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ESTIMATES FOR THE CONTROLS OF THE WAVE EQUATION WITH A POTENTIAL

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Abstract. This article studies the L^2-norm of the boundary controls for the one dimensional linear wave equation with a space variable potential $a = a(x)$. It is known these controls depend on a and their norms may increase exponentially with $\|a\|_{L^\infty}$. Our aim is to make a deeper study of this dependence in correlation with the properties of the initial data. The main result of the paper shows that the minimal L^2-norm controls are uniformly bounded with respect to the potential a, if the initial data have only sufficiently high eigenmodes.

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1. Introduction

For $T > 0$, we consider the one-dimensional boundary controlled linear wave equation with a space variable potential

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + a(x)u(t,x) = 0 & (t,x) \in (0,T)\times(0,1) \\ u(t,0) = 0 & t \in (0,T) \\ u(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1) \\ u_t(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1.1}$$

where the potential $a \in L^\infty(0,1)$ is a real-valued function such that $a(x) > 0$ a. e. in $[0,1]$. Since we are mainly interested in potentials a with large norms, it is convenient to suppose that $\|a\|_{L^\infty} > 1$. Equation (1.1) is said to be *null-controllable in time* $T > 0$ if, for every initial data $\binom{u^0}{u^1} \in \mathcal{H} := L^2(0,1) \times H^{-1}(0,1)$, there exists a control function $v \in L^2(0,T)$ such that the corresponding solution of (1.1) verifies

$$u(T,x) = u_t(T,x) = 0 \qquad (x \in (0,1)). \tag{1.2}$$

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Equation (1.1) represents a potential perturbation of the linear wave equation and it is frequently encountered when studying the stability of stationary solutions for several important systems of partial differential equations (wave-Schrödinger, Maxwell−Schrödinger, Maxwell−Dirac and many others). When a is a constant function, (1.1) is also known as the Klein Gordon equation and plays a fundamental role in quantum field theory. Also, it arises when separation of variables is considered in the multidimensional linear wave equation. The controllability problem for the perturbed wave equation has been studied in the literature and has received a positive answer even in more general contexts (see, for instance, [1, 4, 10, 12, 18, 20, 21]). Usually, the problem is reduced to an observability inequality for the adjoint system, which is obtained using techniques based on non harmonic spectral analysis, multipliers or Carleman type inequalities.

The aim of this paper is to study how the norms of the controls depend on the potential a and the initial data to be controlled. It is known that (see, for instance, [21]) there exist two positive constants M and ω, independent of a, such that the minimal L^2-norm control v corresponding to (1.1) verifies

$$\|v\|_{L^2(0,T)} \le M \exp\left(\omega\sqrt{\|a\|_{L^\infty}}\right) \left\|\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}\right\|_{\mathcal{H}} \qquad \left(\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}\right). \tag{1.3}$$

As indicated by (1.3), the norm of the controls may increase exponentially with the $L^\infty-$norm of the potential a. The fact that estimate (1.3) is optimal can be seen in simple cases with constant potentials a (see, for instance, [7]). Since the growth in (1.3) of the control cost limits practical implementation for large potentials, it is of interest to see if there are initial data which can be controlled with a lower cost. Following this idea, the main result of this article shows that, given any $T > 2$, there exists a constant $M > 0$, depending only on T, such that, given any $a \in L^\infty(0, 1)$, there exists an infinite dimensional space $\mathcal{H}_1 \subset \mathcal{H}$ with the property that

$$\|v\|_{L^2(0,T)} \le M \left\|\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}\right\|_{\mathcal{H}} \qquad \left(\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}_1\right), \tag{1.4}$$

where v is the minimal L^2-norm control corresponding to the initial data $\left(\begin{smallmatrix} u^0 \\ u^1 \end{smallmatrix}\right)$. Inequality (1.4) shows that the cost to control the initial data in $\mathcal{H}_1$ is bounded independently of the potential a. Moreover, we can give a characterization of the space $\mathcal{H}_1$, which consists of elements from $\mathcal{H}$ containing only sufficiently high eigenmodes (depending on the magnitude of the potential a):

$$\mathcal{H}_1 = \left\{ \sum_{n\in\mathbb{Z}^*} a_n \Phi_n \;\middle|\; a_n = 0 \text{ for } |n| \le \mathcal{N} := \rho\|a\|_{L^\infty} \ln^2(\|a\|_{L^\infty}) \right\}, \tag{1.5}$$

where ρ is an absolute positive constant and Φ_n are the normalized eigenfunctions in $\mathcal{H}$ of the differential operator corresponding to (1.1). Roughly speaking, our result shows that the perturbing potential has a lesser impact on the controllability properties of the high frequencies and the exponentially large cost of the controls is due only to the action on the low frequencies. The technique used in this paper reduces the controllability problem to a moment problem which is solved by constructing a biorthogonal sequence $(\theta_n)_{n\in\mathbb{Z}^*}$ to the family of exponential functions $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$, where $(i\lambda_n t)_{n\in\mathbb{Z}^*}$ are the eigenvalues of the differential operator corresponding to (1.1). We prove that the elements θ_n of index $|n| > \rho\|a\|_{L^\infty} \ln^2(\|a\|_{L^\infty})$ are uniformly bounded in a. This allows us to deduce the desired controllability result (1.4).

We remark that similar results have been obtained in [11], with a different technique and a smaller space of uniformly (with respect to a) controllable initial data. Indeed, in [11] the number $\mathcal{N}$ from (1.5) is of the order of $\exp(\rho\|a\|_{L^\infty})$. Finally, let us mention that the value of $\mathcal{N}$ in (1.5) is, probably, not optimal. In fact, we conjecture that the optimal value of $\mathcal{N}$ is of the order of $\sqrt{\|a\|_{L^\infty}}$. It has been proved to be so in a particular case treated in [13], in which the potential a is constant and the spectrum can be explicitly computed. However, in [13] the controllability time T is not the optimal one and should be taken large enough.

The rest of the paper is organized as follows. In Section 2 we present the spectral analysis of the differential operator corresponding to (1.1) and we transform our control problem into a moment problem. Section 3 gives the construction of the biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ and its main properties. In the last section we prove the main result (1.4).

2. The moment problem

In this section we recall some well-known properties concerning the boundary null-controllability problem for the linear wave equation with a potential and we characterize the control problem by using the moment theory. To do that we need the following variational result.

Lemma 2.1. *Let $T>0$ and the initial data $\binom{u^0}{u^1}\in\mathcal{H}:=L^2(0,1)\times H^{-1}(0,1)$. The function $v\in L^2(0,T)$ is a control which drives to zero the solution of (1.1) in time T if and only if, the following relation holds*

$$\int_0^T v(t)\overline{\varphi}_x(t,1)\mathrm{d}t=\langle u^1,\,\varphi(0,\cdot)\rangle_{H^{-1},H_0^1}\\
-\int_0^1 u^0(x)\overline{\varphi_t}(0,x)\mathrm{d}x \qquad \left(\begin{pmatrix}\varphi^0\\ \varphi^1\end{pmatrix}\in H_0^1(0,1)\times L^2(0,1)\right), \tag{2.1}$$

where $\binom{\varphi}{\varphi_t}$ is the solution of the following adjoint backward problem

$$\begin{cases}\varphi_{tt}(t,x)-\varphi_{xx}(t,x)+a(x)\varphi(t,x)=0 & (t,x)\in(0,T)\times(0,1)\\ \varphi(t,0)=\varphi(t,1)=0 & t\in(0,T)\\ \varphi(T,x)=\varphi^0(x) & x\in(0,1)\\ \varphi_t(T,x)=\varphi^1(x) & x\in(0,1),\end{cases} \tag{2.2}$$

and $\langle\cdot,\,\cdot\rangle_{H^{-1},H_0^1}$ denotes the duality product between the spaces $H^{-1}(0,1)$ and $H_0^1(0,1)$.

Proof. If we multiply in (1.1) by $\overline{\varphi}$ and we integrate by parts over $(0,T)\times(0,1)$, we obtain that $v\in L^2(0,T)$ is a null-control for (1.1) if and only if it verifies (2.1). □

Let $A_0:\mathcal{D}(A_0)\to L^2(0,1)$ be the unbounded operator in $L^2(0,1)$ defined by

$$\mathcal{D}(A_0)=H^2(0,1)\cap H_0^1(0,1), \tag{2.3}$$

and

$$A_0u=-u_{xx}+au,\qquad (u\in\mathcal{D}(A_0)). \tag{2.4}$$

Let $\langle\,,\,\rangle_0$ be the canonical inner product in $L^2(0,1)$. In $H_0^1(0,1)$ we introduce the inner product

$$\langle f,g\rangle_1=\int_0^1 f_x(x)\overline{g}_x(x)\mathrm{d}x+\int_0^1 a(x)f(x)\overline{g}(x)\mathrm{d}x\qquad (f,g\in H_0^1(0,1)). \tag{2.5}$$

We remark that the usual norm in H_0^1 (obtained from (2.5) with $a\equiv 0$) and the norm $\|\cdot\|_1$ are equivalent. We have the following result.

Proposition 2.2. *Let $A_0:D(A_0)\to L^2(0,1)$ be the operator defined by (2.3) and (2.4) and $a\in L^\infty(0,1)$, $a(x)>0$ a. e. in $[0,1]$. Then the eigenvalues of A_0 can be ordered to form a strictly increasing sequence $(\mu_n)_{n\in\mathbb{N}^*}$ satisfying*

$$\left|\mu_n-n^2\pi^2\right|\le\|a\|_{L^\infty}\qquad (n\in\mathbb{N}^*), \tag{2.6}$$

and the corresponding eigenfunctions $(\varphi_n)_{\in\mathbb{N}^*}$ form an orthonormal basis in $L^2(0,1)$.

Proof. See, for instance, ([18], Prop. 3.5.5). □

Since $(\mathcal{D}(A_0), A_0)$ is an unbounded, maximal monotone and auto-adjoint operator in $L^2(0,1)$ with compact resolvent, for each $\alpha \geq 0$ we can define the fractional space

$$\mathcal{D}(A_0^\alpha) = \left\{ u = \sum_{n\geq 1} a_n \varphi_n \in L^2(0,1) \;\middle|\; \sum_{n\geq 1} |a_n|^2 \mu_n^{2\alpha} < +\infty \right\}.$$

We note that

$$\mathcal{D}(A_0^0) = L^2(0,1),$$
$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = H_0^1(0,1),$$
$$\mathcal{D}\left(A_0^{-\frac{1}{2}}\right) = H_0^{-1}(0,1),$$
$$\mathcal{D}(A_0) = H^2(0,1) \cap H_0^1(0,1).$$

Let us define the space $X = H_0^1(0,1) \times L^2(0,1)$ endowed with the inner product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{1,0} = \langle f_1, g_1\rangle_1 + \langle f_2, g_2\rangle_0 \qquad \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in X \right). \tag{2.7}$$

By denoting $W = \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}$, equation (2.2) is equivalent with

$$\begin{cases} W_t + AW = 0 \\ W(T) = W^0 = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \end{cases} \tag{2.8}$$

where $A : \mathcal{D}(A) \to X$ is the operator defined by

$$\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1), \tag{2.9}$$

and

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ A_0 u \end{pmatrix} \qquad ((u,v) \in \mathcal{D}(A)). \tag{2.10}$$

We have the following result.

Proposition 2.3. *Let $A : \mathcal{D}(A) \to X$ be the operator defined by (2.9) and (2.10). The eigenvalues of the operator A are given by the family $(i\lambda_n)_{|n|\geq 1}$, where*

$$\lambda_n = \operatorname{sgn}(n)\sqrt{\mu_{|n|}} \qquad (n \in \mathbb{Z}^*), \tag{2.11}$$

and the corresponding eigenfunctions are

$$\phi_n = \frac{1}{\sqrt{2}} \operatorname{sgn}(n) \begin{pmatrix} \frac{1}{i\lambda_n} \\ 1 \end{pmatrix} \varphi_{|n|} \qquad (n \in \mathbb{Z}^*), \tag{2.12}$$

where $\varphi_{|n|}$ are given by Proposition 2.2. Moreover, the vectors $(\phi_n)_{|n|\geq 1}$ form an orthonormal basis in X.

Proof. Let us first determine the eigenvalues of A. If $\lambda \in \mathbb{C}$ and $\Phi = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A)$ are such that $A\Phi = \lambda\Phi$ we obtain from the definition of A that

$$\begin{cases} v = -\lambda u \\ A_0 = -\lambda^2 u. \end{cases}$$

From Proposition 2.2 it follows that the eigenvalues of A are $(i\lambda_n)_{|n|\geq 1}$, where

$$\lambda_n = \mathrm{sgn}(n)\,\sqrt{\mu_{|n|}}, \qquad (n \in \mathbb{Z}^*),$$

and $\mu_{|n|}$ are the eigenvalues of the operator A_0. The corresponding eigenfunctions are given by

$$\phi_n = \frac{1}{\sqrt{2}}\mathrm{sgn}(n)\begin{pmatrix} \frac{1}{i\lambda_n} \\ 1 \end{pmatrix}\varphi_{|n|} \qquad\qquad (n \in \mathbb{Z}^*).$$

For any $n, m \in \mathbb{Z}^*$ we have that

$$\begin{aligned}\langle \phi_n, \phi_m\rangle_X &= \frac{\mathrm{sgn}(nm)}{2}\left\langle \frac{1}{i\lambda_n}\varphi_{|n|}, \frac{1}{i\lambda_m}\varphi_{|m|}\right\rangle_1 + \frac{\mathrm{sgn}(nm)}{2}\left\langle \varphi_{|n|}, \varphi_{|m|}\right\rangle_0 \\ &= \frac{\mathrm{sgn}(nm)}{2\lambda_n\lambda_m}\left\langle A_0\varphi_{|n|}, \varphi_{|m|}\right\rangle_0 + \frac{\mathrm{sgn}(nm)}{2}\delta_{|n||m|} = \frac{\mathrm{sgn}(nm)\mu_{|n|}}{2\lambda_n\lambda_m}\delta_{|n||m|} + \frac{\mathrm{sgn}(nm)}{2}\delta_{|n||m|} = \delta_{\mathrm{nm}},\end{aligned}$$

and the proof ends. $\square$

The distribution of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ is very important for the controllability properties of (1.1). It can be shown that there is a gap between two consecutive eigenvalues λ_n and λ_{n+1}, which does not depend of a, if n is large enough.

Remark 2.4. For every $|n| > [\frac{\sqrt{\|a\|_{L^\infty}}}{\pi}]$ we have from (2.6) that

$$\begin{aligned}|\lambda_{n+1} - \lambda_n| &= \sqrt{\mu_{|n+1|}} - \sqrt{\mu_{|n|}} \geq \sqrt{\pi^2(n+1)^2 - \|a\|_{L^\infty}} - \sqrt{\pi^2 n^2 + \|a\|_{L^\infty}} \\ &\geq \frac{(2n+1)\pi^2 - 2\|a\|_{L^\infty}}{(2n+1)\pi + 2\sqrt{\|a\|_{L^\infty}}} \geq \frac{n\pi^2 - \|a\|_{L^\infty}}{n\pi + \sqrt{\|a\|_{L^\infty}}}.\end{aligned}$$

If $0 < \delta < \pi$ is an arbitrary number then we have that

$$|\lambda_{n+1} - \lambda_n| \geq \frac{n\pi^2 - \|a\|_{L^\infty}}{n\pi + \sqrt{\|a\|_{L^\infty}}} \geq \pi - \delta := \gamma_0, \tag{2.13}$$

when $|n| > N = [\frac{\|a\|_{L^\infty} + \sqrt{\|a\|_{L^\infty}}(\pi-\delta)}{\pi\delta}]$.

Let us now introduce the space $\mathcal{H} = L^2(0,1) \times H^{-1}(0,1)$ endowed with the inner product (we recall that the potential a is a positive function in [0,1]):

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right\rangle_{0,-1} = \langle f_1, g_1\rangle_0 + \langle A_0^{-1} f_2, g_2\rangle_0 \qquad \left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}\right). \tag{2.14}$$

Remark 2.5. A straightforward computation shows that the family $(isgn(n)\lambda_n\phi_n)_{n\in\mathbb{Z}^*}$ forms an orthonormal basis in $\mathcal{H}$.

When control problems for (1.1) are studied, the behavior of the derivative of the eigenfunctions in the extremity $x = 1$ plays a fundamental role. We have the following result:

Proposition 2.6. *We have that*

$$\sup_{n\geq 1} \frac{1}{\lambda_n}|\varphi_n'(1)| < \infty, \tag{2.15}$$

$$\inf_{n\geq 1} \frac{1}{\lambda_n}|\varphi_n'(1)| > 0. \tag{2.16}$$

Proof. See, for instance, [18], Proposition 8.2.1. $\square$

The following lemma transforms the control problem into a moment problem by using the Fourier expansion of the solution of (1.1).

Lemma 2.7. *Problem* (1.1) *is null-controllable in time* $T > 0$ *if and only if, for each initial data* $\binom{u^0}{u^1} \in \mathcal{H}$ *with the Fourier expansion*

$$\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \sum_{n\in\mathbb{Z}^*} a_n^0 i\mathrm{sgn}(n)\lambda_n\phi_n, \tag{2.17}$$

there exists $v \in L^2(0,1)$ *such that*

$$\int_0^T v(t)\mathrm{e}^{-i\lambda_n t}\mathrm{d}t = \sqrt{2}\, i\lambda_n \frac{a_n^0}{\overline{\varphi'_{|n|}}(1)} \qquad (n \in \mathbb{Z}^*). \tag{2.18}$$

Proof. Since $(\phi_n)_{n\in\mathbb{Z}^*}$ is a basis in $H_0^1(0,1)\times L^2(0,1)$, from Lemma 2.1 it follows that (1.1) is null-controllable if and only if, there exists $v \in L^2(0,T)$ such that (2.1) holds for any initial data $\binom{\varphi^0}{\varphi^1}$ which is an eigenfunction of the operator A. If $\binom{\varphi^0}{\varphi^1} = \phi_n$, the corresponding solution of (2.8) is given by $\binom{\varphi}{\varphi_t}(t) = \mathrm{e}^{i\lambda_n(t-T)}\phi_n$. In this case, for any initial data $\binom{u^0}{u^1}$ of the form (2.17), we obtain that

$$\int_0^T v(t)\overline{\varphi}_x(t,1)\mathrm{d}t = -\frac{\mathrm{sgn}(n)}{i\lambda_n\sqrt{2}}\overline{\varphi'_{|n|}}(1)\int_0^T v(t)\mathrm{e}^{-i\lambda_n(t-T)}\mathrm{d}t$$

and

$$\langle u^1,\, \varphi(0,\cdot)\rangle_{H^{-1},H_0^1} - \int_0^1 u^0(x)\overline{\varphi_t}(0,x)\mathrm{d}x = -\mathrm{sgn}(n)a_n^0\mathrm{e}^{i\lambda_n T}.$$

The last two relations give (2.18) and the proof ends. □

The notion of biorthogonality is useful in the study of moment problems like (2.18). We recall that a sequence $(\theta_m)_{m\in\mathbb{Z}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(\mathrm{e}^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t)\mathrm{e}^{-i\lambda_n t}\mathrm{d}t = \delta_{\mathrm{mn}} \qquad (m, n \in \mathbb{Z}^*)\,. \tag{2.19}$$

It is easy to see from (2.18) that, if $(\theta_m)_{m\in\mathbb{Z}^*}$ is a biorthogonal sequence to the family of exponential functions $\left(\mathrm{e}^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, then a solution v of (2.18) will be given by

$$v(t) = \sqrt{2}\sum_{n\in\mathbb{Z}^*} \frac{i\lambda_n \mathrm{e}^{i\lambda_n\frac{T}{2}}}{\overline{\varphi'_{|n|}}(1)} a_n^0\theta_n\left(t - \frac{T}{2}\right) \qquad (t \in (0,T)). \tag{2.20}$$

We remark that the expression in the right hand side of (2.20) is formal. In order to show that it represents an element from $L^2(0,T)$ we need to prove first that there exists a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ to the family $\left(\mathrm{e}^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, to evaluate its L^2-norm and to conclude that the series is convergent in $L^2(0,T)$.

3. Construction of the biorthogonal sequence

This section is devoted to construct and evaluate a biorthogonal sequence to the family $\Lambda = \left(\mathrm{e}^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, for any $T > 2$, where $(\lambda_n)_{n\in\mathbb{Z}^*}$ are the eigenvalues of the operator A given in Proposition 2.3. This will be done in Theorem 3.5 below. In order to prove it we need to introduce some new tools and to show some previous results.

Firstly, let us introduce some notation. In the sequel, $C > 0$ denotes a generic positive constant which may change from line to line but it is always independent of the parameters of the problem. For $f \in L^1(\mathbb{R})$, the Fourier transform of f, denoted by $\widehat{f}$, is defined by

$$\widehat{f}(x) = \int_{\mathbb{R}} \mathrm{e}^{-itx} f(t)\mathrm{d}t \qquad (x \in \mathbb{R}).$$

Let $R > 0$. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t) = \begin{cases} \exp\left(\dfrac{R^3}{t^2 - R^2}\right) & \text{if} \qquad |t| < R \\ 0 & \text{if} \qquad |t| \geq R. \end{cases} \tag{3.1}$$

The function Φ belongs to $C^\infty(\mathbb{R})$, is non-negative and his support is $[-R, R]$.

As in ([11], Prop. 2) we can evaluate the Fourier transform of Φ_R. Since we need a more precise and explicit estimate, we present it in detail.

Proposition 3.1. *Let $R > 0$ and Φ be the function defined in (3.1). With $C_R = (\hat{\Phi}(0))^{-1}$ let*

$$\Phi_R(t) = C_R \Phi(t). \tag{3.2}$$

We have that the Fourier transform $\widehat{\Phi}_R$ of Φ_R is an entire function of exponential type R which verifies

$$\hat{\Phi}_R(0) = 1, \tag{3.3}$$

$$|\widehat{\Phi}_R(x)| < 1 \qquad (x \in \mathbb{R}^*), \tag{3.4}$$

$$|\widehat{\Phi}_R(x)| \leq 3\exp\left(-\frac{R}{4}\sqrt{x}\right) \qquad (x > x_0 = 256). \tag{3.5}$$

Proof. For $x \in \mathbb{R}^*$, we have that

$$\left|\widehat{\Phi}_R(x)\right| = C_R \left|\int_{-R}^{R} \exp\left(\frac{R^3}{t^2 - R^2}\right) \exp\left(-itx\right) \mathrm{d}t\right| = 2C_R \left|\int_0^R \exp\left(\frac{R^3}{t^2 - R^2}\right) \cos(tx)\mathrm{d}t\right| < 1,$$

and (3.4) holds. Since

$$C_R^{-1} = \int_{-R}^{R} \exp\left(\frac{R^3}{t^2 - R^2}\right) \mathrm{d}t \geq \int_{-\frac{R}{2}}^{\frac{R}{2}} \exp\left(\frac{R^3}{t^2 - R^2}\right) \mathrm{d}t \geq \int_{-\frac{R}{2}}^{\frac{R}{2}} \exp\left(\frac{R^3}{\left(\frac{R}{2}\right)^2 - R^2}\right) \mathrm{d}t = R\exp\left(-\frac{4}{3}R\right),$$

we obtain that

$$C_R \leq \frac{1}{R}\exp\left(\frac{4}{3}R\right) \cdot \tag{3.6}$$

Let $x \geq 256$. We evaluate $\widehat{\Phi}_R$ by changing the contour of integration.

Let $\epsilon(x) = \sqrt{1 - \frac{1}{\sqrt{x}}} < 1$ and define, in the complex plane, the curves:

$$\gamma_1 : \left[0, \frac{1}{2\sqrt[4]{x}}\right] \to \mathbb{C}, \quad \gamma_1(s) = -R\epsilon(x) - R\, s\, i, \quad s \in \left[0, \frac{1}{2\sqrt[4]{x}}\right]$$

$$\gamma_2 : [-\epsilon(x), \epsilon(x)] \to \mathbb{C}, \quad \gamma_2(s) = R\, s - R\frac{1}{2\sqrt[4]{x}}\, i, \quad s \in [-\epsilon(x), \epsilon(x)]$$

$$\gamma_3 : \left[-\frac{1}{2\sqrt[4]{x}}, 0\right] \to \mathbb{C}, \quad \gamma_3(s) = R\epsilon(x) + R\, s\, i, \quad s \in \left[-\frac{1}{2\sqrt[4]{x}}, 0\right].$$

Remark that, for any $x \geq 256$,

$$(R\epsilon(x))^2 + \left(\frac{R}{2\sqrt[4]{x}}\right)^2 = R^2\left(1 - \frac{1}{\sqrt{x}} + \frac{1}{4\sqrt{x}}\right) = R^2\left(1 - \frac{3}{4\sqrt{x}}\right) < R^2.$$

Hence, the curves γ_1, γ_2 and γ_3 are contained in $D = \{z \in \mathbb{C} : |z| < R\}$. Moreover, we have that

$$\begin{aligned}
\widehat{\Phi}_R &= C_R \int_{-R}^{R} \Phi(t) \exp(-i\, xt) \mathrm{d}t \\
&= C_R \int_{-R}^{-R\epsilon(x)} \Phi(t) \exp(-i\, xt) \mathrm{d}t + C_R \int_{\gamma_1} \Phi(z) \exp(-i\, xz) \mathrm{d}z + C_R \int_{\gamma_2} \Phi(z) \exp(-i\, xz) \mathrm{d}z \\
&\quad + C_R \int_{\gamma_3} \Phi(z) \exp(-i\, xz) \mathrm{d}z + C_R \int_{R\epsilon(x)}^{R} \Phi(t) \exp(-i\, xt) \mathrm{d}t := I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We evaluate each of the five integrals I_i, $1 \leq i \leq 5$. We have that

$$|I_1| = \left| C_R \int_{-R}^{-R\epsilon(x)} \Phi(t) \exp(-i\, xt) \mathrm{d}t \right| \leq C_R \int_{-R}^{-R\epsilon(x)} \Phi(t) \mathrm{d}t.$$

Since, for $t \leq 0$, Φ is an increasing function, we obtain that

$$|I_1| \leq C_R\, (1 - \epsilon(x))\, R \exp\left(\frac{R^3}{R^2(\epsilon^2(x) - 1)}\right) \leq C_R R \frac{1}{\sqrt{x}} \exp(-R\sqrt{x}).$$

From the above inequality and (3.6) we deduce that

$$|I_1| \leq \frac{1}{\sqrt{x}} \exp\left[\left(\frac{4}{3} - \frac{3\sqrt{x}}{4}\right) R\right] \exp\left(-\frac{R\sqrt{x}}{4}\right) \leq \frac{1}{4} \exp\left(-\frac{R\sqrt{x}}{4}\right) \qquad (x \geq 256). \tag{3.7}$$

We also have

$$|I_5| = \left| C_R \int_{R\epsilon(x)}^{R} \Phi(t) \exp(-i\, xt) \mathrm{d}t \right| \leq \frac{1}{4} \exp\left(-\frac{R\sqrt{x}}{4}\right) \qquad (x \geq 256). \tag{3.8}$$

We evaluate I_2. For $x > 256$ we have that

$$
\begin{aligned}
|I_2| &= \left|C_R \int_{\gamma_1} \Phi(z)\exp(-i\,xz)\mathrm{d}z\right| = \left|-RC_R\, i \int_0^{\frac{1}{2\sqrt[4]{x}}} \Phi\left(-R\epsilon(x) - R\,s\,i\right)\exp\left(-Rsx + Rx\epsilon(x)\,i\right)\mathrm{d}s\right| \\
&\le RC_R \int_0^{\frac{1}{2\sqrt[4]{x}}} \left|\Phi\left(-R\epsilon(x) - R\,s\,i\right)\right|\exp\left(-Rsx\right)\mathrm{d}s \\
&= RC_R \int_0^{\frac{1}{2\sqrt{x}}} \exp\left(R\,\mathrm{Re}\frac{1}{\left(\epsilon(x)+s\,i\right)^2 - 1}\right)\exp(-Rsx)\mathrm{d}s \\
&\quad + RC_R \int_{\frac{1}{2\sqrt{x}}}^{\frac{1}{2\sqrt[4]{x}}} \left|\Phi\left(-R\epsilon(x) - R\,s\,i\right)\right|\exp\left(-Rsx\right)\mathrm{d}s := I_2^1 + I_2^2.
\end{aligned}
$$

Since, for any $z \in \overline{D} = \left\{(a+b\,i)R \;\middle|\; a,b\in\mathbb{R},\ a^2+b^2 \le 1\right\}$ we have that

$$
\begin{aligned}
|\Phi(z)| &= \left|\exp\left(\frac{R}{(a+b\,i)^2-1}\right)\right| = \exp\left(\mathrm{Re}\frac{R}{a^2-b^2-1+2ab\,i}\right) = \exp\left(R\frac{a^2-b^2-1}{(a^2-b^2-1)^2+4a^2b^2}\right) \\
&\le \exp\left(-\frac{2Rb^2}{(a^2-b^2-1)^2+4a^2b^2}\right),
\end{aligned}
$$

it follows that

$$
|\Phi(z)| \le 1 \qquad (z \in \overline{D}). \tag{3.9}
$$

We evaluate each of the two integrals I_2^i, $1 \le i \le 2$. From (3.9) we have that

$$
I_2^2 = RC_R \int_{\frac{1}{2\sqrt{x}}}^{\frac{1}{2\sqrt[4]{x}}} \left|\Phi\left(-R\epsilon(x) - R\,s\,i\right)\right|\exp\left(-Rsx\right)\mathrm{d}s \le RC_R \int_{\frac{1}{2\sqrt{x}}}^{\frac{1}{2\sqrt[4]{x}}} \exp\left(-Rsx\right)\mathrm{d}s.
$$

Since, for $s \ge 0$, Φ is a decreasing function, we obtain that

$$
I_2^2 \le RC_R \int_{\frac{1}{2\sqrt{x}}}^{\frac{1}{2\sqrt[4]{x}}} \exp\left(-\frac{R}{2}\sqrt{x}\right)\mathrm{d}s = RC_R\left(\frac{1}{2\sqrt[4]{x}} - \frac{1}{2\sqrt{x}}\right)\exp\left(-\frac{R}{2}\sqrt{x}\right)\cdot \tag{3.10}
$$

For $x > 1$ and $s \in \left[0, \frac{1}{2\sqrt{x}}\right]$ we have that

$$
\mathrm{Re}\frac{1}{\left(\epsilon(x)+s\,i\right)^2-1} \le -\frac{\sqrt{x}}{3}\cdot \tag{3.11}
$$

From (3.11) we have that

$$
\begin{aligned}
I_2^1 &= RC_R \int_0^{\frac{1}{2\sqrt{x}}} \exp\left(R\,\mathrm{Re}\frac{1}{\left(\epsilon(x)+s\,i\right)^2-1}\right)\exp(-Rsx)\mathrm{d}s \le RC_R \int_0^{\frac{1}{2\sqrt{x}}} \exp\left(-\frac{R}{3}\sqrt{x}\right)\mathrm{d}s \\
&= \frac{RC_R}{2\sqrt{x}}\exp\left(-\frac{R}{3}\sqrt{x}\right)\cdot
\end{aligned} \tag{3.12}
$$

By using (3.10) and (3.12) we obtain that

$$
|I_2| = I_2^1 + I_2^2 \le \frac{RC_R}{2\sqrt[4]{x}}\exp\left(-\frac{R}{3}\sqrt{x}\right)\cdot
$$

From the above inequality and (3.6) we deduce that

$$|I_2| \leq \frac{1}{2\sqrt[4]{x}} \exp\left[\left(\frac{4}{3} - \frac{1}{12}\sqrt{x}\right) R\right] \exp\left(-\frac{R}{4}\sqrt{x}\right) \leq \frac{1}{4} \exp\left(-\frac{R}{4}\sqrt{x}\right) \qquad (x \geq 256). \tag{3.13}$$

In a similar way

$$|I_4| = \left| C_R \int_{\gamma_3} \Phi(z) \exp(-i\, xz) \mathrm{d}z \right| \leq \frac{1}{4} \exp\left(-\frac{R}{4}\sqrt{x}\right) \qquad (x \geq 256). \tag{3.14}$$

Next we evaluate I_3. From (3.9) we have that

$$\begin{aligned} |I_3| &= \left| C_R \int_{\gamma_2} \Phi(z) \exp(-i\, xz) \mathrm{d}z \right| = \left| RC_R \int_{-\epsilon(x)}^{\epsilon(x)} \Phi\left(R\, s - R\frac{1}{2\sqrt[4]{x}}\, i\right) \exp\left(-i\, Rsx - Rx\frac{1}{2\sqrt[4]{x}}\right) \mathrm{d}s \right| \\ &\leq RC_R \int_{-\epsilon(x)}^{\epsilon(x)} \exp\left(-Rx\frac{1}{2\sqrt[4]{x}}\right) \mathrm{d}s \leq 2RC_R\epsilon(x) \exp\left(-\frac{R}{2}\sqrt[4]{x^3}\right). \end{aligned} \tag{3.15}$$

From the above inequality and (3.6) we deduce that

$$|I_3| \leq 2 \exp\left[\left(\frac{4}{3} - \frac{1}{2}\sqrt[4]{x^3} + \frac{1}{4}\sqrt{x}\right) R\right] \exp\left(-\frac{R}{4}\sqrt{x}\right) \leq 2 \exp\left(-\frac{R}{4}\sqrt{x}\right) \qquad (x \geq 256). \tag{3.16}$$

By using (3.7), (3.8), (3.13), (3.14) and (3.16) we obtain that

$$|\widehat{\Phi}_R(x)| = |C_R \int_{-R}^{R} \Phi(t) \exp(-i\, xt) \mathrm{d}t| = |I_1 + I_2 + I_3 + I_4 + I_5| \leq 3 \exp\left(-\frac{R}{4}\sqrt{x}\right) \qquad (x \geq 256),$$

and the proof is complete. □

The following theorem gives and evaluates a biorthogonal sequence to a quite general family of exponential functions and therefore may be of independent interest. Its aim is to show that the norm of some elements of the biorthogonal sequence can be bounded independently of the small gap existing between the lowest exponents.

Theorem 3.2. *Let $N > 1$ be an integer, $\Lambda_1 = (\nu_n)_{1 \leq n \leq N}$ and $\Lambda_2 = (\nu_m)_{m > N}$ be two increasing sequences of real numbers with the properties*

$$\inf_{\substack{m \neq l \\ m,l > N}} |\nu_m - \nu_l| = \gamma > 0, \tag{3.17}$$

$$\nu_N < \nu_{N+1}, \tag{3.18}$$

and

$$\nu_n \neq \nu_k \qquad \forall n \neq k,\ 1 \leq k, n \leq N. \tag{3.19}$$

Let $\varepsilon \in (0, 1)$ and

$$\mathcal{N} = \min\left\{ m > N \;\middle|\; \nu_m - \nu_N > \frac{400N}{\varepsilon^2} \ln^2 N \right\}. \tag{3.20}$$

Then, for any $T > \frac{\pi}{\gamma}$ there exists a biorthogonal sequence $(\theta_m)_{m>N}$ to the family of exponential functions $(\mathrm{e}^{i\nu_m t})_{m>N}$ in $L^2(-T - \varepsilon, T + \varepsilon)$ which is orthogonal to the family $(\mathrm{e}^{i\nu_n t})_{1 \leq n \leq N}$. Moreover, for any sequence $(a_n)_n$ in l^2, we have that

$$\int_{-T-\varepsilon}^{T+\varepsilon} \left| \sum_{m>N} a_m \theta_m(t) \right|^2 \mathrm{d}t \leq \frac{Tc(\varepsilon, N)}{T^2 - \frac{\pi^2}{\gamma^2}} \sum_{N<m\leq\mathcal{N}} |a_m|^2 + \frac{T \exp(-2)}{4\pi \left(T^2 - \frac{\pi^2}{\gamma^2}\right)} \sum_{m>\mathcal{N}} |a_m|^2, \tag{3.21}$$

where $c(\varepsilon, N)$ is a constant which depends on ε and N but it is independent of γ and T.

Proof. Let $\varepsilon \in (0,1)$. From (3.17) and Ingham's Theorem (see [9]) we deduce that, for every $T > \frac{\pi}{\gamma}$ there exists two positive constants $K_1(\gamma,T)$ and $K_2(\gamma,T)$ such that

$$K_1(\gamma,T)\sum_{m>N}|a_m|^2 \le \int_{-T}^{T}\left|\sum_{m>N} a_m \mathrm{e}^{i\nu_m t}\right|^2 \mathrm{d}t \le K_2(\gamma,T)\sum_{m>N}|a_m|^2, \tag{3.22}$$

for every sequence $(a_m)_{m>N} \in l^2$. Theorem 4.1 from [14] gives the following estimates for the above constants

$$K_1(\gamma,T) = \frac{4\pi}{T}\left(T^2 - \frac{\pi^2}{\gamma^2}\right),\ K_2(\gamma,T) = 8\max\left\{\frac{4T^2}{\pi^2} + \frac{1}{\gamma^2}, \frac{2}{\gamma}\right\}. \tag{3.23}$$

From ([19], Thm. 2, p. 151) and (3.22), it follows that there exists a biorthogonal sequence $(\psi_m)_{m>N}$ to the family of exponential functions $(\mathrm{e}^{i\nu_n t})_{n>N}$ in $L^2(-T,T)$ with the following property

$$\int_{-T}^{T}\left|\sum_{m>N} b_m\psi_m\right|^2 \mathrm{d}t \le \frac{1}{K_1(\gamma,T)}\sum_{m>N}|b_m|^2, \tag{3.24}$$

for every sequence $(b_m)_{m>N} \in l^2$.

We'll modify the biorthogonal sequence $(\psi_m)_{m>N}$ such that it becomes orthogonal to the family $(\mathrm{e}^{i\nu_n t})_{1\le n\le N}$. For each $1 \le n \le N$ we define the function $\widehat{\rho}_n : \mathbb{C} \to \mathbb{C}$ given by

$$\widehat{\rho}_n(z) = \widehat{\varPhi}_{\frac{\varepsilon}{2}}\left(\frac{z-\nu_n}{N}\right) \qquad (z \in \mathbb{C}),$$

where $\widehat{\varPhi}_{\frac{\varepsilon}{2}}$ is the Fourier transform of the function $\varPhi_{\frac{\varepsilon}{2}}$ defined by (3.2). We remark that

$$\begin{aligned}|\widehat{\rho}_n(z)| &= \left|\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \exp\left(\frac{\left(\frac{\varepsilon}{2}\right)^3}{t^2-\left(\frac{\varepsilon}{2}\right)^2}\right)\exp\left(-i\,t\frac{z-\nu_n}{N}\right)\mathrm{d}t\right| \\ &= \frac{\varepsilon}{2}\left|\int_{-1}^{1}\exp\left(\frac{\frac{\varepsilon}{2}}{s^2-1}\right)\exp\left(-i\,s\frac{\varepsilon}{2}\frac{z-\nu_n}{N}\right)\mathrm{d}s\right| \le \varepsilon\exp\left(\frac{\varepsilon}{2N}|z|\right)\cdot\end{aligned} \tag{3.25}$$

Hence, $\widehat{\rho}_n$ is an entire function of exponential type at most $\frac{\varepsilon}{2N}$.

We denote

$$P(z) = \prod_{n=1}^{N}(1-\widehat{\rho}_n^2(z)) \qquad (z \in \mathbb{C}). \tag{3.26}$$

From (3.25) we deduce that there exists $C > 0$ such that

$$|P(z)| \le C\exp(\varepsilon|z|) \qquad (z \in \mathbb{C}). \tag{3.27}$$

Moreover, by using (3.4) we obtain that

$$|P(x)| \le \left|\prod_{n=1}^{N}\left(1-\widehat{\varPhi}^2_{\frac{\varepsilon}{2}}\left(\frac{x-\nu_n}{N}\right)\right)\right| \le 1 \qquad (x \in \mathbb{R}). \tag{3.28}$$

For every $m > N$, we define the function $\widehat{\theta}_m : \mathbb{C} \to \mathbb{C}$ given by

$$\widehat{\theta}_m(z) = \frac{\widehat{\psi}_m(z)P(z)}{P(\nu_m)} \qquad (z \in \mathbb{C}), \tag{3.29}$$

and let θ_m be the inverse Fourier transform of $\widehat{\theta}_m$,

$$\theta_m(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{ixt}\widehat{\theta}_m(x)\,\mathrm{d}x.$$

We remark that $\widehat{\theta}_m$ is an entire function of exponential type less than $T+\varepsilon$, since, according to (3.27), P is an entire function of exponential type ε and $\widehat{\psi}_m$ is an entire function of exponential type T.

According to Paley-Wiener's Theorem, θ_m has the property that that $\operatorname{supp}\theta_m \subset (-T-\varepsilon, T+\varepsilon)$. Also, we remark that, for any $m \geq N$, we have that

$$\widehat{\theta}_m(\nu_n) = \delta_{\mathrm{mn}} \qquad (n \geq 1). \tag{3.30}$$

Consequently we have that $(\theta_m)_{m>N}$ is a biorthogonal sequence to the family $(\mathrm{e}^{i\nu_m t})_{m>N}$ which is orthogonal to the family $(\mathrm{e}^{i\nu_n t})_{1\leq n\leq N}$ in $L^2(-T-\varepsilon, T+\varepsilon)$. Nextly, we estimate the norm of this biorthogonal sequence.

From (3.28), (3.29) and Plancherel's Theorem, for every sequence $(a_m)_{m>N} \in l^2$, we obtain that

$$\int_{-T-\varepsilon}^{T+\varepsilon}\left|\sum_{m>N} a_m\theta_m(t)\right|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{m>N} a_m\widehat{\theta}_m(x)\right|^2 \mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{m>N} a_m\frac{P(x)}{P(\nu_m)}\widehat{\psi}_m(x)\right|^2 \mathrm{d}x$$

$$\leq \frac{1}{2\pi}\int_{\mathbb{R}}\left|\sum_{m>N} \frac{a_m}{P(\nu_m)}\widehat{\psi}_m(x)\right|^2 \mathrm{d}x = \int_{-T}^{T}\left|\sum_{m>N} \frac{a_m}{P(\nu_m)}\psi_m(t)\right|^2 \mathrm{d}t.$$

From the above estimate and (3.24), we deduce that

$$\int_{-T-\varepsilon}^{T+\varepsilon}\left|\sum_{m>N} a_m\theta_m(t)\right|^2 \mathrm{d}t \leq \frac{1}{K_1(\gamma,T)}\sum_{m>N}\frac{|a_m|^2}{|P(\nu_m)|^2}. \tag{3.31}$$

Let us show that, for any $m \geq \mathcal{N}$, we have that $|P(\nu_m)| > \exp(-2)$. From (3.20) and since $N > 1$ we have that, for any $m \geq \mathcal{N}$,

$$\frac{\nu_m - \nu_n}{N} \geq \frac{\nu_m - \nu_N}{N} \geq \frac{400\ln^2 N}{\varepsilon^2} > 256 \qquad (1 \leq n \leq N).$$

From (3.5) we deduce that

$$|P(\nu_m)| \geq \prod_{n=1}^{N}\left|1 - 9\exp\left(-\frac{\varepsilon}{4}\sqrt{\frac{\nu_m-\nu_n}{N}}\right)\right| = \exp\left(\sum_{n=1}^{N}\ln\left|1-9\exp\left(-\frac{\varepsilon}{4}\sqrt{\frac{\nu_m-\nu_n}{N}}\right)\right|\right) \quad (m > \mathcal{N}).$$

Using (3.20), we have that, for any $N > 1$ and $m \geq \mathcal{N}$,

$$\exp\left(-\frac{\varepsilon}{4}\sqrt{\frac{\nu_m-\nu_n}{N}}\right) \leq \frac{1}{N^5} < \frac{1}{18}. \tag{3.32}$$

Since

$$\ln(1-t) \geq -2t \qquad \left(t \in \left(0, \frac{1}{2}\right)\right),$$

from (3.32) we deduce that, for any $m \geq \mathcal{N}$

$$|P(\nu_m)| \geq \exp\left(-18\sum_{n=1}^{N}\exp\left(-\frac{\varepsilon}{4}\sqrt{\frac{\nu_m-\nu_n}{N}}\right)\right) \geq \exp\left(-18N\exp\left(-\frac{\varepsilon}{4}\sqrt{\frac{\nu_m-\nu_N}{N}}\right)\right).$$

From the above inequality and (3.20) we obtain that, for any $m \geq \mathcal{N}$,

$$|P(\nu_m)| \geq \exp\{-18N\exp(-5\ln N)\} \geq \exp(-2). \tag{3.33}$$

Since $\nu_m \neq \nu_n$ for $N < m < \mathcal{N}$ and $1 \leq n \leq N$, by taking into account (3.4), it follows that there exists $c(\varepsilon, N)$ such that

$$|P(\nu_m)| \geq c(\varepsilon, N). \tag{3.34}$$

From (3.31), (3.33) and (3.34) we obtain that

$$\int_{-T-\varepsilon}^{T+\varepsilon} \left| \sum_{m>N} a_m \theta_m(t) \right|^2 \mathrm{d}t \leq \frac{c(\varepsilon, N)}{K_1(\gamma, T)} \sum_{N<m<\mathcal{N}} |a_m|^2 + \frac{\exp(-2)}{K_1(\gamma, T)} \sum_{m \geq \mathcal{N}} |a_m|^2. \tag{3.35}$$

The proof ends by taking into account (3.35) and (3.23). □

Remark 3.3. Let us emphasize that the constant in front of the last sum in (3.21) does not depend of N. This shows that the high biorthogonal elements are uniformly bounded independently of the gap of the family Λ_1.

As in ([18], Prop. 8.4.1), we can evaluate the distance between an exponential function and a family of exponential functions. Although this result is known, we choose to present it here, together with its proof, since we need explicit estimates of the constants appearing in this evaluation.

Lemma 3.4. *Let $T > 0$, $I \subset \mathbb{N}$ and $\Lambda = (\nu_n)_{n\in I} \subset \mathbb{R}$ be a sequence with the property that there exists $C_1, C_2 > 0$ such that*

$$C_1 \sum_{n\in I} |a_n|^2 \leq \int_{-T}^{T} \left| \sum_{n\in I} a_n \mathrm{e}^{i\nu_n t} \right|^2 \mathrm{d}t \leq C_2 \sum_{n\in I} |a_n|^2, \tag{3.36}$$

for every sequence $(a_n)_{n\in I} \in l^2$. Let $\mu \in \mathbb{R}$ be such that

$$\inf_{n\in I} |\mu - \nu_n| = d > 0. \tag{3.37}$$

Then, for every $\varepsilon \in (0, T]$ and for every sequence $(a_n)_{n\in I} \in l^2$, we have

$$\left\| \mathrm{e}^{i\mu t} - \sum_{n\in I} a_n \mathrm{e}^{i\nu_n t} \right\|_{L^2(-T-\varepsilon, T+\varepsilon)} \geq \sqrt{T+\varepsilon}\,\frac{\sigma\sqrt{C_1}}{2\sqrt{2C_2}}, \tag{3.38}$$

where $\sigma = \begin{cases} \frac{d^2\varepsilon^2}{\pi^2} & \text{if } \mathrm{d}\varepsilon \in (0, \frac{\pi}{2}) \\ \frac{1}{4} & \text{if } \mathrm{d}\varepsilon \geq \frac{\pi}{2}. \end{cases}$

Proof. Let $\varepsilon \in (0, T]$ and let $(a_n)_{n\in I}$ be an l^2 sequence. We consider the functions

$$q(t) = \mathrm{e}^{i\mu t} - \sum_{n\in I} a_n \mathrm{e}^{i\nu_n t} \qquad (t \in (-T-\varepsilon, T+\varepsilon)), \tag{3.39}$$

$$r(t) = q(t) - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-i\mu s} q(t+s)\mathrm{d}s \qquad (t \in (-\varepsilon, \varepsilon)). \tag{3.40}$$

A simple computation shows that

$$r(t) = \sum_{n\in I} a_n \left(\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{i(\nu_n-\mu)s}\mathrm{d}s - 1 \right) \mathrm{e}^{i\nu_n t}. \tag{3.41}$$

We remark that

$$\left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{i(\nu_n-\mu)s}\mathrm{d}s - 1\right| = \left|\frac{\sin(\nu_n-\mu)\varepsilon}{(\nu_n-\mu)\varepsilon} - 1\right| \geq \begin{cases} \dfrac{d^2\varepsilon^2}{\pi^2} & \text{if} \quad \mathrm{d}\varepsilon \in \left(0, \dfrac{\pi}{2}\right) \\ \dfrac{1}{4} & \text{if} \quad \mathrm{d}\varepsilon \geq \dfrac{\pi}{2} \end{cases} := \sigma, \tag{3.42}$$

where for the last inequality the following two estimates are used

$$1 - \frac{\sin x}{x} \geq \begin{cases} \dfrac{x^2}{\pi^2} & \text{if} \quad 0 < x < \dfrac{\pi}{2} \\ \dfrac{1}{\pi} & \text{if} \quad x \geq \dfrac{\pi}{2}. \end{cases}$$

Relation (3.42), combined with (3.36) and (3.41), implies that

$$\int_{-T}^{T} |r(t)|^2 \mathrm{d}t \geq \sigma^2 C_1 \sum_{n\in I} |a_n|^2. \tag{3.43}$$

On the other hand, from (3.40), by applying the Cauchy−Schwartz inequality, it follows that

$$\int_{-T}^{T} |r(t)|^2 \mathrm{d}t \leq 2\int_{-T}^{T} |q(t)|^2 \mathrm{d}t + \frac{1}{2\varepsilon^2}\int_{-T}^{T}\left|\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-i\mu s} q(t+s)\mathrm{d}s\right|^2 \mathrm{d}t$$

$$\leq 2\int_{-T}^{T} |q(t)|^2 \mathrm{d}t + \frac{1}{\varepsilon}\int_{-T}^{T}\int_{-\varepsilon}^{\varepsilon} |q(t+s)|^2 \,\mathrm{d}s\mathrm{d}t = 2\int_{-T}^{T} |q(t)|^2 \mathrm{d}t + \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-T+s}^{T+s} |q(\tau)|^2 \,\mathrm{d}\tau\mathrm{d}s$$

$$\leq 2\int_{-T}^{T} |q(t)|^2 \mathrm{d}t + \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-T-\varepsilon}^{T+\varepsilon} |q(\tau)|^2 \,\mathrm{d}\tau\mathrm{d}s \leq 4\int_{-T-\varepsilon}^{T+\varepsilon} |q(t)|^2 \,\mathrm{d}t.$$

The above inequality, combined with (3.43), implies that

$$\sum_{n\in I} |a_n|^2 \leq \frac{4}{\sigma^2 C_1}\int_{-T-\varepsilon}^{T+\varepsilon} |q(t)|^2 \mathrm{d}t.$$

From the last estimate we obtain that

$$\int_{-T-\varepsilon}^{T+\varepsilon} |q(t)|^2 \mathrm{d}t \geq \frac{\sigma^2 C_1}{8C_2}\int_{-T-\varepsilon}^{T+\varepsilon}\left|\sum_{n\in I} a_n \mathrm{e}^{i\nu_n t}\right|^2 \mathrm{d}t. \tag{3.44}$$

In (3.44) we have taken into account that $\varepsilon \leq T$ and we have used the following form of the second inequality in (3.36) (with the same constant C_2)

$$\int_{-2T}^{2T}\left|\sum_{n\in I} a_n \mathrm{e}^{i\nu_n t}\right|^2 \mathrm{d}t \leq 2C_2 \sum_{n\in I} |a_n|^2.$$

If $\left\|\sum_{n\in I} a_n \mathrm{e}^{i\nu_n t}\right\|_{L^2(-T-\varepsilon, T+\varepsilon)} \geq \sqrt{T+\varepsilon}$, from (3.44) we have

$$\|q\|^2_{L^2(-T-\varepsilon, T+\varepsilon)} \geq \frac{\sigma^2 C_1}{8C_2}(T+\varepsilon).$$

If $\left\|\sum_{n\in I} a_n e^{i\nu_n t}\right\|_{L^2(-T-\varepsilon,T+\varepsilon)} < \sqrt{T+\varepsilon}$, from (3.39) we have

$$\|q\|_{L^2(-T-\varepsilon,T+\varepsilon)} \geq \left\|e^{i\mu t}\right\|_{L^2(-T-\varepsilon,T+\varepsilon)} - \left\|\sum_{n\in I} a_n e^{i\nu_n t}\right\|_{L^2(-T-\varepsilon,T+\varepsilon)} \geq \sqrt{2(T+\varepsilon)} - \sqrt{T+\varepsilon}.$$

Since $\sqrt{2}-1 > \sigma\dfrac{\sqrt{C_1}}{2\sqrt{2C_2}}$, from the last two estimates we obtain that (3.38) holds. □

Now we have all the ingredients needed to prove our main result on the existence of a biorthogonal sequence to the entire family $(e^{i\lambda_n t})_{|n|\geq 1}$ in $L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$, where λ_n are the eigenvalues given by (2.11).

Theorem 3.5. *Given $a \in L^\infty(0,1)$ with $a \geq 0$ a.e. in $(0,1)$ and $\|a\|_{L^\infty} > 1$, let $(i\lambda_n)_{|n|\geq 1}$ be eigenvalues of the operator given by (2.9) and (2.10). Let $T > 1$, $\varepsilon \in (0,1)$, $\delta \in \left(0, \frac{1}{2\pi}\right)$, $\gamma_0 \in (0,\pi)$ be given by (2.13),*

$$N = \left[\frac{\|a\|_{L^\infty} + \sqrt{\|a\|_{L^\infty}(\pi-\delta)}}{\pi\delta}\right], \tag{3.45}$$

and

$$\mathcal{N} = \min\left\{m > N \mid \lambda_m - \lambda_N > \frac{400N}{\varepsilon^2}\ln^2 N\right\}. \tag{3.46}$$

There exists a biorthogonal sequence $(\eta_m)_{|m|\geq 1}$ to the family of exponentials $(e^{i\lambda_n t})_{|n|\geq 1}$ in $L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$ such that, for any finite sequence $(a_m)_{|m|\geq 1}$, we have that

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left|\sum_{|m|\geq 1} a_m \eta_m(t)\right|^2 dt \leq L(N,T,\gamma_0,\varepsilon) \sum_{1\leq |m|\leq \mathcal{N}} |a_m|^2 + \frac{T\exp(-2)}{4\pi\left(\frac{T^2}{4} - \frac{\pi^2}{\gamma_0^2}\right)} \sum_{|m|>\mathcal{N}} |a_m|^2, \tag{3.47}$$

where $L(N,T,\gamma_0,\varepsilon)$ is a positive constant which depends on ε, T, γ_0 and N.

Proof. Firstly we construct a biorthogonal sequence $(\theta_m)_{|m|>N}$ to the family $\Lambda_2 = (e^{i\lambda_n t})_{|n|>N}$ in $L^2(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon)$ which is orthogonal to the family $\Lambda_1 = (e^{i\lambda_n t})_{1\leq|n|\leq N}$.

According to Remark 2.4, the exponents of the family $\left(e^{i\lambda_n t}\right)_{|n|>N}$ have a gap γ_0. Therefore we can apply Theorem 3.2 to obtain that, for $T > \frac{\pi}{\gamma_0}$ and for every $\varepsilon > 0$, there exists a biorthogonal sequence $(\theta_m)_{|m|>N}$ to the family of exponential functions $(e^{i\lambda_m t})_{|m|>N}$ in $L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$ which is orthogonal to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$, and we have that

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left|\sum_{|m|>N} a_m \theta_m(t)\right|^2 dt \leq \frac{Tc(\varepsilon,N)}{\frac{T^2}{4} - \frac{\pi^2}{\gamma_0^2}} \sum_{N<|m|\leq\mathcal{N}} |a_m|^2 + \frac{T\exp(-2)}{4\pi\left(\frac{T^2}{4} - \frac{\pi^2}{\gamma_0^2}\right)} \sum_{|m|>\mathcal{N}} |a_m|^2, \tag{3.48}$$

for any sequence $(a_n)_n$ in l^2, where $c(\varepsilon,N)$ is a constant which depends on ε and N but it is independent of γ_0 and T.

Secondly, we construct a biorthogonal sequence $(\zeta_k)_{1\leq|k|\leq N}$ to the family $\Lambda_1 = \left(e^{i\lambda_n t}\right)_{1\leq|n|\leq N}$ which is orthogonal to the family $\Lambda_2 = \left(e^{i\lambda_m t}\right)_{|m|>N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$. For each $1\leq|k|\leq N$ we define $G_k \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ as follows:

$$G_k(t) = \begin{cases} \dfrac{e^{i\lambda_k t} - P_{E_{\Lambda_2}} e^{i\lambda_k t}}{\left\|e^{i\lambda_k t} - P_{E_{\Lambda_2}} e^{i\lambda_k t}\right\|^2_{L^2\left(-\frac{T}{2}-\frac{\varepsilon}{2}, \frac{T}{2}+\frac{\varepsilon}{2}\right)}} & \text{if} \quad t \in \left(-\dfrac{T}{2}-\dfrac{\varepsilon}{2}, \dfrac{T}{2}+\dfrac{\varepsilon}{2}\right) \\ 0 & \text{if} \quad t \in \left(-\infty, -\dfrac{T}{2}-\dfrac{\varepsilon}{2}\right] \cup \left[\dfrac{T}{2}+\dfrac{\varepsilon}{2}, \infty\right), \end{cases} \tag{3.49}$$

where $P_{E_{\Lambda_2}}\mathrm{e}^{i\lambda_k t}$ is the orthogonal projection of $\mathrm{e}^{i\lambda_k t}$ over the closure E_{Λ_2} of the linear span of the family $\left(\mathrm{e}^{i\lambda_m t}\right)_{|m|>N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$. From the projection's orthogonality properties we have that, for any $1\le|k|\le N$,

$$\begin{aligned}&\widehat{G}_k(\lambda_k)=1\\&\widehat{G}_k(\lambda_m)=0 \qquad (|m|>N).\end{aligned} \tag{3.50}$$

From (2.13) and Ingham's Theorem (see [9]) we deduce that, for every $\frac{T}{2}>\frac{\pi}{\gamma_0}$ there exist two constants $K_1(\gamma_0,T)=\frac{8\pi}{T}\left(\frac{T^2}{4}-\frac{\pi^2}{\gamma_0^2}\right)$, $K_2(\gamma_0,T)=8\max\left\{\frac{T^2}{\pi^2}+\frac{1}{\gamma_0^2},\frac{2}{\gamma_0}\right\}>0$ such that

$$K_1(\gamma_0,T)\sum_{|m|>N}|a_m|^2\le\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{|m|>N}a_m\mathrm{e}^{i\lambda_m t}\right|^2\mathrm{d}t\le K_2(\gamma_0,T)\sum_{|m|>N}|a_m|^2, \tag{3.51}$$

for every sequence $(a_m)_m\in l^2$.

By using (3.51) and the fact that $\inf\limits_{\substack{|m|>N\\1\le|n|\le N}}|\lambda_m-\lambda_n|\ge\gamma_0$ we can apply Lemma 3.4 to obtain that for each $1\le|k|\le N$ and $x\in\mathbb{R}$

$$\begin{aligned}|\widehat{G}_k(x)|&=\frac{1}{\left\|\mathrm{e}^{i\lambda_k t}-P_{E_{\Lambda_2}}\mathrm{e}^{i\lambda_k t}\right\|^2_{L^2\left(-\frac{T}{2}-\frac{\varepsilon}{2},\frac{T}{2}+\frac{\varepsilon}{2}\right)}}\left|\int_{-\frac{T}{2}-\frac{\varepsilon}{2}}^{\frac{T}{2}+\frac{\varepsilon}{2}}(\mathrm{e}^{i\lambda_k t}-P_{E_{\Lambda_2}}\mathrm{e}^{i\lambda_k t})\mathrm{e}^{-itx}\mathrm{d}t\right|\\&\le\frac{\sqrt{T+\varepsilon}}{\|\mathrm{e}^{i\lambda_k t}-P_{E_{\Lambda_2}}\mathrm{e}^{i\lambda_k t}\|_{L^2\left(-\frac{T}{2}-\frac{\varepsilon}{2},\frac{T}{2}+\frac{\varepsilon}{2}\right)}}\le\frac{\sqrt{T+\varepsilon}}{\sqrt{\frac{T}{2}+\frac{\varepsilon}{2}}\frac{\sigma\sqrt{K_1(\gamma_0,T)}}{2\sqrt{2K_2(\gamma_0,T)}}},\end{aligned}$$

where $\sigma=\begin{cases}\frac{1}{4\pi^2}\gamma_0^2\varepsilon^2 & \text{if } \gamma_0\varepsilon\in(0,\pi)\\ \frac{1}{4} & \text{if } \gamma_0\varepsilon\in[\pi,\infty).\end{cases}$

Thus, we deduce that

$$|\widehat{G}_k(x)|\le\frac{4\sqrt{K_2(\gamma_0,T)}}{\sqrt{K_1(\gamma_0,T)}\gamma_0^2\varepsilon^2}:=K(\gamma_0,T,\varepsilon) \qquad (1\le|k|\le N). \tag{3.52}$$

Since the family $\left(\mathrm{e}^{i\lambda_n t}\right)_{1\le|n|\le N}$ is finite it follows that for every $\varepsilon>0$ there exist $K_1(N,\varepsilon)$ and $K_2(N,\varepsilon)$ such that

$$K_1(N,\varepsilon)\sum_{1\le|n|\le N}|a_n|^2\le\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\left|\sum_{1\le|n|\le N}a_n\mathrm{e}^{i\lambda_n t}\right|^2\mathrm{d}t\le K_2(N,\varepsilon)\sum_{1\le|n|\le N}|a_n|^2, \tag{3.53}$$

for any finite sequence $(a_n)_n$.

From ([19], Thm. 2, p. 151) and (3.53), it follows that there exists a biorthogonal sequence $(\Psi_k)_{1\le|k|\le N}$ to the family of exponential functions $\Lambda_1=\left(\mathrm{e}^{i\lambda_n t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)$ with the following property

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}|\Psi_k(t)|^2\,\mathrm{d}t\le\frac{1}{\sqrt{K_1(N,\varepsilon)}} \qquad (1\le|k|\le N). \tag{3.54}$$

We modify the sequence $(\Psi_k)_{1\le|k|\le N}$ such that to become orthogonal to $\Lambda_2=\left(\mathrm{e}^{i\lambda_m t}\right)_{|m|>N}$. To this aim, for every $1\le|k|\le N$, we define the function $\zeta_k=\Psi_k*G_k$. Since $\mathrm{supp}(\Psi_k)\subset\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]$ and $\mathrm{supp}(G_k)\subset\left[-\frac{T}{2}-\frac{\varepsilon}{2},\frac{T}{2}+\frac{\varepsilon}{2}\right]$ it follows that $\mathrm{supp}(\zeta_k)\subset\left[-\frac{T}{2}-\varepsilon,\frac{T}{2}+\varepsilon\right]$. Moreover, from the fact that $(\Psi_k)_{1\le|k|\le N}$ is a biorthogonal sequence to the family of exponential functions $\Lambda_1 = \left(\mathrm{e}^{i\lambda_n t}\right)_{1\leq|n|\leq N}$ in $L^2\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ and (3.50) we deduce that, for any $1 \leq |k| \leq N$, we have

$$\widehat{\zeta}_k(\lambda_m) = 0 \qquad (|m| > N). \tag{3.55}$$

Hence, $(\zeta_k)_{1\leq|k|\leq N}$ is a biorthogonal sequence to the family Λ_1 in $L^2(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon)$ which is orthogonal to the family Λ_2 in $L^2(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon)$.

By applying Plancherel's Theorem and from (3.52), (3.54) we obtain that

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left| \sum_{1\leq|k|\leq N} a_k \zeta_k(t) \right|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \sum_{1\leq|k|\leq N} a_k \widehat{\zeta}_k(x) \right|^2 \mathrm{d}x$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \sum_{1\leq|k|\leq N} |a_k| \|\widehat{G}_k\|_{L^\infty} |\widehat{\Psi}_k(x)| \right|^2 \mathrm{d}x \leq \frac{K^2(\gamma_0, T, \varepsilon)}{2\pi} \int_{\mathbb{R}} \left(\sum_{1\leq|k|\leq N} |a_k|^2 \right) \left(\sum_{1\leq|k|\leq N} |\widehat{\Psi}_k(x)|^2 \right) \mathrm{d}x$$

$$= K^2(\gamma_0, T, \varepsilon) \left(\sum_{1\leq|k|\leq N} |a_k|^2 \right) \left(\sum_{1\leq|k|\leq N} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} |\Psi_k(t)|^2 \mathrm{d}t \right) \leq \frac{2NK^2(\gamma_0, T, \varepsilon)}{\sqrt{K_1(N, \varepsilon)}} \sum_{1\leq|k|\leq N} |a_k|^2,$$

for any finite sequence $(a_k)_{1\leq|k|\leq N}$. Thus, we deduce that

$$\int_{-\frac{T}{2}-\varepsilon}^{\frac{T}{2}+\varepsilon} \left| \sum_{1\leq|k|\leq N} a_k \zeta_k(t) \right|^2 \mathrm{d}t \leq \frac{2NK^2(\gamma_0, T, \varepsilon)}{\sqrt{K_1(N, \varepsilon)}} \sum_{1\leq|k|\leq N} |a_k|^2, \tag{3.56}$$

for any finite sequence $(a_k)_{k\in F}$.

Finally, we can construct the desired biorthogonal sequence to the whole family $(\mathrm{e}^{i\lambda_n t})_{|n|\geq 1}$. Indeed, from (3.30) and (3.55) we obtain that $(\eta_m)_{|m|\geq 1} = (\zeta_k)_{1\leq|k|\leq N} \cup (\theta_m)_{|m|>N}$ is a biorthogonal sequence to the family of exponential functions $(\mathrm{e}^{i\lambda_n t})_{|n|\geq 1}$ in $L^2\left(-\frac{T}{2}-\varepsilon, \frac{T}{2}+\varepsilon\right)$. Moreover from (3.48) combined with (3.56), we deduce that (3.47) holds and the proof of the theorem is complete. □

Remark 3.6. It is easy to see that there exist two positive constants C_1 and C_2 such that

$$C_1\|a\|_{L^\infty} \leq N \leq C_2\|a\|_{L^\infty}. \tag{3.57}$$

Since, from (3.46) we have that

$$|\lambda_{\mathcal{N}} - \lambda_N| > \frac{400N}{\varepsilon^2} \ln^2 N \geq |\lambda_{\mathcal{N}-1} - \lambda_N|,$$

by taking into account (2.6) and (3.57), we deduce that there exist two positive constants ρ_1 and ρ_2 such that

$$\rho_1\|a\|_{L^\infty} \ln^2 \|a\|_{L^\infty} \leq \mathcal{N} \leq \rho_2\|a\|_{L^\infty} \ln^2 \|a\|_{L^\infty}.$$

4. Control results

This section is devoted to study how the control norm depends on the potential a in (1.1). The main result is the following

Theorem 4.1. *Let $T>2$. There exists $M=M(T)>0$ such that, for any $a\in L^\infty(0,1)$, $a(x)\geq 0$ a. e in $(0,1)$ with $\|a\|_{L^\infty}>1$ and $\binom{u_0}{u_1}\in\mathcal{H}_1$, there exists a control $v\in L^2(0,T)$ for (1.1) with the property that*

$$\|v\|_{L^2(0,T)}\leq M\left\|\begin{pmatrix}u^0\\u^1\end{pmatrix}\right\|_{\mathcal{H}},\tag{4.1}$$

where the linear space $\mathcal{H}_1$ is defined by

$$\mathcal{H}_1=\left\{\begin{pmatrix}u_0\\u_1\end{pmatrix}\in\mathcal{H}\;\middle|\;\begin{pmatrix}u_0\\u_1\end{pmatrix}=\sum_{|n|>\mathcal{N}}a_n i\,\mathrm{sgn}(n)\lambda_n\phi_n\right\},\tag{4.2}$$

and $\mathcal{N}$ is given by (3.46) from Theorem 3.5.

Proof. Given $a\in L^\infty(0,1)$, let $(\eta_m)_{|m|\geq 1}$ be the biorthogonal given by Theorem 3.5. Let $\binom{u^0}{u^1}\in\mathcal{H}_1$ be given by

$$\begin{pmatrix}u_0\\u_1\end{pmatrix}=\sum_{|n|>\mathcal{N}}a_n i\,\mathrm{sgn}(n)\lambda_n\phi_n,\tag{4.3}$$

and let $v\in L^2(0,T)$ be the control defined as in (2.20) from Lemma 2.7. We have that

$$\left\|\begin{pmatrix}u^0\\u^1\end{pmatrix}\right\|^2_{\mathcal{H}}=\sum_{|n|>\mathcal{N}}|a_n^0|^2.\tag{4.4}$$

To estimate the norm of v we analyze the right hand side of (2.20). By using (3.47) we have that

$$\int_0^T|v(t)|^2\mathrm{d}t=\int_0^T\left|\sqrt{2}\sum_{n>\mathcal{N}}\frac{i\lambda_n\mathrm{e}^{i\lambda_n\frac{T}{2}}}{\overline{\varphi'_{|n|}}(1)}a_n^0\theta_n\left(t-\frac{T}{2}\right)\right|^2\mathrm{d}t\leq\frac{\widetilde{M}}{\inf_{|n|>\mathcal{N}}\left|\frac{\varphi_n'(1)}{\lambda_n}\right|^2}\sum_{|n|>\mathcal{N}}|a_n^0|^2,$$

where $\widetilde{M}$ is a positive constant depending only of T. From the last inequality and (2.16) we deduce that there exists $M=M(T)>0$ such that

$$\int_0^T|v(t)|^2\mathrm{d}t\leq M\sum_{|n|>\mathcal{N}}|a_n^0|^2.$$

The above estimate, combined with (4.4), implies (4.1) and the proof is complete. □

Remark 4.2. The main novelty of Theorem 4.1 is the fact that the constant M which bounds the L^2-norm of the controls does not depend on the potential a, if the initial data belong to $\mathcal{H}_1$. Hence, when $\|a\|_{L^\infty}$ is very large, the controllability cost of initial data in the space $\mathcal{H}_1$ is much smaller that the one of arbitrary initial data in $\mathcal{H}$. On the other hand, we should notice that the space $\mathcal{H}_1$ becomes smaller and smaller when $\|a\|_{L^\infty}$ tends to infinity.

Remark 4.3. From Theorem 3.5 we can deduce as above that, given $T>2$ and $a\in L^\infty(0,1)$, $a(x)\geq 0$ a. e in $(0,1)$ with $\|a\|_{L^\infty}>1$, we can find a constant $L=L(T,\|a\|_{L^\infty})$ such that, for each $\binom{u^0}{u^1}\in\mathcal{H}$, there exists a control $v\in L^2(0,T)$ for (1.1) verifying

$$\|v\|_{L^2(0,T)}\leq L\left\|\begin{pmatrix}u^0\\u^1\end{pmatrix}\right\|_{\mathcal{H}}.\tag{4.5}$$

Moreover, we have that

$$L(T,\|a\|_{L^\infty}) \leq M\exp\left(\omega\|a\|_{L^\infty}\right), \tag{4.6}$$

where $M = M(T)$ and ω are positive constants independent of a. This indicates that, outside the space $\mathcal{H}_1$, the control cost may increase exponentially with $\|a\|_{L^\infty}$.

Remark 4.4. There are initial data $\left(\begin{smallmatrix} u^0 \\ u^1 \end{smallmatrix}\right) \in \mathcal{H}\backslash\mathcal{H}_1$ which can be controlled with controls bounded independently of the potential function a. A trivial example is provided by the initial data

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \sum_{|n|\geq 1} a_n i\,\mathrm{sgn}(n)\lambda_n\phi_n \in \mathcal{H},$$

where the Fourier coefficients a_n verify

$$|a_n| \leq \left(L(T,\|a\|_{L^\infty})\right)^{-1} \qquad (n\in\mathbb{Z}^*). \tag{4.7}$$

In (4.7) the constant $L(T,\|a\|_{L^\infty})$ is the one from (4.5) and, consequently, there exists a control $v\in L^2(0,T)$ for (1.1) uniformly bounded in a. We have chosen to state our Theorem 4.1 in terms of the space $\mathcal{H}_1$ given by (4.2), since we think that it is of interest to emphasize that all the sufficiently high frequency initial data have this uniform boundedness property.

5. Further comments and open problems

We close this paper with some comments and open problems:

- **Possible extensions to other type of equations.** Results similar to Theorem 4.1 can be obtained for other types of linear hyperbolic equations. Indeed, since our principal tool is Theorem 3.2, which refers to a quite general family of exponential functions $\left(\mathrm{e}^{i\nu_n t}\right)_{n\geq 1}$, any problem with a spectrum verifying (3.17)−(3.19) can be treated much in the same way. Roughly speaking, we have to show that the eigenvalues $i\nu_n$ of the corresponding differential operator have a strictly positive asymptotic gap. This is the case, for instance, of the linear Schrödinger or beam equations in which a variable potential has been introduced. Perturbations of the classical one dimensional hyperbolic equations with some integral operators (as, for instance, in [2,5]) can be considered, too. Also, some classes of hyperbolic systems like the ones studied in ([15], Sect. 4) can be similarly treated.
  The properties studied in this paper, characterized by a linear dependence of the controls on the initial data, are probably not true in the case of nonlinear equations and systems. On the other hand, an interesting question is related to their validity for parabolic equations. In the case of hyperbolic systems, the presence of a potential function affects primarily the gap of the low part of the spectrum of the corresponding differential operator. A direct consequence of this is the increment of the control cost which becomes of the order of $\exp(C\sqrt{\|a\|_{L^\infty}})$. It is known that, in the case of the linear heat equation with a potential a, the control cost may be estimated from above by a constant of the order of $\exp[C(T\|a\|_{L^\infty} + \|a\|_{L^\infty}^{2/3})]$ (see, for instance, [3,6,8]). For instance, a positive potential a makes diffusion happen faster and consequently produces an increment of the control cost. However, by taking into account the strong dissipative effect of the heat equation, one can easily deduce that the controls of the sufficiently high frequency initial data are bounded independently of the potential a. Hence, the same conclusion as in the hyperbolic case holds for the linear heat equation with a potential, but its justification follows directly from the dissipative character of this equation.

- **Multidimensional problems.** Generally, the techniques used in this paper are not appropriate for the study of multidimensional problems where the controls are functions of more than one variable, both temporal and spatial. However, in some simple cases in which separation of variables can be performed and the problem can be reduced to a 1-dimensional one, some interesting results concerning their behavior may be obtained. For instance, if we consider the wave equation in a rectangular two-dimensional domain $\Omega \subset \mathbb{R}^2$ with a constant potential function $a \in \mathbb{R}_+$, similar arguments allow to show that the cost to control sufficiently high frequency initial data is bounded independently of the potential a. If this result extends to potential functions in separable variables of the form $a(x,y) = a_1(x)a_2(y)$, $(x,y) \in \Omega$ or to more general domains is an open problem. Let us mention that an argument related to the one in Theorem 4.1 allowed us to give a Fourier characterization of the controllable spaces of initial data and to construct particular controls for them in the case of the wave equation without potential posed in the two dimensional unit square $(0,1) \times (0,1)$ with controls acting on one or two adjacent edges. By separating the variables x and y, we reduced the problem to the one dimensional wave equation (1.1) in the $x-$variable with a constant potential $a = m^2\pi^2$ depending on the fixed frequency $m \in \mathbb{N}^*$ of oscillation in the $y-$axis direction. In this particular case we manage to show a result similar to the one in Theorem 4.1 with $\mathcal{N} = \sigma m = \frac{\sigma}{\pi}\sqrt{\|a\|_{L^\infty}}$ in the definition (4.2) of the space $\mathcal{H}_1$, where σ is any positive constant. Notice that, in this particular problem, $\mathcal{N}$ is of order of $\sqrt{\|a\|_{L^\infty}}$. We think that, in the general case of equation (1.1) with variable potential, a similar result should be true, but this is still an open problem.
- **Possible applications.** The main result obtained in this paper may have a particular relevance for those models in which one is interested in controlling a limited but high frequency range of oscillations. This is the case, for instance, of the high frequency narrow band dental drill noise [17] or the microwaves in fiber-optic communications [16]. According to our Theorem 4.1 one can design control devices active on some high frequency range of interest with a cost independent of the low frequencies oscillations presented in the system.

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