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A CONCEPT OF INNER PREDERIVATIVE FOR SET-VALUED MAPPINGS AND ITS APPLICATIONS

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Abstract. We introduce a class of positively homogeneous set-valued mappings, called inner prederivatives, serving as first order approximants to set-valued mappings. We prove an inverse mapping theorem involving such prederivatives and study their stability with respect to variational perturbations. Then, taking advantage of their properties we establish necessary optimality conditions for the existence of several kind of minimizers in set-valued optimization. As an application of these last results, we consider the problem of finding optimal allocations in welfare economics. Finally, to emphasize the interest of our approach, we compare the notion of inner prederivative to the related concepts of set-valued differentiation commonly used in the literature.

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1. Introduction

In [20], Ioffe introduced a class of set-valued mappings dedicated to the local approximation of nonsmooth single-valued functions. It consisted of closed-valued positively homogeneous mappings called *prederivatives*. In this paper, we present an extension of Ioffe's concept of *inner prederivatives* to the set-valued framework, *i.e.*, when the mappings we are dealing with are no longer single-valued but set-valued. We will use such inner prederivatives as first order approximants to set-valued mappings and show that they share a wide scope of applications with derivatives, in particular they may lead to several results fitting into the paradigm of differentiation. As such, we will prove an inverse set-valued mapping theorem (see Thm. 2.5) involving inner prederivatives and we will investigate as well their stability with respect to variational perturbations; more precisely, we will show that the limit (in some sense) of a sequence of set-valued mappings admitting inner prederivatives admits also an inner prederivative (see Thm. 3.1). Finally, we will establish necessary optimality conditions for set optimization problems (see Thm. 4.3) in terms of inner prederivatives and, taking inspiration from the economic model presented in [6], we will propose an application of this result to welfare economics. As we will discuss it at the end of this paper, our very motivation to consider such prederivatives is that they are not only less restrictive and much easier to handle than most of the related concepts of set-valued differentiation

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commonly used in the literature but they also often lead to more accurate results of variational analysis and set-valued optimization.

Throughout, X and Y stand for two Banach spaces. A set-valued mapping F acting from X to the subsets of Y is indicated by $F : X \rightrightarrows Y$, the set $\operatorname{gph} F$ stands for the graph of F and is defined by $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ while the domain of F is the set $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$. The inverse of the mapping F, denoted by $F^{-1} : Y \rightrightarrows X$, is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$. The zero element of a vector space Z being denoted by 0_Z, we recall that a set-valued mapping $H : X \rightrightarrows Y$ is said to be positively homogeneous if $H(0_X) \ni 0_Y$ and $H(\lambda x) = \lambda H(x)$ for all $x \in X$ and $\lambda > 0$ (with the convention $\lambda \emptyset = \emptyset$ for every real number λ). The closure (respectively the interior) of a subset A of a normed space is denoted by cl A (respectively int A). Finally, the closed unit ball of a metric space E is denoted by $\mathbb{B}_E$ and $\mathbb{B}_r(a)$ is the closed ball of radius r centered at a.

The concept of inner prederivative we introduce in this paper reads as follows.

Definition 1.1 (Inner prederivatives)**.** Consider a set-valued mapping $F : X \rightrightarrows Y$. Let $H : X \rightrightarrows Y$ be a positively homogeneous set-valued mapping and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. We say that H is an inner prederivative of F at $\bar{x}$ for $\bar{y}$ if, for all $\delta > 0$, there exists a neighborhood U of $\bar{x}$ such that

$$H(x - \bar{x}) \subset F(x) - \bar{y} + \delta\|x - \bar{x}\|\mathbb{B}_Y, \text{ for all } x \in U. \tag{1.1}$$

Obviously, when such a prederivative H exists, it is not unique since any positively homogeneous mapping H' such that $H'(x) \subset H(x)$ for all $x \in X$ is also an inner prederivative of F at $\bar{x}$ for $\bar{y}$. From now on, we will denote the (possibly empty) set of all the inner prederivatives of a mapping F at $\bar{x}$ for $\bar{y}$ by $\mathcal{H}_{(F|(\bar{x},\bar{y}))}$.

Note that, in the case when $F \equiv f$ is a Fréchet differentiable single-valued function, the derivative of f at $\bar{x}$ is a single-valued inner prederivative of f at $\bar{x}$ for $f(\bar{x})$. Now, we would like to go back to the origins of the notion presented in Definition 1.1. A set-valued positively homogeneous mapping $T : X \rightrightarrows Y$ is an inner prederivative of a single-valued function $f : X \to Y$ at $\bar{x}$, in the sense of Ioffe [20], if

$$T(h) \subset \bigcup_{0<t<1} t^{-1}(f(\bar{x} + th) - f(\bar{x})) + r(h)\|h\|\mathbb{B}_Y, \text{ with } \lim_{\|h\|\to 0} r(h) = 0. \tag{1.2}$$

The following lemma makes clearer the link between our concept of inner prederivative and the one introduced by Ioffe.

Lemma 1.2. *Consider a set-valued mapping $F : X \rightrightarrows Y$, a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and let $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$. Then, for every $x \in X$ and $\delta > 0$, there exists $t \in (0, 1)$ such that*

$$H(x - \bar{x}) \subset t^{-1}\big(F(\bar{x} + t(x - \bar{x})) - \bar{y}\big) + \delta\|x - \bar{x}\|\mathbb{B}_Y. \tag{1.3}$$

Proof. Let $x \in X$. For all $\delta > 0$ there exists $a > 0$ such that

$$H(u - \bar{x}) \subset F(u) - \bar{y} + \delta\|u - \bar{x}\|\mathbb{B}_Y, \text{ for all } u \in \mathbb{B}_a(\bar{x}). \tag{1.4}$$

Moreover, there exists a number $t \in (0, 1)$ such that $\bar{x} + t(x - \bar{x}) \in \mathbb{B}_a(\bar{x})$, thus, according to (1.4) we have

$$H(\bar{x} + t(x - \bar{x}) - \bar{x}) \subset F(\bar{x} + t(x - \bar{x})) - \bar{y} + \delta\|\bar{x} + t(x - \bar{x}) - \bar{x}\|\mathbb{B}_Y.$$

It follows that $tH(x - \bar{x}) \subset F(\bar{x} + t(x - \bar{x})) - \bar{y} + t\delta\|x - \bar{x}\|\mathbb{B}_Y$ which gives us (1.3). □

Of course, the question when inner prederivatives in the sense of Definition 1.1 exist cannot be avoided but, before discussing it, we need to collect some background material. Recall that a set-valued mapping $T : X \rightrightarrows Y$ is said to be a *convex process* (or a sublinear mapping) if T is positively homogeneous and is, in addition, *superadditive*, *i.e.*,

$$T(x) + T(x') \subset T(x + x'), \text{ for all } x, x' \in X;$$

where we make the convention that $A + \emptyset = \emptyset$ for any set A.

Moreover, following the notation in [31], we denote by $\mathcal{N}_\infty$ the set of all subsequences of $\mathbb{N}$ containing all n beyond some given integer and given a set-valued mapping $F : X \rightrightarrows Y$, we let

$$\liminf_{x\to\bar{x}} F(x) = \{y \in Y \mid \forall x_n \to \bar{x}, \exists N \in \mathcal{N}_\infty,\ y_n \to y \text{ with } y_n \in F(x_n) \text{ for all } n \in N\}.$$

The mapping F is said to be *inner semicontinuous* at $\bar{x}$ when $\liminf_{x\to\bar{x}} F(x) \supseteq F(\bar{x})$. When Y is a finite dimensional space and F is closed-valued an alternative definition of the inner semicontinuity of F at $\bar{x}$ (the proof of which can be found in [31]) is given by

$$\forall \rho, \varepsilon > 0,\ \exists \alpha > 0 \text{ such that } F(\bar{x}) \cap \rho\mathbb{B}_Y \subset F(x) + \varepsilon\mathbb{B}_Y,\ \text{ for all } x \in \mathbb{B}_\alpha(\bar{x}). \tag{1.5}$$

Finally, recall that a single-valued map $f : X \to Y$ is a *selection* of a set-valued mapping $F : X \rightrightarrows Y$ if for every $x \in X$, $f(x) \in F(x)$. In [32], it has been proved that any convex set-valued mapping (*i.e.*, with convex graph) $F : X \rightrightarrows Y$ which is compact-valued and such that $F(0_X) \ni 0_Y$ admits a linear selection. Let us mention that the existence of linear selections has also been established in several important special cases (see, *e.g.*, the works of Ioffe [20] and Páles [27]).

The following fundamental lemma is known as the Radström's cancellation law (see [30], Lem. 1).

Lemma 1.3. *Let A, B and C be given sets in a real normed linear space. Suppose B closed and convex, C bounded, and that $A + C \subset B + C$ then $A \subset B$.*

Obviously, any mapping $H : X \rightrightarrows Y$ such that $H(0) := \{0\}$ and $H(x) = \emptyset$ for all $x \in X\backslash\{0\}$ is an inner prederivative of any mapping F at any point $(\bar{x}, \bar{y})$ of its graph. Still, we have to provide examples of nontrivial inner prederivatives. This is the purpose of the next proposition dealing with the existence of inner prederivatives in the sense of Definition 1.1. It provides a reasonably large scope of examples of mappings admitting such prederivatives.

Proposition 1.4 (Existence of inner prederivatives)**.** *Consider a set-valued mapping $F : X \rightrightarrows Y$ and a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then F admits an explicit inner prederivative at $\bar{x}$ for $\bar{y}$ in each of the following cases:*

(i) *F is a convex process. In this case, $F \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$.*

(ii) *The space Y is finite dimensional, F is closed-valued and convex-valued in a neighborhood of $\bar{x}$; in addition, F is inner semicontinuous at $\bar{x}$ and $\bar{y} \in \operatorname{int} F(\bar{x})$. In this case, $\kappa\|\cdot\|\mathbb{B}_Y \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$ for some positive real number κ.*

(iii) *The set $\mathcal{A}$ of linear selections of F is nonempty and $F(\bar{x}) = \{\bar{y}\}$. In this case, the mapping H defined by $H(x) = \bigcup_{a\in\mathcal{A}} a(x), \forall x \in X$ is an element of $\mathcal{H}_{(F|(\bar{x},\bar{y}))}$.*

Proof.

(i) Since F is a convex process, for all $x \in X$, $F(x - \bar{x}) + F(\bar{x}) \subset F(x)$. Hence for all $\delta > 0, F(x - \bar{x}) \subset F(x) - \bar{y} + \delta\|x - \bar{x}\|\mathbb{B}_Y$. Consequently, the mapping $H := F$, which is obviously positively homogeneous, is an inner prederivative of F at $\bar{x}$ for $\bar{y}$.

(ii) Because $\bar{y} \in \operatorname{int} F(\bar{x})$ there is a positive constant $\hat{\kappa}$ such that $\bar{y} + \hat{\kappa}\mathbb{B}_Y \subset F(\bar{x})$. Moreover there exists $\rho > 0$ such that $\bar{y} + \hat{\kappa}\mathbb{B}_Y \subset \rho\mathbb{B}_Y$. Hence $\bar{y} + \hat{\kappa}\mathbb{B}_Y \subset F(\bar{x}) \cap \rho\mathbb{B}_Y$. Take $0 < \varepsilon < \hat{\kappa}$; according to the inner semicontinuity of F at $\bar{x}$ (see assertion (1.5)), there is a constant $\alpha > 0$ such that $\bar{y} + \hat{\kappa}\mathbb{B}_Y \subset F(x) + \varepsilon\mathbb{B}_Y$, for all $x \in \mathbb{B}_\alpha(\bar{x})$. Hence, setting $\kappa := \hat{\kappa} - \varepsilon$ we get

$$\bar{y} + \kappa\mathbb{B}_Y + \varepsilon\mathbb{B}_Y \subset F(x) + \varepsilon\mathbb{B}_Y,\ \ \forall x \in \mathbb{B}_\alpha(\bar{x}).$$

According to the Radström's cancellation law, we infer

$$\bar{y} + \kappa\mathbb{B}_Y \subset F(x),\ \ \forall x \in \mathbb{B}_\alpha(\bar{x}).$$

Consequently, for all $\delta > 0$ there is a constant $0 < a < \min\{\alpha, 1\}$ such that

$$\kappa\|x - \bar{x}\|\mathbb{B}_Y \subset F(x) - \bar{y} + \delta\|x - \bar{x}\|\mathbb{B}_Y,\ \forall x \in \mathbb{B}_a(\bar{x}).$$

that is, the positively homogeneous mapping $H(\cdot) = \kappa\|\cdot\|\mathbb{B}_Y$ is an inner prederivative of F at $\bar{x}$ for $\bar{y}$.

(iii) Let $H(x) = \bigcup_{a\in\mathcal{A}} a(x)$, for all $x \in X$; a straightforward computation shows that H is a positively homogeneous set-valued mapping. Moreover, for all $a \in \mathcal{A}$ and $x \in X$ one has $a(x - \bar{x}) = a(x) - a(\bar{x}) \in F(x) - \bar{y}$. Hence,

$$H(x - \bar{x}) = \bigcup_{a\in\mathcal{A}} a(x - \bar{x}) \subset F(x) - \bar{y},\ \text{for all } x \in X.$$

It follows that $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$. □

Now, we will show that the existence of a particular inner prederivative for a given set-valued mapping can lead to an open mapping theorem. Recall that a set-valued mapping F is said to be *open* at $\bar{x}$ for $\bar{y}$, where $\bar{y} \in F(\bar{x})$, if $\bar{x} \in \operatorname{int}\operatorname{dom} F$ and for every neighborhood U of $\bar{x}$ the set $F(U)$ is a neighborhood of $\bar{y}$. Note that F is open at $\bar{x}$ for $\bar{y}$ whenever it is *linearly open* (or, equivalently, has the covering property) at $\bar{x}$ for $\bar{y}$, *i.e.*, if there is a constant $\kappa \geq 0$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x + \kappa r \operatorname{int}\mathbb{B}_X) \supset (F(x) + r\operatorname{int}\mathbb{B}_Y) \cap V,\ \text{for all } x \in U, r > 0. \tag{1.6}$$

It is worth mentioning that a set-valued mapping is linearly open at $\bar{x}$ for $\bar{y}$ if and only if it is *metrically regular* at $\bar{x}$ for $\bar{y}$ (see Def. 2.3) and that openness and linear openness agree whenever the mapping has a closed and convex graph (see, *e.g.*, Robinson-Ursescu's theorem in [12]). For a comprehensive study of the concepts of linear openness and metric regularity, the reader could refer, for instance, to [12, 24].

Theorem 1.5 (Open mapping theorem)**.** *Consider a set-valued mapping $F : X \rightrightarrows Y$, a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and a positive number κ. Assume that there is a neighborhood U of $\bar{x}$ such that F is closed-valued and convex-valued on U. Let $H(\cdot) = \kappa\|\cdot\|\mathbb{B}_Y$ be an inner prederivative of F at $\bar{x}$ for $\bar{y}$. Then F is open at $\bar{x}$ for $\bar{y}$.*

Proof. We intend to show that

$$\text{for every } a > 0 \text{ there is } b > 0 \text{ such that } \bar{y} + b\mathbb{B}_Y \subset F(\bar{x} + a\mathbb{B}_X). \tag{1.7}$$

Let $0 < \kappa' < \kappa$ and take $\delta > 0$ such that $\kappa' + \delta \leq \kappa$. There exists a constant $a > 0$ such that

$$\kappa'\|x - \bar{x}\|\mathbb{B}_Y + \delta\|x - \bar{x}\|\mathbb{B}_Y \subset F(x) - \bar{y} + \delta\|x - \bar{x}\|\mathbb{B}_Y,\ \forall x \in \mathbb{B}_a(\bar{x}).$$

With no loss of generality, one may assume that $\mathbb{B}_a(\bar{x}) \subset U$. Now, take $x_0 \in \bar{x} + a\mathbb{B}_X$ such that $x_0 \neq \bar{x}$. Then, the set $F(x_0)$ being closed and convex, from the Radström's cancellation law (see Lem. 1.3), we get $\kappa'\|x_0 - \bar{x}\|\mathbb{B}_Y \subset F(x_0) - \bar{y}$. Consequently, for every sufficiently small positive a we get

$$\bar{y} + b\mathbb{B}_Y \subset F(\bar{x} + a\mathbb{B}_X), \tag{1.8}$$

where $b := \kappa'\|x_0 - \bar{x}\|$. Since $a' > a$ implies $F(\bar{x} + a\mathbb{B}_X) \subset F(\bar{x} + a'\mathbb{B}_X)$, relation (1.8) holds for any positive constant a and we have proved (1.7), *i.e.*, the mapping F is open at $\bar{x}$ for $\bar{y}$. □

It is well-known that if a single-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ $(m \leq n)$, is continuous around $\bar{x}$ and differentiable at $\bar{x}$ then f is open whenever its jacobian at $\bar{x}$ is of full rank m. One can easily note that the inner prederivative $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$ we consider in Theorem 1.5 satisfies as well the surjectivity assumption.

The remaining of the paper is as follows; in Section 2 we prove an inverse mapping theorem involving inner prederivatives. Then, we investigate in Section 3 their stability with respect to variational perturbations. In Section 4 we establish optimality conditions in terms of inner prederivatives leading to applications in welfare economics. Finally, in Section 5, to emphasize the interest of our approach we compare the concept of inner prederivative with other set-valued differentiation notions.

2. Inverse mapping theorem

We start this section by recalling the classical inverse function theorem.

Theorem 2.1 (Classical inverse function theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a neighborhood of a point $\bar{x}$. If its jacobian $\nabla f(\bar{x})$ is nonsingular then there are a neighborhood U of $\bar{x}$ and a neighborhood V of $f(\bar{x})$ such that $f : U \to V$ has a continuous inverse $f^{-1} : V \to U$ which is differentiable for all $y \in V$ and its jacobian satisfies $\nabla(f^{-1})(y) = [\nabla f(f^{-1}(y))]^{-1}$ for all $y \in V$.*

Theorem 2.1 may be viewed as a consequence of Dini's implicit function theorem (see [11]) published at the end of the 19th century but was also independently proved by Goursat [18] in 1903. Since then, a tremendous amount of research has been dedicated to this topic; some dealing with set-valued mappings and set-valued concepts of derivatives, see for instance [2–4, 16, 25]. Our purpose here is to show that, when it comes to establishing an inverse mapping theorem, inner prederivatives behave pretty much like derivatives.

Before stating our inverse set-valued mapping theorem we need the following definitions and results.

Definition 2.2 (Outer norm)**.** Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping. The outer norm of H is

$$\|H\|^+ = \sup_{\|x\| \le 1} \sup_{y \in H(x)} \|y\|,$$

with the convention that $\sup\limits_{y \in \emptyset} \|y\| = -\infty$.

It is easily seen that $\|H\|^+ = \inf\{\kappa > 0 \mid H(\mathbb{B}_X) \subset \kappa \mathbb{B}_Y\}$. In particular, $\|H\|^+ < \infty$ if and only if there exists a positive constant κ such that $H(x) \subset \kappa\|x\|\mathbb{B}_Y$, $\forall x \in X$.

Definition 2.3. A set-valued mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa \ge 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)), \text{ for all } (x, y) \in U \times V.$$

The metric regularity of a mapping is closely tied to a property of its inverse called the *Aubin property*, also known as *pseudo-Lipschitz continuity*, see [3, 12, 24].

Proposition 2.4. *A set-valued mapping $F : X \rightrightarrows Y$ is metrically regular at $\bar{x}$ for $\bar{y}$ with a constant κ if and only if its inverse $F^{-1} : Y \rightrightarrows X$ has the Aubin property at $\bar{y}$ for $\bar{x}$ with constant κ, i.e., there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$F^{-1}(y) \cap U \subset F^{-1}(y') + \kappa\|y - y'\|, \text{ for all } y, y' \in V.$$

In the following statement, under suitable assumptions, we prove that the inverse F^{-1} of a metrically regular set-valued mapping F inherits the same differential properties as F in terms of inner prederivatives. Moreover, we show that an inner prederivative of F^{-1} at a reference point $(\bar{y}, \bar{x}) \in \operatorname{gph} F^{-1}$ is the inverse of an inner prederivative of F at $(\bar{x}, \bar{y})$. These properties are the very essence of the classical inverse mapping theorem.

Theorem 2.5 (Inverse mapping theorem)**.** *Consider a set-valued mapping $F : X \rightrightarrows Y$ and a pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ such that F is metrically regular at $\bar{x}$ for $\bar{y}$. Let $H : X \rightrightarrows Y$ be an inner prederivative of F at $\bar{x}$ for $\bar{y}$ such that $\|H^{-1}\|^+ < \infty$. Then the mapping H^{-1} is an inner prederivative of F^{-1} at $\bar{y}$ for $\bar{x}$.*

Proof. Since $\|H^{-1}\|^+ < \infty$ there exists a positive number $\kappa > 0$ such that $\|H^{-1}\|^+ < \kappa$, *i.e.*, $H^{-1}(y) \subset \kappa\|y\|\mathbb{B}_X$, for all $y \in Y$. Moreover, the mapping F being metrically regular at $\bar{x}$ for $\bar{y}$, its inverse F^{-1} has the Aubin property at $\bar{y}$ for $\bar{x}$, *i.e.*, there are positive constants L, α and γ such that

$$F^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x}) \subset F^{-1}(y') + L\|y - y'\|\mathbb{B}_X, \text{ for all } y, y' \in \mathbb{B}_\gamma(\bar{y}). \tag{2.1}$$

Fix $\delta > 0$; because $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$ there exists a positive number a such that

$$H(x-\bar{x}) \subset F(x) - \bar{y} + \frac{\delta}{L\kappa}\|x-\bar{x}\|\mathbb{B}_Y, \quad \forall x \in \mathbb{B}_a(\bar{x}). \tag{2.2}$$

With no loss of generality we may assume that $a < \min\{\alpha, (L\kappa\gamma)/\delta\}$. Take $b > 0$ such that $b < \min\left\{\frac{a}{\kappa}, \gamma - \frac{\delta a}{L\kappa}\right\}$ and fix $y \in \mathbb{B}_b(\bar{y})$; we are going to prove that

$$H^{-1}(y-\bar{y}) \subset F^{-1}(y) - \bar{x} + \delta\|y-\bar{y}\|\mathbb{B}_X. \tag{2.3}$$

If $H^{-1}(y-\bar{y}) = \emptyset$ then there is nothing to prove. Otherwise, take $\tilde{x} \in H^{-1}(y-\bar{y})$ and set $x := \tilde{x} + \bar{x}$. It follows that

$$x - \bar{x} \in H^{-1}(y-\bar{y}) \subset \kappa\|y-\bar{y}\|\mathbb{B}_X. \tag{2.4}$$

Therefore, $\|x-\bar{x}\| \le \kappa b < a$ and $x \in \mathbb{B}_a(\bar{x})$. Hence, using (2.2), we have

$$y - \bar{y} \in F(x) - \bar{y} + \frac{\delta}{L\kappa}\|x-\bar{x}\|\mathbb{B}_Y.$$

Thus, there is $v \in \mathbb{B}_Y$ such that $y + \frac{\delta}{L\kappa}\|x-\bar{x}\|v \in F(x)$. Consequently,

$$x \in F^{-1}\left(y + \frac{\delta}{L\kappa}\|x-\bar{x}\|v\right) \cap \mathbb{B}_\alpha(\bar{x}).$$

Since $a < \frac{L\kappa\gamma}{\delta}$ and $b < \gamma - \frac{\delta a}{L\kappa}$ it follows that $y + \frac{\delta}{L\kappa}\|x-\bar{x}\|v \in \mathbb{B}_\gamma(\bar{y})$.

Hence, from (2.1), we infer that $x \in F^{-1}(y) + \frac{\delta}{\kappa}\|x-\bar{x}\|\mathbb{B}_X$. Therefore, thanks to (2.4), it comes

$$\tilde{x} = x - \bar{x} \in F^{-1}(y) - \bar{x} + \delta\|y-\bar{y}\|\mathbb{B}_X.$$

As a consequence, for all $\delta > 0$ there is a positive constant b such that

$$H^{-1}(y-\bar{y}) \subset F^{-1}(y) - \bar{x} + \delta\|y-\bar{y}\|\mathbb{B}_X, \text{ for all } y \in \mathbb{B}_b(\bar{y}),$$

which completes the proof. □

3. Stability of inner prederivatives

In the very simple case of a sequence of real differentiable functions $f_n : \mathbb{R} \to \mathbb{R}, n = 1, 2, \ldots$, it is well known that if f_n converges pointwise to f and f_n' (the sequence of derivatives of the functions f_n) converges uniformly to a function g then f is differentiable and $f' = g$. In this section we prove a similar result for a sequence of set-valued mappings admitting inner prederivatives. Before stating our result we need to recall a few definitions.

The lower limit, in the sense of Painlevé-Kuratowski, of a sequence A_n of subsets of a normed space E is defined by:

$$\liminf_{n\to\infty} A_n := \bigcap_{\varepsilon>0} \bigcup_{N>0} \bigcap_{n\ge N} (A_n + \varepsilon\mathbb{B}_E).$$

We will say that the sequence A_n is *lower PK-convergent* to A if $A \subset \liminf_{n\to\infty} A_n$. Recall that $\mathcal{N}_\infty$ denotes the set of all subsequences of $\mathbb{N}$ containing all n beyond some given integer; an alternative formulation of the lower limit is then given by:

$$\liminf_{n\to\infty} A_n = \{x \in E \mid \exists N \in \mathcal{N}_\infty, \forall n \in N, \exists x_n \in A_n \text{ with } x_n \to x\}.$$

If A and B are two subsets of a normed space E, the *excess* of A over B is defined by the formula

$$e(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|,$$

where we make the convention that $e(\emptyset,B)=0$ when $B\neq\emptyset$ and $e(\emptyset,B)=\infty$ if $B=\emptyset$. Note that a straighforward computation shows that $e(A,B)=\inf\{\varepsilon>0 \mid A\subset B+\varepsilon\mathbb{B}_E\}$.

Theorem 3.1. *Consider a sequence of set-valued mappings $F_n : X \rightrightarrows Y, n=1,2,\ldots$ and a mapping $F: X\rightrightarrows Y$. Let $H_n : X\rightrightarrows Y, n=1,2,\ldots$ and $H : X\rightrightarrows Y$ be positively homogeneous set-valued mappings. Take $\bar{x}\in X$ and let $\bar{y}_n$ be a sequence in Y converging to some element $\bar{y}\in Y$. We make the following assumptions:*

(1) *For any $\delta>0$, there is $a>0$ such that, for all positive integer n*

$$H_n(x-\bar{x}) \subset F_n(x) - \bar{y}_n + \delta\|x-\bar{x}\|\mathbb{B}_Y,\ \forall x\in\mathbb{B}_a(\bar{x}).$$

(2) *There is a positive number α such that*

(2.1) $\lim_{n\to\infty} \sup_{x\in\mathbb{B}_\alpha(\bar{x})} e(F_n(x),F(x)) = 0.$

(2.2) *$H_n(x)$ lower PK-converges to $H(x)$ for all $x\in\alpha\mathbb{B}_X$.*

(2.3) *$F(x)$ is compact for all $x\in\mathbb{B}_\alpha(\bar{x})$.*

Then, $(\bar{x},\bar{y})\in \operatorname{gph} F$ and the mapping H is an inner prederivative of F at $\bar{x}$ for $\bar{y}$.

Proof. Fix $\delta>0$, according to the first assumption, there is a constant $a>0$ such that

$$H_n(x-\bar{x}) \subset F_n(x) - \bar{y}_n + \delta\|x-\bar{x}\|\mathbb{B}_Y,\quad \text{for all } x\in\mathbb{B}_a(\bar{x}),\ n\in\mathbb{N}\backslash\{0\}. \tag{3.1}$$

Take an arbitrary positive number ε, thanks to assumption (2.1):

$$\exists N_\varepsilon\in\mathbb{N},\ \forall n\geq N_\varepsilon,\ \sup_{x\in\mathbb{B}_\alpha(\bar{x})} e(F_n(x),F(x)) < \varepsilon. \tag{3.2}$$

Morevover, from relation (3.1), we get $H_n(0_X)\subset F_n(\bar{x})-\bar{y}_n$ for all positive n and since $0_Y\in H_n(0_X)$ it follows that $\bar{y}_n\in F_n(\bar{x})$ for all $n\in\mathbb{N}\backslash\{0\}$.

Therefore assertion (3.2) yields $\bar{y}_n\in F(\bar{x})+\varepsilon\mathbb{B}_Y$ for all $n\geq N_\varepsilon$, thus, $\bar{y}\in \operatorname{cl}(F(\bar{x})+\varepsilon\mathbb{B}_Y)$. We infer, from the compactness of the set $F(\bar{x})$, that $\bar{y}\in F(\bar{x})+\varepsilon\mathbb{B}_Y$. Furthermore, ε being an arbitrary positive number, one has

$$\bar{y}\in\bigcap_{\varepsilon>0}(F(\bar{x})+\varepsilon\mathbb{B}_Y) = \operatorname{cl} F(\bar{x}),\ \textit{i.e.}, \bar{y}\in F(\bar{x}).$$

Now let $\varepsilon'>0$; using relation (3.1) together with assumption (2.1) and making a smaller if necessary so that $a<\alpha$, there is a positive integer $N_{\varepsilon'}$ such that

$$H_n(x-\bar{x})+\bar{y}_n \subset F(x) + (\varepsilon'+\delta\|x-\bar{x}\|)\mathbb{B}_Y,\ \text{for all } x\in\mathbb{B}_a(\bar{x}) \text{ and all } n\geq N_{\varepsilon'}.$$

Therefore,

$$\liminf_{n\to\infty}(H_n(x-\bar{x})+\bar{y}_n) \subset \operatorname{cl}\big(F(x)+(\varepsilon'+\delta\|x-\bar{x}\|)\mathbb{B}_Y\big),\ \ \forall x\in\mathbb{B}_a(\bar{x}). \tag{3.3}$$

Because $F(x)$ is compact, the set $F(x)+(\varepsilon'+\delta\|x-\bar{x}\|)\mathbb{B}_Y$ is closed. Thus, taking into account assumption (2.2), we obtain

$$H(x-\bar{x})+\bar{y}\subset F(x)+\delta\|x-\bar{x}\|\mathbb{B}_Y+\varepsilon'\mathbb{B}_Y,\ \ \forall x\in\mathbb{B}_a(\bar{x}).$$

The latter holding true for any positive ε' we get

$$H(x-\bar{x})\subset\bigcap_{\varepsilon'>0}\big(F(x)-\bar{y}+\delta\|x-\bar{x}\|\mathbb{B}_Y+\varepsilon'\mathbb{B}_Y\big),\ \ \forall x\in\mathbb{B}_a(\bar{x}).$$

Hence, for all $x \in \mathbb{B}_a(\bar{x})$,

$$H(x-\bar{x}) \subset \mathrm{cl}\big(F(x)-\bar{y}+\delta\|x-\bar{x}\|\mathbb{B}_Y\big) = F(x)-\bar{y}+\delta\|x-\bar{x}\|\mathbb{B}_Y.$$

In other terms, H is an inner prederivative of F at $\bar{x}$ for $\bar{y}$. □

Remark 3.2. Assumption (1) in Theorem 3.1 asserts that each element of the sequence F_n admits an inner prederivative H_n at $\bar{x}$ for some point $\bar{y}_n$ with a certain uniformity reflected by the fact that the neighborhood of $\bar{x}$ in which the assertion holds does not depend on n. Assertion (2.1) shares some similarities with the concept of Fisher convergence of sets (see, *e.g.*, [14]). Indeed, if a sequence of subsets A_n of a normed space is *Fisher convergent* to a subset A then it must satisfy $\lim_n e(A_n, A) = 0$. Finally, it is easily seen that when $\dim Y < \infty$ we do not need F to be compact-valued in a neighborhood of $\bar{x}$ (but only closed-valued in such a neighborhood) to prove Theorem 3.1.

4. Optimality conditions in set-valued optimization

In this section, we first derive necessary optimality conditions for several classes of minimizers of an unconstrained set-valued optimization problem by means of inner prederivatives. Subsequently, taking advantage of these results, we study the problem of finding optimal allocations in an economy involving a finite number of firms and customers as it has been modeled in [6].

For starters, the optimization problem we are interested in is the following:

$$(P):\begin{cases}\min F(x)\\ \text{s.t. } x\in X,\end{cases}$$

where the mapping $F: X \rightrightarrows Y$ is the set-valued objective function of the problem (P). From now on, the space Y is ordered by a nonempty, closed and convex cone $C \subset Y$ through a binary relation $\leq$ defined by $y_1 \leq y_2$ if and only if $y_2 - y_1 \in C$.

Recall that a pair $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$ is a *Pareto minimizer* of the problem (P) whenever $(\bar{y}-C)\cap F(X) \subset \bar{y}+C$; if the cone C is pointed, *i.e.*, if $C \cap (-C) = \{0_Y\}$, this condition becomes $(\bar{y}-C)\cap F(X) = \{\bar{y}\}$. Moreover, $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$ is said to be a *strong minimizer* of the problem (P) whenever $F(X) \subset \bar{y}+C$ while it is called a *weak minimizer* of (P) if $(\bar{y}-\mathrm{int}(C))\cap F(X) = \emptyset$. Obviously, the nonempty interior requirement of the ordering cone C could be a serious limitation to the use of the latter concept even in the finite dimensional setting. Hence, it seems natural to consider less restrictive relaxed minimizers by using other notions of interiors. In this section, we will utilize the *relative interior* (see [26,29]), the *pseudo relative interior* (see, *e.g.*, [7]) and the *quasi relative interior* introduced by Borwein and Lewis in [8] of the closed and convex cone C denoted, respectively, by $\mathrm{ri}(C), \mathrm{pri}(C)$ and $\mathrm{qri}(C)$.

For every nonempty closed and convex subset K of a Banach space, the inclusion $\mathrm{ri}(K) \subset \mathrm{pri}(K) \subset \mathrm{qri}(K)$ holds always true (see, for instance, [7]). In particular, let us recall that when working in finite dimensions, $\mathrm{ri}(K) \neq \emptyset$ and $\mathrm{ri}(K) = \mathrm{pri}(K) = \mathrm{qri}(K)$. One can also note that, while several natural ordering cones have empty relative interior in the infinite-dimensional framework, it is proved in [8] that $\mathrm{qri}(K) \neq \emptyset$ in separable Banach spaces. For more details on these concepts the reader could refer to [7,8,19,26,29] and the references therein. Throughout, we will use the symbol ρ to denote, in an abstract way, the notions of interior we consider here and we will write $\rho \in \{\mathrm{int}, \mathrm{ri}, \mathrm{pri}, \mathrm{qri}\}$ to indicate that the set $\rho(C)$ can be indistinctly equal to $\mathrm{int}(C), \mathrm{ri}(C), \mathrm{pri}(C)$ or $\mathrm{qri}(C)$. This notation allows us to define a general notion of relaxed minimizers of the problem (P).

Definition 4.1 (Relaxed minimizers)**.** Let $\rho \in \{\mathrm{int}, \mathrm{ri}, \mathrm{pri}, \mathrm{qri}\}$ be such that $\rho(C) \neq \emptyset$. We say that a pair $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$ is a ρ-minimizer of the problem (P) if $(\bar{y}-\rho(C))\cap F(X) = \emptyset$.

Of course, when $\rho(C) = \mathrm{int}(C)$, Definition 4.1 agrees with the usual definition of weak minimizers. Note also that the idea of considering such relaxed minimizers (also known as relative minimizers) is by no means new; indeed, using generalized differentiation techniques, Bao and Mordukhovich studied their existence a few years ago in [5].

Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping and let $(\bar{x}, \bar{y})$ be an element of the graph of our objective set-valued mapping F. To establish our optimality conditions we will need the following condition to be fulfilled:

$$\exists \alpha > 0,\ \ \bar{y} + H(x - \bar{x}) \subset F(X),\ \forall x \in I\!B_\alpha(\bar{x}). \tag{4.1}$$

The next lemma provides us with some sufficient conditions for an inner prederivative $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$ to satisfy (4.1).

Lemma 4.2. *Let $H : X \rightrightarrows Y$ be an inner prederivative of the set-valued objective mapping F at $\bar{x}$ for $\bar{y}$. Then H satisfies condition (4.1) whenever one of the following assertions holds:*

(i) *The mapping F is a convex process and $H \equiv F$.*
(ii) *The mapping F is linearly open at $\bar{x}$ for $\bar{y}$ and $\|H\|^+ < \infty$.*

Proof. **(i).** From Proposition 1.4 we know that $F \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$. Let $x \in X$, because F is a convex process $F(x - \bar{x}) + F(\bar{x}) \subset F(x) \subset F(X)$. Consequently, since $H = F$ and $\bar{y} \in F(\bar{x})$, we get $\bar{y} + H(x - \bar{x}) \subset F(X)$. Thus, relation (4.1) holds.

(ii). Since $\|H\|^+ < \infty$ there exists $\theta \geq 0$ such that $H(x) \subset \theta\|x\| I\!B_Y$, for all $x \in X$. Moreover, because the mapping F is linearly open there are positive constants κ, α and β such that

$$F(x + \kappa r \operatorname{int} I\!B_X) \supset (F(x) + r \operatorname{int} I\!B_Y) \cap I\!B_\beta(\bar{y}),\ \text{for all } x \in I\!B_\alpha(\bar{x}), r > 0. \tag{4.2}$$

Now, fix $\delta > 0$. The mapping H being an inner prederivative of F at $\bar{x}$ for $\bar{y}$ there is a positive constant $a < \alpha$ such that

$$\bar{y} + H(x - \bar{x}) \subset F(x) + \delta\|x - \bar{x}\| I\!B_Y,\ \text{for all } x \in I\!B_a(\bar{x}). \tag{4.3}$$

Making a smaller if necessary, one may assume that $\theta a \leq \beta$. Take any $y \in H(x - \bar{x})$ (if there is not such an element y then there is nothing to prove). We get

$$\|\bar{y} + y - \bar{y}\| = \|y\| \leq \theta\|x - \bar{x}\| \leq \theta a \leq \beta,$$

hence $\bar{y} + y \in I\!B_\beta(\bar{y})$. It follows that $\bar{y} + H(x - \bar{x}) \subset I\!B_\beta(\bar{y})$, thus, taking (4.3) into account,

$$\bar{y} + H(x - \bar{x}) \subset \big(F(x) + \delta\|x - \bar{x}\| I\!B_Y\big) \cap I\!B_\beta(\bar{y}).$$

From (4.2) we infer that $\bar{y} + H(x - \bar{x}) \subset F(X), \forall x \in I\!B_a(\bar{x})$ and the proof is complete. □

The following theorem establishes necessary optimality conditions in terms of inner prederivatives. From now on, we assume that $\rho \in \{\operatorname{int}, \operatorname{ri}, \operatorname{pri}, \operatorname{qri}\}$ is such that $\rho(C) \neq \emptyset$.

Theorem 4.3 (Optimality conditions)**.** *Let the pair $(\bar{x}, \bar{y})$ be in the graph of the set-valued objective function F. Let $H \in \mathcal{H}_{(F|(\bar{x},\bar{y}))}$ satisfy property (4.1).*

(i) *If $(\bar{x}, \bar{y})$ is a ρ-minimizer of the problem (P) then,*

$$\exists\, a > 0,\ \forall x \in I\!B_a(\bar{x}),\ H(x - \bar{x}) \cap -\rho(C) = \emptyset. \tag{4.4}$$

(ii) *If $(\bar{x}, \bar{y})$ is a Pareto minimizer of the problem (P) then,*

$$\exists\, a > 0,\ \forall x \in I\!B_a(\bar{x}),\ H(x - \bar{x}) \cap (-C) \subset C. \tag{4.5}$$

If the cone C is pointed the necessary condition (4.5) becomes

$$\exists\, a > 0,\ \forall x \in I\!B_a(\bar{x}),\ H(x - \bar{x}) \cap (-C) \subset \{0_Y\}. \tag{4.6}$$

(iii) *If* $(\bar{x}, \bar{y})$ *is a strong minimizer of the problem* (P) *then,*

$$\exists\, a > 0,\ \forall x \in I\!B_a(\bar{x}),\ H(x - \bar{x}) \subset C. \tag{4.7}$$

Proof.

(i) Let $(\bar{x}, \bar{y})$ be a relaxed minimizer of the problem (P). Assume that assertion (4.4) does not hold, *i.e.*,

$$\forall a > 0,\ \exists x \in I\!B_a(\bar{x}),\ H(x - \bar{x}) \cap -\rho(C) \neq \emptyset. \tag{4.8}$$

According to assumption (4.1) there exists a positive constant α such that

$$H(x - \bar{x}) \subset F(X) - \bar{y},\ \forall x \in I\!B_\alpha(\bar{x}). \tag{4.9}$$

Moreover, from (4.8), there are $x \in I\!B_\alpha(\bar{x})$ and $y \in H(x - \bar{x})$ such that $y \in -\rho(C)$. Using (4.9), it follows that $y \in F(X) - \bar{y}$. Consequently, $y \in (F(X) - \bar{y}) \cap -\rho(C)$, a contradiction since $(\bar{x}, \bar{y})$ is a ρ-minimizer of the problem (P). We conclude that (4.4) holds and the proof of assertion (i) is complete.

(ii) Let $(\bar{x}, \bar{y})$ be a Pareto minimizer of the problem (P); first we prove assertion (4.5). Thanks to (4.1) there exists $a > 0$ such that $H(x - \bar{x}) \subset F(X) - \bar{y}$ for all $x \in I\!B_a(\bar{x})$. Fix $x \in I\!B_a(\bar{x})$ and take $y \in H(x - \bar{x}) \cap (-C)$ (if there is no such y then we are done), it follows that $y \in (F(X) - \bar{y}) \cap (-C)$ and, because $(\bar{x}, \bar{y})$ is a Pareto minimizer of $(P), y \in C$. which proves assertion (4.5). In the case when C is a pointed cone, assertion (4.5) yields $H(x - \bar{x}) \cap (-C) \subset C \cap (-C) = \{0_Y\}$ and the proof is complete.

(iii) Let $(\bar{x}, \bar{y})$ be a strong minimizer of the problem (P). Using (4.1), there exists a positive constant a such that $H(x - \bar{x}) \subset F(X) - \bar{y} \subset C$, for all $x \in I\!B_a(\bar{x})$. □

Now, we would like to highlight the fact that our necessary conditions fit into the paradigm of optimality criterions for both single-valued and set-valued optimization problems. First, let us consider the special case when the mapping F is a Fréchet differentiable single-valued function $f : \mathbb{R}^n \to \mathbb{R}$, $C = \mathbb{R}^+$ and $\rho = \mathrm{int}$. Then, the derivative $\nabla f(\bar{x})$ of the function f at $\bar{x}$ is obviously a bounded inner prederivative of f at $\bar{x}$ for $f(\bar{x})$. Hence, relations (4.4), (4.6) and (4.7) reduce to the following standard necessary optimality condition for the minimization problem of the function f over $\mathbb{R}^n$: if the point $\bar{x} \in \mathbb{R}^n$ is optimal then

$$\exists a > 0,\ \forall x \in I\!B_a(\bar{x}),\ \nabla f(\bar{x})(x - \bar{x}) \geq 0,\ i.e., \nabla f(\bar{x}) \equiv 0.$$

Closer to our framework are the works of Luc [22,23] who derived a first order necessary condition for a pair in the graph of a set-valued mapping F to be a weak minimizer. More precisely, Luc proved that if $(x, y) \in \mathrm{gph}\, F$ is a local weak minimizer of the problem (P) then $DF(x, y)(u) \cap -\mathrm{int}(C) = \emptyset$, for all $u \in X$; where $DF(x, y)$ denotes the well-known (set-valued) contingent derivative of F at (x, y) (see the monograph of Aubin and Frankowska [3]). More recently, Jahn and Rauh [21] established necessary optimality conditions in terms of contingent epiderivatives; a concept of single-valued derivative for set-valued mappings they introduced in the same paper. They proved that if an element $(\bar{x}, \bar{y})$ in the graph of the mapping $F : D \subset X \rightrightarrows Y$ is a weak minimizer of the problem (P) (where D is the feasible set) then $\mathit{\Delta} F(\bar{x}, \bar{y})(x - \bar{x}) \notin -\mathrm{int}(C)$ for all $x \in D$, where $\mathit{\Delta} F(\bar{x}, \bar{y})$ stands for the contingent epiderivative of F at $(\bar{x}, \bar{y})$. Jahn and Rauh also proved the following necessary optimality condition for a pair $(\bar{x}, \bar{y}) \in \mathrm{gph}\, F$ to be a strong minimizer: $\mathit{\Delta} F(\bar{x}, \bar{y})(x - \bar{x}) \in C$, $\forall x \in D$. The similarities between the conditions established by Luc and the ones by Jahn and Rauh with our necessary conditions in Theorem 4.3 are obvious. In addition, let us mention that the same kind of necessary optimality conditions for set maximization problems were established by Corley [9] in terms of contingent derivatives. As a conclusion, in our opinion, Theorem 4.3 fits well into the theory of optimality conditions.

Taking advantage of our previous results, we consider the special case when the set-valued mapping F is a convex process and establish a necessary optimality condition for a constrained set-valued optimization problem. This particular case is worth mentioning since it leads to applications in welfare economics as shown later on in this very section.

Proposition 4.4. *Let $T : D \subset X \rightrightarrows Y$ be a convex process where $D \subset \operatorname{dom} T$ and let $(\bar{x}, \bar{y}) \in \operatorname{gph} T \cap (D \times Y)$. Consider the following set optimization problem:*

$$\begin{cases} \min\ T(x) \\ \text{s.t. } x \in D. \end{cases} \tag{4.10}$$

If $(\bar{x}, \bar{y})$ is a ρ-minimizer of (4.10) then

$$\forall x \in D,\ \ T(x - \bar{x}) \cap -\rho(C) = \emptyset. \tag{4.11}$$

Proof. Let $(\bar{x}, \bar{y}) \in \operatorname{gph} T \cap (D \times Y)$ be a ρ-minimizer of (4.10). First, we claim that for every $x \in D$, $T(x - \bar{x}) \subset T(D) - \bar{y}$. Indeed let $x \in D$; if $T(x - \bar{x}) = \emptyset$ there is nothing to prove. Otherwise, take $y \in T(x - \bar{x})$ then because T is a convex process, $T(x - \bar{x}) + T(\bar{x}) \subset T(x)$ thus, $y + \bar{y} \in T(D)$ and we have proved our claim. The rest of the proof is as in Theorem 4.3. Assuming that (4.11) does not hold there are $x_0 \in D$ and $y_0 \in T(x_0 - \bar{x})$ such that $y_0 \in -\rho(C)$. Thanks to the above claim $y_0 \in T(D) - \bar{y}$ hence, $(\bar{y} - \rho(C)) \cap T(D) \neq \emptyset$, a contradiction. ☐

In [17], Gaydu *et al.* established optimality conditions for set-valued optimization problems using pseudo strict prederivatives, a concept introduced by Pang in [28] (see [17], Thm. 4.6 and [17], Lem. 4.7). It turns out that the optimality conditions proved in this paper (Thm. 4.3 and Prop. 4.4) are much easier to use in practice that the ones provided in [17]. This is due to the very structure of inner prederivatives which allows us to significantly improve many results as we will discuss it in Section 5.

From now on E is a Banach space, called the *commodity space*, ordered by a nonempty and convex cone $K \subset E$ which we assume is solid, *i.e.*, $\operatorname{int}(K) \neq \emptyset$.

We consider an economy

$$\mathcal{E} = (P_1, \dots, P_p, C_1, \dots, C_n, W),$$

involving p firms and n customers. We associate to each firm $i = 1, \dots, p$ a *production set* $P_i \subset E$ while each consumer $j = 1, \dots, n$ possesses a *consumption set* $C_j \subset E$. The so-called *net demand set* $W \subset E$ describes natural situations that may happen when the initial endowment is not exactly known. A vector $x = (x_1, \dots, x_p) \in P_1 \times \ldots \times P_p$ corresponds to a production strategy and a vector $y = (y_1, \dots, y_n) \in C_1 \times \ldots \times C_n$ is a consumption plan. A pair $(x, y) \in P_1 \times \ldots \times P_p \times C_1 \times \ldots \times C_n$ is called an admissible state of the economy $\mathcal{E}$.

We make the assumption that, for every $i = 1, \dots, p$, the production sets are closed and convex. The closedness of P_i $(i = 1, \dots, p)$ guarantees the stability of the production sets in the sense that if the firm i is able to produce a sequence of commodities $x_i^1, x_i^2, \dots, x_i^k, \dots$ then it should be able to produce $x_i = \lim_{k \to \infty} x_i^k$ if there exists such an element x_i. The convexity of the production sets reflects the divisibility of the commodities. Because a firm may stop producing, for every $i = 1, \dots, p$, one has $P_i \ni 0_E$. Finally, we assume that the consumption sets $C_j, j = 1, \dots, n$, are closed and convex cones.

Definition 4.5 (Feasible allocation)**.** Let (x, y) be an admissible state of the economy $\mathcal{E}$. We say that the pair (x, y) is a feasible allocation of $\mathcal{E}$ whenever

$$\sum_{j=1}^{n} y_j - \sum_{i=1}^{p} x_i \in W.$$

From now on, we endow the space E^n with the strict order $<_{K^n}$ defined by $y <_{K^n} y'$ if and only if $y' - y \in \operatorname{int}(K^n)$. If y and y' are two consumption plans in $C_1 \times \ldots \times C_n$ the relation $y <_{K^n} y'$ means that each customer $j = 1, \dots, n$ strictly prefers y_j over y'_j. This notation allows us to introduce the concept of weak Pareto optimal allocations.

Definition 4.6 (Weak Pareto optimal allocation)**.** Let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$. The pair $(\bar{x}, \bar{y})$ is said to be a weak Pareto optimal allocation of $\mathcal{E}$ if there is no feasible allocation (x, y) such that $y <_{K^n} \bar{y}$.

In other terms, a weak Pareto optimal allocation is a feasible allocation that cannot be strictly improved upon. In the remaining of this section, it is our purpose to show that the study of weak Pareto optimal allocations of the economy $\mathcal{E}$ reduces to the solving of a set optimization problem. To this end, we consider the set-valued mapping $G : E^{p+1} \rightrightarrows E^n$ defined by

$$G(x, w) := \left\{ y \in C_1 \times \ldots \times C_n \mid w = \sum_{j=1}^{n} y_j - \sum_{i=1}^{p} x_i \right\}. \tag{4.12}$$

We associate to the economy $\mathcal{E}$ the following set optimization problem

$$\begin{cases} \min G(x, w) \\ \text{s.t. } (x, w) \in \Omega := P_1 \times \ldots \times P_p \times W. \end{cases} \tag{4.13}$$

The following proposition specifies the relationship between the weak Pareto optimal allocations of the economy $\mathcal{E}$ and the weak minimizers of the set optimization problem (4.13).

Proposition 4.7. *Let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$ and let $\bar{w} \in W$ be such that $\bar{w} = \sum_{j=1}^{n} \bar{y}_j - \sum_{i=1}^{p} \bar{x}_i$. The following assertions are equivalent:*

(i) *The pair $(\bar{x}, \bar{y})$ is a weak Pareto optimal allocation of $\mathcal{E}$.*

(ii) *The triplet $(\bar{x}, \bar{w}, \bar{y})$ is a weak minimizer of the problem (4.13).*

Proof.

(i) ⇒ (ii). Let $(\bar{x}, \bar{y})$ be a weak Pareto optimal allocation of $\mathcal{E}$; obviously, $(\bar{x}, \bar{w}, \bar{y}) \in \operatorname{gph} G \cap (\Omega \times E^n)$. Let us assume that $(\bar{x}, \bar{w}, \bar{y})$ is not a weak minimizer of (4.13) then, there are $(x, w) \in \Omega$ and $y \in G(x, w)$ such that $y \in \bar{y} - \operatorname{int}(K^n)$. It follows that the pair (x, y) is a feasible allocation of the economy $\mathcal{E}$ (due to the fact that $y \in G(x, w)$) such that $y <_{K^n} \bar{y}$, a contradiction. Consequently, $(\bar{x}, \bar{w}, \bar{y})$ is a weak minimizer of (4.13).

(ii) ⇒ (i). Let $(\bar{x}, \bar{w}, \bar{y})$ be a weak minimizer of the problem (4.13). We assume that $(\bar{x}, \bar{y})$ is not a weak Pareto optimal allocation of the economy $\mathcal{E}$. Hence, there exists a feasible allocation (x, y) of $\mathcal{E}$ such that $y <_{K^n} \bar{y}$, *i.e.*, $y \in \bar{y} - \operatorname{int}(K^n)$. Moreover, there is $w \in W$ such that $w = \sum_{j=1}^{n} y_j - \sum_{i=1}^{p} x_i$, thus, $y \in G(x, w)$. It follows that $y \in (\bar{y} - \operatorname{int}(K^n)) \cap G(\Omega)$; a contradiction since $(\bar{y} - \operatorname{int}(K^n)) \cap G(\Omega) = \emptyset$. □

To be able to use the results we proved at the beginning of the present section we need the following proposition.

Proposition 4.8. *The mapping G, as defined in (4.12), is a convex process.*

Proof. Obviously, $0_{E^n} \in G(0_{E^{p+1}})$. Let $(x, w) \in E^{p+1}$ and take $\lambda > 0$. If $(x, w) \notin \operatorname{dom} G$ then $G(x, w) = \emptyset$ and a straightforward computation shows that $G(\lambda x, \lambda w) = \emptyset$; it follows that $\lambda G(x, w) = G(\lambda x, \lambda w)$. Whenever $(x, w) \in \operatorname{dom} G$ for all $y \in G(x, w)$ we have, $\lambda w = \sum_{j=1}^{n} \lambda y_j - \sum_{i=1}^{p} \lambda x_i$, *i.e.*, $\lambda y \in G(\lambda x, \lambda w)$. Therefore, $\lambda G(x, w) \subset G(\lambda x, \lambda w)$.

Conversely, since $(x, w) \in \operatorname{dom} G$, $(\lambda x, \lambda w) \in \operatorname{dom} G$. Then for all $y \in G(\lambda x, \lambda w)$ one has $w = \sum_{j=1}^{n} \frac{1}{\lambda} y_j - \sum_{i=1}^{p} x_i$; which yields $y \in \lambda G(x, w)$. Consequently, $G(\lambda x, \lambda w) \subset \lambda G(x, w)$ and the mapping G is positively homogeneous.

Now, take two pairs (x, w) and (x', w') in E^{p+1}. We claim that

$$G(x, w) + G(x', w') \subset G(x + x', w + w'). \tag{4.14}$$

If $(x, w) \notin \operatorname{dom} G$ or $(x', w') \notin \operatorname{dom} G$ then we are done. Otherwise, take $y \in G(x, w)$ and $y' \in G(x', w')$ we get

$$w + w' = \sum_{j=1}^{n} (y_j + y'_j) - \sum_{i=1}^{p} (x_i + x'_i),$$

i.e., $y + y' \in G(x + x', w + w')$ and the proof is complete. □

Next is our last result providing a necessary condition for a feasible allocation of the economy $\mathcal{E}$ to be a weak Pareto optimal allocation. It relies on the optimality conditions obtained when the objective mapping is a convex process; in that case, an inner prederivative (at any point of its graph) is given by the mapping itself.

Proposition 4.9. *Let* $(\bar{x}, \bar{y})$ *be a feasible allocation of the economy* $\mathcal{E}$ *and let* $\bar{w} := \sum_{j=1}^{n} \bar{y}_j - \sum_{i=1}^{p} \bar{x}_i$. *If* $(\bar{x}, \bar{y})$ *is a weak Pareto optimal allocation of* $\mathcal{E}$ *then for every* $(x, w) \in \Omega$,

$$G(x - \bar{x}, w - \bar{w}) \cap -\operatorname{int}(K^n) = \emptyset. \tag{4.15}$$

Proof. Let $(\bar{x}, \bar{y})$ be a weak Pareto optimal allocation of $\mathcal{E}$. According to Proposition 4.7, the triplet $(\bar{x}, \bar{w}, \bar{y})$ is a weak minimizer of the problem (4.13), where $\bar{w} := \sum_{j=1}^{n} \bar{y}_j - \sum_{i=1}^{p} \bar{x}_i$. Because G is a convex process we can apply Proposition 4.4 with $\rho = \operatorname{int}$ and we obtain (4.15). □

5. Comparison with other concepts of differentiation

In order to emphasize the interest of introducing such a new concept of inner prederivative we will compare it with some well-known existing notions of derivatives for set-valued mappings. For the sake of consistency, we will focus on set-valued differentiation concepts involving a nonunique positively homogeneous set-valued mapping. Therefore, we will not consider in this section the graphical derivatives relying on cones as the contingent derivative [1, 3, 13], the adjacent derivative [15] or the coderivative (see, *e.g.*, [24]). For each of the concepts we will mention here, we will endeavor to point out the benefits brought by the notion of prederivative we introduce in this paper.

First of all, it makes sense to recall the definition of *inner* T*-differentiability* [10, 28] since it is one of the closest notions to our concept of inner prederivative. In [28], Pang extended several kinds of prederivatives introduced by Ioffe in [20] to the set-valued framework. Among them, was the following notion of inner T-differentiability.

Definition 5.1 (Inner T-differentiability)**.** Let $T : X \rightrightarrows Y$ be a positively homogeneous set-valued map. We say that $S : X \rightrightarrows Y$ is inner T-differentiable at $\bar{x}$ if for any $\delta > 0$ there is a neighborhood V of $\bar{x}$ such that

$$S(\bar{x}) \subset S(x) - T(x - \bar{x}) + \delta \|x - \bar{x}\| \mathbb{B} \text{ for all } x \in V. \tag{5.1}$$

Obviously, when T is a single-valued function, the inner T-differentiability in the sense of Pang forces (but is not equivalent to) the existence of an inner prederivative in the sense of Definition 1.1. In the general case (*i.e.*, when T is a set-valued mapping), we are not aware of the existence of any "classical differentiation" results involving the inner T-differentiability in the sense of Definition 5.1. It is our belief that this is due to the very definition of this concept and in particular to inclusion (5.1). Indeed, the prederivative T in (5.1) lies in the right-hand side of the inclusion which leads to very few information on the structure of T whereas inclusion (1.1) gives us a quite good description of the inner prederivative H. In our opinion, this lack of knowledge makes the inner T-differentiability a difficult notion to handle, moreover as stated by Pang himself in ([28], Ex. 4.9) it does not enjoy as many properties as the other concepts of derivative and, as such, is not taken into consideration to establish classical theorems in variational analysis.

Next is a notion of strict differentiability for set-valued maps introduced by Azé [4] in Banach spaces.

Definition 5.2. We say that a multifunction $F : X \rightrightarrows Y$ is strictly differentiable at $\bar{x} \in \operatorname{dom} F$ if there exists a closed convex process $L : X \rightrightarrows Y$ such that, for each $\varepsilon > 0$, there exist $r > 0, \alpha > 0$ such that

$$\forall x \in I\!B_r(\bar{x}), \forall u \in \alpha I\!B_X, \;\; e(L(u) \cap I\!B_Y + F(x),\, F(x+u)) < \varepsilon \|u\|. \tag{5.2}$$

First, we claim that Definition 5.2 forces Definition 1.1. Indeed, if a set-valued mapping F is strictly differentiable at $\bar{x} \in \operatorname{dom} F$ then according to Definition 5.2 for all $\varepsilon > 0$ there exists $\alpha \in (0,1)$ such that

$$L(x-\bar{x}) \cap I\!B_Y + F(\bar{x}) \subset F(x) + \varepsilon\|x-\bar{x}\| I\!B_Y,\ \forall x \in I\!B_\alpha(\bar{x}).$$

Consequently, for any $\bar{y} \in F(\bar{x})$ we get

$$L(x-\bar{x}) \cap \|x-\bar{x}\| I\!B_Y \subset F(x) - \bar{y} + \varepsilon\|x-\bar{x}\| I\!B_Y,\ \forall x \in I\!B_\alpha(\bar{x}).$$

The set-valued mapping $H : x \mapsto L(x) \cap \|x\| I\!B_Y$ being obviously positively homogeneous it is an inner prederivative of F at $\bar{x}$ for $\bar{y}$ in the sense of Definition 1.1. Hence if a set-valued mapping F is strictly differentiable at $\bar{x}$ it admits an inner prederivative at $\bar{x}$ for $\bar{y}$, where $\bar{y}$ is any point in $F(\bar{x})$. It is also interesting to compare the inversion theorem proved by Azé using his concept of strict differentiability with Theorem 2.5. Azé's theorem reads as follows:

Theorem 5.3 (Azé [4])**.** *Assume that the multifunction F is closed and strictly differentiable at $\bar{x} \in \operatorname{dom} F$ and that L is onto, then there exist $r > 0, \eta > 0$ such that*

$$F(\bar{x}) + \eta I\!B_Y \subset F(\bar{x} + r I\!B_X).$$

Moreover, F^{-1} is pseudo-Lipschitz near $(\bar{x}, \bar{y})$ for each $\bar{y} \in F(\bar{x})$ that is there exist $\eta_0, r_0, l_0 > 0$ such that $\forall y_1, y_2 \in y_0 + \eta_0 I\!B_Y$

$$e(F^{-1}(y_1) \cap (\bar{x} + r_0 I\!B_X), F^{-1}(y_2) \cap (\bar{x} + 2r_0 I\!B_X)) \leq l_0 \|y_1 - y_2\|.$$

One can immediately note that contrary to our inverse mapping theorem (Thm. 2.5) and to most results fitting into the pattern of the inverse function theorem paradigm Azé's inversion theorem does not provide any differentiability property that should be satisfied by the inverse mapping F^{-1}. In this regard, Theorem 2.5 seems more comprehensive since it establishes that the inverse mapping F^{-1} inherits the differentiability properties of F and provides an explicit definition of the inner prederivative of F^{-1}.

The last notions of differentiability for set-valued maps we would like to mention are the *pseudo-differentiability* and the *quasi-peridifferentiability* due to Nachi and Penot (see [25], Defs. 3.1 and 3.4). For the convenience of the reader we recall their definitions below.

Definition 5.4 (Pseudo-differentiability)**.** Let X and Y be normed vector spaces, a multivalued function $F : X \rightrightarrows Y$ is said to be pseudo-differentiable at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if it is lower semicontinuous at $(\bar{x}, \bar{y})$ and if there exist some neighborhood V of $\bar{y}$ and a continuous linear map $a : X \to Y$ called a derivative of F at $(\bar{x}, \bar{y})$ such that for any $\varepsilon > 0$ there exists $\beta > 0$ such that

$$F(x) \cap V \subset F(\bar{x}) + a(x-\bar{x}) + \varepsilon\|x-\bar{x}\| I\!B_Y,\ \forall x \in I\!B_\delta(\bar{x}). \tag{5.3}$$

Definition 5.5 (Quasi-peridifferentiability)**.** Let X and Y be normed vector spaces, a multivalued function $F : X \rightrightarrows Y$ is said to be quasi-peridifferentiable at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exists some continuous linear map $a : X \to Y$ such that for any $\varepsilon > 0$ there exist $\beta, \delta > 0$ for which

$$F(x) \cap \mathbb{B}_\beta(\bar{y}) - a(x - x') \subset F(x') + \varepsilon \|x - x'\| \mathbb{B}_Y, \ \forall x, x' \in \mathbb{B}_\delta(\bar{x}). \tag{5.4}$$

Obviously, any set-valued mapping F that is quasi-peridifferentiable at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ admits an inner predervative at $\bar{x}$ for $\bar{y}$. Indeed, such a set-valued mapping F satisfies $F(\bar{x}) \cap \mathbb{B}_\beta(\bar{y}) - a(\bar{x} - x') \subset F(x') + \varepsilon \|\bar{x} - x'\| \mathbb{B}_Y, \ \forall x' \in \mathbb{B}_\delta(\bar{x})$; therefore,

$$a(x - \bar{x}) \subset F(x) - \bar{y} + \varepsilon \|x - \bar{x}\| \mathbb{B}_Y, \ \forall x \in \mathbb{B}_\delta(\bar{x}).$$

Consequently, the linear map a is a single-valued inner prederivative of F at $\bar{x}$ for $\bar{y}$.

Using these differentiation concepts for set-valued maps Nachi and Penot established the following inversion theorem.

Theorem 5.6 (Nachi, Penot [25])**.** *Let X and Y be Banach spaces and let $F : X_0 \rightrightarrows Y$ be a set-valued mapping defined on some open subset X_0 of X with closed nonempty values. Suppose F is quasi-peridifferentiable at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and such that some derivative $a \in \mathcal{L}(X, Y)$ of F at $(\bar{x}, \bar{y})$ is invertible. In addition, suppose F is pseudo-differentiable at $(\bar{x}, \bar{y})$ with $F(\bar{x}) = \{\bar{y}\}$. Then F^{-1} is pseudo-differentiable at $(\bar{y}, \bar{x})$ with derivative a^{-1}.*

In Theorem 5.6, the authors need the mapping F to satisfy two different differentiability properties; in addition F must be single-valued at $\bar{x}$. Moreover, the inverse mapping F^{-1} does not inherit the differentiability properties of F. For all these reasons, together with the fact that our concept of inner prederivative is weaker than the ones used in Theorem 5.6, we believe that Theorem 2.5 expands the scope of applications of Nachi and Penot's theorem.

As a conclusion, we can say that the concept of inner prederivative is less restrictive and easier to handle than many existing notions of differentiation for set-valued mappings and it is powerful enough to lead to important results of variational analysis and set-valued optimization. There is no doubt that its range of applications has not been fully explored yet and it is our belief that inner prederivatives can be of interest, for instance, to establish an implicit mapping theorem for set-valued mappings but also to approximate set-valued maps and set iterative methods for solving inclusions.

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