence (a_ε). To estimate the second term on the right hand side of (6.32), note first that on $Q_{\varepsilon,i} \setminus Q_{\varepsilon,i-1}$ we have

$$\nabla^2 v_{\varepsilon,i} = \nabla^2 \gamma_{\varepsilon,i}(\hat{v}_\varepsilon - \tilde{v}_\varepsilon) + 2\nabla\gamma_{\varepsilon,i} \otimes (\nabla\hat{v}_\varepsilon - \nabla\tilde{v}_\varepsilon) + \gamma_{\varepsilon,i}\nabla^2\hat{v}_\varepsilon + (1-\gamma_{\varepsilon,i})\nabla^2\tilde{v}_\varepsilon.$$

Hence, since $0 \leq \gamma_{\varepsilon,i} \leq 1$ and $\|\nabla\gamma_{\varepsilon,i}\|_{L^\infty} \leq C(\varepsilon a_\varepsilon)^{-1}$, $\|\nabla^2\gamma_{\varepsilon,i}\|_{L^\infty} \leq C(\varepsilon a_\varepsilon)^{-2}$, in view of (2.5) we may deduce that

$$\begin{aligned} G_\varepsilon(u_\varepsilon, v_{\varepsilon,i}, Q_{\varepsilon,i} \setminus Q_{\varepsilon,i-1}) &= \frac{1}{2\sqrt{2}} \int_{Q_{\varepsilon,i}\setminus Q_{\varepsilon,i-1}} \frac{(v_{\varepsilon,i} - 1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v_{\varepsilon,i}) \mathrm{d}x \\ &\leq C \int_{Q_{\varepsilon,i}\setminus Q_{\varepsilon,i-1}} \bigg(\frac{1}{\varepsilon}(\hat{v}_\varepsilon - \tilde{v}_\varepsilon)^2 + \frac{1}{\varepsilon}(\tilde{v}_\varepsilon - 1)^2 + \frac{1}{\varepsilon a_\varepsilon^4}(\hat{v}_\varepsilon - \tilde{v}_\varepsilon)^2 \\ &\qquad + \frac{\varepsilon}{a_\varepsilon^2}|\nabla\hat{v}_\varepsilon - \nabla\tilde{v}_\varepsilon|^2 + \varepsilon^3\|\nabla^2\hat{v}_\varepsilon\|^2 + \varepsilon^3\|\nabla^2\tilde{v}_\varepsilon\|^2 \bigg) \mathrm{d}x. \end{aligned}$$

Summing up over i and averaging we then may find an index $i(\varepsilon) \in \{1, \cdots, b_\varepsilon\}$ such that

$$\begin{aligned} G_\varepsilon(u_\varepsilon, v_{\varepsilon,i(\varepsilon)}, Q) &\leq \frac{1}{b_\varepsilon} \sum_{i=1}^{b_\varepsilon} G_\varepsilon(u_\varepsilon, v_{\varepsilon,i}, Q) \\ &= \frac{1}{b_\varepsilon} \sum_{i=1}^{b_\varepsilon} \left(G_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_{\varepsilon,i-1}) + G_\varepsilon(u_\varepsilon, v_{\varepsilon,i}, Q_{\varepsilon,i} \setminus Q_{\varepsilon,i-1}) + G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,i}) \right) \\ &\leq G_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q) + \frac{C}{b_\varepsilon} \int_{Q\setminus Q_{\varepsilon,0}} \bigg(\frac{1}{\varepsilon}(\hat{v}_\varepsilon - \tilde{v}_\varepsilon)^2 + \frac{1}{\varepsilon}(\tilde{v}_\varepsilon - 1)^2 + \frac{1}{\varepsilon a_\varepsilon^4}(\hat{v}_\varepsilon - \tilde{v}_\varepsilon)^2 \\ &\quad + \frac{\varepsilon}{a_\varepsilon^2}|\nabla\hat{v}_\varepsilon - \nabla\tilde{v}_\varepsilon|^2 + \varepsilon^3\|\nabla^2\hat{v}_\varepsilon\|^2 + \varepsilon^3\|\nabla^2\tilde{v}_\varepsilon\|^2 \bigg) \mathrm{d}x + G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,0}) \\ &\leq G_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q) + C\bigg(\|\hat{v}_\varepsilon - \tilde{v}_\varepsilon\|^2_{L^2(Q)} + \|\tilde{v}_\varepsilon - 1\|^2_{L^2(Q)} + \frac{1}{a_\varepsilon^4}\|\hat{v}_\varepsilon - \tilde{v}_\varepsilon\|^2_{L^2(Q)} \\ &\quad + \frac{\varepsilon^2}{a_\varepsilon^2}\left(\|\nabla\hat{v}_\varepsilon\|^2_{L^2(Q)} + \|\nabla\tilde{v}_\varepsilon\|^2_{L^2(Q)} \right) + \varepsilon^4 \left(\|\nabla^2\hat{v}_\varepsilon\|^2_{L^2(Q)} + \|\nabla^2\tilde{v}_\varepsilon\|^2_{L^2(Q)} \right) \bigg) + G_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon, Q \setminus Q_{\varepsilon,0}). \end{aligned} \tag{6.34}$$

Thus, if we set $v_\varepsilon := v_{\varepsilon,i(\varepsilon)}$, gathering (6.27), (6.28), (6.30), (6.31), (6.33), and (6.34), in view of the choice of the sequence (a_ε), we finally achieve

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, Q) \leq \int_Q f(x, u, \nabla u) + \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q) \\ &\leq \int_Q f(x, u, \nabla u)\mathrm{d}x + (1+\eta)\int_{S_u\cap Q} \theta(y, \nu)\mathrm{d}\mathcal{H}^{n-1}(y) + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u \cap Q). \end{aligned}$$

Repeating this argument on each subcube Q_{ij}, we may find sequences (v_ε^{ij}) in $W^{2,2}(Q_{ij})$ converging to 1 in $L^1(Q_{ij})$ such that $v_\varepsilon^{ij} = \tilde{v}_\varepsilon$ in a neighborhood of ∂Q_{ij} and such that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon^{ij}, Q_{ij}) &\leq \int_{Q_{ij}} f(x,u,\nabla u)\mathrm{d}x + \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon^{ij}, Q_{ij})\\ &\leq \int_{Q_{ij}} f(x,u,\nabla u)\mathrm{d}x + (1+\eta)\int_{S_u\cap Q_{ij}} \theta(x,\nu)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap Q_{ij}).\end{aligned} \tag{6.35}$$

Note that by construction the functions v_ε^{ij} coincide in a small layer around the boundaries of the cubes. Moreover, they are one-dimensional in normal direction and have derivative zero in tangential direction. Hence, if we define $w_{\varepsilon,\rho}$ on B_ε as

$$w_{\varepsilon,\rho}(x) := \begin{cases} v_\varepsilon^{ij}(x) & \text{if } x\in Q_{ij},\\ \tilde{v}_\varepsilon(x) & \text{if } x\in B_\varepsilon\setminus\bigcup U_\rho,\end{cases}$$

we obtain a sequence in $W^{2,2}(B_\varepsilon)$, which we finally extend to Ω by taking a cut-off function γ_ε between K_ε and $K_{2\varepsilon}$ and defining $v_{\varepsilon,\rho}$ as

$$v_{\varepsilon,\rho}(x) := \gamma_\varepsilon(p(x))w_{\varepsilon,\rho}(x) + (1-\gamma_\varepsilon(p(x))).$$

Again, by construction, for every ρ we have

$$\limsup_{\varepsilon\to 0}\int_\Omega v_{\varepsilon,\rho}^2 f(x,u_\varepsilon,\nabla u_\varepsilon)\mathrm{d}x \leq \int_\Omega f(x,u,\nabla u)\mathrm{d}x. \tag{6.36}$$

Thus, it remains to estimate $G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, \Omega)$. Since $\mathcal{L}^n(\partial Q_{ij}) = 0$ and the number of cubes in U_ρ is finite, we may write

$$\begin{aligned}G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, \Omega) &= G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, \Omega\setminus B_\varepsilon) + G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, (B_\varepsilon\setminus A_\varepsilon)\cap\bigcup U_\rho\right)\\ &\quad + G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, (B_\varepsilon\setminus A_\varepsilon)\setminus\bigcup \overline{U_\rho}\right) + G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, A_\varepsilon).\end{aligned}$$

We then estimate the terms separately. Clearly, the meaningful contribution to the energy will come from the second term. Indeed, the first term does not give any contribution, since $v_{\varepsilon,\rho}\equiv 1$ on $\Omega\setminus B_\varepsilon$, which directly gives $G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, \Omega\setminus B_\varepsilon) = 0$. Moreover, as in (6.27), for the last term we deduce that

$$G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, A_\varepsilon) \leq \frac{\xi_\varepsilon}{\varepsilon}\mathcal{H}^{n-1}(K_\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0.$$

To compute the contribution of the second term, we sum up (6.35) over i,j and get

$$\begin{aligned}\limsup_{\varepsilon\to 0} G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, (B_\varepsilon\setminus A_\varepsilon)\cap\bigcup U_\rho\right) &\leq \sum_{i=1}^N\sum_{j=1}^{N(i)}\left((1+\eta)\int_{S_u\cap Q_{ij}}\theta(x,\nu)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap Q_{ij})\right)\\ &\leq (1+\eta)\int_{S_u\cap\Omega}\theta(x,\nu)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap\Omega).\end{aligned} \tag{6.37}$$

Hence, it remains to estimate $G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, (B_\varepsilon\setminus A_\varepsilon)\setminus\bigcup\overline{U}_\rho\right)$. To do so, we write $B_\varepsilon\setminus A_\varepsilon$ as the disjoint union of the sets

$$D_\varepsilon := \{x\in\mathbb{R}^n:\ p(x)\in K_\varepsilon,\ \xi_\varepsilon\leq d(x)\leq \xi_\varepsilon+\varepsilon T\}$$

and

$$E_\varepsilon := \{x\in\mathbb{R}^n:\ p(x)\in K_{2\varepsilon}\setminus K_\varepsilon,\ d(x)\leq \xi_\varepsilon+\varepsilon T\}$$

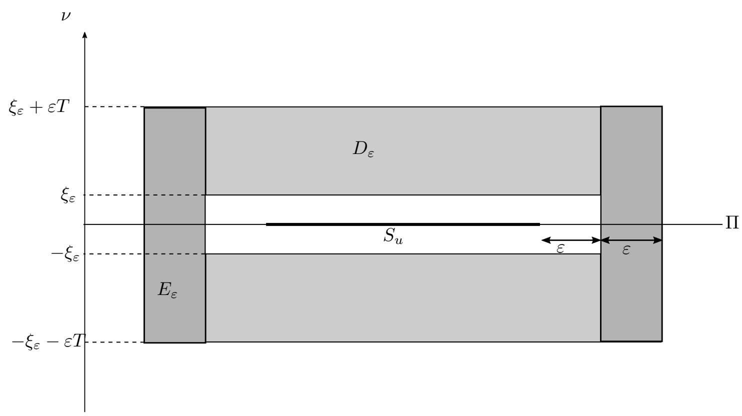

FIGURE 2. The sets D_ε and E_ε.

(see Fig. 2). Clearly, we have that $p(x) \in K_\varepsilon$ for all $x \in \bigcup \overline{U_\rho}$, which gives

$$G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, (B_\varepsilon \setminus A_\varepsilon) \setminus \bigcup \overline{U_\rho}\right) = G_\varepsilon\left(u_\varepsilon, v_{\varepsilon,\rho}, D_\varepsilon \setminus \bigcup \overline{U_\rho}\right) + G_\varepsilon(u_\varepsilon, v_{\varepsilon,\rho}, E_\varepsilon).$$

We start by estimating the second term writing E_ε once more as the disjoint union of the sets

$$V_\varepsilon := \{x \in \mathbb{R}^n : \; p(x) \in K_{2\varepsilon} \setminus K_\varepsilon, \; \xi_\varepsilon \le d(x) \le \xi_\varepsilon + \varepsilon T\}$$

and

$$W_\varepsilon := \{x \in \mathbb{R}^n : \; p(x) \in K_{2\varepsilon} \setminus K_\varepsilon, \; d(x) \le \xi_\varepsilon\}.$$

Note that $\nabla^2 v_{\varepsilon,\rho} = D_p^{\mathrm{T}} \nabla^2 \gamma_\varepsilon D_p$ on W_ε, where D_p is the Jacobian matrix of p. Thus, since $\|D_p\|_{L^\infty} \le 1$, $\|\nabla^2 \gamma_\varepsilon\|_{L^\infty} \le \frac{C}{\varepsilon^2}$, and $0 \le \gamma_\varepsilon \le 1$, using (2.5) we achieve

$$\begin{aligned}\int_{W_\varepsilon} \frac{(v_{\varepsilon,\rho}-1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v_{\varepsilon,\rho})\mathrm{d}x &\le C \int_{W_\varepsilon} \frac{1}{\varepsilon} + \varepsilon^3 \|D_p^{\mathrm{T}} \nabla^2 \gamma_\varepsilon D_p\|^2 \mathrm{d}x \\ &\le C \int_{K_{2\varepsilon}\setminus K_\varepsilon} \left(\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \frac{1}{\varepsilon}\mathrm{d}t \right) \mathrm{d}\mathcal{H}^{n-1}(y) = C\mathcal{H}^{n-1}(K_{2\varepsilon}\setminus K_\varepsilon) 2\frac{\xi_\varepsilon}{\varepsilon},\end{aligned}$$

which again tends to zero as $\varepsilon \to 0$ due to the boundedness of $\mathcal{H}^{n-1}(K_{2\varepsilon} \setminus K_\varepsilon)$. Instead, on V_ε we have $v_{\varepsilon,\rho} = \gamma_\varepsilon(p(x)) f_\eta\left(\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right) + (1 - \gamma_\varepsilon(p(x)))$, which gives

$$\begin{aligned}\int_{V_\varepsilon} \frac{(v_{\varepsilon,\rho}-1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v_{\varepsilon,\rho})\mathrm{d}x &\le C \int_{K_{2\varepsilon}\setminus K_\varepsilon} \int_{\xi_\varepsilon}^{\xi_\varepsilon + \varepsilon T} \left(\frac{1}{\varepsilon} \left(\gamma_\varepsilon(y) f_\eta \left(\frac{t-\xi_\varepsilon}{\varepsilon} \right) - \gamma_\varepsilon(y) \right)^2 \right. \\ &\quad + \varepsilon^3 \left\| D_p(y+t\nu)^T \nabla^2 \gamma_\varepsilon(y) D_p(y+t\nu) \left(f_\eta\left(\frac{t-\xi_\varepsilon}{\varepsilon}\right) - 1 \right) \right. \\ &\quad + \frac{1}{\varepsilon} f_\eta'\left(\frac{t-\xi_\varepsilon}{\varepsilon}\right) \big((\nabla \gamma_\varepsilon(y) D_p(y+t\nu)^{\mathrm{T}} \nabla d(y+t\nu) \\ &\quad + \nabla d(y+t\nu)^T \nabla \gamma_\varepsilon(y) D_p(y+t\nu) \big) \\ &\quad \left.\left. + \gamma_\varepsilon(y) f_\eta''\left(\frac{t-\xi_\varepsilon}{\varepsilon}\right) \frac{\nu \otimes \nu}{\varepsilon^2} \right\|^2 \mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(y) \right. \\ &\le \frac{C}{\varepsilon} \varepsilon \mathcal{H}^{n-1}(K_{2\varepsilon} \setminus K_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0,\end{aligned}$$

where to deduce the last inequality we have used that $\|\nabla\gamma_\varepsilon\|_{L^\infty} \leq \frac{C}{\varepsilon}$. Finally, since $v_{\varepsilon,\rho} = f_\eta\left(\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right)$ on $D_\varepsilon \setminus \bigcup \overline{U_\rho}$, using the same change of variables as in (6.28), we get

$$\int_{D_\varepsilon\setminus\bigcup\overline{U_\rho}} \frac{(v_{\varepsilon,\rho}-1)^2}{\varepsilon} + \varepsilon^3\phi^2(x,\nabla^2 v_\varepsilon,\rho)\mathrm{d}x \leq C\bigg(\int_{(K_\varepsilon\cap S_u)\setminus\bigcup\overline{U'_\rho}}\int_0^T (f_\eta-1)^2+(f''_\eta)^2\mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(y) + \int_{K_\varepsilon\setminus S_u}\int_0^T (f_\eta-1)^2+(f''_\eta)^2\mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(y)\bigg). \tag{6.38}$$

In view of the definition of f_η, for the second term in (6.38) we deduce that

$$\int_{K_\varepsilon\setminus S_u}\int_0^T (f_\eta-1)^2+(f''_\eta)^2\mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(y) \leq (\sqrt{2}+\eta)\mathcal{H}^{n-1}(K_\varepsilon\setminus S_u) \to 0 \quad \text{as } \varepsilon\to 0.$$

To estimate the first term in (6.38), we notice that for all $y\in(K_\varepsilon\cap S_u)\setminus\bigcup\overline{U'_\rho}$, there exists a cube Q such that $\operatorname{dist}(Q',\partial\overline{S_u})<\rho$ and $y\in Q'$, which gives

$$\operatorname{dist}(y,\partial\overline{S_u}) \leq \operatorname{diam}(Q') + \operatorname{dist}(Q',\partial\overline{S_u}) \leq 2\operatorname{dist}(Q',\partial\overline{S_u}) < 2\rho,$$

where in the last step we have again used (6.22). Hence, we get

$$\int_{(K_\varepsilon\cap S_u)\setminus\bigcup\overline{U'_\rho}}\int_0^T (f_\eta-1)^2+(f''_\eta)^2\mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(y) \leq (\sqrt{2}+\eta)\mathcal{H}^{n-1}\left(\{y\in S_u:\ \operatorname{dist}(y,\partial\overline{S_u})<2\rho\}\right) \to 0 \quad \text{as } \rho\to 0. \tag{6.39}$$

Further, we have

$$v_{\varepsilon,\rho}\to 1 \text{ in } L^1(\Omega) \quad \text{as } \varepsilon,\rho\to 0. \tag{6.40}$$

Thus, in view of (6.36), (6.37), (6.39), and (6.40) we can find $\rho(\varepsilon)>0$ with $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that the sequence $v_\varepsilon := v_{\varepsilon,\rho(\varepsilon)}$ satisfies $v_\varepsilon\to 1$ in $L^1(\Omega)$ and

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,\Omega) \leq \int_\Omega f(x,u,\nabla u)\mathrm{d}x + (1+\eta)\int_{S_u\cap\Omega}\theta(x,\nu)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap\Omega).$$

Hence, thanks to the arbitrariness of η, (v_ε) is the desired recovery sequence.

Step 2: Let $u\in\mathcal{W}$ and $\overline{S_u} = \Omega\cap(\cup_{i=1}^r K_i)$, where K_i are pairwise disjoint closed and convex sets contained in a hyperplane Π_i such that $\Pi_i\neq\Pi_j$ for $i\neq j$. Adopting the same notation as in the first step, we write ν_i, $d_i(x)$, $p_i(x)$ for the normal to Π_i, the distance from Π_i and the orthogonal projection onto Π_i and for every $h>0$ we define the sets

$$K_h^i := \{x\in\Pi_i:\ \operatorname{dist}(x,K_i)\leq h\}.$$

Arguing as in the first step we fix $\eta>0$, choose $\xi_\varepsilon = o(\varepsilon)$ and define the sets

$$A_\varepsilon^i := \left\{x\in\mathbb{R}^n:\ p_i(x)\in K_\varepsilon^i,\ d_i(x)\leq\xi_\varepsilon\right\}$$

and

$$B_\varepsilon^i := \left\{x\in\mathbb{R}^n:\ p_i(x)\in K_{2\varepsilon}^i,\ d_i(x)\leq\xi_\varepsilon+\varepsilon T_i\right\},$$

where T_i is chosen according to η in the optimal profile problem in such a way to obtain sequences (u_ε^i), (v_ε^i) satisfying

$$\limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon^i,v_\varepsilon^i,B_\varepsilon^i\setminus A_\varepsilon^i) \leq (1+\eta)\int_{S_u\cap K_i\cap\Omega}\theta(x,\nu_i)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u\cap K_i\cap\Omega).$$

Notice that for ε sufficiently small the sets B_ε^i are pairwise disjoint as well as the sets A_ε^i. More precisely, we set $\delta_0 := \min\{\mathrm{dist}(K_i, K_j) :\ 1 \le i, j \le r\}$ and $T := \max\{T_i :\ 1 \le i \le r\}$, and we choose $\varepsilon_0 > 0$ sufficiently small such that

$$T < \frac{\delta_0 - 2\xi_\varepsilon}{\varepsilon} \quad \forall\, \varepsilon < \varepsilon_0. \tag{6.41}$$

Then, for $\varepsilon < \varepsilon_0$ both

$$\hat{A}_\varepsilon := \bigcup_{i=1}^r A_\varepsilon^i \quad \text{and} \quad \hat{B}_\varepsilon := \bigcup_{i=1}^r B_\varepsilon^i$$

consist of pairwise disjoint sets. Hence, if for $\varepsilon < \varepsilon_0$ we define u_ε, v_ε by

$$u_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in A_\varepsilon^i, \\ u(x) & \text{if} x \in \Omega \setminus \hat{A}_\varepsilon, \end{cases}$$

$$v_\varepsilon(x) := \begin{cases} v_\varepsilon^i(x) & \text{if } x \in B_\varepsilon^i, \\ 1 & \text{if } x \in \Omega \setminus \hat{B}_\varepsilon, \end{cases}$$

then (6.41) ensures that u_ε and v_ε are well defined. Moreover, $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. Finally, summing up over i, using the computations of the first step, gives

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega) &\le \limsup_{\varepsilon\to 0} \int_\Omega v_\varepsilon^2 f(x, u_\varepsilon, \nabla u_\varepsilon)\mathrm{d}x \\
&\quad + \limsup_{\varepsilon\to 0} \Big(G_\varepsilon(u_\varepsilon, v_\varepsilon, \hat{A}_\varepsilon) + G_\varepsilon(u_\varepsilon, v_\varepsilon, \hat{B}_\varepsilon \setminus \hat{A}_\varepsilon) + G_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega \setminus \hat{B}_\varepsilon) \Big) \\
&\le \int_\Omega f(x, u, \nabla u)\mathrm{d}x \\
&\quad + \left(\sum_{i=1}^r \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon, A_\varepsilon^i) + \sum_{i=1}^r \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon, B_\varepsilon^i \setminus A_\varepsilon^i) \right) \\
&\le \int_\Omega f(x, u, \nabla u)\mathrm{d}x + (1+\eta) \int_{S_u\cap\Omega} \theta(x, \nu_u)\mathrm{d}\mathcal{H}^{n-1} + 2\eta(1+\eta)\mathcal{H}^{n-1}(S_u \cap \Omega).
\end{aligned}$$

Thus we deduce the thesis by the arbitrariness of η. □

Remark 6.3. *(Addition of the fidelity term)* Let $g \in L^2(\Omega)$. Then the so-called fidelity term $\int_\Omega |u-g|^2\mathrm{d}x$ is only lower semicontinuous with respect to the strong convergence in $L^1(\Omega)$. Therefore, we cannot appeal to the stability of Γ-convergence under continuous perturbations to deduce that

$$\mathcal{F}_\varepsilon(u, v) + \int_\Omega |u-g|^2\mathrm{d}x \xrightarrow{\Gamma} \mathcal{F}(u, 1) + \int_\Omega |u-g|^2\mathrm{d}x \tag{6.42}$$

with respect to the strong $\big(L^1(\Omega) \times L^1(\Omega)\big)$-topology. Nevertheless, the lower semicontinuity of the fidelity term ensures that

$$\liminf_{\varepsilon\to 0} \left(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \int_\Omega |u_\varepsilon - g|^2\mathrm{d}x \right) \ge \mathcal{F}(u, v) + \int_\Omega |u-g|^2\mathrm{d}x,$$

for every $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. On the other hand, the sequence $(u_\varepsilon, v_\varepsilon)$ as in Proposition 6.2 still provides a recovery sequence for $\mathcal{F}(u, 1) + \int_\Omega |u-g|^2\mathrm{d}x$. This is due to the fact that by adding the fidelity term the domain of the Γ-limit reduces to

$$GSBV^2(\Omega) \cap L^2(\Omega) \times \{v = 1 \text{ a.e.}\}.$$

Indeed, if $u \in GSBV^2(\Omega)$ is such that

$$\mathcal{F}(u,1) + \int_\Omega |u-g|^2 \mathrm{d}x < +\infty,$$

then $u \in L^2(\Omega)$. Thus the sequence (u_ε) as in Proposition 6.2 converges to u in $L^2(\Omega)$, which in turn implies that

$$\limsup_{\varepsilon \to 0} \left(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \int_\Omega |u_\varepsilon - g|^2 \mathrm{d}x \right) \le \mathcal{F}(u,1) + \int_\Omega |u-g|^2 \mathrm{d}x.$$

Hence, we deduce that (6.42) holds true.

7. Convergence of minimization problems

In this section we study the existence of minimizing pairs $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ for a suitable modification of $\mathcal{F}_\varepsilon$ and their asymptotic behavior. Indeed, if $\eta_\varepsilon > 0$ is chosen in a way such that $\eta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$ and if for $(u,v) \in W^{1,2}(\Omega) \times W^{2,2}(\Omega)$ we define the functionals

$$\mathcal{F}_\varepsilon(u,v) + \eta_\varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |u-g|^2 \mathrm{d}x, \tag{7.1}$$

where $g \in L^2(\Omega)$ is given, then for every fixed $\varepsilon > 0$ the functionals defined in (7.1) are coercive with respect to the weak $\big(W^{1,2}(\Omega) \times W^{2,2}(\Omega)\big)$-topology. Further, the following theorem holds true.

Theorem 7.1. *Let $g \in L^2(\Omega)$. Then, for every $\varepsilon > 0$ there exists a pair of minimizers $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ for the problem*

$$M_\varepsilon := \inf \left\{ \mathcal{F}_\varepsilon(u,v) + \eta_\varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |u-g|^2 \mathrm{d}x : \ (u,v) \in W^{1,2}(\Omega) \times W^{2,2}(\Omega) \right\}.$$

Moreover, (up to subsequences) $(\hat{u}_\varepsilon, \hat{v}_\varepsilon) \to (u,1)$ in $L^1(\Omega) \times L^1(\Omega)$, where u is a solution of

$$M := \min \left\{ \mathcal{F}(u,1) + \int_\Omega |u-g|^2 \mathrm{d}x : \ u \in GSBV^2(\Omega) \right\}, \tag{7.2}$$

and if $n = 1$ then $u \in SBV^2(\Omega)$. Finally, $M_\varepsilon \to M$ as $\varepsilon \to 0$.

Proof. Appealing to ([15], Thm. 4.4), in view of the growth conditions on f and ϕ, we may directly deduce the coercivity of the functionals defined in (7.1). Indeed, for $\varepsilon > 0$ fixed let (u_k, v_k) be a minimizing sequence. Then we get

$$\begin{aligned} +\infty &> \sup_k \left(\mathcal{F}_\varepsilon(u_k, v_k) + \eta_\varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |u_k - g|^2 \mathrm{d}x \right) \\ &\ge \sup_k \left(m_1 \int_\Omega |\nabla u_k|^2 v^2 \mathrm{d}x + \eta_\varepsilon \int_\Omega |\nabla u_k|^2 \mathrm{d}x \right. \\ &\quad \left. + \frac{1}{2\sqrt{2}} \int_\Omega \frac{(v_k - 1)^2}{\varepsilon} + c_1^2 \varepsilon^3 \|\nabla^2 v_k\|^2 \mathrm{d}x + \int_\Omega |u-g|^2 \mathrm{d}x \right), \end{aligned}$$

where the coercivity of the latter functional has been shown in ([15], Thm. 4.4). More precisely, the authors proved that there exists a pair $(u,v) \in W^{1,2}(\Omega) \times W^{2,2}(\Omega)$ such that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega)$, $v_k \rightharpoonup v$ in $W^{2,2}(\Omega)$ and $v \nabla u \in L^2(\Omega)$. Then, since ϕ is continuous and convex in the second variable, we may directly deduce that

$$\int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v) \mathrm{d}x \le \liminf_{k \to +\infty} \int_\Omega \frac{(v_k - 1)^2}{\varepsilon} + \varepsilon^3 \phi^2(x, \nabla^2 v_k) \mathrm{d}x,$$

while Ioffe's Theorem (see, *e.g.*, [5], Thm. 5.8) ensures that

$$\int_\Omega v^2 f(x,u,\nabla u)\mathrm{d}x \le \liminf_{k\to+\infty}\int_\Omega v_k^2 f(x,u_k,\nabla u_k)\mathrm{d}x.$$

Hence, the existence of a minimizing pair $(\hat{u}_\varepsilon,\hat{v}_\varepsilon)$ follows by the direct methods. Moreover, Remark 6.3 and the requirement $\eta_\varepsilon/\varepsilon \to 0$ ensure that

$$\mathcal{F}_\varepsilon(u,v)+\eta_\varepsilon\int_\Omega|\nabla u|^2\mathrm{d}x+\int_\Omega|u-g|^2\mathrm{d}x \xrightarrow{\Gamma} \mathcal{F}(u,1)+\int_\Omega|u-g|^2\mathrm{d}x \tag{7.3}$$

in the strong $\big(L^1(\Omega)\times L^1(\Omega)\big)$-topology. Indeed, we can repeat the same constructions as in Theorem 5.1 and Proposition 6.2 now choosing $\xi_\varepsilon=\sqrt{\eta_\varepsilon\varepsilon}$ to see that the perturbation term does not affect the Γ-convergence result. Finally, assume that $(u_\varepsilon,v_\varepsilon)$ is a sequence in $L^1(\Omega)\times L^1(\Omega)$ satisfying

$$\sup_\varepsilon\left(\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)+\eta_\varepsilon\int_\Omega|\nabla u_\varepsilon|^2\mathrm{d}x+\int_\Omega|u_\varepsilon-g|^2\mathrm{d}x\right)<+\infty.$$

Then, appealing again to the growth conditions on ϕ and f and to the interpolation inequality, we find that also

$$\sup_\varepsilon\left(\mathcal{AT}_\varepsilon(u_\varepsilon,v_\varepsilon)+\eta_\varepsilon\int_\Omega|\nabla u|^2\mathrm{d}x+\int_\Omega|u-g|^2\mathrm{d}x\right)<+\infty,$$

where $\mathcal{AT}_\varepsilon$ is as in (2.10). Then ([7], Thm. 1.2) yields the equicoercivity of the functionals defined in (7.1). Finally, the convergence of minimizers and of the corresponding minimization problems follows by the fundamental property of Γ-convergence. □

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