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ON THE STRUCTURE OF MULTIFACTOR OPTIMAL PORTFOLIO STRATEGIES

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Abstract. The paper studies problem of optimal portfolio selection. It is shown that, under some mild conditions, near optimal strategies for investors with different performance criteria can be constructed using a limited number of fixed processes (mutual funds), for a market with a larger number of available risky stocks. This implies dimension reduction for the optimal portfolio selection problem: all rational investors may achieve optimality using the same mutual funds plus a saving account. This result is obtained under mild restrictions for the utility functions without any assumptions on regularity of the value function. The proof is based on the method of dynamic programming applied indirectly to some convenient approximations of the original problem that ensure certain regularity of the value functions. To overcome technical difficulties, we use special time dependent and random constraints for admissible strategies such that the corresponding HJB (Hamilton−Jacobi−Bellman) equation admits "almost explicit" solutions generating near optimal admissible strategies featuring sufficient regularity and integrability.

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1. Introduction

We study an optimal portfolio selection problem. These dynamic portfolio selection problems are usually studied in the framework of optimal stochastic control; see, *e.g.*, books of Krylov [16] and Fleming and Rishel [12]. There are many works devoted to different modifications of the portfolio problem (see, *e.g.*, Merton [24] and review in Karatzas and Shreve [14]). To suggest a strategy, one needs to forecast future market scenarios (or the probability distributions). Unfortunately, the nature of financial markets is such that the choice of a hypothesis about the future distributions is not easy to justify.

To overcome limited predictability of the market parameters, some special methods were developed for the financial models. One of these tools is the so-called Mutual Fund Theorem which, in the classical version, says that the distribution of the risky assets in the optimal portfolio does not depend on the investor's risk preferences (or performance criteria). This implies dimension reduction for the optimal portfolio selection problem: all

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rational investors may achieve optimality using the same mutual fund plus a saving account. Clearly, calculation of the optimal portfolio is easier in this case. So far, this property has no analog in classical stochastic control.

The Mutual Fund Theorem was established first for the discrete time single period mean variance portfolio selection problem, *i.e.*, for the problem with quadratic criteria (see e.g. [13], Chap. 4). This result was a cornerstone of the modern portfolio theory; in particular, the Capital Assets Pricing Model (CAPM) is based on it. For the multi-period discrete time setting, some versions of the Mutual Fund Theorem were obtained so far for problems with quadratic criteria only (Li and Ng [18], Dokuchaev [7]). For the continuous time setting, the Mutual Fund Theorem was obtained for portfolio selection problems for more general utilities. The Mutual Fund Theorem holds for utility functions $U(x) = \delta^{-1}x^\delta$ and $U(x) = \log(x)$ for the case of random totally unhedgeable coefficients, *i.e.*, for the case of random coefficients independent on the driving Brownian motion (Karatzas and Shreve [14]). It is also known that the Mutual Fund Theorem does not hold for power utilities if the coefficients depend on the driving Wiener process (see, *e.g.*, Brennan [2]). Khanna and Kulldorff [15] proved that the Mutual Fund Theorem theorem holds for a general utility function $U(x)$ in the case of non-random coefficients, and for a setting with consumption. Dokuchaev [9] extended this result on the case of random totally unhedgeable coefficients. Lim and Zhou [20] found some cases where the Mutual Fund Theorem holds for problems with quadratic criteria. Dokuchaev and Haussmann [3] found that the Mutual Fund Theorem holds if the scalar value $\int_0^T |\theta(t)|^2 \mathrm{d}t$ is non-random, where $\theta(t)$ is the market price of the risk process. In maximin setting, the Mutual Fund Theorem was established in Dokuchaev ([6], 2013). Schachermayer *et al.* [24] found sufficient conditions for the Mutual Fund Theorem expressed via replicability of the European type claims $F(Z(T))$, where $F(\cdot)$ is a deterministic function and $Z(t)$ is the discounted wealth generated by the log-optimal discounted wealth process. The required replicability has to be achieved by trading of the log-optimal mutual fund with discounted wealth $Z(t)$. It can be summarized that the Mutual Fund Theorem was established so far only for several special optimal portfolio selection problems and special market models.

It appears that there are market models where the classical Mutual Fund Theorem does not hold but the following relaxed version of this theorem holds: the optimal portfolios with different risk preferences can be constructed using μ mutual funds only for a market with $n > \mu$ risky stocks. This μ can be regarded as a dimension of the market; in this sense, a market is one dimensional if the classical Mutual Fund Theorem holds. So far, this feature was studied for few special settings only. In particular, single period CAPM models models were studied in a setting where a number of mutual funds were used to compensate skewness and consumption (so-called three-moment CAPM, multi-beta models, or multifactor CAPM); see, *e.g.*, Merton [21], Poncet [23], Fama [10], Nguyen *et al.* [22]. A diffusion model where optimality can be achieved for strategies using two mutual funds was discussed in Ingersoll [13], Chapter 13. In this book, the optimal strategy was expressed *via* solution of the Hamilton−Jacobi−Bellman (HJB) equation (the Bellman equation) for the value function as a quotient of partial derivatives of the value function. However, the existence and regularity of these derivatives is difficult to ensure, since the underlying HJB equation is degenerate ... In addition, it is difficult to ensure that the resulting stochastic process representing the strategy satisfies reasonable conditions on the growths such as integrability. Moreover, it may happen that the quotient found from the HJB equation is not smooth enough to ensure solvability of the closed loop Itô equations for the wealth process. By these reasons, existence, admissibility, and regularity of the two mutual funds strategy was not yet established. In theory, this could be overcome by an alternative martingale approach mentioned briefly in Remark 3.7 in Schachermayer *et al.* [24]; however, this approach requires replicability of certain claims and does not cover a model with non-hedgeable Wiener processes.

In this paper, we consider a diffusion market model with non-hedgeable Wiener processes and non-hedgeable factors such that the classical Mutual Fund Theorem does not hold. We consider a market with n stocks, with $n + N$ independent driving Wiener processes, including N non-hedgeable Wiener processes, and with a large number of non-hedgeable factor processes defining the evolution of the market prices. We found that, for a wide class of utilities, a near optimal (*i.e.*, ε-optimal) portfolio can be constructed using $\mu < n$ mutual funds only (Thm. 3.2 below). The number μ is defined by the number of the non-hedgeable factors correlated with the stock prices, or by the complexity of correlations in the model, rather than by the number of stocks or by the total number of random factors.

The main result (Thm. 3.2) is obtained under very mild restrictions for the utility functions without any assumptions on regularity of the value function. The proof is based on the method of dynamic programming applied indirectly to some convenient approximations of the original problem that ensure certain regularity of the value functions; the range for the strategies is approximated by bounded sets, and the utility function is approximated by smooth and bounded functions. This approach has some obstacles: the HJB equations with bounded admissible controls does not allow explicit solutions. To overcome these difficulties, we use special time dependent and random constraints for admissible strategies such that the corresponding HJB equation admits "almost explicit" solutions generating near optimal admissible strategies featuring sufficient regularity and integrability.

The results of this paper were presented in Quantitative Methods in Finance conference in Sydney, Australia, 2010.

2. Model setting

We are given a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure that describes a prior probability distribution.

We assume that the market evolution is driven by a pair of standard independent Wiener processes $w(\cdot) = (w_1(\cdot), \ldots, w_n(\cdot))$ and $\widehat{w}(\cdot) = (\widehat{w}_1(\cdot), \ldots, \widehat{w}_N(\cdot))$ with the values in R^n and R^N respectively. Let $\mathcal{F}_t$ be the filtration generated by $(w(t), \widehat{w}(t))$.

We consider the market model similar to the model used in Dokuchaev ([6], 2013). We assume that the market consists of a risk free asset or bank account with price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, where $n < +\infty$ is given.

We assume that

$$B(t) = B(0) \exp\left(\int_0^t r(s)\mathrm{d}s\right), \tag{2.1}$$

where $r(t)$ is a $\mathcal{F}_t$-adapted random process of the risk-free interest rate (or the short rate). We assume that $B(0) = 1$. The process $B(t)$ will be used as numeraire.

The prices of the stocks evolve according to

$$\mathrm{d}S_i(t) = S_i(t)\left(a_i(t)\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(t)\mathrm{d}w_j(t)\right), \quad t > 0, \tag{2.2}$$

where $a_i(t)$ are the appreciation rates, $\sigma_{ij}(t)$ are the volatility coefficients. The initial price $S_i(0) > 0$ is a given non-random constant.

We assume that $r(t)$, $a_i(t)$, and $\sigma_{ij}(t)$ are uniformly bounded $\mathcal{F}_t$-adapted measurable random processes.

We denote by $S(t) \triangleq (S_1(t), \ldots, S_n(t))^\top$ and $a(t) = (a_1(t), \ldots, a_n(t))^\top$ the corresponding vector valued processes with the values in $\mathbf{R}^n$, and a matrix process $\sigma(t) \triangleq \{\sigma_{ij}(t)\}_{i,j=1}^n$ with the values in $\mathbf{R}^{n\times n}$.

Let $\widetilde{S}(t) = (\widetilde{S}_1(t), \ldots, \widetilde{S}_n(t))^\top \triangleq B(t)^{-1}S(t)$ be the vector of discounted prices. Let $\widetilde{a}(t) = a(t) - r(t)\mathbf{1}$, where $\mathbf{1} \triangleq (1, 1, \ldots)^\top \in \mathbf{R}^n$.

We assume that the inverse matrix $\sigma(t)^{-1}$ is defined and bounded and $r(t) \geq 0$.

Wealth and strategies

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$, $X(0) = X_0$. Let $\widetilde{X}(t) \triangleq B(t)^{-1}X(t)$ be the discounted wealth.

Let the process $P_0(t)$ be the wealth invested in the bond, and let $P_i(t)$ be the wealth invested in the ith stock, $i = 1, \ldots, n$. The values of P_i can be negative, in the case of a short position in ith asset.

Let $\pi_i(t) = B(t)^{-1}P_i(t)$. In this case, the process $\pi_0(t)$ represents the quantity of the bonds, or the discounted wealth invested in the bond, $\pi_i(t)$, $i \geq 1$, is the discounted wealth invested in the ith stock.

We assume that

$$\pi_0(t) + \sum_{i=1}^{n} \pi_i(t) = \widetilde{X}(t). \tag{2.3}$$

We denote by π the vector process $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^\top$, $t \geq 0$.

The portfolio is said to be self-financing, if

$$\mathrm{d}X(t) = \sum_{i=1}^{n} \frac{P_i(t)}{S_i(t)} \mathrm{d}S_i(t) + \frac{P_0(t)}{B(t)} \mathrm{d}B(t).$$

It can be rewritten as

$$\mathrm{d}X(t) = \sum_{i=1}^{n} \pi_i(t)^\top \widetilde{S}_i(t)^{-1} \mathrm{d}S_i(t) + \pi_0(t)\mathrm{d}B(t).$$

It follows that for such portfolios

$$\mathrm{d}\widetilde{X}(t) = \sum_{i=1}^{n} \pi_i(t)\widetilde{S}_i(t)^{-1}\mathrm{d}\widetilde{S}_i(t) = \pi(t)^\top(\widetilde{a}(t)\mathrm{d}t + \sigma(t)\mathrm{d}w(t)), \tag{2.4}$$

so π alone suffices to specify the portfolio; see, *e.g.*, Dokuchaev [5], p. 78.

Let D be the range of the process $\widetilde{X}(t)$. We will consider two settings: with $D = (0, +\infty)$ and with $D = \mathbf{R}$.

We consider a class Σ of admissible strategies consisting of all $\mathcal{F}_t$-adapted processes $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))$: $[0, T] \times \Omega \to \mathbf{R}^n$ such that the following holds:

- If $D = \mathbf{R}$ then $\sup_{t,\omega} |\pi(t,\omega)| < +\infty$;
- If $D = (0, +\infty)$ then $\sup_{t,\omega} |\pi(t,\omega)|\widetilde{X}(t)^{-1} < +\infty$.

By these definitions, if $D = (0, +\infty)$, then $X(t) > 0$ for any $\pi \in \Sigma$.

3. The main result

Let $T > 0$ and $X_0 > 0$ be given.

Let $\mathcal{U}$ be the set of all continuous functions $U(\cdot) : D \to \mathbf{R}$ such that if $D = \mathbf{R}$ then there exists $c_1 > 0$ and $c > 0$ such that $|U(x)| \leq c_1(1 + |x|)^c$ for all x. If $D = (0, +\infty)$, then we assume that $|U(x)| \leq c_1(|x|^{-c} + |x|^c)$ for some $c_1 > 0$ and $c > 0$.

The case where $D = (0, +\infty)$ is included with the purpose to allow important utility functions with singularity at $x = 0$ such as $U(x) = \ln x$ or $U(x) = -1/x$.

For the sake of generality, we do not exclude non-differentiable or non-concave U. However, discontinuous functions are not allowed. In particular, step functions used in Dokuchaev and Zhou [4] for the so-called goal achieving problems are not allowed. In addition, our setting does not cover utilities with the exponential growth such as $U(x) = -\mathrm{e}^{-cx}$ for $D = \mathbf{R}$, $c > 0$.

For $U(\cdot) \in \mathcal{U}$, set

$$V(\pi) \triangleq \mathbf{E}U(\widetilde{X}(T)).$$

We will study the problem

$$\text{Maximize} \quad V(\pi) \quad \text{over} \quad \pi(\cdot) \in \Sigma. \tag{3.1}$$

Starting from now, we assume that the coefficients $(\widetilde{a}, \sigma)$ are such that there exist integers $m \geq 0, M \geq 0, N \geq 0$ and continuous functions

$$\mathbf{a} : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^n, \qquad \mathbf{v} : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^{n\times n}$$

and functions

$$\begin{aligned} f^\eta : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^m, \qquad & \beta^\eta : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^{m\times n}, \\ & \widehat{\beta}^\eta : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^{m\times N}, \\ f^\zeta : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^M, \qquad & \widehat{\beta}^\zeta : \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^{M\times N} \end{aligned}$$

such that

$$\widetilde{a}(t) = \mathbf{a}(\eta(t), \zeta(t), t), \qquad \sigma(t) = \mathbf{v}(\eta(t), \zeta(t), t),$$

where $\eta(t)$ and $\zeta(t)$ are stochastic processes that take values in $\mathbf{R}^m$ and $\mathbf{R}^M$ respectively and such that they satisfy Itô equations

$$\begin{aligned} \mathrm{d}\eta(t) &= f^\eta(\eta(t), \zeta(t), t)\mathrm{d}t + \beta^\eta(\eta(t), \zeta(t), t)\mathrm{d}w(t) + \widehat{\beta}^\eta(\eta(t), \zeta(t), t)\mathrm{d}\widehat{w}(t), \\ \mathrm{d}\zeta(t) &= f^\zeta(\eta(t), \zeta(t), t)\mathrm{d}t + \widehat{\beta}^\zeta(\eta(t), \zeta(t), t)\mathrm{d}\widehat{w}(t). \end{aligned}$$

Here $\widehat{w}(\cdot)$ is a Wiener process with values in $\mathbf{R}^N$ that is independent of $w(\cdot)$.

The cases where $m = 0$, $M = 0$, or $N = 0$, are not excluded; they represent models where the corresponding vector processes are absent.

We denote by $|\cdot|$ the Euclidean norm for vectors, the Frobenius norm for matrices, and the similar norm for elements of the spaces formed as Cartesian products of spaces of matrices or vectors such as $\mathbf{R}^n \times \mathbf{R}^{n\times n}$, *etc.*

We assume that the following conditions are satisfied:

- There exists a constant $C > 0$ such that

$$\begin{aligned} &|F(y_1, z_1, t) - F(y_2, z_2, t)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \\ &|F(y, z, t)| \leq C(1 + |y| + |z|) \qquad \forall y_1, y_2, z_1, z_2, y, z, t, \end{aligned}$$

  where $F = (\mathbf{a}, \mathbf{v}, f^\eta, \beta^\eta, \widehat{\beta}^\eta, f^\zeta, \widehat{\beta}^\zeta)$.
- We assume that there exists a constant $\bar{c} > 0$ such that $A(y,z,t)A(y,z,t)^\top \geq \bar{c} I_{m+M}$, where I_{m+M} is the unit matrix in $\mathbf{R}^{(m+M)\times(m+M)}$, and where the matrix $A \in \mathbf{R}^{(m+M)\times(n+N)}$ is formed as

$$A = \begin{pmatrix} \beta^\eta & \widehat{\beta}^\eta \\ 0_{M\times n} & \widehat{\beta}^\zeta \end{pmatrix}.$$

Definition 3.1. Let $L \geq 1$ be an integer. Consider a set of $\mathcal{F}_t$-adapted processes $\mathcal{M}_1(t), \ldots, \mathcal{M}_L(t)$ with the values in $\mathbf{R}^n$. Let $\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_L}$ be the class of all processes $\pi(\cdot) \in \Sigma$ such that there exist $\mathcal{F}_t$-adapted one-dimensional processes $\{\nu_k(t),\ k = 1, \ldots, L\}$ such that

$$\pi(t) = \sum_{k=1}^{L} \nu_k(t)\mathcal{M}_k(t). \tag{3.2}$$

Let $\mathcal{Q} = (\sigma\sigma^\top)^{-1}$, let β^η_k be the kth column of the matrix β^η, and let $\mu = \min(m+1, n)$.

Theorem 3.2. *Consider a set* $\{\mathcal{M}_1(t),\dots,\mathcal{M}_\mu(t)\}$ *of* $\mathcal{F}_t$*-adapted processes with values in* $\mathbf{R}^n$ *defined as*

$$\mathcal{M}_k(t) = (\sigma(t)^\top)^{-1}\beta_k^\eta(\eta(t),\zeta(t),t), \quad k \le \mu-1,$$
$$\mathcal{M}_\mu(t) = \mathcal{Q}(t)\widetilde{a}(t).$$

For this set, for any $U(\cdot)\in\mathcal{U}$,

$$\sup_{\pi\in\Sigma} V(\pi) = \sup_{\pi\in\Sigma_{\mathcal{M}_1,\dots,\mathcal{M}_\mu}} V(\pi). \tag{3.3}$$

For the special case of $\mu = 1$, $m = 0$, $N = 0$ (*i.e.*, where the corresponding vector processes are absent), Theorem 3.2 represents the relaxed version of the classical Mutual Fund Theorem obtained in Khanna and Kulldorf [15] in a setting with consumption and with less general utility functions. For the case where $\mu = 1$, $m = 0$, $N > 0$, Theorem 3.2 represents a version of the Mutual Fund Theorem from Dokuchaev [9]. A special case where $N = 0$ and $M = 0$ corresponds to the model mentioned in Remark 3.7 in Schachermayer *et al.* [24]. A special case where $m = 1$ and $M = 0$ and where the value function is regular enough corresponds to the model from Ingersoll [13], Chapter 13.

4. The implications of Theorem 3.2

Let us discuss the implications and economic interpretation of Theorem 3.2. Representation (3.2) can be interpreted as a distribution of the stock portfolio among μ mutual funds; each vector $\mathcal{M}_k(t)$ can be interpreted as a distribution of the stock portfolio for a mutual fund. Since the selection of $\{\mathcal{M}_k(t)\}$ is independent on $U(\cdot)$, Theorem 3.2 represents a relaxed version of the Mutual Fund Theorem.

The statement of Theorem 3.2 can be reformulated as follows: there exist *near optimal* (ε-optimal, suboptimal) strategies in the class $\Sigma_{\mathcal{M}_1,\dots,\mathcal{M}_\mu}$, meaning that, for any $U(\cdot)\in\mathcal{U}$ and any $\varepsilon > 0$, there exists a strategy $\pi_{U,\varepsilon}\in\Sigma_{\mathcal{M}_1,\dots,\mathcal{M}_\mu}$ represented as (3.2) such that

$$V(\pi_{U,\varepsilon}) \ge \sup_{\pi\in\Sigma} V(\pi) - \varepsilon.$$

This has a clear economic interpretation: all investors with different utilities can construct near optimal strategies by investing in μ mutual funds only, even if $n \gg \mu$, $M \gg \mu$, and $N \gg \mu$.

In Theorem 3.2, the vector $\mathcal{M}_\mu(t)$ represents the so-called log-optimal portfolio; sometimes, it is called the mean-variance portfolio. For $k < \mu$, the vectors $\mathcal{M}_k(t)$ represent some hedging portfolios used to compensate correlations in the market.

The processes $\nu_k(t) = \nu_{U,\varepsilon,k}(t)$ for the near optimal strategies presented in (3.2) depends on $U(\cdot)$. These processes are expressed in the proof of Theorem 3.2 below *via* derivatives of the smooth approximations of the value functions that are solutions of some auxiliary HJB equations. These equations selected such that their solutions have the required regularity. We emphasize that the statement of Theorem 3.2 itself does not require solvability and regularity of the HJB equations.

Under very mild conditions on the utility functions, Theorem 3.2 allows to reduce the original investment problem for a market with n tradable risky assets to an equivalent problem for a market with μ tradable assets. Let us show this. Consider a matrix process $\mathcal{M}(t) = (\mathcal{M}_1(t),\dots,\mathcal{M}_\mu(t))$ with the values in $\mathbf{R}^{\mu\times n}$ formed from the rows $\mathcal{M}_k(t)^\top$. Let $\widetilde{a}_\xi(t) = \mathcal{M}(t)\widetilde{a}(t)$ and $\sigma_\xi(t) = \mathcal{M}(t)\sigma(t)$. Let us consider a process $\xi(t) = \{\xi_k(t)\}_{k=1}^{\mu}$ with the values in $\mathbf{R}^\mu$ defined by the equation

$$\mathrm{d}\xi(t) = \Xi(t)(\widetilde{a}_\xi(t)\mathrm{d}t + \sigma_\xi(t)\mathrm{d}w(t)), \quad \xi_k(0) = 1 \quad k = 1,\dots,\mu.$$

Here $\Xi(t)$ is a diagonal matrix in $\mathbf{R}^{\mu\times\mu}$ with the diagonal elements $\Xi_{kk}(t) = \xi_k(t)$, $k = 1,\dots,\mu$.

Let $\nu(t) = \{\nu_k(t)\}_{k=1}^{\mu}$ be an $\mathcal{F}_t$-adapted process with the values in $\mathbf{R}^{\mu}$. Let $\pi(t) = \sum_{k=1}^{\mu} \nu_k(t)\xi_k(t) = \mathcal{M}(t)^\top \nu(t)$, and let $\widetilde{X}(t)$ be the corresponding discounted wealth. It follows from the definitions that

$$\mathrm{d}\widetilde{X}(t) = \nu(t)^\top [\widetilde{a}_\xi(t) + \sigma_\xi(t)\mathrm{d}w(t)] = \nu(t)^\top \Xi(t)^{-1}\mathrm{d}\xi(t).$$

Comparing this with (2.4), we obtain that $\nu(t)$ can be considered as a portfolio self-financing strategy for a market with the discounted prices $\{\xi_k(t)\}$. Therefore, Theorem 3.2 allows to replace the original investment problem for a market with n stocks by an equivalent problem for a market with μ stocks. This could be useful if $\mu \ll n$.

Remark 4.1. It can be shown that Theorem 3.2 implies that $|\theta(t)| = |\theta_\xi(t)|$, where $\theta(t) = \sigma(t)^{-1}\widetilde{a}(t)$ is the market price of risk of the original market, and where $\theta_\xi(t)$ is the market price of risk for the reduced market defined as $\theta_\xi(t) = \widehat{\sigma}_\xi(t)^{-1}\widetilde{a}_\xi(t)$, where $\widehat{\sigma}_\xi(t)$ is a $\mu \times \mu$-dimensional matrix such that $\widehat{\sigma}_\xi(t)\widehat{\sigma}_\xi(t)^\top = \mathcal{M}(t)\sigma(t)\sigma(t)^\top \mathcal{M}(t)^\top$. Clearly, if $n = \mu$ then the equality $|\theta(t)| = |\theta_\xi(t)|$ holds for any non-degenerate matrix $\mathcal{M}(t)$. However, it is interesting to note that, for $n > \mu$, this equality requires that $\mathcal{M}(t)$ contains a row proportional to $(\mathcal{Q}^{-1}\widehat{a}(t))^\top$ (*i.e.*, such as described in Thm. 3.2); otherwise, simple counterexamples can be found easily. This illustrates again a special role of the log-optimal portfolio $\mathcal{M}_\mu$.

Some examples.

It can be noted that our model covers the case where $(\widetilde{a}(t), \sigma(t)) = F(\widetilde{S}(t), \eta(t), \zeta(t), t)$, for some deterministic function $F : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^M \times [0, T] \to \mathbf{R}^n \times \mathbf{R}^{n\times n}$. It suffices to include the vector $\widetilde{S}(t)$ or some of its components as a part of the vector $\eta(t)$.

Example 4.2. Consider a market model where the volatility and the appreciation rate for stock prices depend on a market index or indicator defined by all prices presented in this market. Let $m = 1$ and let the market index be $\eta(t) = F(S(t))$, for some deterministic function $F : \mathbf{R}^n \to \mathbf{R}$, $n > 1$; For instance, one can consider $\eta(t) = \sum_{i=1}^{n} S_i(t)$. Then $\mu = 2$. By Theorem 3.2, a suboptimal strategy can be achieved by investing in two mutual funds for all risk preferences.

Example 4.3. Consider a market model such that the volatilities and the appreciation rates for stock prices depend on a set of major market indices such as Dow Jones, FTSE, Hang Seng, *etc.* Further, assume that the movement of the stocks $S_1, \ldots, S_n$ has some impact on one particular index, say, on Hang Seng index. For instance, assume that these stocks are included in this index. This model can be described as follows: the vector $(\eta(t), \zeta(t))$ represents the set of market indexes, $m = 1$, and the one dimensional process η represents the Hang Seng index. In this case, $\mu = 2$. By Theorem 3.2, a near optimal strategy can be achieved by investing in two mutual funds for all risk preferences.

Example 4.4. In the previous example, assume that the dynamics of the stocks $S_1, \ldots, S_n$ affects m market indexes, say, Dow Jones, Hang Seng, and some other indexes. In this case, we can use the model with this m and with $\mu = \min(m+1, n)$. By Theorem 3.2, a near optimal strategy can be achieved by investing in μ mutual funds for all risk preferences.

5. Proofs

5.1. Reformulation with constrained strategies

Definition 5.1. Let $K > 0$. Let $\Sigma(K)$ be the class of all strategies $\pi(\cdot) \in \Sigma$ such that

- If $D = \mathbf{R}$ then $\sup_{t,\omega} \pi(t,\omega)^\top \sigma(t,\omega)\sigma(t,\omega)^\top \pi(t,\omega) \le K$; and
- If $D = (0, +\infty)$ then $\sup_{t,\omega} \pi(t,\omega)^\top \sigma(t,\omega)\sigma(t,\omega)^\top \pi(t,\omega)\widetilde{X}(t)^{-1} \le K$.

In addition, let $\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_L}(K) = \Sigma_{M_1,\ldots,\mathcal{M}_L} \cap \Sigma(K)$, for a set $\mathcal{M}_1, \ldots, \mathcal{M}_L$ of $\mathcal{F}_t$-adapted processes with values in $\mathbf{R}^n$.

Clearly, $\Sigma = \cup_{K>0}\Sigma(K)$ and $\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_L} = \cup_{K>0}\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_L}(K)$. Therefore, it suffices to prove that

$$\sup_{\pi\in\Sigma(K)} V(\pi) = \sup_{\pi\in\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_\mu}(K)} V(\pi) \quad \forall K>0. \tag{5.1}$$

In this case, (5.1) implies (3.3).

Further, Theorem 3.2 holds if $m+1>n$. In this case, it suffices to take processes $\mathcal{M}_k(t) = (0,\ldots,0,1,0,\ldots,0)$, with kth component equal to one, $k\le n$. Obviously, any $\pi(t)$ can be represented as a linear combination of these vectors. Therefore, it suffices to assume that $\mu = m+1<n$.

Let us prove (5.1). Starting from now, we assume that $K>0$ is given and $\mu=m+1<n$.

5.2. Some auxiliary lemmas

Let $\Delta(y,z,t) \stackrel{\Delta}{=} \{u\in\mathbf{R}^n:\ u^\top \mathbf{v}(y,z,t)\mathbf{v}(y,z,t)^\top u\le K\}$.

Let a matrix $\widehat{A}(u,y,z,t)$ that takes values in $\mathbf{R}^{(1+m+M)\times(n+N)}$ be defined as

$$\widehat{A}(u,y,z,t) = \begin{pmatrix} u^\top\mathbf{v} & 0_{1\times N} \\ \beta^\eta & \widehat{\beta}^\eta \\ 0_{M\times n} & \widehat{\beta}^\zeta \end{pmatrix}. \tag{5.2}$$

Lemma 5.2. *Let $\Gamma=\{\xi\in\mathbf{R}^{1+m+M}:\ |\xi|=1\}$. For any (y,z,t),*

$$\inf_{\xi\in\Gamma}\sup_{u\in\Delta(y,z,t)} \xi^\top\widehat{A}(u,y,z,t)\widehat{A}(u,y,z,t)^\top\xi>0.$$

Proof of Lemma 5.2. It suffices to replace the supremum over u by the supremum over $u=\widehat{u}$ such that $\widehat{u}$ is on the boundary of Δ and $\beta^\eta\mathbf{v}^\top\widehat{u}=0$. Clearly, this $\widehat{u}$ exists since $n>1$ and $m<n-1$. In this case,

$$\widehat{A}(\widehat{u},y,z,t)\widehat{A}(\widehat{u},y,z,t)^\top = \begin{pmatrix} \widehat{u}^\top\mathbf{v}\mathbf{v}^\top\widehat{u} & 0_{1\times(m+M)} \\ 0_{(m+M)\times 1} & AA^\top\end{pmatrix} = \begin{pmatrix} K & 0_{1\times(m+M)} \\ 0_{(m+M)\times 1} & AA^\top\end{pmatrix}.$$

By the assumptions on A, it follows that there exists a constant $c_1>0$ such that

$$\widehat{A}(\widehat{u},y,z,t)\widehat{A}(\widehat{u},y,z,t)^\top \ge c_1 I_{1+m+M} \qquad \forall y,z,t,$$

where I_{1+m+M} is the unit matrix in $\mathbf{R}^{(1+m+M)\times(1+m+M)}$. Hence

$$\sup_{u\in\Delta(y,z,t)} \widehat{A}(u,y,z,t)\widehat{A}(u,y,z,t)^\top \ge c_1 I_{1+m+M} \qquad \forall y,z,t.$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let $\alpha\in\mathbf{R}$, $\mathrm{b}\in\mathbf{R}^n$, $c>0$ be given. Consider the problem*:

$$\textit{Maximize}\quad -\alpha|p|^2+p^\top\mathrm{b}\quad \textit{over}\quad p\in\mathbf{R}^n\quad \textit{subject to}\quad |p|^2\le c. \tag{5.3}$$

Then an optimal solution p exists and the following holds:

(1) *If $\alpha<0,\mathrm{b}=0$, then any p such that $|p|^2=c$ is optimal.*
(2) *If either $\alpha\ge 0$ or $\alpha<0,\mathrm{b}\ne 0$, then the optimal solution can be selected such that there exists $k=k(\alpha,\mathrm{b},c)\in\mathbf{R}$ such that $p=k\mathrm{b}$.*

Proof. Existence of optimal p follows from the fact that the domain $\{p: \ |p| \le c\}$ is compact. Statement (i) is obvious. Let us prove statement (ii). If $\alpha = 0$ and $\mathrm{b} \neq 0$ then $p = \sqrt{c}\mathrm{b}/|\mathrm{b}|$ is optimal.

If $\alpha = 0$ and $\mathrm{b} = 0$ then $p = \mathrm{b} = 0$ is optimal along with all other admissible p.

Let $\alpha > 0$. It suffices to consider the case $\alpha = 1/2$ only.

Clearly, the maximum of the function $g(p) = -|p|^2/2 + p^\top \mathrm{b}$ is achieved for $p = \mathrm{b}$. It follows that if $|b|^2 \le c$ then $p = \mathrm{b}$ is an optimal solution.

If $|\mathrm{b}|^2 > c$ then $p = \sqrt{c}\mathrm{b}/|\mathrm{b}|$ is an optimal solution. It can be seen from the following:

$$\max_{p:|p|^2\le c} g(p) = \max_{s\in[0,\sqrt{c}]} \max_{p:|p|=s} g(p).$$

Obviously, $\max_{p:|p|=s} g(p) = -s^2/2 + |\mathrm{b}|s$ and it is achieved for $p(s) = s\mathrm{b}/|\mathrm{b}|$. The maximum of $-s^2/2 + |\mathrm{b}|s$ over $s \in [0, \sqrt{c}]$ is achieved for $s = \sqrt{c}$. Hence $p = \sqrt{c}\mathrm{b}/|\mathrm{b}|$ is an optimal solution for this case.

Finally, let $\alpha < 0$ and $\mathrm{b} \neq 0$. Clearly, $p = \sqrt{c}\mathrm{b}/|\mathrm{b}|$ is optimal again in this case. This completes the proof of the Lemma 5.3. □

5.3. Near optimality of constrained Markov strategies

Portfolio selection problem (3.1) can be rewritten as

$$\begin{aligned}
&\text{Maximize} \quad \mathbf{E}U(\widetilde{X}(T)) \quad \text{over} \quad \pi(\cdot) \in \Sigma \quad \text{subject to}\\
&\mathrm{d}\widetilde{X}(t) = \pi(t)^\top[\mathbf{a}(\eta(t),\zeta(t),t)\mathrm{d}t + \mathbf{v}(\eta(t),\zeta(t),t)\mathrm{d}w(t)],\\
&\mathrm{d}\eta(t) = f^\eta(\eta(t),\zeta(t),t)\mathrm{d}t + \beta^\eta(\eta(t),\zeta(t),t)\mathrm{d}w(t) + \widehat{\beta}^\eta(\eta(t),\zeta(t),t)\mathrm{d}\widehat{w}(t),\\
&\mathrm{d}\zeta(t) = f^\zeta(\eta(t),\zeta(t),t)\mathrm{d}t + \widehat{\beta}^\zeta(\eta(t),\zeta(t),t)\mathrm{d}\widehat{w}(t),
\end{aligned} \tag{5.4}$$

given $X(0), \eta(0), \zeta(0)$.

It can be seen that, to Markovianize the problem, it suffices to use the state variables $\widetilde{X}(t), \eta(t)$, and $\zeta(t)$.

The following is an adaptation of Definition 3.1.3 from Krylov [16], p. 131.

Definition 5.4. Let Σ_M be the class of all $\mathcal{F}_t$-adapted processes $\pi(\cdot) \in \Sigma$ such that there exists a measurable function $u : \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^M \times [0,T] \to \mathbf{R}^n$ such that

$$\begin{aligned}
\pi(t) &= u(\widetilde{X}(t),\eta(t),\zeta(t),t) &&\text{if} \quad D = \mathbf{R},\\
\pi(t) &= u(\widetilde{X}(t),\eta(t),\zeta(t),t)\widetilde{X}(t) &&\text{if} \quad D = (0,+\infty).
\end{aligned}$$

A process $\pi(\cdot) \in \Sigma_M$ is said to be a *Markov* strategy.

Remark 5.5. Note that, by the definition of a Markov strategy, the function $u(\cdot)$ is such that the closed-loop solution $(\widetilde{X}(t),\eta(t),\zeta(t))$ of Ito equation exists in the class of $\mathcal{F}_t$-adapted process. Therefore, it may happen that a measurable and bounded function $u(\cdot)$ does not define a Markov strategy.

Let $\Sigma_M(K) = \Sigma_M \cap \Sigma(K)$. Clearly, $\Sigma_M = \cup_{K>0}\Sigma_M(K)$.

5.4. The proof of Theorem 3.2

Note that the matrix A defined by (5.2) represents the diffusion coefficient for the system of Ito equations in (5.4) for Markov strategies.

Let us first prove the theorem for some special cases.

Proof for bounded U, $U'(x)$, $U''(x)$ and for $D=\mathbf{R}$

Let us assume that $D=\mathbf{R}$ and the function U is bounded in D together with the derivatives $U'(x)$ and $U''(x)$. Set

$$J(x,y,z,t) \triangleq \sup_{\pi(\cdot)\in\Sigma(K)} \mathbf{E}\Big\{U(\widetilde{X}(T))\Big|(\widetilde{X}(t),\eta(t),\zeta(t))=(x,y,z)\Big\}. \tag{5.5}$$

It follows from Lemma 5.2 and Theorem 5.2.5 from Krylov [16], p. 225, that

$$J(x,y,z,t) \triangleq \sup_{\pi(\cdot)\in\Sigma_M(K)} \mathbf{E}\left\{U(\widetilde{X}(T))\Big|(\widetilde{X}(t),\eta(t),\zeta(t))=(x,y,z)\right\}. \tag{5.6}$$

The Bellman equation formally satisfied by the value function $J=J(x,y,z,t)$ is

$$G(t,x,y,z,J'_t,J'_\xi,J''_{\xi\xi})=0, \qquad J(x,y,z,T)=U(x). \tag{5.7}$$

Here J'_ξ is the gradient of J with respect to the vector $\xi=(x,y,z)$, $J''_{\xi\xi}$ is the matrix second order derivative with respect to the vector $\xi=(x,y,z)$. The function $G:[0,T]\times\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^M\times\mathbf{R}^{1+m+M}\times\mathbf{R}^{(1+m+M)\times(1+m+M)}\to\mathbf{R}$ is defined as

$$G(t,x,y,z,J'_t,J'_\xi,J''_{\xi\xi})=\sup_{u\in\Delta} G_0(t,u,x,y,z,J'_t,J'_\xi,J''_{\xi\xi})+G_1(t,x,y,z,J'_t,J'_\xi,J''_{\xi\xi}),$$

where

$$G_0(t,u,x,y,z,J_t,J'_\xi,J''_{\xi\xi})=J'_x u^\top\mathbf{a}+\frac{1}{2}J''_{xx}u^\top\mathbf{v}\mathbf{v}^\top u+\operatorname{tr}\left[J''_{xy}u^\top\mathbf{v}\beta^{\eta\top}\right]$$

and

$$G_1(t,x,y,z,J'_t,J'_\xi,J''_{\xi\xi})=J'_t+J'_y f^\eta+J'_z f^\zeta+\frac{1}{2}\operatorname{tr}\left[J''_{yy}\left(\beta^\eta\beta^{\eta\top}+\widehat{\beta}^\eta\widehat{\beta}^{\eta\top}\right)\right]+\operatorname{tr}\left[J''_{yz}\widehat{\beta}^\eta\widehat{\beta}^{\zeta\top}\right]+\frac{1}{2}\operatorname{tr}\left[J''_{zz}\widehat{\beta}^\zeta\widehat{\beta}^{\zeta\top}\right].$$

In this equation, $x\in D$; the set Δ and the coefficients depend on (y,z,t).

Note that $\Delta(y,z,t)$ is a convex set for all K,y,z,t.

By Lemma 5.2 and by Theorem 4.7.4 from Krylov [16], p. 206, there exists a unique solution J that is bounded in any bounded domain together with the derivatives presented in this equation. By Lemma 5.2 again and by Theorem 4.7.7 from Krylov [16], p. 209, it follows that the function J defined by (5.5) is the solution of (5.7); in other words, the Verification Theorem holds. The Bellman equation does not include generalized derivatives mentioned in Theorem 4.7.7 from Krylov [16] because of the existence of locally bounded derivatives.

Remark 5.6. Technically, Theorems 4.7.4 and 4.7.7 from Krylov [16] do not cover the case of non-constant $\Delta=\Delta(y,z,t)$. However, the extension on this case is straightforward for our special setting. For instance, one can consider the processes $p(t)=(\sigma(t)^\top)^{-1}\pi(t)$ to be the strategies instead of $\pi(t)$. In this case, the restriction $\{\pi(t):\ \pi(t)\in\Delta\}$ is replaced by the restriction $\{p(t):\ |p(t)|\le K\}$.

Let $\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)$, where $\mathbf{v}_j$ is the jth column of the matrix $\mathbf{v}$, and let $\beta^\eta=(\beta^\eta_1,\ldots,\beta^\eta_n)$, where β^η_j is the jth column of the matrix $\beta_\eta=\{\beta^\eta_{ki}\}_{k,i=1}^{m,n}$. We have that

$$\operatorname{tr}\left[J''_{xy}u^\top\mathbf{v}\beta^{\eta\top}\right]=\sum_{i=1}^n u^\top\mathbf{v}_i J''_{xy}\beta^\eta_i=u^\top\sum_{i=1}^n\mathbf{v}_i J''_{xy}\beta^\eta_i=u^\top\sum_{i=1}^n\mathbf{v}_i\sum_{k=1}^m J''_{xy_k}\beta^\eta_{ki}=u^\top\sum_{k=1}^m J''_{xy_k}\sum_{i=1}^n\mathbf{v}_i\beta^\eta_{ki}.$$

It follows that, for a given (u,x,y,z,t),

$$G_0(t,u,x,y,z,J_t,J'_\xi,J''_{\xi\xi})=J'_x u^\top\mathbf{a}+\tfrac{1}{2}J''_{xx}u^\top\mathbf{v}\mathbf{v}^\top u+u^\top\sum_{k=1}^m J''_{xy_k}\sum_{i=1}^n\mathbf{v}_i\beta^\eta_{ki}.$$

The maximum for G_0 in u is achieved for $\widehat{u} = {\mathbf{v}^{-1}}^\top p$, where $p = \mathbf{v}^\top u$ is a solution of the optimization problem

$$\text{Maximize} \qquad -\nu|p|^2 + p^\top b \quad \text{over} \quad p \in \mathbf{R}^n \quad \text{subject to} \quad |p| \le K. \tag{5.8}$$

Here $\nu = \nu(x, y, z, t)$ and $b = b(x, y, z, t)$ are defined as

$$\nu = -\frac{1}{2} J''_{xx}, \quad b = b(x, y, z, t) = J'_x \mathbf{v}^{-1}\mathbf{a} + \sum_{k=1}^m J''_{xy_k} \sum_{i=1}^n \mathbf{v}^{-1}\mathbf{v}_i \beta^\eta_{ki}.$$

By Lemma 5.3, problem (5.8) has an optimal solution

$$p(x, y, z, t) = \kappa(x, y, z, t) b(x, y, z, t),$$

where $\kappa(\cdot) : \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^M \times [0, T] \to \mathbf{R}$ can be selected to be a measurable function; its selection depends on K. Hence the maximum of G_0 is achieved for

$$\widehat{u} = \widehat{u}(x, y, z, t) = \kappa {\mathbf{v}^{-1}}^\top b = \kappa \left({\mathbf{v}^{-1}}^\top J'_x \mathbf{v}^{-1}\mathbf{a} + {\mathbf{v}^{-1}}^\top \sum_{k=1}^m J''_{xy_k} \sum_{i=1}^n \mathbf{v}^{-1}\mathbf{v}_i \beta^\eta_{ki} \right). \tag{5.9}$$

Let $Q(y, z, t) = (\mathbf{v}(y, z, t)\mathbf{v}(y, z, t)^\top)^{-1}$. Equation (5.9) can be rewritten as

$$\widehat{u} = \kappa \left(J'_x Q\mathbf{a} + \sum_{k=1}^m J''_{xy_k} \sum_{i=1}^n Q\mathbf{v}_i \beta^\eta_{ki} \right). \tag{5.10}$$

Further, let $(\mathbf{v}^\top)^{-1} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$, where $\mathbf{q}_j$ is the jth column of the matrix $(\mathbf{v}^\top)^{-1}$. We have

$$Q\mathbf{v} = (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v} = (\mathbf{v}^\top)^{-1}\mathbf{v}^{-1}\mathbf{v} = (\mathbf{v}^\top)^{-1} = (\mathbf{q}_1, \dots, \mathbf{q}_n).$$

Hence $Q\mathbf{v}_i = \mathbf{q}_i$, $\sum_{i=1}^n Q\mathbf{v}_i\beta^\eta_{ki} = \sum_{i=1}^n q_i \beta^\eta_{ki} = (\mathbf{v}^\top)^{-1}\beta^\eta_k$, and the maximum of G_0 is achieved for

$$\widehat{u}(x, y, z, t) = \sum_{k=1}^{m+1} \bar{H}_k(x, y, z, t)\psi_k(y, z, t), \tag{5.11}$$

where

$$\psi_k(y, z, t) = (\mathbf{v}(y, z, t)^\top)^{-1}\beta^\eta_k(y, z, t), \quad k \le m, \qquad \psi_{m+1}(y, z, t) = Q(y, z, t)\mathbf{a}(y, z, t),$$

and

$$\begin{aligned} \bar{H}_k(x, y, z, t) &= \kappa(x, y, z, t) J''_{xy_k}(x, y, z, t), \quad k \le m, \\ \bar{H}_{m+1}(x, y, z, t) &= \kappa(x, y, z, t) J'_x(x, y, z, t). \end{aligned} \tag{5.12}$$

Assume that the function $\widehat{u}(x, y, z, t)$ is regular enough in x to ensure solvability of the closed equation (5.4), for instance, it is Lipschitz in x uniformly in (y, z, t). In this case, the strategy $\widehat{\pi}(t) = \widehat{u}(\widetilde{X}(t), \eta(t), \zeta(t), t)$ is optimal and belongs to the class $\Sigma_M(K)$. Moreover, $\pi(t) = \sum_{k=1}^{m+1} \nu_k(t)\mathcal{M}_k(t)$, where

$$\begin{aligned} \mathcal{M}_k(t) &= \psi_k(\eta(t), \zeta(t), t) = (\sigma(t)^\top)^{-1}\beta^\eta_k(\eta(t), \zeta(t), t), \quad k \le m, \\ \mathcal{M}_{m+1}(t) &= \psi_{m+1}(\eta(t), \zeta(t), t) = \mathcal{Q}\widetilde{a}(t). \end{aligned}$$

Here q_j is the jth column of the matrix $(\sigma(t)^\top)^{-1} = (q_1, \dots, q_n)$, and

$$\nu_k(t) = \bar{H}_k(\widetilde{X}(t), \eta(t), \zeta(t), t), \quad k \le m, \qquad \nu_{m+1}(t) = \bar{H}_{m+1}(\widetilde{X}(t), \eta(t), \zeta(t), t).$$

Remark 5.7. The selection of $\{\mathcal{M}_k\}$ is independent of K and $U(\cdot)$. The selection of $\kappa(x,y,z,t)$ and $\{\bar{H}_k\}$ depends on K and $U(\cdot)$.

Therefore, equality (5.1) for the case where $D=\mathbf{R}$ holds for this case of regular enough $\widehat{u}$. Moreover, the strategy $\widehat{\pi}\in\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_{m+1}}(K)$ is optimal in $\Sigma(K)$ for this case.

In the general case, it cannot be guaranteed that the function $\widehat{u}(x,y,z,t)$ providing the maximum for G_0 is regular enough in x to ensure solvability of the closed loop equation (5.4). In this case, we have to approximate $\widehat{u}$ by regular enough functions. We will follow Chapter 5 from Krylov [16], with some simplifications that are possible because of the following features of our special setting: (a) The maximum for G_0 is achieved for $\widehat{u}$ that has the special form (5.11); (b) The regularity of $\bar{H}_k(x,y,z,t)$ in x is sufficient.

For $R>0$, let $\mathrm{C}_R=\mathrm{S}_R\times[0,T]$, where S_R is the origin-centered ball with the radius R in $\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^M$. We will consider large enough $R\to+\infty$ and small enough $\varepsilon\to 0$, $\varepsilon>0$.

Let $\widetilde{H}_{k,\varepsilon}(x,y,z,t)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\bar{H}_k(x+q,y,z,t)\mathrm{d}q$, and let

$$\begin{aligned} u_{\varepsilon,R}(x,y,z,t)&=\textstyle\sum_{k=1}^{m+1}\widetilde{H}_{k,\varepsilon}(x,y,z,t)\psi_k(y,z,t), &&(x,y,z,t)\in\mathrm{C}_R,\\ u_{\varepsilon,R}(x,y,z,t)&=0, &&(x,y,z,t)\notin\mathrm{C}_R.\end{aligned}$$

It follows from the definitions that $u_{\varepsilon,R}(x,y,z,t)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\widehat{u}(x+q,y,z,t)\mathrm{d}q$, for any (x,y,z) from the interior of $\mathrm{S_R}$ and for small enough ε. Hence

$$u_{\varepsilon,R}(x,y,z,t)\to\widehat{u}(x,y,z,t)\quad\text{as}\quad\varepsilon\to 0\quad\text{for a.e. }(x,y,z,t)\in\mathrm{C}_R. \tag{5.13}$$

Since the set $\Delta(y,z,t)$ is convex and contains zero vector, we have that $u_{\varepsilon,R}(x,y,z,t)$ takes the values in $\Delta(y,z,t)$.

Consider the set of closed-loop strategies

$$\pi_{\varepsilon,R}(t)=u_{\varepsilon,R}(\widetilde{X}_\varepsilon(t),\eta(t),\zeta(t),t).$$

Here $\widetilde{X}_\varepsilon(t)$ is the corresponding discounted wealth. By the definitions, these strategies belong to $\Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_{m+1}}(K)$. Let us show that they are Markov strategies.

Let $\tau_{\varepsilon,R}$ be the first exit time of the process $(\widetilde{X}_\varepsilon(t),\eta(t),\zeta(t))$ from C_R. Since the functions $u_{\varepsilon,R}(x,y,z,t)$ are bounded, they take values in $\Delta(y,z,t)$, and, for every $\varepsilon>0$, there exists $c>0$ such that

$$|u_{\varepsilon,R}(x_1,y,z,t)-u_{\varepsilon,R}(x_2,y,z,t)|\le c|x_1-x_2|\quad\forall x_1,x_2,y,z,t,\quad(x_i,y,z,t)\in\mathrm{C}_R,i=1,2.$$

Therefore, the existence of an unique strong solution of closed equation (5.4) is ensured for the strategy $\pi_{\varepsilon,R}(t)=u_{\varepsilon,R}(\widetilde{X}(t),\eta(t),\zeta(t),t)$ up to the time $\tau_{\varepsilon,R}$. To prove (5.1), it suffices to show that

$$\sup_{\pi\in\Sigma(K)}V(\pi)=\sup_{\varepsilon>0,R>0}V(\pi_{\varepsilon,R}). \tag{5.14}$$

Let us prove (5.14).

For a function $u(x,y,z,t)$, set

$$\rho^u(x,y,z,t)=G_0(t,\widehat{u}(x,y,z,t),x,y,z,J_t,J'_\xi,J''_{\xi\xi})-G_0(t,u(x,y,z,t),x,y,z,J_t,J'_\xi,J''_{\xi\xi}).$$

This equation can be rewritten as

$$\rho^u(x,y,z,t)=J'_x\widehat{u}^\top\mathbf{a}+\frac{1}{2}J''_{xx}\widehat{u}^\top\mathbf{v}\mathbf{v}^\top\widehat{u}+\widehat{u}^\top\sum_{k=1}^m J''_{xy_k}\sum_{i=1}^n\mathbf{v}_i\beta^\eta_{ki}-\left\{J'_xu^\top\mathbf{a}+\frac{1}{2}J''_{xx}u^\top\mathbf{v}\mathbf{v}^\top u+u^\top\sum_{k=1}^m J''_{xy_k}\sum_{i=1}^n\mathbf{v}_i\beta^\eta_{ki}\right\}. \tag{5.15}$$

Hence

$$|\rho^u(x,y,z,t)| \le (|\widehat{u}(x,y,z,t) - u(x,y,z,t)| + \frac{1}{2}|J''_{xx}||\widehat{u}(x,y,z,t)^\top \mathbf{v}\mathbf{v}^\top \widehat{u}(x,y,z,t) - u(x,y,z,t)^\top \mathbf{v}\mathbf{v}^\top u(x,y,z,t)|)h_1(x,y,z,t),$$

where

$$h_1(x,u,z,t) = |J'_x||\mathbf{a}| + \sum_{k=1}^m |J''_{xy_k}| \sum_{i=1}^n |\mathbf{v}_i||\beta^\eta_{ki}|.$$

Applying an obvious inequality $|\widehat{u}^\top \mathbf{v}\mathbf{v}^\top \widehat{u} - u^\top \mathbf{v}\mathbf{v}^\top u| \le |(\widehat{u}-u)^\top \mathbf{v}\mathbf{v}^\top(\widehat{u}+u)|$, we obtain that

$$|\rho^u(x,y,z,t)| \le |\widehat{u}(x,y,z,t) - u(x,y,z,t)|h(x,y,z,t),$$

where

$$h(x,u,z,t) = h_1(x,y,z,t) + \frac{1}{2}|J''_{xx}||\mathbf{v}\mathbf{v}^\top|(|\widehat{u}| + |u|).$$

As was mentioned already, by Lemma 5.2 and by Theorem 4.7.4 from Krylov [16], p. 206, J is bounded in any bounded domain together with the derivatives presented in this equation. Hence the function $h(x,u,z,t)$ is bounded on C_R.

Let $g(t) = \rho^{u_{\varepsilon,R}}(\widetilde{X}_\varepsilon(t), \eta(t), \zeta(t))$.

By Itô formula, we have that

$$\mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}>T\}}U(\widetilde{X}(\tau_{\varepsilon,R})) + \mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}\le T\}}J(\widetilde{X}(\tau_{\varepsilon,R}), \eta(\tau_{\varepsilon,R}), \zeta(\tau_{\varepsilon,R}), \tau_{\varepsilon,R})) = J(X_0, \eta(0), \zeta(0), 0) - \mathbf{E}\int_0^{\tau_{\varepsilon,R}} g(t)\mathrm{d}t.$$

Hence

$$J(X_0, \eta(0), \zeta(0), 0) = \mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}>T\}}U(\widetilde{X}(\tau_{\varepsilon,R})) + r_1 + r_2,$$

where

$$r_1 = \mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}\le T\}}J(\widetilde{X}(\tau_{\varepsilon,R}), \eta(\tau_{\varepsilon,R}), \zeta(\tau_{\varepsilon,R}), \tau_{\varepsilon,R})), \qquad r_2 = \mathbf{E}\int_0^{\tau_{\varepsilon,R}} g(t)\mathrm{d}t.$$

It suffices to show that, for any $\delta > 0$, there exists ε and R such that

$$J(X_0, \eta(0), \zeta(0), 0) \le \mathbf{E}U(\widetilde{X}(\tau_{\varepsilon,R})) + \delta. \tag{5.16}$$

Let $\delta > 0$ be given.

Let $\widehat{\tau}_R = T \wedge \inf\{t \ge 0 : \ \eta(t)^2 + \zeta(t)^2 \ge R^2\}$. Clearly, $\tau_{\varepsilon,R} \le \widehat{\tau}_R$. Since we have assumed that the matrix $AA^\top > 0$ is uniformly non-degenerate, we have that $\mathbf{P}(\widehat{\tau}_R \le T) \to 0$ as $R \to +\infty$ uniformly in $\varepsilon > 0$.

By Corollary 1 from Zakai [25], it follows that, for any $m > 0$,

$$\sup_{y,z}(\mathbf{E}|\widetilde{X}_\varepsilon(\tau_{\varepsilon,R})|^m + \sup_{\pi\in\Sigma(K)} \mathbf{E}|\widetilde{X}(T,\pi) - \widetilde{X}_\varepsilon(\tau_{\varepsilon,R})|^m) < +\infty,$$

where $\widetilde{X}(T,\pi)$ is the discounted terminal wealth for the strategy π given that

$$\widetilde{X}(\tau_{\varepsilon,R},\pi) = \widetilde{X}_\varepsilon(\tau_{\varepsilon,R}), \quad \eta(\tau_{\varepsilon,R}) = y, \quad \zeta(\tau_{\varepsilon,R}) = z.$$

By the assumptions on U, it follows that $\sup_\varepsilon \mathbf{E}|J(\widetilde{X}(\tau_{\varepsilon,R}),\eta(\tau_{\varepsilon,R}),\zeta(\tau_{\varepsilon,R}),\tau_{\varepsilon,R}))|^2 < +\infty$. Hence $r_1 \to 0$ as $R \to +\infty$ uniformly in $\varepsilon > 0$. By the assumptions on U again, we obtain that $\mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}\geq T\}}U(\widetilde{X}(\tau_{\varepsilon,R})) \to 0$ as $R \to +\infty$ uniformly in $\varepsilon > 0$. Hence $\mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}>T\}}U(\widetilde{X}(\tau_{\varepsilon,R})) \to \mathbf{E}U(\widetilde{X}(\tau_{\varepsilon,R}))$ as $R \to +\infty$ uniformly in $\varepsilon > 0$.

It follows that there exists $R = \widehat{R}$ such that

$$|r_1| \leq \delta/3, \qquad \mathbf{E}\mathbb{I}_{\{\tau_{\varepsilon,R}>T\}}U(\widetilde{X}(\tau_{\varepsilon,R})) \geq \mathbf{E}U(\widetilde{X}(\tau_{\varepsilon,R})) - \delta/3 \qquad \forall \varepsilon > 0.$$

By the Lebesgue's Dominated Convergence Theorem, it follows that $r_2 \to 0$ as $\varepsilon \to 0$ for the given $R = \widehat{R}$. Let $\varepsilon = \widehat{\varepsilon}$ be selected such that $|r_2| \leq \delta/3$. It follows that (5.16) holds. Hence (5.14) holds. This completes the proof of equality (5.1) for the case where $D = \mathbf{R}$ and where the functions U, $U'(x)$, and $U''(x)$ are bounded in D.

5.4.1. The proof for bounded U, $U'(x)$, $U''(x)$ and for $D = (0,+\infty)$

Let us assume that $D = (0,+\infty)$ and the function U is bounded in D together with the derivatives $U'(x)$ and $U''(x)$. We consider the change of variables $q(t) = \ln \widetilde{X}(t)$. Using the Ito formula, we obtain that this change of variables transfers the corresponding control problem as

$$\begin{aligned}
&\text{Maximize} \quad \mathbf{E}U(\mathrm{e}^{q(T)}) \quad \text{over} \quad \widetilde{\pi}(\cdot) \quad \text{subject to}\\
&\mathrm{d}q(t) = \widetilde{\pi}(t)^\top \mathbf{a}(\eta(t),\zeta(t),t)\mathrm{d}t - \frac{1}{2}\widetilde{\pi}(t)^\top \mathbf{v}(\eta(t),\zeta(t),t)\mathbf{v}(\eta(t),\zeta(t),t)^\top\widetilde{\pi}(t) + \widetilde{\pi}(t)\mathbf{v}(\eta(t),\zeta(t),t)\mathrm{d}w(t)],\\
&\mathrm{d}\eta(t) = f^\eta(\eta(t),\zeta(t),t)\mathrm{d}t + \beta^\eta(\eta(t),\zeta(t),t)\mathrm{d}w(t) + \widehat{\beta}^\eta(\eta(t),\zeta(t),t)\mathrm{d}\widehat{w}(t),\\
&\mathrm{d}\zeta(t) = f^\zeta(\eta(t),\zeta(t),t)\mathrm{d}t + \widehat{\beta}^\zeta(\eta(t),\zeta(t),t)\mathrm{d}\widehat{w}(t),
\end{aligned} \tag{5.17}$$

given $X(0), \eta(0), \zeta(0)$. We consider here maximization over the strategies $\widetilde{\pi}$ from the class $\Sigma(K)$ defined for $D = \mathbf{R}$, *i.e.*, such that $\sup_{t,\omega}|\widetilde{\pi}(t,\omega)| < +\infty$.

The proof of equality (5.1) repeats the proof given above for $D = \mathbf{R}$ with few modifications. Instead of (5.5), we use

$$J(x,y,z,t) \triangleq \sup_{\pi(\cdot)\in\Sigma_M(K)} \mathbf{E}\left\{U(\mathrm{e}^{q(T)})\Big|q(t) = x,\, \eta(t) = y,\, \zeta(t) = z\right\}.$$

Here $x \in \mathbf{R}$ and $q(t) = \ln(\widetilde{X}(t))$; the maximization is over the class $\Sigma(K)$ defined for $D = \mathbf{R}$. The Bellman equation for J is defined similarly to the Bellman equation for $D = \mathbf{R}^n$, with G_0 replaced by $G_0 - \frac{1}{2}J'_x u^\top \mathbf{v}\mathbf{v}^\top u$. Respectively, ν in (5.8) has to be defined as $\nu = -\frac{1}{2}(-J'_x + J''_{xx})$. This gives the proof of (5.1) where $D = (0,+\infty)$ and the functions U, $U'(x)$, and $U''(x)$ are bounded in D.

Proof for the general case.

Consider now the case where either $D = \mathbf{R}$ or $D = (0,+\infty)$ and where the functions U, $U'(x)$, and $U''(x)$ are not necessarily bounded in D.

Let $\delta > 0$, $K > 0$, and $\bar{\pi} \in \Sigma(K)$ be given.

For $L > 0$, let $\bar{U}_L(x) = \max(-L, \min(U(x), L))$, and let $\bar{V}_L(\pi)$ be defined similarly to $V(\pi)$ with U replaced by $\bar{U}_L$. Let us select $L > 0$ such that $|\bar{V}_L(\pi) - V(\pi)| \leq \delta/5$ for all $\pi \in \Sigma(K)$; by the assumptions on $\Sigma(K)$, this L exists. Further, for $L_1 > 0$, $\rho > 0$, let a function $\widetilde{U} = \widetilde{U}_{L,L_1,\rho} : D \to \mathbf{R}$ be such that $|\widetilde{U}(x)| \leq L + 1$ for all $x \in D$, $|\widetilde{U}(x) - \bar{U}_L(x)| \leq \rho$ if $|x| < L_1$, and such the derivatives $\widetilde{U}'(x)$ and $\widetilde{U}''(x)$ are bounded in D. This function can be obtained *via* convolution of $\bar{U}_L$ with a smoothing averaging kernel, for instance, such as described in Krylov [16], Section II.1. Let $\widetilde{V}(\pi)$ be defined similarly to $V(\pi)$ with U replaced by $\widetilde{U}$. By the assumptions on $\Sigma(K)$, there exists $L_1 > 0$, $\rho > 0$ and $\widetilde{U}(x)$ such that $|\widetilde{V}(\pi) - \bar{V}_L(\pi)| \leq \delta/5$ for all $\pi \in \Sigma(K)$.

By the theorem proved above for the utilities with the properties featured by $\widetilde{U}$, there exists $\widehat{\pi} \in \Sigma_{\mathcal{M}_1,\ldots,\mathcal{M}_\mu}(K)$ such that $\widetilde{V}(\widehat{\pi}) \geq \widetilde{V}(\bar{\pi}) - \delta/5$. In addition, we have that

$$V(\widehat{\pi}) \geq \bar{V}_L(\widehat{\pi}) - \frac{\delta}{5} \geq \widetilde{V}(\widehat{\pi}) - \frac{2\delta}{5} \geq \widetilde{V}(\bar{\pi}) - \frac{3\delta}{5} \geq \bar{V}_L(\bar{\pi}) - \frac{4\delta}{5} \geq V(\bar{\pi}) - \delta.$$

Since $\bar{\pi}$ and δ were selected arbitrary, the proof of (5.1) follows for the general case.

Finally, the proof of Theorem 3.2 follows from (5.1).

Remark 5.8. For a typical case, we have that $\kappa(\widetilde{X}(t), \eta(t), \zeta(t), t) = -J''_{x,x}(\widetilde{X}(t), \eta(t), \zeta(t), t)^{-1}$ if $D = \mathbf{R}$, or $\kappa(q(t), \eta(t), \zeta(t), t) = (J'_x(\widetilde{X}(t), \eta(t), \zeta(t), t) - J''_{x,x}(\widetilde{X}(t), \eta(t), \zeta(t), t))^{-1}$ if $D = (0, +\infty)$. It happens when the strategy $\pi(t) = \widehat{u}(\widetilde{X}(t), \eta(t), \zeta(t), t)$ belongs to the class $\Sigma_M(K)$ and such that $\pi(t)^\top \sigma(t)\sigma(t)^\top \pi < K$. We use the constraints $\pi(t)^\top \sigma(t)\sigma(t)^\top \pi(t) \le K$ as an auxiliary class of near optimal (suboptimal) admissible strategies; the final result does not require these constraints.

Remark 5.9. To calculate the processes $\nu_k(t)$, one have to find J from a HJB equation. Analytical solutions of these equations are rarely feasible; however, numerical methods for them are well developed; see, *e.g.* Barles and Jakobsen [1] and the review in Kushner [17].

6. Conclusion

The Mutual Fund Theorem defines the distribution of risky assets for the optimal strategy. If this theorem holds, then the distribution is the same for all risk preferences, and the strategy selection can be reduced to the selection of a one dimensional process of the total investment in risky assets. This interesting feature is presented in portfolio theory only and does not have an analog for the general theory of stochastic optimal control. The efforts in the existing literature are mostly concentrated on the extension of the list of models where the Mutual Fund Theorem holds. The current paper suggests a relaxed version of this theorem to cover models where the classical Mutual Fund Theorem does not hold. We found conditions that ensure that the optimal strategy can be represented as a linear combination of μ fixed processes (or μ Mutual Funds), for a wide class of risk preferences, for a model with $n \gg \mu$ stocks. The number μ is defined by the number of correlations in the model rather than by the number of stocks.

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