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STOCHASTIC HOMOGENIZATION OF PLASTICITY EQUATIONS

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Abstract. In the context of infinitesimal strain plasticity with hardening, we derive a stochastic homogenization result. We assume that the coefficients of the equation are random functions: elasticity tensor, hardening parameter and flow-rule function are given through a dynamical system on a probability space. A parameter $\varepsilon > 0$ denotes the typical length scale of oscillations. We derive effective equations that describe the behavior of solutions in the limit $\varepsilon \to 0$. The homogenization procedure is based on the fact that stochastic coefficients "allow averaging": For one representative volume element, a strain evolution $[0, T] \ni t \mapsto \xi(t) \in \mathbb{R}^{d\times d}_s$ induces a stress evolution $[0, T] \ni t \mapsto \Sigma(\xi)(t) \in \mathbb{R}^{d\times d}_s$. Once the hysteretic evolution law Σ is justified for averages, we obtain that the macroscopic limit equation is given by $-\nabla \cdot \Sigma(\nabla^s u) = f$.

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1. Introduction

In its history, mathematics has often been inspired by questions from continuum mechanics: Given a body of metal and given a force acting on it, what is the deformation that the body of metal is experiencing? Euler has been inspired by this question; much later, the development of linear and non-linear elasticity theory provided excellent models (and mathematical theories) for non-permanent deformations. In contrast, the description of permanent deformations with plasticity models is much less developed. The only well-established plasticity models are based on infinitesimal strain theories, ad-hoc decomposition rules of the strain tensor and flow rules for the plastic deformation tensor.

Homogenization theory is, in its origins, concerned with the following question: How does a heterogeneous material (composed of different materials) behave effectively? Can we characterize an effective material such that a heterogeneous medium (consisting of a very fine mixture) behaves like the effective material? This homogenization question has a positive answer in the context of linear elasticity: effective coefficients can be computed and bounds for these effective coefficients are available. The situation is quite different for plasticity models: Results have been obtained only in the last ten years. The effective model cannot be reduced to one macroscopic set of differential equations. The effective system either remains a two-scale model or, as we do here, must be formulated with a hysteretic stress-strain map.

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With only two exceptions, so far, homogenization results in plasticity treat essentially the same system: Infinitesimal strains and an additive decomposition of the strain tensor are used, some hardening effect is included, and the homogenization is performed in a periodic setting. The two exceptions are [6, 19]: In [6], no hardening effect is used and the limit system is much more involved. In [19], stochastic coefficients are permitted, but at the expence of a one-dimensional setting. The present article is based on [9] and provides the third exception: We treat a model with stochastic coefficients in dimensions 2 and 3.

We mention at this point the more abstract approach in the framework of energetic solutions, see [12, 13], and its application in gradient plasticity in [8].

Plasticity equations.

We study a bounded domain $Q \subset \mathbb{R}^d$, $d \in \{2,3\}$, occupied by a heterogeneous material, and its evolution in a time interval $(0,T) \subset \mathbb{R}$. For a parameter $\varepsilon > 0$, we consider on $Q \times (0,T)$ the plasticity system

$$\begin{aligned} -\nabla \cdot \sigma^\varepsilon &= f, & \sigma^\varepsilon &= C_\varepsilon^{-1} \mathrm{e}^\varepsilon, \\ \nabla^s u^\varepsilon &= \mathrm{e}^\varepsilon + p^\varepsilon, & \partial_t p^\varepsilon &\in \partial \Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon). \end{aligned} \tag{1.1}$$

The first relation is the quasi-static balance of forces in the body, f is a given load, σ the stress tensor. The second relation is Hooke's law which relates linearly the stress σ with the elastic strain e. The third relation is the additive decomposition of the infinitesimal strain $\nabla^s u = (\nabla u + (\nabla u)^T)/2$. The fourth relation is the flow rule for the plastic strain p, it uses the subdifferential $\partial \Psi_\varepsilon$ of a convex function Ψ_ε. Kinematic hardening is introduced with the positive tensor B_ε.

Hardening is an experimental fact in metals. From the physical point of view, the model without hardening ("perfect plasticity") allows for arbitrarily large deformations once the yield stress is reached. From the analytical point of view, hardening simplifies the mathematical treatment considerably: Standard function spaces can be used, while in perfect plasticity the space $BD(Q)$ of bounded deformations must be used (measure-valued shear bands can occur). In particular, the *a priori* estimates from Theorem 1.10 and Lemma 2.10 are due to the hardening assumption. We refer to [1, 7] for the modelling.

Our interest here is to study coefficients $B = B_\varepsilon$ (hardening), $C = C_\varepsilon$ (elasticity tensor), and $\Psi = \Psi_\varepsilon$ (convex flow rule function) that depend on the parameter $\varepsilon > 0$. We imagine ε to be the spatial length scale of the heterogeneities. Since the coefficients depend on ε, also the solution $(u, \sigma, e, p) = (u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ depends on ε.

We consider only positive and symmetric coefficient tensors, using the following setting: We denote by $\mathbb{R}^{d\times d}_s \subset \mathbb{R}^{d\times d}$ the space of symmetric matrices, $\mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s)$ is the space of linear mappings on $\mathbb{R}^{d\times d}_s$. For every $\varepsilon > 0$ and almost every $x \in Q$, the tensors $C_\varepsilon(x), B_\varepsilon(x) \in \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s)$ are assumed to be symmetric. Furthermore, for constants $\gamma, \beta > 0$, we assume the positivity and boundedness

$$\gamma\, |\xi|^2 \le \xi : (C_\varepsilon(x)\, \xi) \le \frac{1}{\gamma}\, |\xi|^2\,, \qquad \beta\, |\xi|^2 \le \xi : (B_\varepsilon(x)\, \xi) \le \frac{1}{\beta}\, |\xi|^2 \tag{1.2}$$

for every $\xi \in \mathbb{R}^{d\times d}_s$, a.e. $x \in Q$, and every $\varepsilon > 0$.

System (1.1) is accompanied by a Dirichlet boundary condition $u^\varepsilon = U$ on $\partial Q \times (0,T)$ and an initial condition for the plastic strain tensor (for simplicity, we assume here a vanishing initial plastic deformation). Finally, the load f must be imposed. We consider data

$$U \in H^1(0,T; H^1(Q;\mathbb{R}^d)), \quad f \in H^1(0,T; L^2(Q,\mathbb{R}^d)), \quad p^\varepsilon|_{t=0} \equiv 0. \tag{1.3}$$

The fundamental task of homogenization theory is the following: If $u^\varepsilon \rightharpoonup u$ converges in some topology as $\varepsilon \to 0$, what is the equation that characterizes u?

Known homogenization results and the needle-problem approach.

The periodic homogenization of system (1.1) was performed in the last 10 years. The effective two-scale limit system was first stated in [2]. The rigorous derivation of the limit system (under different assumptions on the coefficients) was obtained by Visintin with two-scale convergence methods [23–25], by Alber and Nesenenko with phase-shift convergence [3,16], and by Veneroni together with the second author with energy methods [20]. By the same authors, some progress was achieved regarding the monotone flow rule and a simplification of proofs in [22]. We refer to these publications also for a further discussion of the periodic homogenization of system (1.1).

The non-periodic homogenization of system (1.1) is much less treated. In particular, we are not aware of any stochastic homogenization result (with the exception of [19], but the analysis of the one-dimensional case is much simpler, since the stress variable can be obtained by a simple integration from the force f).

For the non-periodic case, a partial homogenization result has been obtained in [9]. That contribution is based on the needle-problem approach, which has its origin in [21]. The present article is based on [9] and we therefore describe in the next paragraph the needle-problem approach in more detail.

In the *needle-problem approach,* homogenization is seen as a two-step procedure. We describe the two steps of the needle approach here with the scalar model $-\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f$ for a deformation $u^\varepsilon : Q \to \mathbb{R}$. Step 1 is concerned with cell-problems: One verifies that, on a representative elementary volume (an REV, the unit square in periodic homogenization) and for a vanishing load, the material behaves in a well-defined way: An input (here: the averaged gradient ξ of the solution across the REV) results in a certain output (here: the averaged stress $\sigma(\xi) = a^*\xi$ for a matrix a^*). Step 2 is concerned with arbitrary domains Q and arbitrary loads f. The conclusion of Step 2 (which can be justified *e.g.* with the needle-problem approach) is the following: If the REV-analysis provides the material law $\xi \mapsto \sigma(\xi)$, then the behavior of the material on the macroscopic scale is characterized by $-\nabla \cdot (\sigma(\nabla u)) = f$ in Q (in our example by $-\nabla \cdot (a^*\nabla u) = f$). In [21], these methods are developed and the two-step scheme is illustrated with the linear model: The assumption of an averaging property on simplices implies the homogenization on the macroscopic scale with the corresponding law.

We emphasize that the idea to decouple the homogenization procedure into two steps is not new. In periodic homogenization, the periodic cell problem provides the effective parameters; in stochastic homogenization the splitting appears with a cell problem that is posed on the entire space $\mathbb{R}^d$. In the theory of elliptic equations, the construction of the corrector function plays the role of Step 1. But the splitting into two steps can be traced back even further, to the definition of H-convergence [15], early stochastic homogenization results [17], or the homogenization of integral functionals [5,14]. The view-point in the needle-problem approach is extreme: We choose not even to ask why Step 1 can be carried out in a specific situation — we assume that averaging occurs. With this view-point, one may replace Step 1 also by the determination of material laws from laboratory or numerical experiments.

The needle-problem approach focusses on Step 2. Assuming that the averaged behavior of samples are described by some effective law, the effective behavior on arbitrary samples with arbitrary loads is derived.

In [9], we performed Step 2 (using the needle-problem approach) in the context of plasticity. Our assumption was that the material parameters allow averaging: solutions on simplices with affine boundary data $x \mapsto \xi \cdot x$ and vanishing forces $f \equiv 0$ have convergent stress averages: in the limit $\varepsilon \to 0$, stress integrals converge to some deterministic quantity $\varSigma(\xi)$. Due to memory effects in plasticity problems, one has to find for every evolution of strains $\xi = \xi(t)$ an evolution of stresses $\varSigma(\xi)(t) = \varSigma(\xi(.))(t)$. The result in [9] is a homogenization result under this averaging assumption: For general domains Q, general boundary data U and general forces f, the effective problem for every limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ reads

$$-\nabla \cdot \varSigma(\nabla^s u) = f \qquad \text{in} \qquad Q \times (0,T). \tag{1.4}$$

The present work focuses on Step 1, *i.e.* we will rigorously prove the averaging and admissibility condition for stationary ergodic coefficients.

Let us briefly describe the relation between the needle-problem approach (used here) with classical stochastic homogenization results (as in [10, 11, 17]): We believe that our result on the stochastic homogenization of plasticity equations could also be obtained along the classical route. In such a proof, one would first obtain a two-scale effective problem in the variables (x, t, ω). In a second step, one can realize that the dependence on x can be disintegrated: The two-scale system can be written in the form (1.4), if the hysteretic stress operator $\varSigma$

is defined through a stochastic cell problem in the variables (t, ω). In the needle-problem approach, we keep these two aspects separated: The abstract result "averaging property for Σ implies homogenization" of [9] is independent of the stochastic description. The stochastic analysis concerns only the operator Σ and its properties (the work at hand).

The stochastic homogenization result.

In this contribution, we perform the stochastic homogenization of the plasticity system. In particular, we demonstrate that the averaging assumption is satisfied for an evolution operator Σ and that equation (1.4) is the effective plasticity problem. Comparing with other homogenization results for plasticity equations, this means that we obtain a *disintegrated* effective system: equation (1.4) is local in space, it is not a *two-scale* system. The microscopic behavior is synthesized in the operator Σ. The only non-local effect occurs in the time variable, since Σ is an evolution operator.

Definition 1.1 (The structure of the limit problem)**.** Let the domain $Q \subset \mathbb{R}^d$ and $T > 0$ be as above, let Ω be a probability space with ergodic dynamical system as in Section 1.1, let stochastic coefficients C, B and Ψ be as in Assumption 1.8.

(i) Definition of the hysteretic strain-to-stress map $\Sigma : \xi \mapsto \sigma$**.** We consider an input $\xi : [0, T] \to \mathbb{R}^{d\times d}_s$ and solve the following stochastic cell problem with a triplet (p, z, v), where $p \in H^1(0, T; L^2(\Omega; \mathbb{R}^{d\times d}_s))$, $z \in H^1(0, T; L^2_{\mathrm{sol}}(\Omega; \mathbb{R}^{d\times d}))$, $v \in H^1(0, T; L^2_{\mathrm{pot}}(\Omega; \mathbb{R}^{d\times d}))$, and z is symmetric, $z = z^s$:

$$\begin{aligned} \xi = Cz - v^s + p \qquad &\text{a.e. in} \quad [0, T] \times \Omega, \\ \partial_t p \in \partial\Psi(z - Bp) \qquad &\text{a.e. in} \quad [0, T] \times \Omega. \end{aligned} \tag{1.5}$$

For the definition of the function spaces $L^2_{\mathrm{pot}}(\Omega)$ and $L^2_{\mathrm{sol}}(\Omega)$ see (1.12) and (1.14). The solution (p, z, v) defines the operator Σ,

$$\Sigma(\xi)(t) := \int_\Omega z(t, \omega)\, \mathrm{d}\mathcal{P}(\omega). \tag{1.6}$$

(ii) Definition of the effective equation. For boundary data U and loading f as in (1.3), we search for $u \in H^1(0, T; H^1(Q))$ such that

$$\int_0^T \int_Q \Sigma(\nabla^s u) : \nabla\varphi = \int_0^T \int_Q f \cdot \varphi \quad \forall \varphi \in L^2(0, T; H^1_0(Q)). \tag{1.7}$$

Additionally, we demand that the boundary condition $u = U$ on $\partial Q \times (0, T)$ is satisfied in the sense of traces.

Remark 1.2. The argument of the stress function Σ is $\xi = \xi(t)$, in the limit problem (1.7) the stress function is evaluated, for every $x \in Q$, with the argument $\xi(.) = \nabla^s_x u(., x)$. For a more detailed description of the limit problem (1.7) see Definition 1.12. The precise statement of the stochastic cell-problem (1.5) and the corresponding definition of the operator Σ in (1.6) is given in Definition 2.2.

Our stochastic homogenization result follows by applying the main theorem of [9]. Essentially, we only have to verify that, if the coefficient functions of system (1.1) are given by an ergodic stochastic process, then the coefficients "allow averaging": In the limit $\varepsilon \to 0$, averages of the stress (for a homogeneous plasticity system on a simplex with affine boundary data ξ) are given by the operator Σ.

We verify this statement in Sections 2 and 3. The consequence is the following homogenization theorem, which is our main result.

Theorem 1.3 (Stochastic homogenization in plasticity)**.** *Let $Q \subset \mathbb{R}^d$ be a bounded domain, $d \in \{2, 3\}$, $T > 0$. Let τ be an ergodic dynamical system on the probability space $(\Omega, \Sigma_\Omega, \mathcal{P})$ as in Section 1.1, let the stochastic coefficients B, C, Ψ and the data U and f be as in Assumption 1.8. Then, there exists a unique solution u to* *the limit problem* (1.5)–(1.7) *of Definition* 1.1*. For* $\omega \in \Omega$*, let* $(u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ *be weak solutions to* (1.1)*. Then, for a.e.* $\omega \in \Omega$*, as* $\varepsilon \to 0$*,*

$$u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1(0,T;H^1(Q)) \qquad \textit{and} \qquad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u) \quad \textit{weakly in} \qquad H^1(0,T;L^2(Q)).$$

Remark 1.4. The weak solution concept for the ε-problem (1.1) is made precise in Definition 1.9. The unique existence of a solution u^ε for a.e. $\omega \in \Omega$ is guaranteed by Theorem 1.10.

The proof of Theorem 1.3 is concluded in Section 3.4. A sketch of the proof is presented at the end of Section 1.3.

1.1. Setting in stochastic homogenization

We follow the traditional setting in stochastic homogenization, first outlined by Papanicolaou and Varadhan in [17] and by Kozlov in [11], later used by Jikov, Kozlov and Oleinik [10]. Let $(\Omega, \Sigma_\Omega, \mathcal{P})$ be a probability space where we assume that the σ-algebra Σ_Ω is countably generated. This implies that $L^2(\Omega)$ is separable. Let $(\tau_x)_{x\in\mathbb{R}^d}$ be an ergodic dynamical system on $(\Omega, \Sigma_\Omega, \mathcal{P})$. We rely on the following definitions: A family $(\tau_x)_{x\in\mathbb{R}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$ is called a dynamical system on $(\Omega, \Sigma_\Omega, \mathcal{P})$ if it satisfies

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ (group property).
(ii) $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B)$ $\forall x \in \mathbb{R}^d$, $B \in \Sigma_\Omega$ (measure preservation).
(iii) $A:\ \mathbb{R}^d \times \Omega \to \Omega$ $(x,\omega) \mapsto \tau_x\omega$ is measurable (measurability property).

We say that the system $(\tau_x)_{x\in\mathbb{R}^d}$ is ergodic, if for every measurable function $f : \Omega \to \mathbb{R}$ it holds

$$\left[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{R}^d,\ a.e.\ \omega \in \Omega\right] \Rightarrow \left[\exists c_0 \in \mathbb{R} :\ f(\omega) = c_0 \text{ for a.e. } \omega \in \Omega\right]. \tag{1.8}$$

Example 1.5. Let us provide a simple non-trivial example for a stochastic setting: the checker board construction of i.i.d. random variables. We use $Y := [0,1[^d$ with the topology of the torus and the partition of $\mathbb{R}^d$ with unit cubes $\mathcal{C}_z := z + Y$ for $z \in \mathbb{Z}^d$. We consider the sets

$$\begin{aligned}\tilde{\Omega} &:= \left\{u \in L^\infty(\mathbb{R}^d) \,|\, u|_{\mathcal{C}_z} \equiv c_z,\ \text{for some}\ \ c : \mathbb{Z}^d \to [0,1], z \mapsto c_z\right\}\\ \Omega &:= \left\{u \in L^\infty(\mathbb{R}^d) \,|\, \exists \xi \in Y \text{ s.t. } u(\,.-\xi) \in \tilde{\Omega}\right\}.\end{aligned}$$

For $u \in \Omega$ we denote a shift ξ from the above definition as $\xi(u)$. Note that $\Omega = Y \times \bigotimes_{z\in\mathbb{Z}^d}[1,2]$.

The probability measure on Ω corresponding to i.i.d. random variables can be defined with the help of elementary subsets. For an open set $U \subseteq Y$, a number $k \in \mathbb{N}$, and relatively open intervals $I_z := ((a_z, b_z) \cap [0,1]) \subset [0,1]$, $z \in \mathbb{Z}^d$ and $a_z < b_z$, the sets

$$A\left(U, (I_z)_{z\in\mathbb{Z}^d}, k\right) = \left\{u \in \Omega \,|\, \xi(u) \in U,\ u(\,. - \xi(u))|_{\mathcal{C}_z} \in I_z\, \forall z,\ |z| \le k\right\} \tag{1.9}$$

are open and form a basis of the product σ-algebra in Ω. The product measure of the Lebesgue-measures on Y and on $[0,1]$ can then be characterized through such sets $A(\,.\,)$ via

$$\mathcal{P}\left(A\left(U, (I_z)_{z\in\mathbb{Z}^d}, k\right)\right) := |U| \prod_{|z|\le k} |b_z - a_z|.$$

We finally introduce $\tau_x : \Omega \to \Omega$ for every $x \in \mathbb{R}^d$ through $\tau_x u(\,.\,) = u(x + .)$. It is easy to check that the family $(\tau_x)_{x\in\mathbb{R}^d}$ is a dynamical system. Since $\mathcal{P}(A) = \mathcal{P}(\tau_x A)$ for A as in (1.9) and $x \in \mathbb{R}^d$, the dynamical system is measure preserving.

Given $f \in L^2(\Omega)$ and $\omega \in \Omega$, we call $f_\omega : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(\tau_x\omega)$ the ω-realization of f. An important property of ergodic dynamical systems is the fact that spatial averages can be related to expectations. For a quite general version of the ergodic theorem, we refer to [26]. The following simple version is sufficient for our purposes.

Theorem 1.6 (Ergodic theorem)**.** *Let $(\Omega, \Sigma_\Omega, \mathcal{P})$ be a probability space with an ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on Ω. Let $f \in L^1(\Omega)$ be a function and $Q \subset \mathbb{R}^d$ be a bounded open set. Then, for $\mathcal{P}$-almost every $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0} \int_Q f(\tau_{x/\varepsilon}\omega)\,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_Q f_\omega\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x = |Q| \int_\Omega f(\omega)\,\mathrm{d}\mathcal{P}(\omega). \tag{1.10}$$

Furthermore, for every $f \in L^p(\Omega)$, $1 \le p \le \infty$, and a.e. $\omega \in \Omega$, the function $f_\omega(x) = f(\tau_x\omega)$ satisfies $f_\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. For $p < \infty$ holds $f_\omega(\cdot/\varepsilon) = f(\tau_{\cdot/\varepsilon}\omega) \rightharpoonup \int_\Omega f\,\mathrm{d}\mathcal{P}$ weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0$.

For brevity of notation in calculations and proofs, we will often omit the symbol $\mathrm{d}\mathcal{P}$ in Ω-integrals. We assume that the coefficients in (1.1) have the form

$$C_\varepsilon(x) = C(\tau_{\frac{x}{\varepsilon}}\omega), \qquad B_\varepsilon(x) = B\left(\tau_{\frac{x}{\varepsilon}}\omega\right), \qquad \Psi_\varepsilon(\sigma) = \Psi\left(\sigma; \tau_{\frac{x}{\varepsilon}}\omega\right) \tag{1.11}$$

for some functions B, C, and Ψ, see Assumption 1.8.

Using the function spaces

$$L^2_{\mathrm{pot,loc}}(\mathbb{R}^d) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \mid \forall U \text{ bounded domain, } \exists\varphi \in H^1(U;\mathbb{R}^d) : u = \nabla\varphi\right\},$$
$$L^2_{\mathrm{sol,loc}}(\mathbb{R}^d) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \mid \int_{\mathbb{R}^d} u\cdot\nabla\varphi = 0 \ \forall\varphi \in C^1_c(\mathbb{R}^d)\right\},$$

we follow Chapter 7 in [10] and define

$$L^2_{\mathrm{pot}}(\Omega) := \left\{v \in L^2(\Omega;\mathbb{R}^{d\times d}) \,|\, x \mapsto v(\tau_x\omega) \text{ in } L^2_{\mathrm{pot,loc}}(\mathbb{R}^d) \text{ for a.e. } \omega \in \Omega\right\}, \tag{1.12}$$

$$\mathcal{V}^2_{\mathrm{pot}}(\Omega) := \left\{f \in L^2_{\mathrm{pot}}(\Omega) \,|\, \int_\Omega f\,\mathrm{d}\mathcal{P} = 0\right\}, \tag{1.13}$$

$$L^2_{\mathrm{sol}}(\Omega) := \left\{v \in L^2(\Omega;\mathbb{R}^{d\times d}) \,|\, x \mapsto v(\tau_x\omega) \text{ in } L^2_{\mathrm{sol,loc}}(\mathbb{R}^d) \text{ for a.e. } \omega \in \Omega\right\}. \tag{1.14}$$

The three spaces (1.12)–(1.14) are closed subspaces of $L^2(\Omega;\mathbb{R}^{d\times d})$. The latter spaces can be decomposed in an orthogonal sum as $L^2(\Omega;\mathbb{R}^{d\times d}) = \mathcal{V}^2_{\mathrm{pot}}(\Omega) \oplus L^2_{\mathrm{sol}}(\Omega)$, see [10].

Remark 1.7. The periodic homogenization setting is a special case of the stochastic setting, and we recover known results in the periodic case. The cell problem on the periodicity cell is encoded in (1.5) with the help of the spaces $L^2_{\mathrm{pot}}(\Omega)$ and $L^2_{\mathrm{sol}}(\Omega)$ (v^s is a symmetrized gradient and z has a vanishing divergence).

1.2. Solution concepts and existence results

To formulate a stochastic setting, we consider $C, B \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s))$, pointwise symmetric, such that for $\gamma, \beta > 0$ holds

$$\gamma\,|\xi|^2 \le \xi : C(\omega)\xi \le \frac{1}{\gamma}\,|\xi|^2, \qquad \beta\,|\xi|^2 \le \xi : B(\omega)\xi \le \frac{1}{\beta}\,|\xi|^2, \tag{1.15}$$

for every $\xi \in \mathbb{R}^d$ and a.e. $\omega \in \Omega$. Let $\Psi : \mathbb{R}^{d\times d}_s \times \Omega \to (-\infty, +\infty]$, $(\xi,\omega) \mapsto \Psi(\xi,\omega)$ be measurable in $\mathbb{R}^{d\times d}_s \times \Omega$, lower semicontinuous and convex in $\mathbb{R}^{d\times d}_s$ for a.e. $\omega \in \Omega$, and with $\Psi(0,\omega) = 0$ for a.e. $\omega \in \Omega$. We furthermore assume that for a.e. $\omega \in \Omega$ there is $c(\omega) > 0$ such that the convex dual (in the first variable) satisfies

$$|\Psi^*(\sigma;\tau_x\omega) - \Psi^*(\sigma;\tau_y\omega)| \le c(\omega)\ |x-y|\,|\sigma| \qquad \forall\sigma \in \mathbb{R}^{d\times d}_s,\ x, y \in \mathbb{R}^d. \tag{1.16}$$

We note that the above assumption on Ψ implies that no discontinuities are allowed in the flow rule.

Assumption 1.8 (Data)**.** Let $C, B \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{d\times d}_s, \mathbb{R}^{d\times d}_s))$ and $\Psi : \mathbb{R}^{d\times d}_s \times \Omega \to (-\infty, +\infty]$ satisfy (1.15)–(1.16). We consider only parameters $\omega \in \Omega$ such that the ω-realizations $C_\omega(x) := C(\tau_x\omega)$, $B_\omega(x) := B(\tau_x\omega)$ are measurable and such that (1.2) and (1.16) hold. We furthermore assume that U and f satisfy the regularity (1.3) and the compatibility conditions $U|_{t=0} = 0$, $f|_{t=0} = 0$.

Our aim is to study (1.1) with the coefficients defined in (1.11). By slight abuse of notation and omitting the index ω whenever possible, we also write $C_\varepsilon(x) := C_{\varepsilon,\omega}(x) := C(\tau_{\frac{x}{\varepsilon}}\omega)$ and $B_\varepsilon(x) := B_{\varepsilon,\omega}(x) := B(\tau_{\frac{x}{\varepsilon}}\omega)$ as in (1.11). We assume that they satisfy (1.2) and that Ψ_ε satisfies

$$\left|\Psi^*_{\varepsilon,\omega}(\sigma; x_1) - \Psi^*_{\varepsilon,\omega}(\sigma; x_2)\right| \le c(\varepsilon, \omega)\, |x_1 - x_2|\, |\sigma|\,. \tag{1.17}$$

This condition is of a technical nature. It is used only in the proof of the existence result of Theorem 1.10. We remark that the existence result remains valid also without assumption (1.17), as can be shown with the methods of Section 2. Since we do not want to repeat the proof of Theorem 1.10 here, we assume the above Lipschitz condition.

Definition 1.9 (Weak formulation of the ε-problem)**.** We say that $(u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ is a weak solution to the ε-problem (1.1) on Q with boundary condition U if the following is satisfied: There holds $u^\varepsilon = v^\varepsilon + U$ with

$$v^\varepsilon \in H^1(0, T; H^1_0(Q)), \quad \mathrm{e}^\varepsilon, p^\varepsilon, \sigma^\varepsilon \in H^1(0, T; L^2(Q; \mathbb{R}^{d\times d}_s)),$$

equation $-\nabla \cdot \sigma^\varepsilon = f$ of (1.1) holds in the distributional sense and the other relations of (1.1) hold pointwise almost everywhere in $Q \times (0, T)$.

We note that, due to the regularity of σ^ε, every weak solution to (1.1) satisfies

$$\int_0^T \int_Q \sigma^\varepsilon : \nabla^s \varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0, T; H^1_0(Q)). \tag{1.18}$$

Theorem 1.2 of [9] provides the following existence result.

Theorem 1.10 (Existence of solutions to the ε-problem)**.** *Let the coefficient functions C, B, Ψ, the parameter $\omega \in \Omega$, and the data U and f be as in Assumption 1.8. Then, for every $\varepsilon > 0$, there exists a unique weak solution $(u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ to the ε-problem (1.1) in the sense of Definition 1.9. The solutions satisfy the a priori estimate*

$$\|u^\varepsilon\|_{\mathcal{V}^1_1} + \|\mathrm{e}^\varepsilon\|_{\mathcal{V}^1_0} + \|p^\varepsilon\|_{\mathcal{V}^1_0} + \|\sigma^\varepsilon\|_{\mathcal{V}^1_0} \le C, \tag{1.19}$$

in the spaces $\mathcal{V}^1_0 := H^1(0, T; L^2(Q; \mathbb{R}^{d\times d}_s))$ and $\mathcal{V}^1_1 := H^1(0, T; H^1_0(Q))$, the constant $C = C(U, f, \beta, \gamma)$ depends on β and γ from (1.2), but it does not depend on $\varepsilon > 0$ or $\omega \in \Omega$.

1.3. The needle problem approach to plasticity

The main result of [9] is a homogenization theorem. Under the assumption that causal operators Σ and Π satisfy certain admissibility and averaging properties, we obtain the convergence of the ε-solutions u^ε to the solution u of the effective problem (1.4). We next recall the required properties. In the following, we use the space $H^1_*(0, T; \mathbb{R}^{d\times d}_s) := H^1(0, T; \mathbb{R}^{d\times d}_s) \cap \{\xi \mid \xi|_{t=0} = 0\}$ of evolutions with vanishing initial values.

Definition 1.11 (Averaging)**.** We say that a map $F : H^1_*(0, T; \mathbb{R}^{d\times d}_s) \to H^1(0, T; \mathbb{R}^{d\times d}_s)$ defines a *causal operator,* if, for almost every $t \in [0, T]$, the value $F(\xi, t) := F(\xi)(t)$ is independent of $\xi|_{(t,T]}$. We say that the coefficients C_ε, B_ε and Ψ_ε *allow averaging,* if there exist causal operators Σ and Π such that the following property holds: For every simplex $\mathcal{T} \subset Q$, every boundary condition $\xi \in H^1_*(0, T; \mathbb{R}^{d\times d}_s)$ and every additive constant $a \in H^1(0, T; \mathbb{R}^d)$, the corresponding solution $(u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ of the ε-problem (1.1) on $\mathcal{T}$ with $f = 0$

and $U(x,t) = \xi(t)x + a(t)$ satisfies the following: As $\varepsilon \to 0$, for a.e. $t \in (0,T)$, the averages of p^ε and σ^ε converge:

$$\fint_{\mathcal{T}} p^\varepsilon(t) \to \Pi(\xi)(t), \qquad \fint_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t). \tag{1.20}$$

Here, $\fint_{\mathcal{T}} = |\mathcal{T}|^{-1} \int_{\mathcal{T}}$ denotes averages. In particular, we demand that limits of (averages of) stress and plastic strain depend only on the (time-dependent) boundary condition ξ, not on a and not on the simplex $\mathcal{T}$.

Definition 1.12 (Effective equation in the needle problem approach)**.** The effective plasticity problem in the needle problem approach is given by

$$-\nabla \cdot \Sigma(\nabla^s u) = f \quad \text{ in } Q \times (0,T), \tag{1.21}$$

with boundary condition $u = U$ on $\partial Q \times (0,T)$. A function u is a solution to this limit problem if $u = U + v$ holds with $v \in H^1(0,T; H^1_0(Q;\mathbb{R}^d))$ and (1.21) is satisfied in the distributional sense. Regarding the expression $\Sigma(\nabla^s u)$ we note that, for a.e. $x \in Q$, the map $t \mapsto \nabla^s u(x,t)$ is in the space $H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, hence $\Sigma(\nabla^s u)$ is well-defined for almost every point in $Q \times (0,T)$.

The original intention of the needle problem approach is to avoid any cell–problem or corrector result, since these might not be available [21]. Nevertheless, in the application, the causal operator Σ is defined through a kind of cell-problem, see Definition 2.2. We emphasize that this structure is not needed for the result obtained in [9].

The proof of the subsequent Theorem 1.13 is based on a discretization of Q with a triangulation $\mathcal{T}_h$, where the parameter $h > 0$ stands for the mesh-size. Given the triangulation, we consider two auxiliary problems. The first problem is the finite element discretization of the homogenized problem (1.21) with a solution U_h. The second problem is the "needle problem", an approximation of the original equation (1.1), with solution u^ε_h. The needle problem approach is based on the following diagram of convergences:

$$\begin{tikzcd} u^\varepsilon_h \arrow[r, "(1.20)", "\varepsilon"'] \arrow[d, leftrightarrow, "{\varepsilon, h}"'] & U_h \arrow[d, "h"', "\text{admissibility}"] \\ u^\varepsilon & u \end{tikzcd} \tag{1.22}$$

The vertical arrow on the left is obtained from a testing procedure. The horizontal arrow is a consequence of the averaging property, $u^\varepsilon_h \to U_h$ as $\varepsilon \to 0$. The vertical arrow on the right exploits admissibility, compare Definition 1.14 below. The work at hand is concerned with the construction of an operator Σ such that (1.20) holds and such that Σ is admissible in the sense of Definition 1.14 below.

Result of the needle problem approach. In Theorem 1.6 of [9], the abstract operator Σ is assumed to satisfy two conditions: (i) Averaging property. This assumption is recalled in Definition 1.11. (ii) Admissibility. Admissibility is defined in Definition 1.5 of [9] as: The effective problem has a solution.

The existence property of the admissibility condition (ii) can be shown by proving that Galerkin approximations converge to solutions. We formulate a sufficient condition in this spirit in Definition 1.14 below. We therefore obtain from Theorem 1.6 of [9]:

Theorem 1.13 (Needle-approach homogenization theorem in plasticity)**.** *Let $Q \subset \mathbb{R}^d$ be open and bounded, let the data f and U be as in Assumption 1.8, let the coefficients C_ε, B_ε and Ψ_ε be as above, satisfying (1.2). Let the data allow averaging in the sense of Definition 1.11 with causal operators Σ and Π, and let Σ satisfy the admissibility condition of Definition 1.14. Let $(u^\varepsilon, \sigma^\varepsilon, \mathrm{e}^\varepsilon, p^\varepsilon)$ be the weak solutions to the ε-problems (1.1). Then, as $\varepsilon \to 0$, there holds*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(0,T;H^1_0(Q;\mathbb{R}^d)), p^\varepsilon \rightharpoonup \Pi(\nabla^s u), \quad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u) \quad \text{weakly in } H^1(0,T;L^2(Q;\mathbb{R}^{d\times d})),$$

where u is the unique weak solution to the homogenized problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f \qquad on \qquad Q \times (0,T)$$

with boundary condition U in the sense of Definition 1.12.

An assumption that implies admissibility. For arbitrary $h > 0$, we use a polygonal domain $Q_h \subset Q$ and a triangulation $\mathbb{T}_h$ with the properties

$$\begin{aligned} &\mathbb{T}_h := \{\mathcal{T}_k\}_{k\in\Lambda_h} \quad \text{is a triangulation of } Q_h, \quad \operatorname{diam}(\mathcal{T}_k) < h \quad \forall\, \mathcal{T}_k \in \mathbb{T}_h, \\ &Q_h \text{ has the property that } x \in Q, \operatorname{dist}(x, \partial Q) \geq h \text{ implies } x \in Q_h, \end{aligned} \tag{1.23}$$

where $\mathcal{T}_k$ are disjoint open simplices and $\Lambda_h \subset \mathbb{N}$ is a finite set of indices. We always assume that the sequence of meshes is regular in the sense of [4], Section 3.1. As in [21], we consider the finite element space of continuous and piecewise linear functions with vanishing boundary values,

$$Y_h := \left\{ \phi \in H^1_0(Q) \,|\, \phi|_{\mathcal{T}_k} \text{ is affine } \forall\, \mathcal{T}_k \in \mathbb{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h \right\}. \tag{1.24}$$

Discretization of boundary conditions: We may extend the triangulation of Q_h by a finite amount of simplices with diameter not greater than h to obtain a grid $\tilde{\mathbb{T}}_h$ that covers Q in the sense $Q \subset \bigcup_{\mathcal{T}_k\in\tilde{\mathbb{T}}_h} \bar{\mathcal{T}}_k$ and introduce the finite element space $\tilde{Y}_h := \{\phi \in H^1(Q) \,|\, \phi|_{\mathcal{T}_k\cap Q} \text{ is affine } \forall\, \mathcal{T}_k \in \tilde{\mathbb{T}}_h\}$. Denoting by $R_{Q,h}$ the H^1-orthogonal Riesz-projection $H^1(Q) \to \tilde{Y}_h$, we set $U_h := R_{Q,h}(U)$ and observe that $U_h \to U$ converges strongly in $H^1(0,T;H^1(Q))$ as $h \to 0$.

Definition 1.14 (Sufficient condition for admissibility of Σ)**.** We consider a causal operator $\Sigma : H^1_*(0,T;\mathbb{R}^{d\times d}_s) \to H^1(0,T;\mathbb{R}^{d\times d}_s)$. We say that Σ *satisfies the sufficient condition for admissibility* if the following property holds: Let $h \to 0$ be a sequence of positive numbers, let $\mathbb{T}_h$ be a sequence of regular grids satisfying (1.23), and let $v_h \in L^2(0,T;Y_h)$ be a corresponding sequence of solutions to the discretized problems (the existence is guaranteed in [9])

$$\int_Q \Sigma\left(\nabla^s\left(v_h + U_h\right)\right) : \nabla\varphi_h = \int_Q f\varphi_h \qquad \forall \varphi_h \in L^2(0,T;Y_h).$$

Assume furthermore that the solutions converge, $v_h \rightharpoonup v$ weakly in $H^1(0,T;H^1_0(Q))$ as $h \to 0$. Then v is a solution to

$$\int_Q \Sigma\left(\nabla^s\left(v + U\right)\right) : \nabla\varphi = \int_Q f\varphi \qquad \forall \varphi \in L^2(0,T;H^1_0(Q)).$$

Remaining program. Using Theorem 1.13, our stochastic homogenization result of Theorem 1.3 can be shown as follows: For stochastic parameters C_ε, B_ε and Ψ_ε we define causal operators Σ and Π with cell-problems on Ω. For these operators, we only have to check the averaging property of Definition 1.11 and the admissibility condition of Definition 1.14.

2. Stochastic cell problem and definition of Σ

Given a strain evolution ξ, we want to define the corresponding evolution $\Sigma(\xi)$ of plastic stresses. For the strain ξ, we use the function space

$$H^1_*(0,T;\mathbb{R}^{d\times d}_s) := \left\{ \xi \in H^1(0,T;\mathbb{R}^{d\times d}_s) \;\mid \xi|_{t=0} = 0 \right\} \tag{2.1}$$

of evolutions with vanishing initial values. For any function $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$ we consider the ordinary differential equation (inclusion) for $p(t,.) \in L^2(\Omega;\mathbb{R}^{d\times d}_s)$,

$$\partial_t p(t,\omega) \in \partial\Psi\left(z(t,\omega) - B(\omega)\, p(t,\omega)\,;\, \omega\right) \tag{2.2}$$

(equality pointwise a.e.), with the initial condition $p(0,\omega)=0$. In order to close the system, the function $z(t)$ must be determined through $\xi(t)$ and $p(t)$. We search for a map $z(t)\in L^2_{\mathrm{sol}}(\Omega)$, symmetric in every point ω, *i.e.* $z(t,\omega)=z^T(t,\omega)$, such that the equality

$$Cz(t)=\xi(t)+v^s(t)-p(t) \tag{2.3}$$

holds in $L^2(\Omega)$ for a function $v\in L^2(0,T;\mathcal{V}^2_{\mathrm{pot}}(\Omega))$. Throughout this text we use $z^s=(z+z^T)/2$ for the symmetric part of a matrix z; for the symmetric matrix z there holds $z=z^s$. Note that $v\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ does not imply $v^s\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)$. Up to the matrix factor C and the symmetrization, equation (2.3) is a Helmholz decomposition of the field $\xi(t)-p(t)$: Essentially, the given field is decomposed into a gradient field and a solennoidal field. It is therefore plausible that, given $\xi(t)$ and $p(t)$, (2.3) yields $z(t)$ and thus closes the evolution equation (2.2). The rigorous existence result is provided in the following theorem.

Theorem 2.1. *Let C, B and Ψ be as in Assumption 1.8. Then, for $\xi\in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, there exists a unique solution $(p,z,v)\in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_s))\times H^1(0,T;L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^{d\times d}))\times H^1(0,T;\mathcal{V}^2_{\mathrm{pot}}(\Omega;\mathbb{R}^{d\times d}))$ with $z=z^s$ to (2.2)–(2.3) satisfying the* a priori *estimate*

$$\|p\|_{\mathcal{V}^1_0}+\|z\|_{\mathcal{V}^1_0}+\|v\|_{\mathcal{V}^1_0}\le C\,\|\xi\|_{H^1(0,T)}\,, \tag{2.4}$$

where $\mathcal{V}^1_0:=H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_s))$. The solution $(p,z,v)\in(\mathcal{V}^1_0)^3$ depends continuously on $\xi\in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$ with respect to the weak topologies in both spaces.

Theorem 2.1 permits us to define the operators Σ and Π.

Definition 2.2 (The effective plasticity operators)**.** For arbitrary $\xi\in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, let (p,z,v) be the solution of (2.2)–(2.3) with $z=z^s$. We set

$$\Sigma(\xi)(t):=\int_\Omega z(t,\omega)\,\mathrm{d}\mathcal{P}(\omega),\qquad \Pi(\xi)(t):=\int_\Omega p(t,\omega)\,\mathrm{d}\mathcal{P}(\omega). \tag{2.5}$$

We note that the operators $\Sigma,\Pi : H^1_*(0,T;\mathbb{R}^{d\times d}_s)\to H^1(0,T;\mathbb{R}^{d\times d}_s)$ are well defined and continuous by Theorem 2.1.

The rest of this section is devoted to the proof of Theorem 2.1. We proceed as follows: In Section 2.1, we introduce a Galerkin approximation scheme for (2.2)–(2.3), using additionally a regularization of Ψ. In 2.2, we recall some results from the theory of convex functions, in 2.3 we provide a Korn's inequality in the probability space Ω. In Section 2.4 we prove existence and uniqueness of solutions to the approximate problems and show that these solutions satisfy uniform bounds. Finally, in Section 2.5, we show that the solutions of the approximate problems converge to the unique solution of the original system (2.2)–(2.3).

2.1. Galerkin method and regularization

Finite dimensional approximation. In what follows, let $\langle\varphi,\psi\rangle_\Omega:=\int_\Omega\varphi:\psi\,\mathrm{d}\mathcal{P}$ denote the scalar product in $L^2(\Omega):=L^2(\Omega;\mathbb{R}^{d\times d})$. We choose complete orthonormal systems $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ and $\{\tilde e_k\}_{k\in\mathbb{N}}$ of $L^2_{\mathrm{sol}}(\Omega)$ and consider the finite dimensional spaces

$$\begin{aligned}
&\tilde L^2_n(\Omega):=\operatorname{span}\{e_k\}_{k=1,\dots,n}\oplus\operatorname{span}\{\tilde e_k\}_{k=1,\dots,n}\,, && L^2_n(\Omega):=\tilde L_n(\Omega)\oplus\left\{v^s\,|\,v\in\tilde L_n(\Omega)\right\},\\
&\mathcal{V}^2_{\mathrm{pot},n}(\Omega):=\mathcal{V}^2_{\mathrm{pot}}(\Omega)\cap L^2_n(\Omega), && L^2_{\mathrm{sol},n}(\Omega):=L^2_{\mathrm{sol}}(\Omega)\cap L^2_n(\Omega).
\end{aligned}$$

We furthermore set $L^2_s(\Omega):=L^2(\Omega;\mathbb{R}^{d\times d}_s)$ and $L^2_{n,s}(\Omega):=\left\{v^s\,|\,v\in L^2_n(\Omega)\right\}$. Since constants are in $L^2_{\mathrm{sol}}(\Omega)$, we can assume that they are in $L^2_n(\Omega)$ and thus in $L^2_{\mathrm{sol},n}(\Omega)$ for every $n\ge d^2$. We finally introduce the orthogonal projection $P_n : L^2(\Omega)\to L^2_n(\Omega)$ and note that $P_n\varphi\to\varphi$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $n\to\infty$ for every $\varphi\in L^2(\Omega;\mathbb{R}^{d\times d})$.

Definition of regularized convex functionals. In order to prove Theorem 2.1, we consider the family of Moreau−Yosida approximations

$$\Psi^\delta(\sigma,\omega) := \inf_{\xi\in\mathbb{R}^{d\times d}_s} \left\{ \Psi(\xi,\omega) + \frac{|\xi-\sigma|^2}{2\delta} \right\}, \tag{2.6}$$

satisfying (see [18], Exercise 12.23; for the definition of the subdifferential $\partial\Psi^\delta$ see (2.11))

$$\begin{aligned} \Psi^\delta \,:\, \mathbb{R}^{d\times d}_s \to \mathbb{R} \quad & \text{is convex, coercive and continuously differentiable} \\ \partial\Psi^\delta \,:\, \mathbb{R}^{d\times d}_s \to \mathbb{R}^{d\times d}_s \quad & \text{is single valued and globally Lipschitz-continuous} \\ \lim_{\delta\to 0} \Psi^\delta(\sigma;\omega) = \Psi(\sigma;\omega) \qquad & \forall \sigma\in\mathbb{R}^{d\times d}_s, \text{ and a.e. } \omega\in\Omega. \end{aligned} \tag{2.7}$$

Note that the last convergence is monotone, since $\Psi^{\delta_2} \ge \Psi^{\delta_1}$ for all $\delta_2 < \delta_1$. Given Ψ and Ψ^δ, we consider the corresponding functionals

$$\Upsilon, \Upsilon^\delta :\ L^2_s(\Omega) \to \mathbb{R}, \quad \Upsilon(z) := \int_\Omega \Psi(z(\omega))\,\mathrm{d}\mathcal{P}(\omega),\ \Upsilon^\delta(z) := \int_\Omega \Psi^\delta(z(\omega))\,\mathrm{d}\mathcal{P}(\omega). \tag{2.8}$$

We denote by $\Upsilon_n : L^2_{n,s}(\Omega) \to \mathbb{R}$ the restriction of Υ to $L^2_{n,s}(\Omega)$. the subdifferential of Υ_n is $\partial\Upsilon_n$. Accordingly, we can define Υ^δ_n and $\partial\Upsilon^\delta_n$.

The approximate problem for (2.2)–(2.3).

We consider the following problem on discretized function spaces: Given an evolution $\xi \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, we look for

$$p_{\delta,n} \in C^1(0,T;L^2_{n,s}(\Omega)), \qquad z_{\delta,n} \in H^1(0,T;L^2_{\mathrm{sol},n}(\Omega)), \qquad v_{\delta,n} \in H^1(0,T;\mathcal{V}^2_{\mathrm{pot},n}(\Omega)),$$

with the symmetry $z_{\delta,n} = z^s_{\delta,n}$, satisfying

$$\partial_t p_{\delta,n} = \partial\Upsilon^\delta_n\,(z_{\delta,n} - B_n\, p_{\delta,n}) \tag{2.9}$$

and $C_n\, z_{\delta,n} = \xi + v^s_{\delta,n} - p_{\delta,n}$. The last equation can be written as

$$z_{\delta,n} = C_n^{-1}\left(\xi + v^s_{\delta,n} - p_{\delta,n}\right). \tag{2.10}$$

Here, B_n, $C_n : L^2_{n,s}(\Omega) \to L^2_{n,s}(\Omega)$ are bounded positive (and thus invertible) operators defined through

$$\langle B_n\psi, \varphi\rangle_\Omega = \int_\Omega (B\psi) : \varphi, \quad \langle C_n\psi,\varphi\rangle_\Omega = \int_\Omega (C\psi) : \varphi \qquad \forall \varphi,\psi \in L^2_{n,s}(\Omega).$$

We obtain the existence and uniqueness of solutions to (2.9)–(2.10) from the Picard-Lindelöf theorem: We show that the system can be understood as a single ordinary differential equation for $p_{\delta,n}$ with a Lipschitz continuous right hand side, and that the solutions are uniformly bounded.

2.2. Convex functionals

Basic concepts of convex functions. We recall some well known results from convex analysis on a separable Hilbert space X with scalar product "$\cdot$". In the following, $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ is a convex and lower-semicontinuous functional with $\varphi \not\equiv +\infty$. The domain of φ is $\mathrm{dom}(\varphi) := \{\sigma \in X | \varphi(\sigma) < +\infty\}$, and the Legendre−Fenchel conjugate φ^* is defined by

$$\varphi^* : X \to \mathbb{R}\cup\{+\infty\}, \quad \varepsilon \mapsto \sup_{\sigma\in X}\{\varepsilon\cdot\sigma - \varphi(\sigma)\}.$$

The subdifferential $\partial\varphi : \mathrm{dom}(\varphi) \to \mathcal{P}(X)$ is defined by

$$\partial\varphi(\sigma) = \{\varepsilon \in X \mid \varphi(\xi) \geq \varphi(\sigma) + \varepsilon \cdot (\xi - \sigma) \quad \forall \xi \in X\}. \tag{2.11}$$

A multivalued operator $f : \mathrm{dom}(f) \subset X \to \mathcal{P}(X)$ is said to be *monotone* if

$$(\sigma_1 - \sigma_2) \cdot (\varepsilon_1 - \varepsilon_2) \geq 0, \quad \forall \varepsilon_i \in \mathrm{dom}(f), \quad \sigma_i \in f(\varepsilon_i), \ (i = 1, 2).$$

In what follows, we frequently use the following properties of convex functionals [18].

Lemma 2.3. *For every convex and lower semicontinuous function φ on a Hilbert space X with $\varphi \not\equiv +\infty$ holds*

(i) φ^* *is convex, lower-semicontinuous; and* $\mathrm{dom}(\varphi^*) \neq \emptyset$.
(ii) $\partial\varphi,\ \partial\varphi^*$ *are monotone operators.*
(iii) $\varphi(\sigma) + \varphi^*(\varepsilon) \geq \sigma \cdot \varepsilon \qquad \forall \sigma, \varepsilon \in X$.
(iv) $\sigma \in \mathrm{dom}(\varphi)$ *and* $\varepsilon \in \partial\varphi(\sigma) \ \Leftrightarrow\ \varepsilon \in \mathrm{dom}(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon)$.
(v) $\varepsilon \in \mathrm{dom}(\varphi^*)$ *and* $\sigma \in \partial\varphi^*(\varepsilon) \quad \Leftrightarrow\ \varphi(\sigma) + \varphi^*(\varepsilon) = \sigma \cdot \varepsilon$.
(vi) $\varphi^{**} = \varphi$.

We refer to (v) as Fenchel's equality and to (iii) as Fenchel's *in*equality.

Continuity properties of Υ and Υ^δ and subdifferentials.

In order to obtain the subdifferential of the functional $\Upsilon : L^2_s(\Omega) \to \mathbb{R}$ we calculate

$$\begin{aligned} a \in \partial\Upsilon(z) \quad &\Leftrightarrow \quad \Upsilon(z + \psi) \geq \Upsilon(z) + \langle a, \psi \rangle_\Omega \quad \forall \psi \in L^2_s(\Omega) \\ &\Leftrightarrow \ \int_\Omega \Psi(z + \psi) \geq \int_\Omega \Psi(z) + \langle a, \psi \rangle_\Omega \quad \forall \psi \in L^2_s(\Omega) \\ &\Leftrightarrow \ a(\omega) \in \partial\Psi(z(\omega)) \text{ for a.e. } \omega \in \Omega. \end{aligned} \tag{2.12}$$

Similarly, $a \in \partial\Upsilon^\delta(z)$ if and only if $a \in \partial\Psi^\delta(z)$ almost everywhere. Both subdifferentials are therefore single-valued and we may identify $\partial\Upsilon^\delta(z) = \partial\Psi^\delta(z)$. We next determine the subdifferential of the restricted functional Υ^δ_n.

Lemma 2.4. *The functionals Υ^δ_n have a single valued subdifferential in every $z_0 \in L^2_{n,s}(\Omega)$, given through*

$$\partial\Upsilon^\delta_n(z_0) = P_n \partial\Psi^\delta(z_0). \tag{2.13}$$

Proof. Let $a \in \partial\Upsilon^\delta_n(z_0) \subset L^2_{n,s}(\Omega)$ and let id be the identity on $L^2_s(\Omega)$. For arbitrary $\varphi \in L^2_s(\Omega)$ we set $\varphi_n := P_n\varphi$ and $\varphi_o := (id - P_n)\varphi$. We obtain

$$\begin{aligned} \int_\Omega \Psi^\delta(z_0 + t\varphi) = \Upsilon^\delta\left(z_0 + t\varphi_n + t\varphi_o\right) &\geq \Upsilon^\delta_n\left(z_0 + t\varphi_n\right) + t\left\langle \partial\Psi^\delta\left(z_0 + t\varphi_n\right), \varphi_o \right\rangle_\Omega \\ &\geq \Upsilon^\delta_n\left(z_0\right) + t\left\langle a, \varphi_n \right\rangle_\Omega + t\left\langle \partial\Psi^\delta\left(z_0 + t\varphi_n\right), \varphi_o \right\rangle_\Omega \end{aligned}$$

Since Ψ^δ is differentiable and $\partial\Psi^\delta$ is Lipschitz continuous, we obtain from the fact that the subdifferential coincides with the derivative and from the last inequality

$$\left\langle \partial\Psi^\delta(z_0), \varphi \right\rangle_\Omega = \lim_{t\to 0} \frac{1}{t}\left(\int_\Omega \Psi^\delta(z_0 + t\varphi) - \int_\Omega \Psi^\delta(z_0)\right) \geq \left\langle a, \varphi_n \right\rangle_\Omega + \left\langle \partial\Psi^\delta\left(z_0\right), \varphi_o \right\rangle_\Omega.$$

Replacing φ by $-\varphi$ in the above calculations, we obtain $\partial\Psi^\delta(z_0) = a + (id - P_n)\partial\Psi^\delta(z_0)$ or $P_n\partial\Psi^\delta(z_0) = a$. $\square$

The Fenchel conjugate of Υ_n^δ in $L^2_{n,s}(\Omega)$ is

$$\Upsilon_n^{\delta*}(\sigma) := \sup\left\{\int_\Omega \sigma : e \,\mathrm{d}\mathcal{P} - \Upsilon_n^\delta(e) \,|\, e \in L^2_{n,s}(\Omega)\right\}.$$

Since $-\Upsilon_n^\delta(\cdot)$ is coercive in a finite dimensional space, it has compact sublevels in $L^2_n(\Omega)$, and the supremum is indeed attained.

Lemma 2.5. *Let $\Upsilon^{\delta*}$ be the Fenchel conjugate of Υ^δ. For every $p \in L^2_s(\Omega)$ holds*

$$\Upsilon^*(p) = \int_\Omega \Psi^*(p)\,\mathrm{d}\mathcal{P}, \qquad \Upsilon^{\delta*}(p) = \int_\Omega \Psi^{\delta*}(p)\,\mathrm{d}\mathcal{P}, \tag{2.14}$$

and the functionals Υ, Υ^*, Υ^δ and $\Upsilon^{\delta*}$ are convex and weakly lower semicontinuous on $L^2_s(\Omega)$.

Proof. The functional Υ is convex with the conjugate

$$\Upsilon^*(p) := \sup\left\{\langle p, e\rangle_\Omega - \Upsilon(e) \,|\, e \in L^2_s(\Omega)\right\} \qquad \forall p \in L^2_s(\Omega).$$

We first prove (2.14): Let $p \in \operatorname{dom}\Upsilon^* = L^2_s(\Omega)$. Since Υ^* is convex, we know that $\partial\Upsilon^*(p) \neq \emptyset$. Lemma 2.3 (iv) yields for any $\sigma \in \partial\Upsilon^*(p)$ that $\sigma \in \operatorname{dom}\Upsilon$ with $p \in \partial\Upsilon(\sigma)$ and Lemma 2.3 (v) then yields

$$\Upsilon^*(p) + \Upsilon(\sigma) = \langle p, \sigma\rangle_\Omega. \tag{2.15}$$

Since $p \in \partial\Upsilon(\sigma)$, (2.12) yields $p(\omega) \in \partial\Psi(\sigma(\omega);\omega)$ for a.e. $\omega \in \Omega$ and Lemma 2.3 (v) yields $\Psi^*(p)+\Psi(\sigma) = p : \sigma$ a.e.. Integrating the last equality over Ω and comparing with (2.15), we find $\Upsilon^*(p) = \int_\Omega \Psi^*(p)$ since $\Upsilon(\sigma) = \int_\Omega \Psi(\sigma)$. The proof for the second statement in (2.14) is similar.

We now prove the weak lower semicontinuity of Υ^*. Let $\sigma_i \in \operatorname{dom}(\Psi)$, $i \in \mathbb{N}$, be dense in $\operatorname{dom}(\Psi)$. We define Ψ^*_m as the maximum of finitely many functions

$$\Psi^*_m(p) := \max_{i=1,\dots,m} \{p : \sigma_i - \Psi(\sigma_i)\} \qquad \forall p \in \mathbb{R}^{d\times d}_s$$

and note that $\Psi^*_m(p) \le \Psi^*(p)$ for every $p \in \mathbb{R}^{d\times d}_s$. For $z \in L^2_s(\Omega)$ and $i = 1,\dots,m$, we introduce the sets

$$\Omega_i := \{\omega \in \Omega \,|\, \Psi^*_m(z) = z : \sigma_i - \Psi(\sigma_i)\} \setminus \bigcup_{j<i} \Omega_j.$$

Let $(z_n)_n$ be a sequence such that $z_n \rightharpoonup z$ weakly in $L^2_s(\Omega)$. We find that

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega \Psi^*(z_n) &\ge \liminf_{n\to\infty} \sum_{i=1}^m \int_\Omega \Psi^*_m(z_n) = \liminf_{n\to\infty} \sum_i \int_{\Omega_i} \max_{j=1,\dots,m} (z_n : \sigma_j - \Psi(\sigma_j)) \\ &\ge \liminf_{n\to\infty} \sum_i \int_{\Omega_i} (z_n : \sigma_i - \Psi(\sigma_i)) = \sum_i \int_{\Omega_i} (z : \sigma_i - \Psi(\sigma_i)) = \int_\Omega \Psi^*_m(z).\end{aligned}$$

Since $\Psi^*(p) = \lim_{m\to\infty} \Psi^*_m(p)$ for every $p \in \mathbb{R}^{d\times d}_s$ by definition of Ψ^*_m, and since this convergence is monotone, we can apply the monotone convergence theorem and get $\int_\Omega \Psi^*_m(z) \to \int_\Omega \Psi^*(z) = \Upsilon^*(z)$. This yields the weak lower semicontinuity of Υ^*.

Since Ψ is convex and lower semicontinuous, we find $\Psi = \Psi^{**}$ and switching Ψ and Ψ^* in the above argumentation, the weak lower semicontinuity of Υ follows. The statements for Υ^δ and $\Upsilon^{\delta*}$ follow similarly. $\square$

Convergence properties.

We will later need additional lower semicontinuity properties: We have to analyze the behavior of, *e.g.*, $\Upsilon^\delta(u_\delta)$.

Lemma 2.6 (Lower semicontinuity property of Ψ^δ and $\Psi^{\delta*}$)**.** *Let $U_s := \Omega \times (0,s)$ be the space-time cylinder and let $(u_\delta)_\delta$ be a weakly convergent sequence, $u_\delta \rightharpoonup u$ weakly in $L^2(U_s)$ as $\delta \to 0$. Then, for Ψ^δ, Ψ as above, we find*

$$\liminf_{\delta\to 0} \int_{U_s} \Psi^{\delta*}(u_\delta)\,\mathrm{d}\mathcal{P}\,\mathrm{d}t \geq \int_{U_s} \Psi^*(u)\,\mathrm{d}\mathcal{P}\,\mathrm{d}t. \tag{2.16}$$

For every sequence $(u_\delta)_\delta$ with $u_\delta \rightharpoonup u$ weakly in $L^2_s(\Omega)$ we find

$$\liminf_{\delta\to 0} \Upsilon^\delta(u_\delta) \geq \Upsilon(u). \tag{2.17}$$

Proof. The proof of (2.16) is the same as in [20], Lemma 2.6.

Using the definition of Ψ^δ in (2.6), we choose, for every $\delta > 0$, a function $\pi_\delta \in L^2(\Omega;\mathbb{R}^{d\times d})$ such that

$$\int_\Omega \left(\frac{|\pi_\delta - u_\delta|^2}{\delta} + \Psi(\pi_\delta) \right) \mathrm{d}\mathcal{P} \leq \int_\Omega \Psi^\delta(u_\delta)\,\mathrm{d}\mathcal{P} + \delta.$$

Without loss of generality, we may assume $\liminf_{\delta\to 0} \int_\Omega \Psi^\delta(u_\delta)\,\mathrm{d}\mathcal{P} < \infty$. Then we get for a subsequence $\int_\Omega |\pi_\delta - u_\delta|^2 \to 0$ as $\delta \to 0$ and hence $\pi_\delta \rightharpoonup u$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ for this subsequence. Since $\int_\Omega |\pi_\delta - u_\delta|^2$ is positive and $\Upsilon(z) = \int_\Omega \Psi(z)$ is weakly lower semicontinuous, we find (2.17). □

The following lemma uses time-dependent functions and the discretization parameter $n \in \mathbb{N}$.

Lemma 2.7. *Let $s > 0$ and let $p \in L^2(0,s;L^2_s(\Omega))$ and $p_n \in L^2(0,s;L^2_n(\Omega))$ such that $p_n \rightharpoonup p$ weakly in $L^2(0,s;L^2(\Omega;\mathbb{R}^{d\times d}))$ as $n \to \infty$. Then, for $\Upsilon_n^{\delta*}$ and Υ_n^δ as above we find*

$$\liminf_{n\to\infty} \int_0^s \Upsilon_n^\delta(p_n)\,\mathrm{d}t \geq \int_0^s \Upsilon^\delta(p)\,\mathrm{d}t, \qquad \liminf_{n\to\infty} \int_0^s \Upsilon_n^{\delta*}(p_n)\,\mathrm{d}t \geq \int_0^s \Upsilon^{\delta*}(p)\,\mathrm{d}t. \tag{2.18}$$

Furthermore, if $z_n \to z$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ as $n \to \infty$, then

$$\lim_{n\to\infty} \Upsilon_n^\delta(z_n) = \Upsilon^\delta(z). \tag{2.19}$$

Proof. Let $z_n \to z$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Since Ψ^δ is Lipschitz continuous with $\Psi^\delta(0) = 0$, we find because of $\Upsilon_n^\delta(z_n) = \Upsilon^\delta(z_n)$

$$\lim_{n\to\infty} \Upsilon_n^\delta(z_n) = \lim_{n\to\infty} \int_\Omega \Psi^\delta(z_n) = \int_\Omega \Psi^\delta(z)$$

and thus (2.19). For $p_n \rightharpoonup p$ weakly in $L^2(U_s)$ with $p_n \in L^2(0,s;L^2_n(\Omega))$, the first inequality in (2.18) can be proved similarly to the weak lower semicontinuity results of Lemma 2.5, using $\Upsilon_n^\delta(p_n) = \Upsilon^\delta(p_n)$.

For the second inequality in (2.18), we choose finite sets $B_n = \left\{ e_n^i \,|\, i = 1,\ldots,K_n \right\} \subset L^2_n(\Omega)$ with $K_n \geq n$ such that $B_n \subset B_{n+1}$ and $\bigcup_n B_n$ is dense in $L^2(\Omega;\mathbb{R}^{d\times d})$. For fixed $N \in \mathbb{N}$, the interval $[0,s]$ is split into subsets

$$\tilde{\mathbb{T}}_N^i := \left\{ t \in [0,s] \;\middle|\; \max\left\{ \langle e, p(t)\rangle_\Omega - \Upsilon^\delta(e) \,|\, e \in B_N \right\} = \langle e_N^i, p(t)\rangle_\Omega - \Upsilon^\delta(e_N^i) \right\} \tag{2.20}$$

and we set $\mathbb{T}_N^1 := \tilde{\mathbb{T}}_N^1$ and $\mathbb{T}_N^i := \tilde{\mathbb{T}}_N^i \setminus \bigcup_{j<i} \mathbb{T}_N^j$ for $i = 2, \dots, K_N$. For $n \geq N$ we find, decomposing the time integral, taking the maximum, performing the weak limit, and using the definition of $\mathbb{T}_N^i$:

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^s \Upsilon_n^{\delta *}(p_n) &\geq \liminf_{n\to\infty} \sum_{i=1}^{K_N} \int_{\mathbb{T}_N^i} \max\left\{\langle e, p_n(t)\rangle_\Omega - \Upsilon_n^\delta(e) \,|\, e \in B_N\right\} \mathrm{d}t \\
&\geq \liminf_{n\to\infty} \sum_{i=1}^{K_N} \int_{\mathbb{T}_N^i} \left(\left\langle e_N^i, p_n(t)\right\rangle_\Omega - \Upsilon_n^\delta(e_N^i)\right) \mathrm{d}t \\
&= \sum_{i=1}^{K_N} \int_{\mathbb{T}_N^i} \left(\left\langle e_N^i, p(t)\right\rangle_\Omega - \Upsilon^\delta(e_N^i)\right) \mathrm{d}t \\
&\overset{(2.20)}{=} \sum_{i=1}^{K_N} \int_{\mathbb{T}_N^i} \max\left\{\langle e, p(t)\rangle_\Omega - \Upsilon^\delta(e) \,|\, e \in B_N\right\} \mathrm{d}t \\
&= \sup\left\{\int_0^s \left(\langle \tilde{e}, p(t)\rangle_\Omega - \Upsilon^\delta(\tilde{e}(t))\right) \mathrm{d}t \,|\, \tilde{e} \in L^2(0,s;B_N)\right\}.
\end{aligned}$$

This inequality implies, due to density of $\bigcup_N B_N$ in $L^2(\Omega;\mathbb{R}^{d\times d})$,

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^s \Upsilon_n^{\delta *}(p_n) &\geq \sup\left\{\int_0^s \int_\Omega \left(e : p - \Psi^\delta(e)\right) \,|\, e \in L^2(0,s;L^2(\Omega;\mathbb{R}^{d\times d}))\right\} \\
&= \int_0^s \int_\Omega \Psi^{\delta *}(p) = \int_0^s \Upsilon^{\delta *}(p),
\end{aligned}$$

where we used (2.14) in the last equality. We have thus verified the second inequality of (2.18). □

2.3. Properties of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$-functions

Lemma 2.8 (Potentials with small norm)**.** *Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz-domain and let $v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$. Then, for $\mathcal{P}$-a.e. $\omega \in \Omega$ and every $\varepsilon > 0$ there exists $\phi_{\varepsilon,\omega,v} \in H^1(U;\mathbb{R}^n)$ such that $\nabla\phi_{\varepsilon,\omega,v}(x) = v(\tau_{\frac{x}{\varepsilon}}\omega)$ and such that*

$$\lim_{\varepsilon\to 0} \|\phi_{\varepsilon,\omega,v}\|_{L^2(U)} = 0.$$

Proof. Let $v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ and write $v_{\varepsilon,\omega}(x) := v(\tau_{\frac{x}{\varepsilon}}\omega)$. By the ergodic Theorem 1.6, there exists $\Omega_v \subset \Omega$ with $\mathcal{P}(\Omega_v) = 1$ such that for all $\omega \in \Omega_v$ there exists $C_\omega > 0$ with

$$\sup_{\varepsilon>0} \|v_{\varepsilon,\omega}\|_{L^2(U)} \leq C_\omega. \tag{2.21}$$

Let $(\varphi_i)_{i\in\mathbb{N}} \subset L^2(\Omega;\mathbb{R}^{d\times d})$ a countably dense family. For every $i \in \mathbb{N}$ there exists $\Omega_i \subset \Omega$ with $\mathcal{P}(\Omega_i) = 1$ such that for every $\omega \in \Omega_i$

$$\int_U v_{\varepsilon,\omega}(x)\varphi_i(x)\,\mathrm{d}x \to \int_U \left(\int_\Omega v\,\mathrm{d}\mathcal{P}\right)\varphi_i\,\mathrm{d}x = 0 \qquad \text{as } \varepsilon \to 0. \tag{2.22}$$

We define $\tilde{\Omega} := \Omega_v \cup \bigcup_{i\in\mathbb{N}} \Omega_i$. By (2.21) and (2.22) we obtain that $v_{\varepsilon,\omega}(x) \rightharpoonup 0$ as $\varepsilon \to 0$ for all $\omega \in \tilde{\Omega}$.

By the definition of $L^2_{\mathrm{pot}}(\Omega)$ in (1.12), there exists $\phi_{\varepsilon,\omega,v} \in H^1(U)$ such that $\nabla\phi_{\varepsilon,\omega,v}(x) = v(\tau_{\frac{x}{\varepsilon}}\omega)$. By adding a constant, we can achieve $\int_U \phi_{\varepsilon,\omega,v} = 0$. By the Poincaré inequality, it follows that

$$\|\phi_{\varepsilon,\omega,v}\|_{L^2(U)} \leq \|\nabla\phi_{\varepsilon,\omega,v}(x)\|_{L^2(U)} + \left|\int_U \phi_{\varepsilon,\omega,v}\right| = \|\nabla\phi_{\varepsilon,\omega,v}(x)\|_{L^2(U)} = \int_U |v_{\varepsilon,\omega}(x)|^2.$$

Since the family $\phi_{\varepsilon,\omega,v}$ is bounded in $H^1(U)$, it is precompact in $L^2(U)$. We chose $f \in C_c^\infty(U;\mathbb{R}^n)$ and denote by F the solution to the Neumann boundary problem $-\Delta F = f$. We obtain

$$-\lim_{\varepsilon\to 0}\int_U \phi_{\varepsilon,\omega,v}\cdot f = \lim_{\varepsilon\to 0}\int_U \nabla\phi_{\varepsilon,\omega,v} : \nabla F = \lim_{\varepsilon\to 0}\int_U v(\tau_{\frac{x}{\varepsilon}}\omega) : \nabla F(x)\,\mathrm{d}x = 0.$$

Therefore, $\phi_{\varepsilon,\omega,v} \rightharpoonup 0$ in $L^2(U)$. Since $(\phi_{\varepsilon,\omega,v})_{\varepsilon>0}$ is precompact in $L^2(U)$, it follows that $\phi_{\varepsilon,\omega,v} \to 0$ in $L^2(U)$. □

Lemma 2.9 (A Korn's inequality on Ω)**.** *For every $f \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ holds*

$$\|f\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \le 2\,\|f^s\|_{L^2(\Omega;\mathbb{R}^{d\times d})}\,. \tag{2.23}$$

Proof. In what follows, we denote $Q := (-1,1)^d$ and $Q_\eta := (-1+\eta, 1-\eta)^d$ for $\frac{1}{2} > \eta > 0$. We choose $\psi_\eta \in C_c^\infty(Q)$ with $0 \le \psi_\eta \le 1$, $\psi_\eta \equiv 1$ on Q_η and $|\nabla\psi_\eta| < 2\eta^{-1}$.

Let $f \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ and for every $\varepsilon > 0$ and $\omega \in \Omega$ let $\phi_{\varepsilon,\omega,f}$ denote the potential of f_ω from Lemma 2.8. If we denote the characteristic function of $Q\backslash Q_\eta$ by $\chi_{Q\backslash Q_\eta}$, we have the pointwise inequality

$$\left|\,|\nabla\phi_{\varepsilon,\omega,f}|^2 - |\nabla(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2\right| \le \chi_{Q\backslash Q_\eta}\left(2\,|\nabla\phi_{\varepsilon,\omega,f}|^2 + \frac{4}{\eta}\,|\phi_{\varepsilon,\omega,f}|\,|\nabla\phi_{\varepsilon,\omega,f}| + \frac{4}{\eta^2}\,|\phi_{\varepsilon,\omega,f}|^2\right)$$

Using this inequality, we get from the ergodic Theorem 1.6 and Lemma 2.8 for $\mathcal{P}$-a.e. $\omega \in \Omega$

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_Q &\left|\,|\nabla\phi_{\varepsilon,\omega,f}|^2 - |\nabla(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2\right| \\ &\le \lim_{\varepsilon\to 0}\int_{Q\backslash Q_\eta} 2\,|\nabla\phi_{\varepsilon,\omega,f}|^2 + \frac{4}{\eta}\lim_{\varepsilon\to 0}\|\phi_{\varepsilon,\omega,f}\|_{L^2(Q)}\|\nabla\phi_{\varepsilon,\omega,f}\|_{L^2(Q)} + \frac{4}{\eta^2}\lim_{\varepsilon\to 0}\|\phi_{\varepsilon,\omega,f}\|^2_{L^2(Q)} \\ &= 2|Q\backslash Q_\eta|\int_\Omega f^2\,\mathrm{d}\mathcal{P},\end{aligned} \tag{2.24}$$

where we have used that $\phi_{\varepsilon,\omega,f} \to 0$ strongly in $L^2(Q)$. Arguing along the same limes with symmetrized functions, we can show that

$$\lim_{\varepsilon\to 0}\int_Q \left|\,|\nabla^s\phi_{\varepsilon,\omega,f}|^2 - |\nabla^s(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2\right| \le 2|Q\backslash Q_\eta|\int_\Omega (f^s)^2\,\mathrm{d}\mathcal{P}. \tag{2.25}$$

Since $(\phi_{\varepsilon,\omega,f}\psi_\eta) \in H^1_0(Q)$, we can apply Korn's inequality in $\mathbb{R}^n$ and obtain

$$\int_Q |\nabla(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2 \le 2\int_Q |\nabla^s(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2\,. \tag{2.26}$$

Combining (2.24)–(2.26) with the ergodic Theorem 1.6, we obtain that

$$\begin{aligned}|Q|\int_\Omega |f|^2\,\mathrm{d}\mathcal{P} &\overset{1.6}{=} \lim_{\varepsilon\to 0}\int_Q \left(f(\tau_{\frac{x}{\varepsilon}}\omega)\right)^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_Q |\nabla\phi_{\varepsilon,\omega,f}|^2\,\mathrm{d}x \\ &\overset{(2.24)}{\le} \lim_{\varepsilon\to 0}\int_Q |\nabla(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2 + 2|Q\backslash Q_\eta|\int_\Omega f^2\,\mathrm{d}\mathcal{P} \\ &\overset{(2.26)}{\le} \lim_{\varepsilon\to 0} 2\int_Q |\nabla^s(\phi_{\varepsilon,\omega,f}\psi_\eta)|^2 + 2|Q\backslash Q_\eta|\int_\Omega f^2\,\mathrm{d}\mathcal{P} \\ &\overset{(2.25)}{\le} \lim_{\varepsilon\to 0} 2\int_Q |\nabla^s\phi_{\varepsilon,\omega,f}|^2 + (2+4)|Q\backslash Q_\eta|\int_\Omega f^2\,\mathrm{d}\mathcal{P} \\ &\overset{1.6}{\le} |Q|\,2\int_\Omega |f^s|^2\,\mathrm{d}\mathcal{P} + 6|Q\backslash Q_\eta|\int_\Omega f^2\,\mathrm{d}\mathcal{P}.\end{aligned}$$

Since the last estimate holds for every small $\eta > 0$, we obtain inequality (2.23). □

2.4. Solutions to the approximate problem and *a priori* estimates

Lemma 2.10. *There exists a unique solution $p_{\delta,n}$, $z_{\delta,n}$, $v_{\delta,n}$ to problem (2.9)–(2.10) which satisfies the a priori estimate*

$$\|p_{\delta,n}\|_{\mathcal{V}_0^1} + \|z_{\delta,n}\|_{\mathcal{V}_0^1} + \|v_{\delta,n}\|_{\mathcal{V}_0^1} \le c\left(\Upsilon_n^\delta(z_{\delta,n}(0)) + \|\xi\|_{H^1(0,T)}\right), \tag{2.27}$$

with $\mathcal{V}_0^1 := H^1(0,T;L^2(\Omega;\mathbb{R}_s^{d\times d}))$ and c independent of δ and n.

Proof. In the following, all integrals over Ω are with respect to $\mathcal{P}$ and we omit d$\mathcal{P}$ for ease of notation. We will prove the lemma in two steps: we first show that the system (2.9)–(2.10) is equivalent to an ordinary differential equation for $p_{\delta,n}$ with Lipschitz continuous right hand side. Then, we show that the solution admits uniform *a priori* estimates.

Step 1. Existence. In order to study (2.9)–(2.10), we fix $\tilde p \in L^2_{n,s}(\Omega)$ and $\tilde\xi \in \mathbb{R}_s^{d\times d}$, and search for $\tilde v \in \mathcal{V}^2_{\mathrm{pot},n}(\Omega)$ such that

$$\left\langle C_n^{-1}\tilde v^s, \zeta\right\rangle_\Omega = \left\langle C_n^{-1}\tilde p, \zeta\right\rangle_\Omega - \left\langle C_n^{-1}\tilde\xi, \zeta\right\rangle_\Omega \qquad \forall \zeta \in \mathcal{V}^2_{\mathrm{pot},n}(\Omega). \tag{2.28}$$

The Lax-Milgram theorem in combination with Korn's inequality (2.23) yields a unique solution $\tilde v \in \mathcal{V}^2_{\mathrm{pot},n}(\Omega)$ of the last equality. We introduce the mapping $V_{\tilde\xi} : L^2_{n,s}(\Omega) \to \mathcal{V}^2_{\mathrm{pot},n}(\Omega)$ with $V_{\tilde\xi}(\tilde p) = \tilde v$ and note that this operator is linear and bounded. We then look for a solution $p_{\delta,n} \in C^1(0,T;L^2_n(\Omega))$ to the following version of (2.9):

$$\partial_t p_{\delta,n} = \partial\Upsilon_n^\delta\left(C_n^{-1}\left(\xi + V_\xi(p_{\delta,n})^s - p_{\delta,n}\right) - B_n\, p_{\delta,n}\right).$$

Relation (2.13) yields the Lipschitz continuity of $\partial\Upsilon_n^\delta$. Therefore, since also $\partial\Upsilon_n^\delta$, C_n^{-1}, V_ξ^s and B_n are Lipschitz-continuous mappings $L^2_{n,s}(\Omega) \to L^2_{n,s}(\Omega)$, we find a unique solution $p_{\delta,n} \in C^1([0,T];L^2_{n,s}(\Omega))$ of the ordinary differential equation (*a priori* bounds are provided below). We furthermore set $v_{\delta,n} = V_\xi(p_{\delta,n}) \in C^1([0,T];\mathcal{V}^2_{\mathrm{pot},n}(\Omega))$ and $z_{\delta,n} = C_n^{-1}(\xi + v^s_{\delta,n} - p_{\delta,n}) \in H^1(0,T;L^2_{n,s}(\Omega))$. From (2.28) and the definition of $v_{\delta,n}$, it follows that $z_{\delta,n} \in H^1(0,T;L^2_{\mathrm{sol},n}(\Omega))$. Note that $p_{\delta,n}$, $z_{\delta,n}$ and $v_{\delta,n}$ are constructed in such a way that (2.9)–(2.10) holds. The construction shows that the solution is uniquely determined.

Step 2. *A priori* estimates of order 0. We take the time derivative of (2.10), multiply by $z_{\delta,n}$ and integrate over $[0,t]\times\Omega$ for $t \in (0,T]$ to find

$$\begin{aligned}
\int_0^t\int_\Omega \partial_t\xi : z_{\delta,n} &\overset{(2.10)}{=} \int_0^t\int_\Omega \left((C_n\partial_t z_{\delta,n}) : z_{\delta,n} + \partial_t p_{\delta,n} : z_{\delta,n} - \partial_t v^s_{\delta,n} : z_{\delta,n}\right)\\
&= \frac12\int_\Omega \left(p_{\delta,n} : B_n p_{\delta,n} + z_{\delta,n} : C_n z_{\delta,n}\right)\Big|_0^t\\
&\quad + \int_0^t \left\langle \partial_t p_{\delta,n}, z_{\delta,n} - B_n p_{\delta,n}\right\rangle_\Omega - \int_0^t\int_\Omega z_{\delta,n} : \partial_t v_{\delta,n}\\
&\overset{(*)}{=} \frac12\int_\Omega \left(p_{\delta,n} : (Bp_{\delta,n}) + z_{\delta,n} : (Cz_{\delta,n})\right)\Big|_0^t\\
&\quad + \int_0^t \left(\Upsilon_n^{\delta*}\left(\partial_t p_{\delta,n}\right) + \Upsilon_n^\delta\left(z_{\delta,n} - B_n p_{\delta,n}\right)\right).
\end{aligned} \tag{2.29}$$

In $(*)$ we used the orthogonality of potentials and (symmetric) solenoidals, $\int_\Omega z_{\delta,n} : \partial_t v_{\delta,n} = 0$, and Lemma 2.3 (v), written as

$$\langle\partial_t p, z - Bp\rangle_\Omega = \Upsilon_n^\delta(z - Bp) + \Upsilon_n^{\delta*}(\partial_t p) \quad\Leftrightarrow\quad \partial_t p = \partial\Upsilon_n^\delta(z - Bp).$$

***A priori* estimates of order 1.** Taking the time derivative of (2.10), multiplying the result by $\partial_t z_{\delta,n}$ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega \partial_t\xi : \partial_t z_{\delta,n} &= \int_\Omega \partial_t z_{\delta,n} : \partial_t \left(p_{\delta,n} + C_n z_{\delta,n} - v_{\delta,n}\right) + \int_\Omega \left(B_n\partial_t p_{\delta,n} - B_n\partial_t p_{\delta,n}\right) : \partial_t p_{\delta,n} \\ &\overset{(2.9)}{=} \left\langle \partial_t z_{\delta,n} - B_n\partial_t p_{\delta,n}, \partial\Upsilon_n^\delta(z_{\delta,n} - B_n p_{\delta,n})\right\rangle_\Omega + \int_\Omega (B_n\partial_t p_{\delta,n}) : \partial_t p_{\delta,n} \\ &\quad + \int_\Omega (C_n\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} - \int_\Omega \partial_t z_{\delta,n} : \partial_t v_{\delta,n} \\ &\overset{(*)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_n^\delta(z_{\delta,n} - B_n p_{\delta,n}) + \int_\Omega \left((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\right),\end{aligned}$$

where we used $\int_\Omega \partial_t z_{\delta,n} : \partial_t v_{\delta,n} = 0$ in $(*)$. We integrate the last equality over $(0,t)$ for $t \in (0,T]$ and obtain

$$\Upsilon_n^\delta(z_{\delta,n}(0)) + \int_0^t\int_\Omega \partial_t z_{\delta,n} : \partial_t\xi^s \geq \Upsilon_n^\delta(z_{\delta,n}(t) - B_n p_{\delta,n}(t)) + \int_0^t\int_\Omega \left((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\right). \tag{2.30}$$

Since $\Upsilon_n^{\delta*}$ and Υ_n^δ are positive, we can neglect them in (2.29). Applying the Cauchy-Schwarz inequality to the right hand side of (2.29) and then Gronwall's inequality yields an estimate

$$\sup_{t\in[0,T]} \|z_{\delta,n}(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \sup_{t\in[0,T]} \|p_{\delta,n}(t)\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq c\,\|\xi\|_{H^1}.$$

From positivity of Υ_n^δ on the right hand side of (2.30), it follows that

$$\int_0^t\int_\Omega \left((C\partial_t z_{\delta,n}) : \partial_t z_{\delta,n} + (B\partial_t p_{\delta,n}) : \partial_t p_{\delta,n}\right) \leq \Upsilon_n^\delta(z_{\delta,n}(0)) + \|\xi\|_{H^1}.$$

The last two inequalities yield (2.27) for $z_{\delta,n}$ and $p_{\delta,n}$. The inequality for $v_{\delta,n}$ follows from equation (2.10). □

2.5. Proof of Theorem 2.1

Existence. Using the sequence $(p_{\delta,n}, z_{\delta,n}, v_{\delta,n})$ of solutions to (2.9)–(2.10), we can now prove Theorem 2.1. For $n \to \infty$, we find weakly convergent subsequences of $p_{\delta,n}$, $z_{\delta,n}$, $v_{\delta,n}$ in $\mathcal{V}_0^1$ with limits p_δ, z_δ, v_δ. We note that $z_{\delta,n}(0)$ is the unique solution in $L^2_{\mathrm{sol},n}(\Omega)$ to

$$\int_\Omega (C_n z_{\delta,n}(0)) : \psi = \int_\Omega \xi(0) : \psi \qquad \forall \psi \in L^2_{\mathrm{sol},n}(\Omega).$$

Hence, since we consider only ξ with $\xi(0) = 0$, the initial values $z_{\delta,n}(0)$ vanish identically. As a consequence, also $\Upsilon_n^\delta(z_{\delta,n}(0))$ in (2.27) vanishes. The estimate (2.27) therefore implies (2.4) for $(p_\delta, z_\delta, v_\delta)$.

Since $p_{\delta,n}$, $z_{\delta,n}$, $v_{\delta,n}$ satisfy (2.10), the limits p_δ, z_δ, v_δ satisfy

$$C z_\delta = \xi + v_\delta^s - p_\delta. \tag{2.31}$$

We take the limit $n \to \infty$ in (2.29), apply Lemma 2.7 and exploit the vanishing initial data to conclude that the functions p_δ, z_δ, v_δ satisfy

$$\int_0^t\int_\Omega \left(\Psi^{\delta*}(\partial_t p_\delta) + \Psi^\delta(z_\delta - B\,p_\delta)\right) \leq \int_0^t\int_\Omega z_\delta : \partial_t\xi - \frac{1}{2}\int_\Omega \left(p_\delta : (B\,p_\delta) + z_\delta : (C z_\delta)\right)\Big|_0^t. \tag{2.32}$$

In the limit $\delta \to 0$ we find weakly convergent subsequences of p_δ, z_δ, v_δ with the respective weak limits p, z, v satisfying the estimate (2.4). Passing to the limit $\delta \to 0$ in (2.31), we find that (p,z,v) satisfies (2.3). Furthermore, passing to the limit in (2.32), using Lemma 2.6, we find that the functions p, z, v satisfy

$$\int_0^t \int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - B\,p\right)\right) \le \int_0^t \int_\Omega z : \partial_t \xi - \frac{1}{2}\left(\int_\Omega p : (Bp) + \int_\Omega z : (Cz)\right)\Big|_0^t .$$

We thus obtain

$$\begin{aligned}
\int_0^t \int_\Omega \left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - B\,p\right)\right) &= \int_0^t \int_\Omega \left(z : \partial_t \xi - \partial_t p : Bp - \partial_t z : Cz\right) \\
&\overset{(2.3)}{=} \int_0^t \int_\Omega \left(z : C\partial_t z - z : \partial_t v^s + z : \partial_t p - \partial_t p : Bp - \partial_t z : Cz\right) \\
&= \int_0^t \int_\Omega \left(-z : \partial_t v^s + \partial_t p : (z - B\,p)\right) \;=\; \int_0^t \int_\Omega \partial_t p : (z - B\,p)
\end{aligned}$$

for every $t \in (0,T)$. On the other hand, since Lemma 2.3 (iii) yields $\left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - Bp\right)\right) \ge \partial_t p : (z - Bp)$ pointwise a.e., we find

$$\left(\Psi^*\left(\partial_t p\right) + \Psi\left(z - Bp\right)\right) = \partial_t p : (z - Bp)$$

pointwise a.e. in $(0,T) \times \Omega$. The Fenchel equality of Lemma 2.3 (v) then yields (2.2).

Uniqueness and continuity. Let $\xi_1, \xi_2 \in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$. Let $(p_i, z_i, v_i)_{i\in\{1,2\}}$ be two solutions to (2.2)–(2.3) for ξ_1, ξ_2 respectively with the difference $(\tilde p, \tilde z, \tilde v) := (p_1, z_1, v_1) - (p_2, z_2, v_2)$. We integrate $\tilde z : \partial_t\left(\xi_1 - \xi_2\right)$ over Ω and obtain from a calculation similar to (2.29)

$$\begin{aligned}
&\int_\Omega \tilde z : (\xi_1 - \xi_2)\Big|_0^t - \int_0^t \int_\Omega \partial_t \tilde z : (\xi_1 - \xi_2) \\
&\quad= \int_0^t \int_\Omega \tilde z : \partial_t (\xi_1 - \xi_2) = \int_\Omega \tilde z : \partial_t\left(C\tilde z - \tilde p + \tilde v\right) \\
&\quad= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\tilde p : (B\tilde p) + \tilde z : (C\tilde z)\right) \\
&\qquad+ \int_\Omega \left[\left(z_1(t,\omega) - B(\omega)\,p_1(t,\omega)\right) - \left(z_2(t,\omega) - B(\omega)\,p_2(t,\omega)\right)\right]\left(\partial_t p_1 - \partial_t p_2\right).
\end{aligned}$$

From the monotonicity of $\partial\Psi$ (Lem. 2.3(ii)) and $(2.2)_{1,2}$, we find

$$\frac{1}{2}\int_\Omega \left(\tilde p : (B\tilde p) + \tilde z : (C\tilde z)\right)\Big|_0^t \le \int_\Omega \tilde z : (\xi_1 - \xi_2)\Big|_0^t - \int_0^t \int_\Omega \partial_t \tilde z : (\xi_1 - \xi_2)$$

for every $t \in (0,T)$. Compactness of the embedding $H^1(0,T;\mathbb{R}^{d\times d}_s) \subset C([0,T];\mathbb{R}^{d\times d}_s)$ and boundedness of $\partial_t \tilde z$ provide the weak continuity of the mapping $\xi \mapsto (z,p,v)$. At the same time, it implies uniqueness of solutions, *i.e.* $(\tilde p, \tilde z, \tilde v) = (0,0,0)$ for $\xi_1 = \xi_2$. This completes the proof of Theorem 2.1.

3. Proof of the main theorem

3.1. Preliminaries

Lemma 3.1 (A time dependent ergodic theorem)**.** *Let $f \in L^p(0,T;L^p(\Omega))$, $1 \le p < \infty$ and $f_\omega(t,x) := f(t,\tau_x\omega)$. Then, for almost every $\omega \in \Omega$, there holds $f_\omega \in L^p(0,T;L^p_{\mathrm{loc}}(\mathbb{R}^d))$. Furthermore, for almost every $\omega \in \Omega$, there holds*

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q f(t, \tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}x\,\mathrm{d}t = |Q| \int_0^T \int_\Omega f(t,\omega)\,\mathrm{d}\mathcal{P}(\omega)\,\mathrm{d}t. \tag{3.1}$$

Proof. Since the mapping $(x,\omega) \mapsto \tau_x\omega$ is measurable, we find that $\tilde{f}(\omega,t,x) := f(t,\tau_x\omega)$ is $\mathcal{P}\otimes\mathcal{L}\otimes\mathcal{L}^d$-measurable. Since the mappings $\tau_x : \Omega \to \Omega$ are measure preserving, we find for every $x \in \mathbb{R}^d$

$$\int_0^T \int_\Omega |f(t,\omega)|^p \,\mathrm{d}\mathcal{P}(\omega)\mathrm{d}t = \int_0^T \int_\Omega |f(t,\tau_x\omega)|^p \,\mathrm{d}\mathcal{P}(\omega)\mathrm{d}t.$$

Integrating the last equation over $Q \subset \mathbb{R}^d$ and applying Fubini's theorem, we obtain

$$|Q| \int_0^T \int_\Omega |f(t,\omega)|^p \,\mathrm{d}\mathcal{P}(\omega)\mathrm{d}t = \int_\Omega \int_0^T \int_Q |f(t,\tau_x\omega)|^p \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mathcal{P}(\omega).$$

Thus, $\tilde{f}$ has the integrability $\tilde{f} \in L^p(\Omega; L^p(0,T; L^p(Q)))$ and $f_\omega \in L^p(0,T;L^p(Q))$ for almost every $\omega \in \Omega$. In particular, $f_\omega \in L^1(0,T;L^1(Q))$. Setting $F(\omega) := \int_0^T f(t,\omega)\,\mathrm{d}t$, we find as a consequence of Theorem 1.6:

$$\lim_{\varepsilon\to 0} \int_0^T \int_Q f(t,\tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_Q F(\tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}x = |Q| \int_\Omega F\,\mathrm{d}\mathcal{P} = |Q| \int_0^T \int_\Omega f\,\mathrm{d}\mathcal{P}\,\mathrm{d}t.$$

This was the claim in (3.1). □

Lemma 3.2 (Div-curl-lemma)**.** *Let $U \subset \mathbb{R}^d$ be open and bounded with Lipschitz-boundary ∂U. For a sequence $\varepsilon \to 0$ we consider sequences of functions u^ε and v^ε as follows*:

$$\begin{aligned} &u^\varepsilon \in L^2(0,T;L^2(U;\mathbb{R}^{d\times d})) \quad \textit{with} \quad \nabla\cdot u^\varepsilon(t) = 0 \quad \textit{in } \mathcal{D}'(U) \textit{ for a.e. } t\in[0,T],\\ &v^\varepsilon \in L^2(0,T;L^2(U;\mathbb{R}^{d\times d})), \quad v^\varepsilon(t,x) := v(t,\tau_{\frac{x}{\varepsilon}}\omega) \quad \textit{for} \quad v \in L^2(0,T;\mathcal{V}^2_{\mathrm{pot}}(\Omega)) \end{aligned}$$

and some $\omega \in \Omega$. We assume the boundedness $\|u^\varepsilon\|_{L^2(0,T;L^2(U))} \le C_0$. Then, for almost every $\omega \in \Omega$, there holds

$$\lim_{\varepsilon\to 0} \int_0^T \int_U u^\varepsilon : v^\varepsilon = 0. \tag{3.2}$$

Proof. In this proof, we omit the time-dependence of u^ε and v for simplicity of notation, *i.e.* we consider $u^\varepsilon \in L^2(U;\mathbb{R}^{d\times d})$ and $v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$. In the time dependent case, one has to apply Lemma 3.1 instead of the ergodic Theorem 1.6.

We consider a compact set $K \subset U$ and a cut-off function $\psi \in C^\infty(\mathbb{R}^d)$ with $\psi \equiv 1$ on K, $\psi \equiv 0$ on $\mathbb{R}^d\backslash U$, and $0 \le \psi \le 1$. We fix $\omega \in \Omega$ such that $x \mapsto v(\tau_x\omega) \in L^2_{\mathrm{pot,loc}}(\mathbb{R}^d)$ and such that the assertion of Theorem 1.6 holds. Furthermore, we make use of $\phi_{\varepsilon,\omega,v}$ of Lemma 2.8 and observe the limit behavior

$$\begin{aligned} \int_U u^\varepsilon : v^\varepsilon\psi &= \int_U u^\varepsilon : (\nabla\phi_{\varepsilon,\omega,v})\psi = \int_U u^\varepsilon : \nabla_x(\phi_{\varepsilon,\omega,v}\psi) - \int_U u^\varepsilon : (\phi_{\varepsilon,\omega,v}\otimes\nabla_x\psi)\\ &= -\int_U u^\varepsilon : (\phi_{\varepsilon,\omega,v}\otimes\nabla_x\psi) \to 0 \end{aligned} \tag{3.3}$$

as $\varepsilon \to \infty$ due to $\phi_{\varepsilon,\omega,v} \to 0$ of Lemma 2.8 and the boundedness of $\nabla\psi$ and u^ε.

Concerning the integral over $u^\varepsilon : v^\varepsilon(1-\psi)$, we find by the ergodic Theorem 1.6

$$\left|\int_U u^\varepsilon : v^\varepsilon(1-\psi)\right| \le C_0 \|v^\varepsilon\|_{L^2(U\backslash K)} \to C_0 \|v\|_{L^2(\Omega;\mathbb{R}^{d\times d})} |U\backslash K|^{\frac{1}{2}} \tag{3.4}$$

as $\varepsilon \to 0$. Choosing $K \subset U$ large we obtain (3.2). □

3.2. The averaging property of Σ

Theorem 3.3 (Averaging property)**.** *Let the coefficients $B(\omega)$, $C(\omega)$, $\Psi(\,\cdot\,;\omega)$ be as in Assumption 1.8 and let realizations C_ε, B_ε, Ψ_ε be defined by (1.11). Then, for a.e. $\omega\in\Omega$, the coefficients allow averaging in sense of Definition 1.11 with the operators Σ and Π of (2.5).*

Proof. We will prove a slightly stronger result: Given $\xi\in H^1_*(0,T;\mathbb{R}^{d\times d}_s)$, let $(p,\, z,\, v)$ be the unique solution of (2.2)–(2.3) (which exists by Thm. 2.1). Let $\omega\in\Omega$ be such that $p_\omega(t,x) := p(t,\tau_x\omega)$, $z_\omega(t,x) := z(t,\tau_x\omega)$ and $v_\omega(t,x) := v(t,\tau_x\omega)$ satisfy

$$p_\omega\in H^1(0,T;L^2_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_s)),\ z_\omega\in H^1(0,T;L^2_{\text{sol,loc}}(\mathbb{R}^d)),\ v_\omega\in H^1(0,T;L^2_{\text{pot,loc}}(\mathbb{R}^d)).$$

This regularity is valid for a.e. ω as can be seen applying Lemma 3.1 to time derivatives. Furthermore, we choose ω as in Assumption 1.8. For any $\varepsilon>0$ let $\tilde p^\varepsilon(t,x) := p\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$, $\tilde z^\varepsilon(t,x) := z\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$, $\tilde v^\varepsilon(t,x) := v\left(t,\tau_{\frac{x}{\varepsilon}}\omega\right)$ be realizations. Let $\mathcal{T}\subset\mathbb{R}^d$ be a simplex and let u^ε, p^ε, σ^ε be the unique solution to

$$\begin{aligned} -\nabla\cdot\sigma^\varepsilon &= 0,\\ \nabla^s u^\varepsilon &= C_\varepsilon\sigma^\varepsilon + p^\varepsilon\\ \partial_t p^\varepsilon &\in \partial\Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon\,;.\,) \end{aligned}\tag{3.5}$$

on $\mathcal{T}$ with boundary condition

$$u^\varepsilon(x) = \xi\cdot x \qquad \text{on } \partial\mathcal{T}\tag{3.6}$$

and initial condition $p^\varepsilon(0,\cdot)=0$ (we recall $\partial\Psi_\varepsilon(\sigma\,;x) := \partial\Psi(\sigma\,;\tau_{\frac{x}{\varepsilon}}\omega)$). We will prove that the realizations of the stochastic cell solutions and the plasticity solutions on $\mathcal{T}$ coincide in the limit $\varepsilon\to0$; more precisely, we claim that

$$\lim_{\varepsilon\to0}\Big(\|\sigma^\varepsilon-\tilde z^\varepsilon\|_{L^2(0,T;L^2(\mathcal{T}))} + \|p^\varepsilon-\tilde p^\varepsilon\|_{L^2(0,T;L^2(\mathcal{T}))}\Big) = 0.\tag{3.7}$$

Let us first show that (3.7) indeed implies Theorem 3.3: The ergodic theorem in the version of Lemma 3.1 and the definition of Σ and Π in (2.5) imply that $⨍_{\mathcal{T}}\tilde z^\varepsilon(.) \to \int_\Omega z(.) = \Sigma(\xi)(.)$ and $⨍_{\mathcal{T}}\tilde p^\varepsilon(.)\to\int_\Omega p(.) = \Pi(\xi)(.)$ holds in the space $L^2(0,T;\mathbb{R}^{d\times d}_s)$. Equation (3.7) therefore yields $⨍_{\mathcal{T}}\sigma^\varepsilon\to\Sigma(\xi)$ and $⨍_{\mathcal{T}}p^\varepsilon\to\Pi(\xi)$ in $L^2(0,T;\mathbb{R}^{d\times d}_s)$. This provides the averaging property (1.20) of Definition 1.11 (at first, for a subsequence $\varepsilon\to0$ for almost every $t\in(0,T)$, then, since the limit is determined, along the original sequence $\varepsilon\to0$).

Let us now prove (3.7). We will use a testing procedure and energy-type estimates. Due to (2.2)–(2.3), $\tilde z^\varepsilon$, $\tilde p^\varepsilon$ and $\tilde v^\varepsilon$ satisfy the following system of equations on $\mathcal{T}\times(0,T)$

$$\begin{aligned} -\nabla\cdot\tilde z^\varepsilon &= 0,\\ \xi &= C_\varepsilon\tilde z^\varepsilon + \tilde p^\varepsilon - (\tilde v^\varepsilon)^s\,,\\ \partial_t\tilde p^\varepsilon &\in \partial\Psi_\varepsilon(\tilde z^\varepsilon - B_\varepsilon\tilde p^\varepsilon\,;.\,). \end{aligned}\tag{3.8}$$

In what follows we use the notation $|\zeta|^2_{B_\varepsilon} := \zeta : B_\varepsilon\zeta$ and $|\zeta|^2_{C_\varepsilon} := \zeta : C_\varepsilon\zeta$. We take the difference of $(3.5)_1$ and $(3.8)_1$, multiply the result by $(\partial_t u^\varepsilon - \partial_t(\xi\cdot x))$ and integrate over $\mathcal{T}$. We integrate by parts and exploit that boundary integrals vanish due to (3.6),

$$\begin{aligned} 0 &= -\int_{\mathcal{T}}(\tilde z^\varepsilon-\sigma^\varepsilon) : (\partial_t\nabla^s u^\varepsilon - \partial_t\xi)\\ &= \int_{\mathcal{T}}(\tilde z^\varepsilon-\sigma^\varepsilon) : \partial_t\left(C_\varepsilon\tilde z^\varepsilon + \tilde p^\varepsilon - (\tilde v^\varepsilon)^s - C_\varepsilon\sigma^\varepsilon - p^\varepsilon\right)\\ &= \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}}\left[(\tilde z^\varepsilon-\sigma^\varepsilon) : (C_\varepsilon(\tilde z^\varepsilon-\sigma^\varepsilon)) + (\tilde p^\varepsilon - p^\varepsilon) : (B_\varepsilon(\tilde p^\varepsilon-p^\varepsilon))\right] + \int_{\mathcal{T}}(\tilde z^\varepsilon-\sigma^\varepsilon) : \partial_t\tilde v^\varepsilon \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathcal{T}}\left(\partial_t\tilde{p}^\varepsilon-\partial_t p^\varepsilon\right):\left((\tilde{z}^\varepsilon-B_\varepsilon\tilde{p}^\varepsilon)-(\sigma^\varepsilon-B_\varepsilon p^\varepsilon)\right).\\
\in\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}}\left[|\tilde{z}^\varepsilon-\sigma^\varepsilon|^2_{C_\varepsilon}+|\tilde{p}^\varepsilon-p^\varepsilon|^2_{B_\varepsilon}\right]+\int_{\mathcal{T}}(\tilde{z}^\varepsilon-\sigma^\varepsilon):\partial_t\tilde{v}^\varepsilon\\
&+\int_{\mathcal{T}}\left(\partial\Psi_\varepsilon\left(\tilde{z}^\varepsilon-B_\varepsilon\tilde{p}^\varepsilon\right)-\partial\Psi_\varepsilon\left(\sigma^\varepsilon-B_\varepsilon p^\varepsilon\right)\right):\left((\tilde{z}^\varepsilon-B_\varepsilon\tilde{p}^\varepsilon)-(\sigma^\varepsilon-B_\varepsilon p^\varepsilon)\right).
\end{aligned}
\tag{3.9}
$$

In the second line, we used $(3.5)_2$ and $(3.8)_2$. In the third line we used the symmetry of σ^ε and $\tilde{z}^\varepsilon$ to replace $(\tilde{v}^\varepsilon)^s$ by $\tilde{v}^\varepsilon$.

Concerning the second integral on the right hand side of (3.9), note that $\int_0^t\int_{\mathcal{T}}\tilde{z}^\varepsilon:\partial_t\tilde{v}^\varepsilon\to\int_0^t\int_{\mathcal{T}}\int_\Omega z:\partial_t v=0$ by Lemma 3.1 and orthogonality of $L^2_{\mathrm{sol}}(\Omega)$ and $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$. Furthermore, $\int_0^t\int_{\mathcal{T}}\sigma^\varepsilon:\partial_t\tilde{v}^\varepsilon\to 0$ by Lemma 3.2. By monotonicity of $\partial\Psi_\varepsilon$, the last integral on the right hand side of (3.9) is positive. An integration over $(0,t)$ therefore provides

$$
\limsup_{\varepsilon\to 0}\int_{\mathcal{T}}\left[|\tilde{z}^\varepsilon-\sigma^\varepsilon|^2_{C_\varepsilon}+|\tilde{p}^\varepsilon-p^\varepsilon|^2_{B_\varepsilon}\right](t)\le\limsup_{\varepsilon\to 0}\int_0^t\int_{\mathcal{T}}(\tilde{z}^\varepsilon-\sigma^\varepsilon)\partial_t\tilde{v}^\varepsilon=0,\tag{3.10}
$$

where we used that initial data vanish, $\tilde{z}^\varepsilon|_{t=0}=0$ by (3.8) and $\sigma^\varepsilon|_{t=0}=0$ by (3.5) for vanishing p^ε and ξ in $t=0$. We have thus shown (3.7) and hence Theorem 3.3. □

3.3. Admissibility of Σ

Theorem 3.4 (Admissibility). *Let the coefficients $B(\omega)$, $C(\omega)$, $\Psi(\cdot\,;\omega)$ and data U, f be as in Assumption 1.8. Then the causal operator Σ of Definition 2.2 satisfies the sufficient condition for admissibility of Definition 1.14.*

Proof. We have to study solutions u_h of the discretized effective problem with the discretized boundary data $U_h\to U$ strongly in $H^1(0,T;H^1(Q))$ as $h\to 0$. With Σ given through (2.5), let $u_h\in H^1(0,T;H^1(Q))$ be a sequence with $u_h\in U_h+H^1(0,T;Y_h)$, satisfying the discrete system

$$
\int_0^T\int_Q\Sigma(\nabla^s u_h):\nabla\varphi=\int_0^T\int_Q f\cdot\varphi\qquad\forall\varphi\in L^2(0,T;Y_h).\tag{3.11}
$$

We furthermore have the weak convergence $u_h\rightharpoonup u\in H^1(0,T;H^1(Q;\mathbb{R}^d)$ as $h\to 0$ for some $u\in U+H^1(0,T;H^1_0(Q;\mathbb{R}^d))$. Our aim is to show that u solves the effective problem

$$
\int_0^T\int_Q\Sigma(\nabla^s u):\nabla\varphi=\int_0^T\int_Q f\cdot\varphi\qquad\forall\varphi\in L^2(0,T;H^1_0(Q)).\tag{3.12}
$$

Step 1. For every $x\in Q$, we denote by $p_h(t,x,\cdot)$, $z_h(t,x,\cdot)$, $v_h(t,x,\cdot)$ the solutions of (2.2)–(2.3) corresponding to $\xi(t)=\nabla^s u_h(t,x)$. By definition of Σ, there holds $\Sigma(\nabla^s u_h)=\int_\Omega z_h(\omega)\,\mathrm{d}\mathcal{P}(\omega)$. The *a priori* estimate of Theorem 2.1 provides

$$
\|p_h\|_{\mathcal{V}^1_{0,0}}+\|z_h\|_{\mathcal{V}^1_{0,0}}+\|v_h\|_{\mathcal{V}^1_{0,0}}\le C\,\|\nabla^s u\|_{H^1(0,T;L^2(Q))}\,,
$$

where $\mathcal{V}^1_{0,0}:=H^1(0,T;L^2(Q;L^2(\Omega;\mathbb{R}^{d\times d})))$. By this estimate, we obtain the weak convergence in $(\mathcal{V}^1_{0,0})^3$ of a subsequence, again denoted (p_h,z_h,v_h), weakly converging to some limit (p,z,v). The limit satisfies again the linear law (2.3),

$$
Cz=\nabla^s u+v^s-p.\tag{3.13}
$$

Equation (3.11) can be rewritten as

$$
\int_0^T\int_Q\int_\Omega z_h(t,x,\omega)\,\mathrm{d}\mathcal{P}(\omega):\,\nabla\varphi(x)\,\mathrm{d}x=\int_0^T\int_Q f\cdot\varphi\qquad\forall\varphi\in L^2(0,T;Y_h),
$$

and the limit $h \to 0$ provides

$$\int_0^T \int_Q \int_\Omega z : \nabla\varphi = \int_0^T \int_Q f \cdot \varphi \qquad \forall \varphi \in L^2(0,T;H_0^1(Q)). \tag{3.14}$$

Step 2. It remains to verify $\int_\Omega z = \Sigma(\nabla^s u)$. We use $\varphi = \partial_t (u_h - U_h)$ as a test function in (3.11) and exploit the orthogonality $0 = \int_Q \int_\Omega z_h : \partial_t v_h$. We follow the lines of the calculation in (2.29) to obtain

$$\begin{aligned}
&\int_Q f \cdot \partial_t (u_h - U_h) + \int_Q \int_\Omega z_h : \nabla \partial_t U_h \\
&\quad = \int_Q \int_\Omega z_h : \partial_t \nabla^s u_h = \int_Q \int_\Omega [z_h : C\partial_t z_h + z_h : \partial_t p_h - z_h : \partial_t v_h] \\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q \int_\Omega p_h : Bp_h + \int_Q \int_\Omega z_h : Cz_h\right) + \int_Q \int_\Omega \left(\Psi^*(\partial_t p_h) + \Psi(z_h - Bp_h)\right).
\end{aligned} \tag{3.15}$$

Taking weak limits in (3.15) yields

$$\begin{aligned}
&\int_0^T \int_Q \int_\Omega \left(\Psi^*(\partial_t p) + \Psi(z - Bp)\right) \\
&\qquad \le \int_0^T \int_Q f \cdot \partial_t (u - U) + \int_0^T \int_Q \int_\Omega z : \nabla \partial_t U - \frac{1}{2}\left(\int_Q \int_\Omega p : (Bp) + \int_Q \int_\Omega z : (Cz)\right)\Bigg|_0^T.
\end{aligned}$$

Relations (3.13) and (3.14) allow to perform the calculations of (3.15) also for the limit functions. We obtain from the last inequality

$$\int_0^T \int_Q \int_\Omega \left(\Psi^*(\partial_t p) + \Psi(z - Bp)\right) \le \int_0^T \int_Q \int_\Omega \partial_t p : (z - Bp).$$

The Fenchel inequality of Lemma 2.3 (iii) yields $\partial_t p : (z - Bp) \le \Psi^*(\partial_t p) + \Psi(z - Bp)$ pointwise. We can therefore conclude from the Fenchel equality

$$\partial_t p \in \partial\Psi(\sigma - Bp). \tag{3.16}$$

Relations (3.13) and (3.16) imply that z is defined as in the definition of Σ, hence $\int_\Omega z(t,x,.) = \Sigma(\nabla^s u)(t,x,.)$ for every $t \in [0,T]$ and a.e. $x \in Q$. Therefore, (3.14) is equivalent with (3.12) and the theorem is shown. □

3.4. Conclusion of the proof

We can now conclude the proof of our main result, Theorem 1.3. Theorem 3.4 implies that Σ of (2.5) is admissible. Theorem 3.3 yields that, for almost every $\omega \in \Omega$, the coefficients $C_{\varepsilon,\omega}(x)$, $B_{\varepsilon,\omega}(x)$, $\Psi_{\varepsilon,\omega}(\sigma;x)$ allow averaging with limit operator Σ. We can therefore apply Theorem 1.13 and obtain

$$\begin{aligned}
&u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(0,T;H^1(Q;\mathbb{R}^d)) \\
&p^\varepsilon \rightharpoonup \Pi(\nabla^s u), \quad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u) \quad \text{weakly in } H^1(0,T;L^2(Q;\mathbb{R}^{d\times d})),
\end{aligned}$$

where u is the unique weak solution to the homogenized problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f$$

with boundary condition U as in Definition 1.12. Theorem 1.3 is shown.

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