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DYNAMIC PROGRAMMING PRINCIPLE AND ASSOCIATED HAMILTON-JACOBI-BELLMAN EQUATION FOR STOCHASTIC RECURSIVE CONTROL PROBLEM WITH NON-LIPSCHITZ AGGREGATOR

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Abstract. In this work we study the stochastic recursive control problem, in which the aggregator (or generator) of the backward stochastic differential equation describing the running cost is continuous but not necessarily Lipschitz with respect to the first unknown variable and the control, and monotonic with respect to the first unknown variable. The dynamic programming principle and the connection between the value function and the viscosity solution of the associated Hamilton-Jacobi-Bellman equation are established in this setting by the generalized comparison theorem for backward stochastic differential equations and the stability of viscosity solutions. Finally we take the control problem of continuous-time Epstein−Zin utility with non-Lipschitz aggregator as an example to demonstrate the application of our study.

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1. Introduction

The stochastic control theory arose along with the birth of stochastic analysis and developed fast in the last few decades due to its wide applications. Indeed the stochastic control system is a natural and effective way to involve the uncertainty, disturbance and ambiguity appearing in the real-world control problems. Its powerful feature is especially embodied in the mathematical finance problems as we study the pricing of contingent claim and the optimal strategy in the stochastic financial models, which on the contrary promotes the development of stochastic control theory.

In the development of stochastic control theory, the backward stochastic differential equation (BSDE for short) plays a big role. First of all, linear BSDE itself originated from the study of maximum principle for a stochastic control problem in Bismut [2], where it appears as the adjoint equation, and later the application of this pioneer work to mathematical finance was presented by Bismut [3]. The maximum principle reveals that the optimal solution of a stochastic control problem can be depicted by the stochastic Hamiltonian system which is actually a forward-backward stochastic differential equation (FBSDE for short). Furthermore, when the stochastic control system is observed partially or the state equation itself is a stochastic partial differential

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equation, the adjoint equation in this case is a backward stochastic partial differential equation, which was indicated in Bensoussan [1]. The maximum principle for stochastic control system with the diffusion term dependent on control and the control regions not necessarily convex was another milestone in stochastic control theory, which was solved in Peng [22] by using the second-order matrix-valued BSDE to serve as the adjoint equation. We recommend the reader to refer to the monograph [29] by Yong and Zhou, in which comprehensive introduction to the stochastic control theory is presented.

The role of BSDE in stochastic control theory is not restricted to the maximum principle as an adjoint equation. The nonlinear BSDE has even more influence in the stochastic recursive utilities and their associated control problems, thanks to the importance of recursive utilities in modern mathematical finance. The existence and uniqueness of adapted solution to nonlinear BSDE in mathematics was proved by Pardoux and Peng [20]. Later Duffie and Epstein [8] put forward the concept of stochastic differential utility in a conditional expectation form which is equivalent to the nonlinear BSDE. Since then both BSDEs and stochastic control problems in mathematical finance achieved a great progress benefiting from their connections. The reader can refer to El Karoui, Peng and Quenez [9] which concluded early works on BSDEs and their applications to mathematical finance.

The stochastic recursive control problem we concern with was introduced by Peng [24]. Its state equation is a stochastic differential equation (SDE for short):

$$X_s^{t,x;v} = x + \int_t^s b(r, X_r^{t,x;v}, v_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x;v}, v_r)\mathrm{d}B_r \quad \text{ for } x \in \mathbb{R}^n,\ s \geq t. \tag{1.1}$$

The cost functional is associated with the solution to a BSDE on the interval $[t, T]$ coupled with the state process:

$$Y_s^{t,x;v} = h(X_T^{t,x;v}) + \int_s^T f(r, X_r^{t,x;v}, Y_r^{t,x;v}, Z_r^{t,x;v}, v_r)\mathrm{d}r - \int_s^T Z_r^{t,x;v}\mathrm{d}B_r \tag{1.2}$$

and defined as below

$$J(t, x; v) \triangleq Y_t^{t,x;v}, \tag{1.3}$$

where v is an admissible control process in the admissible control set $\mathcal{U}$. The corresponding control problem is to find an optimal $\bar{v} \in \mathcal{U}$ to maximize the cost functional (1.3) for given (t, x). As you can see, FBSDE arises again to depict this recursive control system. Actually, as a popular equation, FBSDEs appear in numerous control and related mathematical finance problems. For the theories and applications of FBSDEs, we recommend the reader to refer to *e.g.* Ma, Protter and Yong [17], Peng and Wu [26], Yong [28] or the classical book [18] by Ma and Yong.

For this stochastic recursive control system (1.1)–(1.3), Peng [24] established the dynamic programming principle in the Lipschitz setting of the aggregator (or generator from BSDE point of view) and connected its value function with the Hamilton-Jacobi-Bellman (HJB for short) equation. Since the recursive utility can be regarded as the solution of BSDE (1.2) with the conditional expectation form, the stochastic recursive control system in form includes the control problem related to stochastic (recursive) differential utilities

$$V_t = \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T f(c_s, V_s)\mathrm{d}s\right], \tag{1.4}$$

where c is the consumption process serving as the control. It is well known that the recursive utility is an extension of the time-additive expected utility. In comparison with the latter, the former's risk aversion and intertemporal substitutability are separated in the aggregator which is "useful in clarifying the determinants of asset prices and presumably for a number of other issues in capital theory and finance" (see [8]).

In our study we aim to relax the Lipschitz restriction to the aggregator, *i.e.* the aggregator $f(c,u)$ in (1.4) is continuous but not necessarily Lipschitz with respect to both the utility variable u and the consumption variable c, moreover, it is of polynomial growth with respect to u in our assumptions. These settings would make much difference in the deduction of the dynamic programming principle and bring much trouble in the verification of conditions for the stability of viscosity solutions which gives the connection between the value function and the viscosity solution of the associated HJB equation. Although there are some further results on stochastic recursive control problem from the dynamic programming principle point of view, such as Peng [25] for non-Markovian framework, Buckdahn and Li [6] for stochastic differential games, Wu and Yu [27] for the cost functional generated by reflected BSDE, Li and Peng [16] for the cost functional generated by BSDE with jumps, Chen and Wu [7] for the state equation with delay, *etc.*, as far as we know there are no existing results in the non-Lipschitz aggregator setting. However, back to the stochastic recursive utilities, the aggregators in many situations are not Lipschitz with respect to the utilities and consumptions. For instance, the aggregator of the well-known continuous-time Epstein−Zin utility has a form

$$f(c,u)=\frac{\delta}{1-\frac{1}{\psi}}(1-\gamma)u\left[\left(\frac{c}{((1-\gamma)u)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}}-1\right], \tag{1.5}$$

where $\delta>0$ is the rate of time preference, $0<\gamma\neq 1$ is the coefficient of relative risk aversion and $0<\psi\neq 1$ is the elasticity of intertemporal substitution. In general, the aggregator $f(c,u)$ in (1.5) is not Lipschitz with respect to c and u but could be monotonic with respect to the latter by appropriate choices of parameters. We notice that a remarkable progress for stochastic recursive control problem with non-Lipschitz aggregator had been made by Kraft, Seifried and Steffensen [13], in which the verification theorem is proved for the non-Lipschitz Epstein−Zin aggregator and explicit solutions to HJB equation are given in some cases. Nevertheless, the dynamic programming principle for non-Lipschitz stochastic recursive control system is still not involved.

Certainly, one important technique to study stochastic recursive control problem in the non-Lipschitz setting is how to deal with the BSDE with non-Lipschitz aggregator. There is much literature devoting to the relaxation of Lipschitz condition of the aggregator $f(t,y,z)$ with respect to the first unknown variable y and/or the second unknown variable z, such as Lepeltier and San Martin [15] for linear growth condition of y and z, Kobylanski [14] for quadratic growth condition of z, Briand and Carmona [4] for polynomial growth condition of y and Pardoux [19]) for arbitrary growth condition of y, to name but a few. As for the monotonic condition of y it was first introduced to BSDE theory by Peng [23] for the infinite horizon BSDE. After that many works adopted the monotonic condition to weaken the Lipschitz assumption or make BSDE more applicable to the related fields, including *e.g.* Hu and Peng [12], Pardoux and Tang [21], Briand, Delyon, Hu, Pardoux and Stoica [5], besides [4, 19, 26] mentioned above. However, to our best knowledge there are no existing results about the dynamic programming principle and associated HJB equation for a stochastic control system involving nonlinear BSDE with the monotonic or other non-Lipschitz aggregators.

This paper generalizes the results in [24] by studying a stochastic recursive control problem, in which the cost functional generated by BSDE with the non-Lipschitz but continuous and monotonic aggregator. We first establish the dynamic programming principle with the help of the backward semigroups and generalized comparison theorem in non-Lipschitz setting, and then connect the value function of our concerned control problem with a viscosity solution of the corresponding HJB equation by means of stability of viscosity solution. Needless to say, the relaxation of Lipschitz condition makes our control problem applicable to more mathematical finance models, including the continuous-time Epstein−Zin utility with non-Lipschitz aggregator.

The rest of this paper is organized as follows. In Section 2, some useful notation is introduced and the necessary preliminaries are clarified. Then we deduce the dynamic programming principle in a non-Lipschitz aggregator setting in Section 3. In Section 4 we establish the relationship between the value function of the control problem and the viscosity solution of the corresponding HJB equation, provided that the aggregator of BSDE is independent of the second unknown variable. Finally, an example from the control problem of continuous-time Epstein−Zin utilities is given in Section 5 to demonstrate the application of our work to mathematical finance.

2. Notation and preliminaries

Given a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $(B_s)_{0\le s\le T}$ be a d-dimensional Brownian motion on the probability space. Denote by $(\mathscr{F}_s)_{0\le s\le T}$ the nature filtration generated by $(B_s)_{0\le s\le T}$ with $\mathscr{F}_0$ containing all $\mathbb{P}$-null sets of $\mathscr{F}$. We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ throughout the paper to denote the Euclidean norm and dotproduct, respectively, and then we define some useful notation.

Definition 2.1. For $q \ge 1$, $0 \le t \le T$, we denote by:

- $L^{2q}(\Omega, \mathscr{F}_t; \mathbb{R}^n)$: the space of all $\mathscr{F}_t$-measurable random variables $\xi : \Omega \to \mathbb{R}^n$ satisfying $\mathbb{E}[|\xi|^{2q}] < \infty$;
- $S^{2q}(t, T; \mathbb{R}^n)$: the space of all jointly measurable processes $\varphi : [t, T] \times \Omega \to \mathbb{R}^n$ satisfying

(i) φ_s is $\mathscr{F}_s$-adapted and φ_s is a.s. continuous for $t \le s \le T$,
(ii) $\mathbb{E}[\sup_{s\in[t,T]} |\varphi_s|^{2q}] < \infty$;

- $M^{2q}(t, T; \mathbb{R}^n)$: the space of all jointly measurable processes $\varphi : [t, T] \times \Omega \to \mathbb{R}^n$ satisfying

(i) φ_s is $\mathscr{F}_s$-adapted for $t \le s \le T$,
(ii) $\mathbb{E}[\int_t^T |\varphi_s|^{2q} \mathrm{d}s] < \infty$.

Next we clarify the set of admissible control processes $\mathcal{U}$ in the control system (1.1)–(1.3) which is defined as below:

$$\mathcal{U} \triangleq \{v |\ v \in M^2(0, T; \mathbb{R}^m) \text{ and takes values in a compact set } U \subset \mathbb{R}^m\}.$$

We make the following assumptions on the coefficients of state equation (1.1).

(H1) Both $b(t, x, v) : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma(t, x, v) : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ are jointly measurable and continuous with respect to t.

(H2) For any $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $v, v' \in U$, there exists a constant $L \ge 0$ such that

$$|b(t, x, v) - b(t, x', v')| + |\sigma(t, x, v) - \sigma(t, x', v')| \le L(|x - x'| + |v - v'|).$$

A standard argument for SDE with Lipschitz condition gives the existence and uniqueness result to the solution of SDE (1.1). To prove the dynamic programming principle, we consider a general SDE with a random variable initial value, and conclude the existence result, uniqueness result and some useful estimates to its solution.

Proposition 2.2. *Assume Conditions* (H1)–(H2). *Given $q \ge 1$, for any $t \in [0, T]$, $v \in \mathcal{U}$, $\eta \in L^{2q}(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, the following SDE*

$$X_s^{t,\eta;v} = \eta + \int_t^s b(r, X_r^{t,\eta;v}, v_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\eta;v}, v_r)\mathrm{d}B_r \tag{2.1}$$

has a unique strong solution $X_\cdot^{t,\eta;v} \in S^{2q}(t, T; \mathbb{R}^n)$.
Moreover, there exists a constant $C > 0$ depending only on L, T, q such that for any $t \le s \le T$, $v, v' \in \mathcal{U}$, $\eta, \eta' \in L^{2q}(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, we have

$$\mathbb{E}\left[\sup_{s\in[t,T]} |X_s^{t,\eta;v}|^{2q}\right] \le C\left(1 + \mathbb{E}\left[|\eta|^{2q} + \int_t^T |v_r|^{2q}\mathrm{d}r\right]\right)$$

and

$$\mathbb{E}^{\mathscr{F}_t}\left[\sup_{s\in[t,T]} |X_s^{t,\eta;v} - X_s^{t,\eta';v'}|^{2q}\right] \le C\left(|\eta - \eta'|^{2q} + \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |v_r - v'_r|^{2q}\mathrm{d}r\right]\right).$$

Then we turn to the assumptions on the coefficients of BSDE (1.2).

(H3) Both $h(x) : \mathbb{R}^n \to \mathbb{R}$ and $f(t,x,y,z,v) : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}$ are jointly measurable, and $f(t,x,y,z,v)$ is continuous with respect to (t,y,v).

(H4) For any $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $v \in U$, there exists a constant $\lambda \geq 0$ such that

$$|h(x) - h(x')| + |f(t,x,y,z,v) - f(t,x',y,z',v)| \leq \lambda(|x - x'| + |z - z'|).$$

(H5) For any $t \in [0,T]$, $x \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z \in \mathbb{R}^d$, $v \in U$, there exists a constant $\mu \in \mathbb{R}$ such that

$$(y - y')\big(f(t,x,y,z,v) - f(t,x,y',z,v)\big) \leq \mu|y - y'|^2.$$

(H6) For a given $p \geq 1$ and any $t \in [0,T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $v \in U$, there exists a constant $\kappa > 0$ such that

$$|f(t,x,y,z,v) - f(t,x,0,z,v)| \leq \kappa(1 + |y|^p).$$

With Conditions (H1)–(H6), the existence and uniqueness of solution of BSDE (1.2) is an existing result and we recommend the reader to refer to [19] for details. Also here we consider a general BSDE coupled with the solution of SDE (2.1), and conclude the existence result, uniqueness result and some useful estimates to its solution.

Proposition 2.3. *Assume Conditions* (H1)–(H6). *Given $q \geq 1$, for any $t \in [0,T]$, $s \in [t,T]$, $v \in \mathcal{U}$, $\eta \in L^{2q}(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, the following BSDE*

$$Y_s^{t,\eta;v} = h(X_T^{t,\eta;v}) + \int_s^T f(r, X_r^{t,\eta;v}, Y_r^{t,\eta;v}, Z_r^{t,\eta;v}, v_r)\mathrm{d}r - \int_s^T Z_r^{t,\eta;v}\mathrm{d}B_r \tag{2.2}$$

has a unique solution $(Y_\cdot^{t,\eta;v}, Z_\cdot^{t,\eta;v}) \in S^{2q}(t,T;\mathbb{R}) \times M^2(t,T;\mathbb{R}^d)$.

Moreover, there exists a constant $C > 0$ depending only on $L, \lambda, \mu, \kappa, T, q$ such that for any $t \leq s \leq T$, $v \in \mathcal{U}$, $\eta, \eta' \in L^{2q}(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, we have

$$|Y_t^{t,\eta;v}|^{2q} \leq C\left(1 + |\eta|^{2q} + \mathbb{E}^{\mathscr{F}_t}\left[\int_t^T |f(r,0,0,0,v_r)|^{2q}\mathrm{d}r\right]\right),$$

$$|Y_t^{t,\eta;v} - Y_t^{t,\eta';v}| \leq C|\eta - \eta'|$$

and

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y_s^{t,\eta;v}|^{2q} + \int_t^T |Y_s^{t,\eta;v}|^{2q-2}|Z_s^{t,\eta;v}|^2\mathrm{d}s\right] \leq C\left(1 + \mathbb{E}\left[|\eta|^{2q} + \int_t^T |f(s,0,0,0,v_s)|^{2q}\mathrm{d}s\right]\right).$$

Remark 2.4. In the proof of Proposition 2.3, we substitute the monotonic condition (H5) for the global Lipschitz condition in the standard deduction by Itô's formula to obtain the same forms of estimates. As for the L^{2q} estimates of solutions, $q \geq 1$, the common localization method is applied in the proof and the reader can refer to *e.g.* Lemma 3.3 in Zhang and Zhao [30].

Just as in the classical situation, the comparison theorem for BSDE (1.2) is necessary to establish the dynamic programming principle, without the exception to the new situation that the aggregator satisfies the continuous and monotonic condition rather than the Lipschitz condition. For the new situation, the following comparison theorem in Fan and Jiang [10] is applicable.

Theorem 2.5 (Comparison theorem in [10]). *Let* $(y,z)\in S^2(0,T;\mathbb{R})\times M^2(0,T;\mathbb{R}^d)$ *and* $(y',z')\in S^2(0,T;\mathbb{R})\times M^2(0,T;\mathbb{R}^d)$ *be the solutions of the following BSDEs*

$$y_t=\xi+\int_t^T f(s,y_s,z_s)\mathrm{d}s-\int_t^T z_s\mathrm{d}B_s$$

and

$$y'_t=\xi'+\int_t^T f'(s,y'_s,z'_s)\mathrm{d}s-\int_t^T z'_s\mathrm{d}B_s,$$

respectively. Assume that the terminal values $\xi,\xi'\in L^2(\Omega,\mathscr{F}_T;\mathbb{R})$ *satisfy* $\xi\le\xi'$ *a.s. and the aggregator* f (resp. f') *satisfies the following conditions:*

(A1) $f(t,y,z)$ *is weakly monotonic with respect to* y*, i.e. there exists a nondecreasing concave function* $\rho:\mathbb{R}^+\to\mathbb{R}^+$ *with* $\rho(0)=0$ *and* $\rho(u)>0$ *for* $u>0$ *such that* $\int_{0^+}\frac{1}{\rho(u)}\mathrm{d}u=+\infty$ *and for any* $t\in[0,T]$, $y_1,y_2\in\mathbb{R}$, $z\in\mathbb{R}^d$,

$$sgn(y_1-y_2)\cdot(f(t,y_1,z)-f(t,y_2,z))\le\rho(|y_1-y_2|)\quad \text{a.s.};$$

(A2) *there exists a continuous, nondecreasing and linear growth function* $\phi:\mathbb{R}^+\to\mathbb{R}^+$ *satisfying* $\phi(0)=0$ *such that for any* $t\in[0,T]$, $y\in\mathbb{R}$, $z_1,z_2\in\mathbb{R}^d$,

$$|f(t,y,z_1)-f(t,y,z_2)|\le\phi(|z_1-z_2|)\quad \text{a.s.};$$

(A3) *for any* $t\in[0,T]$, $f(t,y'_t,z'_t)\le f'(t,y'_t,z'_t)$ (resp. $f(t,y_t,z_t)\le f'(t,y_t,z_t)$).
Then we have

$$y_t\le y'_t\quad \textit{for all } t\in[0,T]\quad \text{a.s.}$$

3. Dynamic programming principle with non-Lipschitz aggregator

In this section, we prove a generalized dynamic programming for the stochastic recursive control problem, in which the aggregator f is not necessarily Lipschitz but continuous and monotonic. To begin with, we introduce the so-called backward semigroup brought forward by Peng in [25].

For given $t\in[0,T]$, $t_1\in(t,T]$, $x\in\mathbb{R}^n$, $v\in\mathcal{U}$ and $\varsigma\in L^2(\Omega,\mathscr{F}_{t_1};\mathbb{R})$, we define

$$G^{t,x;v}_{r,t_1}[\varsigma]\triangleq\hat{Y}^{t,x;v}_r,\ r\in[t,t_1],\tag{3.1}$$

where $(\hat{Y}^{t,x;v}_\cdot,\hat{Z}^{t,x;v}_\cdot)\in S^2(t,t_1;\mathbb{R})\times M^2(t,t_1;\mathbb{R}^d)$ is the solution of the following BSDE on the interval $[t,t_1]$:

$$\hat{Y}^{t,x;v}_s=\varsigma+\int_s^{t_1} f(r,X^{t,x;v}_r,\hat{Y}^{t,x;v}_r,\hat{Z}^{t,x;v}_r,v_r)\mathrm{d}r-\int_s^{t_1}\hat{Z}^{t,x;v}_r\mathrm{d}B_r$$

and $X^{t,x;v}_\cdot$ is the solution of SDE (1.1).

Based on the definition of G, *i.e.* (3.1) with $r=t$ and $t_1=T$, we know from BSDE (1.2) that

$$J(t,x;v)=Y^{t,x;v}_t=G^{t,x;v}_{t,T}\left[h\left(X^{t,x;v}_T\right)\right].\tag{3.2}$$

On the other hand, back to the control system (1.1)–(1.3), the relevant value function of the control problem maximizing the cost functional is defined as below:

$$u(t,x)\triangleq\operatorname{esssup}_{v\in\mathcal{U}}J(t,x;v),\quad (t,x)\in[0,T]\times\mathbb{R}^n.\tag{3.3}$$

In fact, u is still deterministic in our non-Lipschitz setting.

Lemma 3.1. *Assume Conditions* (H1)–(H6)*. Then the value function u defined in* (3.3) *is a deterministic function.*

Proof. To prove this lemma we follow Peng [25]. But due to the lack of Lipschitz condition of $f(t,x,y,z,v)$ with respect to y and v, some changes should be made in this proof.

To begin with, we denote by $(\mathscr{F}_{t,s})_{t\leq s\leq T}$ the natural filtration generated by $(B_s - B_t)_{0\leq t\leq s\leq T}$ and define two subspaces of $\mathcal{U}$:

$$\mathcal{U}^t \triangleq \{v \in \mathcal{U} |\ v_s \text{ is } \mathscr{F}_{t,s} - \text{measurable for } t \leq s \leq T\};$$

$$\bar{\mathcal{U}}^t \triangleq \left\{ v \in \mathcal{U} |\ v_s = \sum_{j=1}^{N} v_s^j I_{A_j},\ \text{where } v^j \in \mathcal{U}^t \text{ and } \{A_j\}_{j=1}^N \text{ is a partition of } (\Omega, \mathscr{F}_t) \right\}.$$

We first prove

$$\operatorname{esssup}_{v\in\mathcal{U}} J(t,x;v) = \operatorname{esssup}_{v\in\bar{\mathcal{U}}^t} J(t,x;v). \tag{3.4}$$

Noticing $\mathcal{U}^t \subset \bar{\mathcal{U}}^t \subset \mathcal{U}$, we only need to prove that "$\leq$" holds. To see this, note that $\bar{\mathcal{U}}^t$ is dense in $\mathcal{U}$ as shown in the proof of Proposition 3.1 in [16], so for any $v \in \mathcal{U}$, there exists a sequence $\{v^n\}_{n=1}^{\infty} \subset \bar{\mathcal{U}}^t$ such that:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_t^T |v_s^n - v_s|^2 \mathrm{d}s\right] = 0.$$

Moreover, we can choose a subsequence from $\{v^n\}_{n=1}^{\infty}$, still denoted by $\{v^n\}_{n=1}^{\infty}$ without loss of any generality, which satisfies

$$\lim_{n\to\infty} v_s^n = v_s \quad \text{a.s.}$$

Applying Itô formula to $|Y_s^{t,x;v} - Y_s^{t,x;v^n}|^2$, together with the monotonic condition, we have

$$\begin{aligned}
&\mathbb{E}[|Y_t^{t,x;v^n} - Y_t^{t,x;v}|^2] + \mathbb{E}\left[\int_t^T |Z_s^{t,x;v^n} - Z_s^{t,x;v}|^2 \mathrm{d}s\right] \\
=&\mathbb{E}[|h(X_T^{t,x;v^n}) - h(X_T^{t,x;v})|^2] + 2\mathbb{E}\Bigg[\int_t^T (Y_s^{t,x;v^n} - Y_s^{t,x;v}) \\
&\times \Big(f(s, X_s^{t,x;v^n}, Y_s^{t,x;v^n}, Z_s^{t,x;v^n}, v_s^n) - f(s, X_s^{t,x;v^n}, Y_s^{t,x;v}, Z_s^{t,x;v^n}, v_s^n)\Big)\mathrm{d}s\Bigg] \\
&+ 2\mathbb{E}\left[\int_t^T (Y_s^{t,x;v^n} - Y_s^{t,x;v}) \times \Big(f(s, X_s^{t,x;v^n}, Y_s^{t,x;v}, Z_s^{t,x;v^n}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n)\Big)\mathrm{d}s\right] \\
&+ 2\mathbb{E}\left[\int_t^T (Y_s^{t,x;v^n} - Y_s^{t,x;v}) \times \big(f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)\big)\mathrm{d}s\right] \\
\leq& \lambda^2\mathbb{E}[|X_T^{t,x;v^n} - X_T^{t,x;v}|^2] + 2\mu\mathbb{E}\left[\int_t^T |Y_s^{t,x;v^n} - Y_s^{t,x;v}|^2\mathrm{d}s\right] \\
&+ 4\lambda^2\mathbb{E}\left[\int_t^T |Y_s^{t,x;v^n} - Y_s^{t,x;v}|^2\mathrm{d}s\right] + \frac{1}{2}\mathbb{E}\left[\int_t^T |X_s^{t,x;v^n} - X_s^{t,x;v}|^2\mathrm{d}s\right] \\
&+ \frac{1}{2}\mathbb{E}\left[\int_t^T |Z_s^{t,x;v^n} - Z_s^{t,x;v}|^2\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T |Y_s^{t,x;v^n} - Y_s^{t,x;v}|^2\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_t^T |f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)|^2\mathrm{d}s\right].
\end{aligned}$$

Then an application of Gronwall's inequality leads to

$$\mathbb{E}[|Y_t^{t,x;v^n} - Y_t^{t,x;v}|^2] \le C_p\mathbb{E}\left[\int_t^T |X_s^{t,x;v^n} - X_s^{t,x;v}|^2 \mathrm{d}s\right] + C_p\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)|^2 \mathrm{d}s\right]. \quad (3.5)$$

Here and in the rest of this paper C_p is a generic constant depending only on given parameters and its values may change from line to line, moreover, we use a bracket immediately after C_p to indicate the relevant parameters when necessary.

By (3.5) and Propositions 2.2, it turns out that

$$\mathbb{E}[|Y_t^{t,x;v^n} - Y_t^{t,x;v}|^2] \le C_p\mathbb{E}\left[\int_t^T |v_s^n - v_s|^2 \mathrm{d}s\right] + C_p\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)|^2 \mathrm{d}s\right]. \quad (3.6)$$

Noticing Conditions (H3), (H4), (H6) and the fact that controls take value in a compact set, we know that

$$|f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)|^2 \le C_p\left(1 + |X_s^{t,x;v}|^2 + |Y_s^{t,x;v}|^{2p} + |Z_s^{t,x;v}|^2\right),$$

which is integrable in $L^2(\Omega \times [t,T];\mathbb{R})$ in view of Propositions 2.2 and 2.3. Thus by the dominated control theorem it yields

$$\lim_{n\to\infty} \mathbb{E}\left[\int_t^T |f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s^n) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s)|^2 \mathrm{d}s\right] = 0.$$

Hence, taking the limits on both sides of (3.6) we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left|Y_t^{t,x;v^n} - Y_t^{t,x;v}\right|^2\right] = 0.$$

Consequently, there exists a subsequence of $\{v^n\}_{n=1}^{\infty}$, still denoted by $\{v^n\}_{n=1}^{\infty}$ without loss of any generality, such that

$$\lim_{n\to\infty} Y_t^{t,x;v^n} = Y_t^{t,x;v} \quad \text{a.s.}$$

Due to the definition of cost functionals and the arbitrariness of $v \in \mathcal{U}$, we have

$$\mathrm{esssup}_{v\in\mathcal{U}}\, J(t,x;v) \le \mathrm{esssup}_{v\in\bar{\mathcal{U}}^t}\, J(t,x;v),$$

and then (3.4) follows.

The next step is to prove

$$\mathrm{esssup}_{v\in\bar{\mathcal{U}}^t}\, J(t,x;v) = \mathrm{esssup}_{v\in\mathcal{U}^t}\, J(t,x;v), \quad (3.7)$$

whose proof is similar to the classical case where $f(t,x,y,z,v)$ satisfies the Lipschitz condition with respect to y and v in [16,25] or [27]. Here we only give a sketch of proof for the reader's convenience.

For any $v \in \bar{\mathcal{U}}^t$, we assume $v_s = \sum_{j=1}^{N} v_s^j I_{A_j}$, where $v^j \in \mathcal{U}^t$ and $\{A_j\}_{j=1}^N$ is a partition of $(\Omega, \mathscr{F}_t)$. By the uniqueness of solution of BSDE (2.2), we know

$$Y_s^{t,x;\sum_{j=1}^{N} v^j I_{A_j}} = \sum_{j=1}^{N} I_{A_j} Y_s^{t,x;v^j}, \quad t \le s \le T.$$

So

$$J(t,x;v) = J\left(t,x;\sum_{j=1}^{N} v_j I_{A_j}\right) = \sum_{j=1}^{N} I_{A_j} J(t,x;v_j).$$

Note that $v_j \in \mathcal{U}^t$, hence $Y_s^{t,x;v_j}$ is $\mathscr{F}_{t,s}$-measurable for $t \le s \le T$. In particular, $J(t,x;v_j)$ is a constant for $j = 1,2,\ldots,N$. Without loss of any generality, we assume $J(t,x;v_1) = \max\{J(t,x;v_1), J(t,x;v_2),\ldots,J(t,x;v_N)\}$. Then

$$J(t,x;v) \le J(t,x;v_1) \le \operatorname{esssup}_{v\in\mathcal{U}^t} J(t,x;v).$$

The arbitrariness of v implies

$$\operatorname{esssup}_{v\in\bar{\mathcal{U}}^t} J(t,x;v) \le \operatorname{esssup}_{v\in\mathcal{U}^t} J(t,x;v).$$

On the other hand, since $\mathcal{U}^t \subset \bar{\mathcal{U}}^t$, the reverse inequality also holds, which yields (3.7).

Therefore, we conclude from (3.4) and (3.7) that

$$\operatorname{esssup}_{v\in\mathcal{U}} J(t,x;v) = \operatorname{esssup}_{v\in\mathcal{U}^t} J(t,x;v). \tag{3.8}$$

Bear in mind that the solution $Y_s^{t,x;v}$ to BSDE (2.2) is $\mathscr{F}_{t,s}$-measurable for $t \le s \le T$ if $v \in \mathcal{U}^t$. So $Y_t^{t,x;v} = J(t,x;v)$ is a constant. Therefore, (3.8) implies that u defined by (3.3) is a deterministic function. □

With Proposition 2.3 we can also obtain two lemmas related to the value function. In fact, their proofs are similar to the counterparts in [16, 25, 27], in which the estimates in Proposition 2.3 are used but the Lipschitz conditions for $f(t,x,y,z,v)$ with respect to y and v are not needed any more.

The first lemma claims the Lipschitz continuity and linear growth of the value function $u(t,x)$ with respect to x.

Lemma 3.2. *For any $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, there exists a constant C such that*

(i) $|u(t,x) - u(t,x')| \le C|x - x'|$,
(ii) $|u(t,x)| \le C(1 + |x|)$.

Proof. By Proposition 2.3, we know that for each $v \in \mathcal{U}$,

$$|J(t,x;v) - J(t,x';v)| \le C|x - x'| \quad \text{and} \quad |J(t,x;v)| \le C(1 + |x|). \tag{3.9}$$

Moreover, for any $\varepsilon > 0$, by the definition of value function (3.3), there exist v and $v' \in \mathcal{U}$ such that

$$J(t,x;v') \le u(t,x) \le J(t,x;v) + \varepsilon \quad \text{and} \quad J(t,x';v) \le u(t,x') \le J(t,x';v') + \varepsilon.$$

The above implies (ii) since

$$-C(1 + |x|) \le J(t,x;v') \le u(t,x) \le J(t,x;v) + \varepsilon \le C(1 + |x|) + \varepsilon.$$

As for (i), note that

$$J(t,x;v') - J(t,x';v') - \varepsilon \le u(t,x) - u(t,x') \le J(t,x;v) - J(t,x';v) + \varepsilon,$$

which implies

$$\begin{aligned} |u(t,x) - u(t,x')| &\le \max\{|J(t,x;v') - J(t,x';v')|, |J(t,x;v) - J(t,x';v)|\} + \varepsilon \\ &\le C|x - x'| + \varepsilon. \end{aligned}$$

Therefore, (i) follows due to the arbitrariness of ε. □

The other lemma connects the cost functional (1.3) with the solution of BSDE (2.2), where the initial state variable of SDE is a random variable.

Lemma 3.3. *For any $t \in [0,T]$, $v \in \mathcal{U}$, $\eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, we have*

$$J(t,\eta;v) = Y_t^{t,\eta;v}. \tag{3.10}$$

Proof. We first prove that (3.10) is true for simple random state variables. For $\eta = \sum_{j=1}^{N} x_j I_{A_j}$, where $x_j \in \mathbb{R}^n$ for $j = 1, 2, \ldots, N$ and $\{A_j\}_{j=1}^N$ is a finite partition of $(\Omega, \mathscr{F}_t)$, the uniqueness of solution of BSDE (2.2), together with the definition of cost functional (1.3), leads to

$$Y_t^{t,\eta;v} = Y_t^{t,\sum_{j=1}^N x_j I_{A_j};v} = \sum_{j=1}^N I_{A_j} Y_t^{t,x_j;v} = \sum_{j=1}^N I_{A_j} J(t,x_j;v) = J\left(t, \sum_{j=1}^N x_j I_{A_j}; v\right) = J(t,\eta;v).$$

Therefore, (3.10) follows for simple random state variables.

For a general $\eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, we can find a sequence of simple random state variables $\{\eta_j\}_{j=1}^\infty$ such that η_j converges to η in $L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$ as $j \to \infty$. Also, by Proposition 2.3 and (3.9), as $j \to \infty$ we have

$$\mathbb{E}[|Y_t^{t,\eta;v} - Y_t^{t,\eta_j;v}|^2] \le C^2 \mathbb{E}[|\eta - \eta_j|^2] \longrightarrow 0$$

and

$$\mathbb{E}[|J(t,\eta;v) - J(t,\eta_j;v)|^2] \le C^2 \mathbb{E}[|\eta - \eta_j|^2] \longrightarrow 0.$$

Then (3.10) holds for any random state variable in $L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$ based on $Y_t^{t,\eta_j;v} = J(t,\eta_j;v)$ for $j = 1, 2, \ldots$ ☐

Moreover, we need the following lemma which plays a big role in the proof of dynamic programming principle.

Lemma 3.4. *For any $t \in [0,T]$, $v \in \mathcal{U}$, $\eta \in L^2(\Omega, \mathscr{F}_t; \mathbb{R}^n)$, we have*

$$u(t,\eta) \ge Y_t^{t,\eta;v} \quad \text{a.s.} \tag{3.11}$$

On the other hand, for any $\varepsilon > 0$, there exists an admissible control $v \in \mathcal{U}$ such that

$$u(t,\eta) \le Y_t^{t,\eta;v} + \varepsilon \quad \text{a.s.} \tag{3.12}$$

Proof. We first prove that Lemma 3.4 holds for any simple random state variable $\zeta = \sum_{j=1}^N x_j I_{A_j}$, where $N \in \mathbb{N}$, $x_j \in \mathbb{R}^n$ and $\{A_j\}_{j=1}^N$ is a partition of $(\Omega, \mathscr{F}_t)$.

For any $v \in \mathcal{U}$, since

$$Y_t^{t,\zeta;v} = \sum_{j=1}^N Y_t^{t,x_j;v} I_{A_j} \le \sum_{j=1}^N u(t,x_j) I_{A_j} = u(t,\zeta),$$

(3.11) is true for the simple random variables. To prove (3.12), we notice that for each x_j, by (3.8) there exists an admissible control $v_j \in \mathcal{U}^t$ such that

$$u(t,x_j) \le Y_t^{t,x_j;v_j} + \varepsilon.$$

Hence taking $v=\sum_{j=1}^{N} v_j I_{A_j} \in \mathcal{U}$ we have

$$Y_t^{t,\zeta;v}+\varepsilon=\sum_{j=1}^{N}(Y_t^{t,x_j;v_j}+\varepsilon)I_{A_j}\geq\sum_{j=1}^{N}u(t,x_j)I_{A_j}=u(t,\zeta).$$

That is to say that both (3.11) and (3.12) are satisfied for the simple random state variables.

For any random state variable $\eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$, there exists a sequence of simple random variables $\{\zeta\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}|\zeta_n-\eta|=0.$$

By Proposition 2.3 and Lemma 3.2 we have for any $v\in\mathcal{U}$,

$$\lim_{n\to\infty}|Y_t^{t,\zeta_n;v}-Y_t^{t,\eta;v}|=0\ \text{ a.s.}\quad\text{and}\quad\lim_{n\to\infty}|u(t,\zeta_n)-u(t,\eta)|=0\ \text{ a.s.}$$

Since $Y_t^{t,\zeta_n;v}\leq u(t,\zeta_n)$ holds for all n, (3.11) follows for η as $n\to\infty$.

Also (3.12) is true for any random state variable $\eta\in L^2(\Omega,\mathscr{F}_t;\mathbb{R}^n)$. To demonstrate this, we choose a simple random variable ζ such that $|\zeta-\eta|<\frac{\varepsilon}{3C}$. In view of Proposition 2.3 and Lemma 3.2 again it yields that for any $v\in\mathcal{U}$,

$$|Y_t^{t,\zeta;v}-Y_t^{t,\eta;v}|\leq\frac{\varepsilon}{3}\ \text{ and }\ |u(t,\zeta)-u(t,\eta)|\leq\frac{\varepsilon}{3}\cdot$$

Moreover, since ζ is a simple random variable, (3.12) holds for ζ and there exists an admissible control $\tilde{v}\in\mathcal{U}$ such that

$$Y_t^{t,\zeta;\tilde{v}}+\frac{\varepsilon}{3}\geq u(t,\zeta).$$

Hence

$$Y_t^{t,\eta;\tilde{v}}\geq-|Y_t^{t,\zeta;\tilde{v}}-Y_t^{t,\eta;\tilde{v}}|+Y_t^{t,\zeta;\tilde{v}}\geq u(t,\zeta)-\frac{2\varepsilon}{3}\geq u(t,\eta)-\varepsilon,$$

which puts an end of proof for Lemma 3.4. □

Now we are well prepared to prove the dynamic programming principle in our settings.

Theorem 3.5 (Dynamic programming principle with non-Lipschitz aggregator)**.** *Assume Conditions* (H1)–(H6)*. Then for any* $0\leq\delta\leq T-t$*, the value function* $u(t,x)$ *defined by* (3.3) *has the following property*:

$$u(t,x)=\operatorname{esssup}_{v\in\mathcal{U}}G_{t,t+\delta}^{t,x;v}\left[u(t+\delta,X_{t+\delta}^{t,x;v})\right].$$

Proof. First of all, we claim that

$$\begin{aligned}u(t,x)&=\operatorname{esssup}_{v\in\mathcal{U}}G_{t,T}^{t,x;v}\left[h(X_T^{t,x;v})\right]\\&=\operatorname{esssup}_{v\in\mathcal{U}}G_{t,t+\delta}^{t,x;v}\left[Y_{t+\delta}^{t,x;v}\right]\\&=\operatorname{esssup}_{v\in\mathcal{U}}G_{t,t+\delta}^{t,x;v}\left[Y_{t+\delta}^{t+\delta,X_{t+\delta}^{t,x;v};v}\right].\end{aligned}\tag{3.13}$$

In the above deductions of (3.13), the first equality comes from the property of the operator G in (3.2) and the definition of the value function $u(t,x)$ in (3.3), the second equality holds due to the uniqueness of solution Y to BSDE (2.2), and the last equality is true since (2.1) has a unique strong solution X.

In the next step we need to use the comparison theorem for BSDE. Bear in mind that the aggregator of BSDE (1.2) is not Lipschitz with respect to the first unknown variable, so the classical comparison theorem does not work. Instead, we apply the generalized comparison theorem with "weakly" monotonic aggregator (Thm. 2.5) to our case. But before using this generalized comparison theorem we first need to show that $u(s, X_s^{t,x;v})$, for $t \le s \le T$ and $v \in \mathcal{U}$, is square integrable (which acts as the terminal value of BSDE). To see this, note that for any $\varepsilon > 0$, by (3.11) and (3.12) there exists $\tilde{v} \in \mathcal{U}$ such that

$$Y_s^{s,X_s^{t,x;v};\tilde{v}} \le u(s, X_s^{t,x;v}) \le Y_s^{s,X_s^{t,x;v};\tilde{v}} + \varepsilon,$$

so we only need to prove $\mathbb{E}[|Y_s^{s,X_s^{t,x;v};\tilde{v}}|^2] < \infty$. Noticing the uniform boundedness of the control processes in $\mathcal{U}$ we use Propositions 2.2 and 2.3 to know that

$$\begin{aligned}\mathbb{E}[|Y_s^{s,X_s^{t,x;v};\tilde{v}}|^2] &\le C_p \left(1 + \mathbb{E}\left[|X_s^{t,x;v}|^2 + \int_s^T |f(r,0,0,0,\tilde{v}_r)|^2 \mathrm{d}r\right]\right) \\ &\le C_p \left(1 + \mathbb{E}\left[|x|^2 + \int_t^T |v_r|^2 \mathrm{d}r + \int_t^T |f(r,0,0,0,\tilde{v}_r)|^2 \mathrm{d}r\right]\right) < \infty.\end{aligned}$$

Hence an application of Theorem 2.5, together with (3.13), yields

$$u(t,x) \le \operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u(t+\delta, X_{t+\delta}^{t,x;v})\right]. \tag{3.14}$$

On the other hand, according to (3.12), for arbitrary ε, there exists an admissible control $\bar{v} \in \mathcal{U}$ such that

$$u(t+\delta, X_{t+\delta}^{t,x;v}) \le Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v};\bar{v}} + \varepsilon. \tag{3.15}$$

Hence we have

$$\begin{aligned} u(t,x) &\ge \operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v};\bar{v}}\right] \\ &\ge \operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u(t+\delta, X_{t+\delta}^{t,x;v}) - \varepsilon\right] \\ &\ge \operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}\left[u(t+\delta, X_{t+\delta}^{t,x;v})\right] - \sqrt{C_p}\varepsilon, \end{aligned} \tag{3.16}$$

with a constant C_p. Here the second inequality in (3.16) is based on (3.15) and Theorem 2.5, and the last inequality comes from a basic estimate of BSDE. To see this, we set $Y_t^{1;v} = G_{t,t+\delta}^{t,x;v}[u(t+\delta, X_{t+\delta}^{t,x;v})]$ and $Y_t^{2;v} = G_{t,t+\delta}^{t,x;v}[u(t+\delta, X_{t+\delta}^{t,x;v}) - \varepsilon]$. Applying Itô's formula to $\mathrm{e}^{-ks}|Y_s^{1;v} - Y_s^{2;v}|^2$, where $t \le s \le t+\delta$ and $k > 0$ is a sufficiently large constant, we have

$$|Y_t^{1;v} - Y_t^{2;v}|^2 \le C_p \mathbb{E}^{\mathscr{F}_t}[|u(t+\delta, X_{t+\delta}^{t,x;v}) - \varepsilon - u(t+\delta, X_{t+\delta}^{t,x;v})|^2] = C_p \mathbb{E}^{\mathscr{F}_t}[\varepsilon^2].$$

Thus

$$\operatorname{esssup}_{v\in\mathcal{U}} Y_t^{1;v} - \operatorname{esssup}_{v\in\mathcal{U}} Y_t^{2;v} \le \operatorname{esssup}_{v\in\mathcal{U}}(Y_t^{1;v} - Y_t^{2;v}) \le \operatorname{esssup}_{v\in\mathcal{U}} |Y_t^{1;v} - Y_t^{2;v}| \le \sqrt{C_p}\varepsilon,$$

which implies

$$\operatorname{esssup}_{v\in\mathcal{U}} Y_t^{1;v} \le \operatorname{esssup}_{v\in\mathcal{U}} Y_t^{2;v} + \sqrt{C_p}\varepsilon,$$

i.e.

$$\operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}[u(t+\delta, X_{t+\delta}^{t,x;v})] \le \operatorname{esssup}_{v\in\mathcal{U}} G_{t,t+\delta}^{t,x;v}[u(t+\delta, X_{t+\delta}^{t,x;v}) - \varepsilon] + \sqrt{C_p}\varepsilon.$$

Therefore, the dynamic programming follows from (3.14) and (3.16), due to the arbitrariness of ε in (3.16). □

4. Viscosity solution of HJB equation

In this section we aim to establish the connection between the value function (3.3) of our concerned stochastic recursive control problem and the viscosity solution of its corresponding HJB equation. For this, we need to assume that the aggregator of BSDE in our concerned recursive control problem is independent of the second unknown variable throughout Section 4, *i.e.* $f(t,x,y,z,v) = f(t,x,y,v)$ for f in BSDE (1.2).

In this situation the HJB equation, a second-order fully nonlinear PDE of parabolic type, has a form:

$$\begin{cases} \frac{\partial}{\partial t}u + H(t,x,u,D_xu,D_x^2u) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = h(x). \end{cases} \tag{4.1}$$

Here D_xu and D_x^2u denote the gradient matrix and the Hessian matrix of u, respectively. The Hamiltonian $H = H(t,x,r,p,A) : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is defined as below:

$$H \triangleq \sup_{v \in U} \left\{ \frac{1}{2}\mathrm{Tr}(\sigma(t,x,v)\sigma^*(t,x,v)A) + \langle p, b(t,x,v)\rangle + f(t,x,r,v) \right\}, \tag{4.2}$$

where $\mathbb{S}^n$ is the matrix space including all $n \times n$ symmetric matrices.

Denote by $C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$ the space of all functions from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}$ whose derivatives up to the first order with respect to time variable and up to the second order with respect to state variable are continuous. Then we recall the definition for the viscosity solution of HJB equation (4.1).

Definition 4.1. A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of HJB equation (4.1), if for any $x \in \mathbb{R}^n$, $u(T,x) \le h(x)$ (resp. $u(T,x) \ge h(x)$), and for any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$, $(t,x) \in [0,T) \times \mathbb{R}^n$, $\varphi - u$ attains a global minimum (resp. maximum) at (t,x) and φ satisfies

$$\frac{\partial}{\partial t}\varphi(t,x) + H(t,x,\varphi,D_x\varphi,D_x^2\varphi) \ge 0$$
$$\left(\text{resp. } \frac{\partial}{\partial t}\varphi(t,x) + H(t,x,\varphi,D_x\varphi,D_x^2\varphi) \le 0\right).$$

We call u the viscosity solution of (4.1) if u is both a viscosity subsolution and a viscosity supersolution.

We need some preliminaries to establish the connection. First, we indicate the continuity of the value function.

Proposition 4.2. *Assume Conditions* (H1)–(H6)*. Then the value function* $u(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *defined in* (3.3) *is continuous with respect to* (t,x)*.*

Note that the Lipschitz continuity of $u(t,x)$ with respect to x is a result of Lemma 3.2. Also we can prove the $\frac{1}{2}$-Hölder continuity of $u(t,x)$ with respect to t in a similar way referring to *e.g.* Theorem 3.2 in [16], which together with Lemma 3.2 implies the continuity of the value function with respect to (t,x). In the proof of $\frac{1}{2}$-Hölder continuity of $u(t,x)$, the only difference between the non-Lipschitz aggregator in our paper and the Lipschitz aggregator in classical cases is that the Lipschitz condition is replaced by the monotonic condition of aggregator in the application of Itô formula, so we leave out the proof here.

Then we define a sequence of smootherized functions f_n, $n \in \mathbb{N}$, based on the aggregator f as follows:

$$f_n(t,x,y,v) \triangleq \ (\rho_n * f(t,x,\cdot,v))(y), \tag{4.3}$$

where $\rho_n : \mathbb{R} \to \mathbb{R}^+$, $n \in \mathbb{N}$, is a family of sufficiently smooth functions with compact support in $[-\frac{1}{n}, \frac{1}{n}]$ and satisfies

$$\int_{\mathbb{R}} \rho_n(a)\mathrm{d}a = 1.$$

Consequently, we have a sequence of BSDEs with the smootherized aggregators f_n, $n \in \mathbb{N}$, on the interval $[t, T]$:

$$Y_s^{t,x,n;v} = h(X_T^{t,x;v}) + \int_s^T f_n(r, X_r^{t,x;v}, Y_r^{t,x,n;v}, v_r)\mathrm{d}r - \int_s^T Z_r^{t,x,n;v}\mathrm{d}B_r \quad \text{for } t \le s \le T. \tag{4.4}$$

With the solutions of BSDEs (4.4), we can define a sequence of stochastic recursive control problems whose cost functional for each $n \in \mathbb{N}$ is

$$J_n(t, x; v) \triangleq Y_t^{t,x,n;v} \quad \text{for } v \in \mathcal{U},\ t \in [0, T],\ x \in \mathbb{R}^n \tag{4.5}$$

and corresponding control problem is to find an optimal $\bar{v} \in \mathcal{U}$ to maximize above cost functional (4.5) for given (t, x). Thus, for each $n \in \mathbb{N}$, the value function of control problem is defined by

$$u_n(t, x) \triangleq \operatorname{esssup}_{v\in\mathcal{U}} J_n(t, x; v) \quad \text{for } t \in [0, T],\ x \in \mathbb{R}^n \tag{4.6}$$

and the Hamiltonian appears like

$$H_n(t, x, r, p, A) \triangleq \sup_{v\in U} \left\{ \frac{1}{2}\mathrm{Tr}(\sigma(t, x, v)\sigma^*(t, x, v)A) + \langle p, b(t, x, v)\rangle + f_n(t, x, r, v) \right\}, \tag{4.7}$$

where $(t, x, r, p, A) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$.

Then we prove the uniform convergence of the smootherized aggregators in a compact subset of their domain utilizing the continuity of the aggregator.

Lemma 4.3. *Assume Conditions* (H3)–(H4). *Then f_n defined in* (4.3) *converges to f, uniformly in every compact subset of $[0, T] \times \mathbb{R}^n \times \mathbb{R} \times U$.*

Proof. Since $\int_{\mathbb{R}} \rho_n(a)da = 1$, we have

$$f_n(t, x, y, v) - f(t, x, y, v) = \int_{\mathbb{R}} (f(t, x, y - a, v) - f(t, x, y, v))\rho_n(a)da.$$

For any given compact set $K \subset [0, T]\times\mathbb{R}^n\times\mathbb{R}\times U$, there exists another compact set $\hat{K}$ such that $(t, x, y-a, v) \in \hat{K}$ for any $(t, x, y, v) \in K$ and $a \in [-1, 1]$. Notice that since $f(t, x, y, v)$ is continuous with respect to (t, y, v) and Lipschitz continuous with respect to x, we know the continuity and further the uniform continuity of $f(t, x, y, v)$ with respect to (t, x, y, v) in the compact set $\hat{K}$. So, for any $\varepsilon > 0$, as n is sufficiently large we have

$$\begin{aligned}
\sup_{(t,x,y,v)\in K} |f_n(t, x, y, v) - f(t, x, y, v)| &\le \sup_{(t,x,y,v)\in K} \int_{|a|\le\frac{1}{n}} |f(t, x, y - a, v) - f(t, x, y, v)|\rho_n(a)\mathrm{d}a \\
&\le \varepsilon \int_{|a|\le\frac{1}{n}} \rho_n(a)\mathrm{d}a \\
&= \varepsilon,
\end{aligned}$$

which implies the desired conclusion. ∎

As a result, we can further get the uniform convergence of the solutions of BSDEs with smootherized aggregators in $L^2(\Omega)$ space.

Lemma 4.4. *Assume Conditions* (H1)–(H6). *Then for any $v \in \mathcal{U}$,*

$$\lim_{n\to\infty} \sup_{(t,x)\in K} \mathbb{E}\left[\left| Y_t^{t,x,n;v} - Y_t^{t,x;v} \right|^2 \right] = 0,$$

where K is an arbitrary compact set in $[0, T]\times\mathbb{R}^n$, $Y_\cdot^{t,x;v}$ and $Y_\cdot^{t,x,n;v}$ are the solutions of BSDEs (1.2) *and* (4.4), *respectively.*

Proof. Firstly, it is obvious that the smootherized aggregator f_n satisfies Conditions (H3)–(H6). Hence, applying Itô's formula to $|Y_s^{t,x,n;v} - Y_s^{t,x;v}|^2$, we have for any $(t,x) \in K$,

$$
\begin{aligned}
\mathbb{E}[|Y_t^{t,x,n;v} - Y_t^{t,x;v}|^2] \le\,& C_p \mathbb{E}\left[\int_t^T |f_n(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2 \mathrm{d}s\right] \\
=\,& C_p \mathbb{E}\left[\int_t^T |f_n - f|^2 I_{\left\{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right\} \cup \left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| \ge N\right\}\right\}} \mathrm{d}s\right] \\
&+ C_p \mathbb{E}\left[\int_t^T |f_n - f|^2 I_{\left\{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| < N\right\} \cap \left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| < N\right\}\right\}} \mathrm{d}s\right] \\
\le\,& C_p \mathbb{E}\left[\int_t^T |f_n - f|^2 I_{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right] + C_p \mathbb{E}\left[\int_t^T |f_n - f|^2 I_{\left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right] \\
&+ C_p \mathbb{E}\left[\int_t^T |f_n - f|^2 I_{\left\{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| < N\right\} \cap \left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| < N\right\}\right\}} \mathrm{d}s\right].
\end{aligned}
$$

Then we define

$$
\begin{aligned}
J_1 &\triangleq \mathbb{E}\left[\int_t^T |f_n(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2 I_{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right], \\
J_2 &\triangleq \mathbb{E}\left[\int_t^T |f_n(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2 I_{\left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right], \\
J_3 &\triangleq \mathbb{E}\left[\int_t^T |f_n(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s) - f(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2 \times I_{\left\{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| < N\right\} \cap \left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}| < N\right\}\right\}} \mathrm{d}s\right],
\end{aligned}
$$

and deal with J_1, J_2 and J_3 in turn.

For J_1, it turns out that

$$
\begin{aligned}
\sup_{(t,x)\in K} J_1 \le& \sup_{(t,x)\in K} 2\mathbb{E}\left[\int_t^T \left(|f_n(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2 + |f(s, X_s^{t,x;v}, Y_s^{t,x;v}, v_s)|^2\right) I_{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right] \\
\le& \sup_{(t,x)\in K} C_p \mathbb{E}\left[\int_t^T (1 + |X_s^{t,x;v}|^2 + |Y_s^{t,x;v}|^{2p}) I_{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right\}} \mathrm{d}s\right] \\
\le& \sup_{(t,x)\in K} C_p \left(\mathbb{E}\left[\int_t^T (1 + |X_s^{t,x;v}|^4 + |Y_s^{t,x;v}|^{4p}) \mathrm{d}s\right]\right)^{\frac{1}{2}} \left(\sup_{(t,x)\in K} \mathbb{P}\left[\sup_{s\in[t,T]} |X_s^{t,x;v}| \ge N\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

To estimate above, we use Chebychev's inequality and Proposition 2.2 to obtain for any $N > 0$,

$$\mathbb{P}\left[\sup_{s\in[t,T]} |X_s^{t,x;v}| \geq N\right] \leq \frac{1}{N^2}\mathbb{E}\left[\sup_{s\in[t,T]} |X_s^{t,x;v}|^2\right] \leq \frac{C_p}{N^2}|x|^2.$$

Thus for any $\delta > 0$, when we take a sufficiently large N, it follows from the boundedness of x in K that

$$\sup_{(t,x)\in K} \mathbb{P}\left[\sup_{s\in[t,T]} |X_s^{t,x;v}| \geq N\right] \leq \delta. \tag{4.8}$$

Moreover, by Propositions 2.2 and 2.3, we know

$$\sup_{(t,x)\in K}\left(\mathbb{E}\left[\int_t^T (1 + |X_s^{t,x;v}|^4 + |Y_s^{t,x;v}|^{4p})\mathrm{d}s\right]\right)^{\frac{1}{2}} < \infty,$$

which together with (4.8) implies that for any given $\varepsilon > 0$, there exists a sufficiently large N_1 such that as $N \geq N_1$, for all $n \in \mathbb{N}$,

$$J_1 \leq \varepsilon, \quad \text{uniformly in the compact set } K.$$

Then we turn to J_2, and by Proposition 2.3 we have

$$\mathbb{E}\left[\sup_{s\in[t,T]} |Y_s^{t,x;v}|^{2p}\right] \leq C_p(1 + |x|^{2p}).$$

Again with a sufficiently large N, the application of Chebychev's inequality in K leads to for any $\delta > 0$,

$$\sup_{(t,x)\in K} \mathbb{P}\left[\sup_{s\in[t,T]} |Y_s^{t,x;v}| \geq N\right] \leq \delta. \tag{4.9}$$

Similar to the treatment of J_1, by (4.9) we can find a sufficiently large N_2 such that as $N \geq N_2$, for all $n \in \mathbb{N}$,

$$J_2 \leq \varepsilon, \quad \text{uniformly in the compact set } K.$$

We take $\hat{N} = N_1 \vee N_2$ and use $\hat{N}$ to prove the uniform convergence of J_3.

To this end, notice for any $v \in \mathcal{U}$,

$$\begin{aligned}\sup_{(t,x)\in K} J_3 &\leq \mathbb{E}\left[\int_t^T \sup_{(t,x)\in K} |f_n - f|^2 I_{\left\{\left\{\sup\limits_{s\in[t,T]} |X_s^{t,x;v}|<\hat{N}\right\}\cap\left\{\sup\limits_{s\in[t,T]} |Y_s^{t,x;v}|<\hat{N}\right\}\right\}}\mathrm{d}s\right] \\ &\leq \int_t^T \sup_{(t,x,y)\in[0,T]\times\bar{B}^n_{\hat{N}}\times[-\hat{N},\hat{N}]} |f_n(t,x,y,v) - f(t,x,y,v)|^2\mathrm{d}s,\end{aligned}$$

where $\bar{B}^n_{\hat{N}}$ is the closed ball with the radium $\hat{N}$ in $\mathbb{R}^n$. By the dominated convergence theorem and Lemma 4.3, as n is sufficiently large, we have

$$J_3 \leq \varepsilon, \quad \text{uniformly in the compact set } K.$$

Therefore, due to the arbitrariness of ε, the claim that $\lim\limits_{n\to\infty} \sup\limits_{(t,x)\in K} \mathbb{E}[|Y_t^{t,x,n;v} - Y_t^{t,x;v}|^2] = 0$ follows, which puts an end of proof. □

The uniform convergence in compact subset of domain holds for the value function as well, which is displayed in next lemma.

Lemma 4.5. *Assume Conditions* (H1)–(H6)*. Then u_n converges to u, uniformly in every compact subset of* $[0,T]\times\mathbb{R}^n$*.*

Proof. Given arbitrary $\varepsilon>0$, for any $t\in[0,T]$ and $x\in\mathbb{R}^n$, we can find $v_1\in\mathcal{U}$ such that

$$u(t,x) < Y_t^{t,x;v_1}+\varepsilon.$$

So

$$u(t,x)-u_n(t,x)=u(t,x)-\sup_{v\in\mathcal{U}} Y_t^{t,x,n;v}\leq Y_t^{t,x;v_1}+\varepsilon-Y_t^{t,x,n;v_1}. \tag{4.10}$$

On the other hand, for above ε, there exists $v_2\in\mathcal{U}$ such that

$$u_n(t,x)\leq Y_t^{t,x,n;v_2}+\varepsilon,$$

which implies

$$u(t,x)-u_n(t,x)\geq u(t,x)-Y_t^{t,x,n;v_2}-\varepsilon\geq Y_t^{t,x;v_2}-Y_t^{t,x,n;v_2}-\varepsilon. \tag{4.11}$$

Since (4.10) and (4.11), we have

$$|u(t,x)-u_n(t,x)|\leq \mathbb{E}\left[|Y_t^{t,x;v_1}-Y_t^{t,x,n;v_1}|\right]+\mathbb{E}\left[|Y_t^{t,x;v_2}-Y_t^{t,x,n;v_2}|\right]+2\varepsilon.$$

Noticing v_1 and v_2 are given admissible controls, by Lemma 4.4 we know that for any compact set $K\subset[0,T]\times\mathbb{R}^n$,

$$\lim_{n\to\infty}\sup_{(t,x)\in K}\mathbb{E}\left[|Y_t^{t,x;v_1}-Y_t^{t,zx,n;v_1}\right]+\mathbb{E}\left[|Y_t^{t,x;v_2}-Y_t^{t,x,n;v_2}|\right]=0.$$

Due to the arbitrariness of ε, we obtain the uniform convergence of the value functions u_n to u in K. □

We have known that f_n converges uniformly to f in the compact subset of their domain, so, by definitions of the Hamiltonians (4.2) and (4.7), it comes without a surprise that the same kind of convergence of the Hamiltonian H_n to H holds as well.

Lemma 4.6. *Assume Conditions* (H1)–(H6)*. Then H_n converges to H, uniformly in every compact subset of their domain.*

Proof. To see this, for any $t\in[0,T]$, $x\in\mathbb{R}^n$, $r\in\mathbb{R}$, $p\in\mathbb{R}^n$, $A\in\mathbb{S}^n$, $v\in U$, we set

$$\mathcal{A}=\frac{1}{2}\mathrm{Tr}(\sigma(t,x,v)\sigma^*(t,x,v)A)+\langle p,b(t,x,v)\rangle+f_n(t,x,r,v)$$

and

$$\mathcal{B}=\frac{1}{2}\mathrm{Tr}(\sigma(t,x,v)\sigma^*(t,x,v)A)+\langle p,b(t,x,v)\rangle+f(t,x,r,v).$$

Noticing

$$H_n-H=\sup_{v\in U}\mathcal{A}-\sup_{v\in U}\mathcal{B}\leq\sup_{v\in U}(\mathcal{A}-\mathcal{B})=\sup_{v\in U}(f_n-f)\leq\sup_{v\in U}|f_n-f|$$

and

$$H-H_n=\sup_{v\in U}\mathcal{B}-\sup_{v\in U}\mathcal{A}\leq\sup_{v\in U}(\mathcal{B}-\mathcal{A})=\sup_{v\in U}(f-f_n)\leq\sup_{v\in U}|f-f_n|,$$

we have

$$|H_n-H|\leq\sup_{v\in U}|f_n(t,x,r,v)-f(t,x,r,v)|.$$

For any compact set $K \subset [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n$ and $(t,x,r,p,A,v)\in K\times U$, note that $(t,x,r,v)\in\hat{K}$ and $\hat{K}$ is a compact set in $[0,T]\times\mathbb{R}^n\times\mathbb{R}\times U$. Hence by Lemma 4.3, we have

$$\begin{aligned}\lim_{n\to\infty}\sup_{(t,x,r,p,A)\in K}|H_n(t,x,r,p,A)-H(t,x,r,p,A)| &\le \lim_{n\to\infty}\sup_{(t,x,r,p,A)\in K}\sup_{v\in U}|f_n(t,x,r,v)-f(t,x,r,v)|\\ &\le \lim_{n\to\infty}\sup_{(t,x,y,v)\in\hat{K}}|f_n(t,x,y,v)-f(t,x,y,v)|=0.\end{aligned}$$

Therefore, the uniform convergence of H_n to H in K follows from above. □

To end the preliminaries, we introduce the stability property of viscosity solutions below (see *e.g.* Lem. 6.2 in Fleming and Soner [11] for details of proof) which provides a method based on the uniform convergence of Hamiltonians to get the connection between the value function and the solution of HJB equation.

Proposition 4.7 (Stability)**.** *Let u_n be a viscosity subsolution* (resp. *supersolution*) *to the following PDE*

$$\frac{\partial}{\partial t}u_n(t,x)+H_n(t,x,u_n(t,x),D_xu_n(t,x),D_x^2u_n(t,x))=0,\quad (t,x)\in[0,T)\times\mathbb{R}^n,$$

where $H_n(t,x,r,p,A):[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\to\mathbb{R}$ is continuous and satisfies the ellipticity condition

$$H_n(t,x,r,p,X)\le H_n(t,x,r,p,Y)\quad \textit{whenever}\quad X\le Y. \tag{4.12}$$

Assume that H_n and u_n converge to H and u, respectively, uniformly in every compact subset of their own domains. Then u is a viscosity subsolution (*resp. supersolution*) *of the limit equation*

$$\frac{\partial}{\partial t}u(t,x)+H(t,x,u(t,x),D_xu(t,x),D_x^2u(t,x))=0.$$

Now we are well prepared to prove the main theorem in this section.

Theorem 4.8. *Assume Conditions* (H1)–(H6)*. Then u defined in* (3.3) *is a viscosity solution of HJB equation* (4.1)*.*

Proof. We divide our proof into two steps.

Step 1. Assume that $|f(t,x,0,v)|$ is uniformly bounded on $[0,T]\times\mathbb{R}^n\times U$.

Note that the uniform boundedness of $|f(t,x,0,v)|$ implies the global Lipschitz of $f_n(t,x,y,v)$ with respect to y. To see this, for any $(t,x,v)\in[0,T]\times\mathbb{R}^n\times U$, $y_1,y_2\in\mathbb{R}$, by (H6) it yields that

$$\begin{aligned}|f_n(t,x,y_1,v)-f_n(t,x,y_2,v)| &=\left|\int_{|a|\le\frac{1}{n}}f(t,x,a,v)\big(\rho_n(y_1-a)-\rho_n(y_2-a)\big)\mathrm{d}a\right|\\ &\le \max_{a\in[-\frac{1}{n},\frac{1}{n}]}|f(t,x,a,v)|\int_{|a|\le\frac{1}{n}}|\rho_n(y_1-a)-\rho_n(y_2-a)|\mathrm{d}a\\ &\le \max_{a\in[-\frac{1}{n},\frac{1}{n}]}\big(|f(t,x,0,v)|+\kappa(1+|a|^p)\big)\int_{|a|\le\frac{1}{n}}C_p(n)|y_1-y_2|\mathrm{d}a\\ &\le C_p(\kappa,n)|y_1-y_2|.\end{aligned}$$

Hence we immediately know from Theorem 7.3 in [25] (which establishes the connection between the value function (3.3) and the solution of HJB equation (4.1) with the Lipschitz continuous aggregator) that $u_n(t,x)$ is the viscosity solution of the following equations:

$$\begin{cases}\dfrac{\partial}{\partial t}u_n+H_n(t,x,u_n,D_xu_n,D_x^2u_n)=0, & (t,x)\in[0,T)\times\mathbb{R}^n,\\ u_n(T,x)=h(x),\end{cases}\tag{4.13}$$

where u_n and H_n are defined by (4.6) and (4.7), respectively.

By Lemmas 4.3–4.6 the uniform convergence of f_n to f, $Y_t^{t,x,n;v}$ to $Y_t^{t,x;v}$, u_n to u and H_n to H holds in every compact subset of their own domains as $n \to \infty$. Moreover, H_n satisfies the ellipticity condition (4.12). Therefore, by the stability of viscosity solution stated in Proposition 4.7, we know that u is a viscosity solution of the limit equation

$$\frac{\partial}{\partial t}u(t,x) + H(t,x,u(t,x),D_x u(t,x),D_x^2 u(t,x)) = 0,$$

where u and H are defined by (3.3) and (4.2), respectively.

As for the terminal value of above equation, *i.e.* $u(T,x) = h(x)$, which can be seen from the definition of the value function. Thereby u is a viscosity solution of HJB equation (4.1).

Step 2. $|f(t,x,0,v)|$ is not necessarily uniformly bounded on $[0,T] \times \mathbb{R}^n \times U$.

We construct a sequence of functions

$$f_m(t,x,y,v) \triangleq f(t,x,y,v) - f(t,x,0,v) + \Pi_m\big(f(t,x,0,v)\big) \quad \text{for } m \in \mathbb{N},$$

where $\Pi_m(x) = \frac{\inf(m,|x|)}{|x|}x$.

With these f_m, we get a family of BSDEs for $m \in \mathbb{N}$ on the interval $[t,T]$:

$$Y_t^{t,x,m;v} = h(X_T^{t,x;v}) + \int_t^T f_m(s,X_s^{t,x;v},Y_s^{t,x,m;v},v_s)\mathrm{d}s - \int_t^T Z_s^{t,x,m;v}\mathrm{d}B_s.$$

Similarly we define the corresponding cost functional

$$J_m(t,x;v) \triangleq Y_t^{t,x,m;v} \quad \text{for } v \in \mathcal{U},\ t \in [0,T],\ x \in \mathbb{R},$$

the value function

$$u_m(t,x) \triangleq \operatorname{esssup}_{v\in\mathcal{U}} J_m(t,x;v) \quad \text{for } t \in [0,T],\ x \in \mathbb{R},$$

and the Hamiltonian

$$H_m(t,x,r,p,A) \triangleq \sup_{v\in U}\left\{\frac{1}{2}\mathrm{Tr}(\sigma(t,x,v)\sigma^*(t,x,v)A) + \langle p, b(t,x,v)\rangle + f_m(t,x,r,v)\right\}$$

for $(t,x,r,p,A) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$.

Since $f_m(t,x,0,v) = \Pi_m\big(f(t,x,0,v)\big)$, $f_m(t,x,0,v)$ is uniformly bounded. Moreover, it is not difficult to verify that f_m satisfies Conditions (H3)–(H6). Hence f_m satisfies the conditions in Step 1. By Step 1 we know that u_m is a viscosity solution of the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}u_m + H_m(t,x,u_m,D_x u_m,D_x^2 u_m) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u_m(T,x) = h(x). \end{cases} \tag{4.14}$$

We then prove that the uniform convergence of f_m to f, $Y_t^{t,x,m;v}$ to $Y_t^{t,x;v}$, u_m to u and H_m to H also holds in every compact subset of their own domains as $m \to \infty$, among which only the proof for the convergence of f_m to f is very different from Lemma 4.3 due to the different definitions of f_m from f_n, and other convergence can be proved similarly according to Lemmas 4.4–4.6 in turn.

In fact, the uniform convergence of f_m to f in every compact subset of $[0,T] \times \mathbb{R}^n \times \mathbb{R} \times U$ is easy to see if we notice that for any given compact set $K \subset [0,T] \times \mathbb{R}^n \times \mathbb{R} \times U$, $f(t,x,y,v)$ is bounded by a positive integer M_K for any $(t,x,y,v) \in K$ due to the continuity of f. Hence, when $m \geq M_K$, we have

$$\sup_{(t,x,y,v)\in K}|f_m(t,x,y,v) - f(t,x,y,v)| = \sup_{(t,x,y,v)\in K}|f(t,x,0,v) - \Pi_m\big(f(t,x,0,v)\big)| = 0,$$

which implies the uniform convergence of f_m to f in every compact subset of their domain as $m \to \infty$.

Finally, using Proposition 4.7 again we know that u satisfies the limit equation of (4.14), which together with the fact $u(T,x)=h(x)$ by the definition of value function shows that u is still a viscosity solution of HJB equation (4.1) even if $|f(t,x,0,v)|$ is not necessary to be uniformly bounded for any $(t,x,v)\in[0,T]\times\mathbb{R}^n\times U$. □

5. Example

As mentioned in Introduction, Duffie and Epstein [8] put forward the stochastic differential utility which can be regarded as the solution of a BSDE. Based on this basic correspondence, we give an example to demonstrate the application of our study to utility.

We start from setting an financial market with two assets which can be traded continuously. One is the bond, a non-risky asset, whose price process P_t^0 is governed by the ordinary differential equation

$$P_t^0 = 1 + \int_0^t r_s P_s^0 \mathrm{d}s. \tag{5.1}$$

The other asset is the stock, a risky asset, whose price process P_t is modeled by the linear SDE

$$P_t = p + \int_0^t P_s b_s \mathrm{d}s + \int_0^t P_s \sigma_s \mathrm{d}B_s, \qquad \text{where } p>0 \text{ is given.} \tag{5.2}$$

In (5.1) and (5.2), $r:[0,T]\to\mathbb{R}$ is the interest rate of the bond, $b:[0,T]\to\mathbb{R}$ is the appreciation rate of the stock and $\sigma:[0,T]\to\mathbb{R}$ is the volatility process, all of which are continuous functions.

An agent may decide what is the optimal investment and consumption at time $t\in[0,T]$. We denote by $\pi:\Omega\times[0,T]\to[-1,1]$ the proportion of the wealth invested into the stock and by $c:\Omega\times[0,T]\to[a_1,a_2]$, $0\le a_1<a_2$, a restricted consumption decision. If X_t denotes the wealth of the agent at time t, then the amount of money invested in the bond is $X_t(1-\pi_t)$. In view of (5.1) and (5.2), the agent's wealth satisfies the following equation:

$$\begin{cases} \mathrm{d}X_t = [r_t X_t + (b_t - r_t)\pi_t X_t - c_t]\mathrm{d}t + X_t\pi_t\sigma_t \mathrm{d}B_t, \\ X_0 = x, \end{cases} \tag{5.3}$$

where $x>0$ is the initial wealth of the agent. It is clear that (5.3) acting as the state equation satisfies Conditions (H1) and (H2).

We assume that the stochastic differential utility preference of the agent is a continuous time Epstein−Zin utility as illustrated in (1.5) and the utility satisfies the following BSDE:

$$\begin{cases} \mathrm{d}V_t = -\dfrac{\delta}{1-\frac{1}{\psi}}(1-\gamma)V_t\left[\left(\dfrac{c_t}{((1-\gamma)V_t)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right]\mathrm{d}t + Z_t\mathrm{d}B_t, \\ V_T = h(X_T), \end{cases} \tag{5.4}$$

where $h:\mathbb{R}\to\mathbb{R}$ is a given Lipschitz continuous function. The optimization objective of the agent is to maximize his/her utility as below:

$$\max_{(\pi,c)\in\mathcal{U}} V_0,$$

where

$$\mathcal{U} \triangleq \{(\pi,c)|\ (\pi,c):[0,T]\times\Omega\to[-1,1]\times[a_1,a_2] \text{ is the } \{\mathscr{F}_t\}_{0\le t\le T}\text{-adapted process}\}$$

is the admissible control set.

Indeed, the aggregator given by (5.4) does not satisfy the Lipschitz condition with respect to the utility and the consumption in most cases. However, our study is applicable in some non-Lipschitz cases. Note that Proposition 3.2 in [13] provides four cases in which the aggregator is monotonic with respect to the utility.

Taking into account the polynomial growth condition (H6) with respect to the utility, we select two cases for further consideration:

$$\begin{aligned} &\text{(i) } \gamma > 1 \qquad \text{and} \qquad \psi > 1; \\ &\text{(ii) } \gamma < 1 \qquad \text{and} \qquad \psi < 1. \end{aligned}$$

Then we can find appropriate power such that the aggregator of (5.4) is continuous and monotonic but non-Lipschitz in $\mathbb{R}$ with respect to the utility in both cases. As for the continuity with respect to the consumption, if $a_1 > 0$, both cases are Lipschitz continuous obviously. In particular, if $a_1 = 0$, only case (i) satisfies the continuous but not Lipschitz continuous condition with respect to the consumption.

Therefore, for all suitable non-Lipschitz situations which satisfy Conditions (H3)–(H6), we can use Theorem 4.8 to know that the value function of the agent is a viscosity solution of the following HJB equation:

$$\begin{cases} \max\limits_{(\pi,c)\in[-1,1]\times[a_1,a_2]} \Bigg\{ w_t(t,x) + [x(r_t + \pi(b_t - r_t)) - c]w_x(t,x) + \dfrac{1}{2}x^2\pi^2\sigma_t^2 w_{xx}(t,x) \\ \qquad\qquad + \dfrac{\delta}{1-\frac{1}{\psi}}(1-\gamma)w(t,x)\left[\left(\dfrac{c}{((1-\gamma)w(t,x))^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right]\Bigg\} = 0, \\ w(T,x) = h(x). \end{cases} \tag{5.5}$$

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