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SEMICLASSICAL GROUND STATE SOLUTIONS FOR A CHOQUARD TYPE EQUATION IN $\mathbb{R}^2$ WITH CRITICAL EXPONENTIAL GROWTH*

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Abstract. In this paper we study a nonlocal singularly perturbed Choquard type equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-2} \left[\frac{1}{|x|^\mu} * \big(P(x)G(u)\big) \right] P(x)g(u)$$

in $\mathbb{R}^2$, where ε is a positive parameter, $\frac{1}{|x|^\mu}$ with $0 < \mu < 2$ is the Riesz potential, $*$ is the convolution operator, $V(x)$, $P(x)$ are two continuous real functions and $G(s)$ is the primitive function of $g(s)$. Suppose that the nonlinearity g is of critical exponential growth in $\mathbb{R}^2$ in the sense of the Trudinger-Moser inequality, we establish some existence and concentration results of the semiclassical solutions of the Choquard type equation in the whole plane. As a particular case, the concentration appears at the maximum point set of $P(x)$ if $V(x)$ is a constant.

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1. Introduction and main results

We are interested in a singularly perturbed nonlocal Choquard type equation

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-N} \left[\dfrac{1}{|x|^\mu} * \big(P(x)G(u)\big) \right] P(x)g(u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $N \geq 2$, $0 < \mu < N$, V and Q are two positive continuous functions, $G(s)$ is the primitive function of $g(s)$. This nonlocal equation arises in many interesting physical situations in quantum theory and plays an important role in describing the finite-range many-body interactions. For example, equation (1.1) was investigated by Pekar in [30] in the quantum theory of a polaron at rest and it was also used by P. Choquard in [21] as an approximation to Hartree-Fock theory of one-component plasma. Mathematically, if $N = 3$, $\mu = 1$, $\varepsilon = 1$, $V(x) = 1$ and $G(s) = |s|^2$, Lieb [21] and Lions [22] considered

$$-\Delta u + u = \left[\frac{1}{|x|^\mu} * |u|^2 \right] u \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

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and investigated the existence and uniqueness of positive solutions by variational methods. [27,28] investigated the qualitative properties of solutions and showed the regularity, positivity and radial symmetry decay behavior at infinity.

For a classical nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = g(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

an interesting issue is the existence of the semiclassical states, *i.e.* existence of solutions for equation (1.3) with small positive parameter ε. From the point of view of physics, semiclassical states are used to describe the transition between Quantum Mechanics and classical Mechanics. The first result in this direction goes back to the pioneering work [18] by Floer and Weinstein. Since then, it has been studied extensively under various hypotheses on the potential and the nonlinearity, see for example [9, 13, 15, 18, 19, 31, 32, 34] and the references therein.

For equation (1.1), the question of the existence of semiclassical solutions for the nonlocal Choquard type equation has been posed more recently ([8], p. 29). It can be observed that if u is a solution, for $x_0 \in \mathbb{R}^N$, then the function $v = u(x_0 + \varepsilon x)$ satisfies

$$-\Delta v + V(x_0 + \varepsilon x)v = \left[\frac{1}{|x|^\mu} * \big(P(x_0 + \varepsilon x)G(v)\big)\right] P(x_0 + \varepsilon x)g(v) \qquad \text{in} \qquad \mathbb{R}^N.$$

This suggests some convergence, as $\varepsilon \to 0$, of the family of solutions to a solution u_0 of the limit problem

$$-\Delta v + V(x_0)v = P^2(x_0)\left[\frac{1}{|x|^\mu} * G(v)\right] g(v) \qquad \text{in} \qquad \mathbb{R}^N. \tag{1.4}$$

One of the interesting problems is to apply the Lyapunov-Schmidt type reduction arguments to study (1.1). However, to the best of our knowledge, little is known about the uniqueness and non-degeneracy of the ground states of the limit problem

$$-\Delta u + u = \left[\frac{1}{|x|^\mu} * G(u)\right] g(u), \qquad \text{in} \qquad \mathbb{R}^N.$$

Only for $N = 3$, $\mu = 1$ and $G(s) = |s|^2$, the uniqueness and non-degeneracy of the ground states were proved by Lenzmann, Wei and Winter in [25, 33]. By assuming that $\inf V > 0$ and $Q(x) = 1$, Wei and Winter also constructed families of solutions by Lyapunov-Schmidt reduction arguments. To study the existence of semiclassical states by pure variational techniques, Cingolani *et al.* in [14] applied the penalization arguments due to Byeon and Jeanjean [10] to show the existence of a family of multipeak solutions located around the minimum points of the potential $V(x)$. Recently, the author of the present paper and his colleagues also applied the penalization arguments in [36] to construct multi-peak solutions for the nonlinear Choquard equation with general assumptions on the nonlinearities g. For $N \geq 3$ and $G(u) = u^p$ with $\frac{2N-\mu}{N} \leq p < \frac{2N-\mu}{N-2}$, under some additional assumptions at infinity on the potential $V(x)$, Moroz and Van Schaftingen [29] used variational methods and developed a novel nonlocal penalization technique to show that the equation (1.1) has a family of solutions concentrating around the local minimum of the function V. In [5, 6], Alves and Yang proved the existence, multiplicity and concentration of solutions for the subcritical Choquard equation by penalization method and Lusternik−Schnirelmann theory.

From the existing results mentioned above, most of the works are set in $\mathbb{R}^N$ with $N \geq 3$ and subcritical growth nonlinearities ($N = 2$ with polynomial growth). For the case $N \geq 3$, by Sobolev imbedding, the subcritical and critical growth mean that the nonlinearity can not exceed the polynomial of degree $2^* = \frac{2N}{N-2}$. However, the case $N = 2$ is special, since the corresponding Sobolev embedding says that $H_0^1(\Omega) \subset L^q(\Omega)$ for all $q \geq 1$, but $H_0^1(\Omega) \nsubseteq L^\infty(\Omega)$, where Ω is a bounded domain. Once that we intend to work with nonlinearity with *critical growth* in the plane, we need to recall the Trudinger-Moser inequality. A version on a bounded domain Ω in $\mathbb{R}^2$

says that, for all $\alpha > 0$ and $u \in H_0^1(\Omega)$, $e^{\alpha u^2} \in L^1(\Omega)$. Moreover, there exists a positive constant C such that

$$\sup_{u \in H_0^1(\Omega)\,:\,\|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2} \le C|\Omega| \qquad \text{if} \quad \alpha \le 4\pi,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. This inequality is optimal in the sense that for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the correspondent supremum is infinite. Thus we say that function $g(s)$ has *critical exponential growth* when it behaves like $e^{\alpha s^2}$ as $|s| \to +\infty$. More exactly,

$$\lim_{|s|\to+\infty} \frac{|g(s)|}{e^{\alpha s^2}} = 0, \quad \forall \alpha > 4\pi, \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{|g(s)|}{e^{\alpha s^2}} = +\infty, \quad \forall \alpha < 4\pi. \tag{1.5}$$

The nonlinearity is said to be of *subcritical growth* if for any $\alpha > 0$,

$$\lim_{|s|\to+\infty} \frac{|g(s)|}{e^{\alpha s^2}} = 0.$$

The above notion of criticality was introduced by Adimurthi and Yadava [1], see also de Figueiredo, Miyagaki and Ruf [16]. To study the elliptic problem in $\mathbb{R}^2$ with critical exponential growth, the Trudinger-Moser inequality on the whole space plays an important role, we refer the readers to [2,23,24] for recent progress in the literature of existence of solutions. However, it seems that there are not so many papers discussing the semiclassical problems for equation with critical exponential growth nonlinearities. For the N-Laplacian equation set in $\mathbb{R}^N$, Alves and Figueiredo [3] studied the multiplicity of semiclassical solutions with Rabinowitz type assumption on the potential. do Ó and Severo [17] also studied a class of quasilinear Schrödinger equations in $\mathbb{R}^2$ with critical exponential growth. The nonlocal Schrödinger equation in the plane was firstly considered in [4], there the authors first established the existence of ground states for a periodic problem with critical exponential growth and then studied the concentration around the global minimum set of the potential $V(x)$.

The purpose of the present paper is to study the existence of semiclassical ground state solutions for a Choquard type equation in $\mathbb{R}^2$ with critical exponential nonlinearities. In fact, we are going to study the equation of the form

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-2} \left[\dfrac{1}{|x|^\mu} * \big(P(x)G(u)\big) \right] P(x)g(u), \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{1.6}$$

where $g(u) = h(u) + f(u)$, h is of subcritical exponential growth at infinity and f is of critical exponential growth. We want to establish some existence results of the semiclassical solutions for equation (1.6) and to describe certain concentration phenomena of these solutions at particular sets characterized by the potentials $V(x)$ and $P(x)$. As a particular case we can observe that the concentration phenomena appears at the maximum point set of $P(x)$.

Since we are going to study the nonlocal type problems with Riesz potential, we would like to recall the famous Hardy−Littlewood−Sobolev inequality.

Proposition 1.1 ([20], Hardy−Littlewood−Sobolev inequality)**.**
Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$. Let $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(s, N, \mu, r)$, independent of f, h, such that

$$\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * f(x) \right] h(x) \le C(s, N, \mu, r) |f|_s |h|_r.$$

Remark 1.2. By Hardy−Littlewood−Sobolev inequality,

$$\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * G(u) \right] G(u)$$

is well defined if $G(u) \in L^s(\mathbb{R}^2)$ for $s > 1$ defined by

$$\frac{2}{s} + \frac{\mu}{2} = 2.$$

That means, we must require

$$G(u) \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2).$$

Let E be the Sobolev space $H^1(\mathbb{R}^2)$ equipped with the standard norm

$$\|u\| := \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2 \right) \right)^{1/2}$$

and $L^s(\mathbb{R}^2)$, $1 \leq s < \infty$, to denote the Lebesgue space with the norms

$$|u|_s := \left(\int_{\mathbb{R}^2} |u|^s \right)^{1/s}, \quad 1 \leq s < \infty.$$

We will use the following Trundiger-Moser type inequality in $H^1(\mathbb{R}^2)$ by Cao [12] in our variational arguments frequently and we may also refer the readers to [1,2,23] for recent progress of the topic of Trundiger-Moser type inequality.

Lemma 1.3. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left[\mathrm{e}^{\alpha |u|^2} - 1 \right] < \infty. \tag{1.7}$$

Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2 \leq M < \infty$, and $\alpha < \alpha_0 = 4\pi$, then there exists a constant C, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} \left[\mathrm{e}^{\alpha |u|^2} - 1 \right] \leq C(M, \alpha). \tag{1.8}$$

In order to state our main results, we need to introduce the following notations

$$\begin{aligned} \kappa_{\min} &:= \min_{x \in \mathbb{R}^2} V(x), \quad \mathcal{V} := \{x \in \mathbb{R}^2 : V(x) = \kappa_{\min}\}, \\ \kappa_{\max} &= |V|_\infty, \quad \kappa_\infty = \liminf_{|x| \to \infty} V(x) < \infty. \\ \tau_{\max} &:= \max_{x \in \mathbb{R}^2} P(x), \quad \mathcal{P} := \{x \in \mathbb{R}^2 : P(x) = \tau_{\max}\}, \\ \tau_{\min} &= \inf_{x \in \mathbb{R}^2} P(x), \quad \tau_\infty = \limsup_{|x| \to \infty} P(x) < \infty. \end{aligned}$$

We suppose that $V, P : \mathbb{R}^2 \to \mathbb{R}$ are two positive continuous and bounded functions with $\kappa_{\min} > 0$, $\tau_{\min} > 0$. For the first case we assume that the potential $V(x)$ and $P(x)$ satisfy

$$\begin{aligned} &\tau_{\max} > \tau_\infty \quad \text{and there exist} \quad R > 0, \quad x^* \in \mathcal{P} \quad \text{such that} \\ &V(x^*) \leq V(x) \quad \text{for all} \quad |x| \geq R. \end{aligned} \tag{$VP1$}$$

If $(VP1)$ holds, then we may assume that $V(x^*) = \min_{x \in \mathcal{P}} V(x)$ and set

$$\mathcal{A}_P := \{x \in \mathcal{P} : V(x) = V(x^*)\} \cup \{x \notin \mathcal{P} : V(x) < V(x^*)\}.$$

Assumptions $(VP1)$ was firstly introduced by Ding and Liu in [15]. Obviously, $\mathcal{A}_P$ is bounded. Moreover, $\mathcal{A}_P = \mathcal{V} \cap \mathcal{P}$ if $\mathcal{V} \cap \mathcal{P} \neq \emptyset$. In particular, $\mathcal{A}_P = \mathcal{P}$ if $V(x)$ is a constant function.

Since $g(u) = h(u) + f(u)$ and we are going to study the existence of positive solutions, we may assume that

$$h(s) = 0, \quad f(s) = 0 \quad \forall s \leq 0.$$

We suppose that the nonlinearity $h : \mathbb{R}^+ \to \mathbb{R}$ is of $\mathcal{C}^1$ verifies the following hypotheses:
There holds

$$h(0) = 0, \quad \lim_{s\to 0} h'(s) = 0. \tag{h_1}$$

The nonlinearity h is of subcritical growth at infinity. Moreover, for any $\alpha > 0$, there exists $C_0 > 0$ such that

$$|h'(s)| \leq \; C_0 \mathrm{e}^{\alpha s^2} \quad \forall s \geq 0. \tag{h_2}$$

There exists $\theta_1 > 2$ such that

$$0 < \theta_1 H(s) \leq 2h(s)s, \quad \forall s > 0, \tag{h_3}$$

where $H(t) = \int_0^t h(s)\mathrm{d}s$. This is the Ambrosetti−Rabinowitz condition for nonlocal problem.

We also assume that

$$s \to h(s) \quad \text{is strictly increasing on } (0, +\infty). \tag{h_4}$$

To obtain the existence and concentration of the semiclassical solutions for the critical exponential case, we need to introduce some additional assumptions on the subcritical term h. Since the imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for any $p \in (2, +\infty)$, by the Hardy−Littlewood−Sobolev inequality, we know there is a best constant S_p such that

$$S_p = \inf_{u\in E, u\neq 0} \frac{\left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \kappa_{\max}|u|^2\right)\right)^{1/2}}{\left(\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * |u|^p\right] |u|^p\right)^{\frac{1}{2p}}},$$

moreover, a standard minimizing argument shows that there exists a positive radial function $u_p \in E$ such that S_p is achieved by u_p, see [28, 29].

We suppose there exists $p > \frac{4-\mu}{2}$, such that

$$H(s) \geq C_p s^p, \quad \forall s \geq 0 \tag{h_5}$$

where

$$C_p > \frac{\left[\frac{4\theta(p-1)}{(2-\mu)(\theta-2)}\right]^{\frac{p-1}{2}} S_p^p}{\tau_{\min} p^{\frac{p}{2}}}.$$

For the nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$, we Suppose that it is of $\mathcal{C}^1$ and verifies the following hypotheses:
There holds

$$f(0) = 0, \quad \lim_{s\to 0} f'(s) = 0. \tag{f_1}$$

The nonlinearity f is of critical growth at infinity. Moreover, there exists C_0 such that

$$|f'(s)| \leq \; C_0 \mathrm{e}^{4\pi s^2}, \quad \forall s \geq 0. \tag{f_2}$$

There exists $\theta_2 > 2$ such that

$$0 < \theta_2 F(s) \leq 2f(s)s, \quad \forall s > 0, \tag{f_3}$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$.

We also assume that

$$s \to f(s) \quad \text{is strictly increasing on } (0, +\infty). \tag{f_4}$$

Remark 1.4. It was observed in [4] that the assumption (h_5) is not optimal in fact, since little is know about the embedding constant S_p. However we would not focus too much on this kind of conditions, since here we are mainly concerned with the existence and the concentration behavior of semiclassical ground states for the Choquard type equation.

We have the following existence and concentration results.

Theorem 1.5. *Suppose that h satisfies $(h_1)-(h_5)$ and f satisfies $(f_1)-(f_4)$. If the potential functions V and P satisfy the condition $(VP1)$, then for any $\varepsilon > 0$ small, problem (1.6) has at least one positive ground state solution u_ε. Moreover,*

(a) *There exists a maximum point $x_\varepsilon \in \mathbb{R}^2$ of u_ε, such that $\lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \mathcal{A}_P) = 0$ and for some $c, C > 0$,*

$$|u_\varepsilon(x)| \le C\exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right)\cdot$$

(b) *Setting $v_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$, $\varepsilon \to 0$, v_ε converges in E to a ground state solution v of*

$$-\Delta v + V(x_0)v = P^2(x_0)\left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

In particular if $\mathcal{V} \cap \mathcal{P} \neq \emptyset$, then $\lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \mathcal{V} \cap \mathcal{P}) = 0$ and up to subsequences, v_ε converges in E to a ground state solution v of

$$-\Delta v + \kappa_{\min} v = \tau_{\max}^2\left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

We have a dual theorem for the results obtained in Theorem 1.5. Suppose that

$$\begin{aligned} &\kappa_{\min} < \kappa_\infty \quad \text{and there exist} \quad R > 0, \quad x^* \in \mathcal{V} \quad \text{such that} \\ &\qquad\qquad P(x^*) \ge P(x) \quad \text{for all} \quad |x| \ge R. \end{aligned} \tag{$VP2$}$$

If $(VP2)$ holds, we may assume that $P(x^*) = \max_{x\in\mathcal{V}} P(x)$ and set

$$\mathcal{A}_V := \{x \in \mathcal{V} : P(x) = P(x^*)\} \cup \{x \notin \mathcal{V} : P(x) > P(x^*)\}.$$

Obviously, $\mathcal{A}_V$ is bounded. Moreover, $\mathcal{A}_V = \mathcal{V} \cap \mathcal{P}$ if $\mathcal{V} \cap \mathcal{P} \neq \emptyset$. In particular, $\mathcal{A}_V = \mathcal{V}$ if $P(x)$ is a constant function. For this dual case, we have the following theorem.

Theorem 1.6. *Suppose that g satisfies $(h_1)-(h_5)$ and f satisfies $(f_1)-(f_4)$. If the potential functions V and P satisfy the condition $(VP2)$, then all the statements in Theorem 1.5 remain true with $\mathcal{A}_P$ replaced by $\mathcal{A}_V$.*

Remark 1.7. The proof of Theorem 1.6 is similar but easier than the one of Theorem 1.5, for this theorem, we do not need to truncate partially the potential to introduce the second auxiliary equation and so the proof will be omitted here. Furthermore one can also adapt the arguments by Rabinowitz [32] and Alves and Yang [5] to establish the existence and concentration results.

We would like to write some more words about the semiclassical problem for the Schrödinger equation. One interesting problem is to consider the case that there is a competition between the external potential and the nonlinear potential. On this topic Wang and Zeng considered in [34] the equation

$$-\varepsilon^2\Delta u + V(x)u = K(x)u^{r-1} + Q(x)u^{t-1} \quad \text{in} \quad \mathbb{R}^N, \tag{1.9}$$

there the authors proved that the concentration points are located on the middle ground of the competing potential functions and in some cases are given explicitly in terms of these functions. Cingolani and Lazzo [13] obtained a multiplicity result involving the set of global minima of a function which provides some kind of global median value between the minimum of V and the maximum of K and Q. We finally mention the paper of Ambrosetti, Malchiodi and Secchi [9] consider the case $Q = 0$. Among other results, they proved that the number of solutions of (1.9) is related with the set of minima of a function given explicitly in terms of V, K, r, and the dimension N. Ding and Liu [15] considered

$$(-i\varepsilon\nabla + A(z))^2 u + V(x)u = Q(x)\left(g(|u|) + |u|^{2^*-2}\right)u, \quad \text{in } \mathbb{R}^N$$

for $u \in H^1(\mathbb{R}^N, \mathbb{C})$, where the function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a continuous magnetic potential associated with a magnetic field B (*i.e. curl*$A = B$), $g(|u|)u$ is a superlinear and subcritical. Under suitable assumptions on the potentials, the authors obtained existence and new concentration phenomena of the semiclassical ground states. However, it is hard to study the competition between the external potential $V(x)$ and the nonlinear potential $P(x)$ of the Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-N}\left[\frac{1}{|x|^\mu} * \big(P(x)G(u)\big)\right]P(x)g(u), \quad \text{in } \mathbb{R}^N,$$

The difficulty is that the nonlinear potentials P in the equation, in some sense, is convoluted by the Riesz potential $\frac{1}{|x|^\mu}$, and so it seems impossible to construct an auxiliary function characterized by V and P. Moreover, to the best knowledge of the author, there seems no existence results for semiclassical problems of the nonlocal Choquard type equation with critical exponential growth in the plane or the upper critical growth due to the Hardy−Littlewood−Sobolev inequality. So one of the motivations of the present paper is to study the interaction between the linear potential V and the nonlinear potential P, and to investigate how the behavior of these potentials will affect the existence and concentration of these solutions.

We will use the following notations:

- C, C_i denote positive constants.
- B_R denote the open ball centered at the origin with radius $R > 0$.
- $C_0^\infty(\mathbb{R}^2)$ denotes the space of the functions infinitely differentiable with compact support in $\mathbb{R}^2$.
- For a mensurable function u, we denote by u^+ and u^- its positive and negative parts respectively, given by

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

- Let E be a real Hilbert space and $I : E \to \mathbb{R}$ be a functional of class $\mathcal{C}^1$. We say that $(u_n) \subset E$ is a Palais-Smale ((PS) for short) sequence at c for I if (u_n) satisfies

$$I(u_n) \to c \text{ and } I'(u_n) \to 0, \text{ as } n \to \infty.$$

Moreover, I satisfies the (PS) condition at c, if any (PS) sequence at c possesses a convergent subsequence.

The paper is organized as follows. In Section 2 we will establish the variational framework and prove some basic lemmas for special bounded (PS) sequences. In Section 3 we study the properties of the ground state solutions of the critical nonlocal problem in $\mathbb{R}^2$ and establish a comparison lemma for the minimax values. In Section 4 we use truncating arguments for the potentials to establish some estimates for the critical values. In the last section, we prove the existence and concentration results for the Choquard equation with critical exponential growth.

2. Variational setting

In this section, we are going to study equation (1.6) *via* variational arguments. Changing variables by $u(x) = v(\epsilon x)$, it is possible to see that (1.6) is equivalent to

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \left[\frac{1}{|x|^\mu} * \big(P(\varepsilon x)G(u)\big)\right] P(\varepsilon x)g(u), \\ u \in H^1(\mathbb{R}^2). \end{cases} \tag{2.1}$$

Instead of (h_1) and (f_1), we introduce the following weaker condition:

$$g(0) = 0, \quad \lim_{s\to 0} \frac{h(s)}{s^{\frac{2-\mu}{2}}} = 0 \tag{h_1'}$$

and

$$f(0) = 0, \quad \lim_{s\to 0} \frac{f(s)}{s^{\frac{2-\mu}{2}}} = 0. \tag{f_1'}$$

The regularity conditions (h_1) and (f_1) of f, h are used to prove a splitting Lemma for nonlocal nonlinearity see Lemma 2.8 and the decay property.

The energy functional associated to equation (2.1) then can be expressed by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \mathbf{G}_P(u),$$

where

$$\mathbf{G}_P(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \big(P(\varepsilon x)G(u)\big)\right] P(\varepsilon x)G(u)$$

and

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^2} \big(|\nabla u|^2 + V(\varepsilon x)|u|^2\big)\right)^{1/2}$$

is an equivalent norm on E. From the growth assumptions on g, f and Remark 1.2, using Lemma 1.3 and the Hölder inequality, we know that $G(u) \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ for any $u \in H^1(\mathbb{R}^2)$. Thus the Hardy$-$Littlewood$-$Sobolev inequality implies that I_ε is well defined on E and belongs to $\mathcal{C}^1$ with its derivative given by

$$\langle I_\varepsilon'(u), \varphi\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla\varphi + V(\varepsilon x)u\varphi) - \mathbf{G}_P'(u)[\varphi] \ \ \forall u, \varphi \in E,$$

where

$$\mathbf{G}_P'(u)[\varphi] := \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \big(P(\varepsilon x)G(u)\big)\right] P(\varepsilon x)g(u)\varphi.$$

The Nehari manifold associated to I_ε will be denote by $\mathcal{N}_\varepsilon$, that is,

$$\mathcal{N}_\varepsilon = \Big\{u \in E : u \neq 0, \langle I_\varepsilon'(u), u\rangle = 0\Big\}.$$

In the next two lemmas we check that the Nehari manifold is bounded away from 0 and the functional I_ε satisfies the geometric conditions of the Mountain-Pass Theorem.

Lemma 2.1. *Suppose that conditions (h_1')–(h_3) and (f_1')–(f_3) are satisfied. There exists $\alpha > 0$, independent of ε, such that*

$$\|u\|_\varepsilon \geq \alpha, \ \ \forall u \in \mathcal{N}_\varepsilon. \tag{2.2}$$

Proof. Without loss of generality, we may assume that $\theta := \theta_1 = \theta_2 > 2$. For any $\delta > 0$, $p > 1$ and $\beta > 1$, there exists $C(\delta, p, \beta) > 0$ such that

$$G(s) < \frac{2}{\theta} g(s)s \leq \delta s^{\frac{4-\mu}{2}} + C(\delta, p, \beta) s^p \left[\mathrm{e}^{\beta 4\pi s^2} - 1 \right], \forall s \in \mathbb{R},$$

then it follows that

$$|G(u)|_{\frac{4}{4-\mu}} \leq C |g(u)u|_{\frac{4}{4-\mu}} \leq \varepsilon C |u|_2^{\frac{4-\mu}{2}} + C(\varepsilon, p, \beta) \big| u^p \left[\mathrm{e}^{\beta 4\pi u^2} - 1 \right] \big|_{\frac{4}{4-\mu}}. \tag{2.3}$$

Since the imbedding $E \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for any $p \in (2, +\infty)$, we know that there exists a constant C_1 such that

$$\begin{aligned} \int_{\mathbb{R}^2} |u|^{\frac{4p}{4-\mu}} \left[\mathrm{e}^{\beta 4\pi u^2} - 1 \right]^{\frac{4}{4-\mu}} &\leq \left(\int_{\mathbb{R}^2} |u|^{\frac{8p}{4-\mu}} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left[\mathrm{e}^{\beta 4\pi u^2} - 1 \right]^{\frac{4}{4-\mu}} \right)^{\frac{1}{2}} \\ &\leq C_1 \|u\|_\varepsilon^{\frac{4p}{4-\mu}} \left(\int_{\mathbb{R}^2} \left[\mathrm{e}^{\left(\frac{4\beta}{4-\mu} 4\pi u^2 \right)} - 1 \right] \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2} \left[\mathrm{e}^{\left(\frac{4\beta}{4-\mu} 4\pi u^2 \right)} - 1 \right] = \int_{\mathbb{R}^2} \left[\mathrm{e}^{\left(\frac{4\beta}{4-\mu} \|u\|_\varepsilon^2 4\pi \frac{u^2}{\|u\|_\varepsilon^2} \right)} - 1 \right],$$

then if $\|u\|_\varepsilon$ is small enough, Lemma 1.3 implies that there exists a constant C_2 such that

$$\int_{\mathbb{R}^2} \left[\mathrm{e}^{\left(\frac{4\beta}{4-\mu} \|u\|_\varepsilon^2 4\pi \frac{u^2}{\|u\|_\varepsilon^2} \right)} - 1 \right] \leq C_2.$$

Thus, by (2.3), there exists C_3 such that

$$|G(u)|_{\frac{4}{4-\mu}} \leq \delta \|u\|_\varepsilon^{\frac{4-\mu}{2}} + C_3 \|u\|_\varepsilon^p.$$

Recall that $P(x)$ is bounded, by the Hardy−Littlewood−Sobolev inequality, we know

$$\mathbf{G}'_P(u)[u] \leq \delta^2 C_4 \|u\|_\varepsilon^{4-\mu} + C_4 \|u\|_\varepsilon^{2p}.$$

Now, since $u \in \mathcal{N}_\varepsilon$, we have

$$\|u\|_\varepsilon^2 = \mathbf{G}'_P(u)[u],$$

thus

$$\|u\|_\varepsilon^2 \leq \delta^2 C_5 \|u\|_\varepsilon^{4-\mu} + C_5 \|u\|_\varepsilon^{2p}.$$

The conclusion then follows immediately. □

We are ready to check that the functional I_ε satisfies the Mountain Pass Geometry.

Lemma 2.2. *Suppose that conditions (h_1')−(h_3) and (f_1')−(f_3) are satisfied.*

(1). *There exist $\rho, \delta_0 > 0$ such that $I_\varepsilon|_S \geq \delta_0 > 0$ for all $u \in S = \{u \in E : \|u\|_\varepsilon = \rho\}$;*
(2). *There is $e \in E$ with $\|e\|_\varepsilon > \rho$ such that $I_\varepsilon(e) < 0$.*

Proof. The proof of (1) is similar to the one of Lemma 2.1, we only prove (2) here. Fix $u_0 \in E$ with $u_0^+(x) = \max\{u_0(x), 0\}$, we set

$$w(t) = \mathbf{G}_P \left(\frac{t u_0}{\|u_0\|_\varepsilon} \right) > 0 \qquad \text{for} \qquad t > 0.$$

By the Ambrosetti–Rabinowitz condition (f_3), we know

$$\frac{w'(t)}{w(t)} \geq \frac{\theta}{t} \quad \text{for} \quad t > 0.$$

Integrating this over $[1, s\|u_0\|_\varepsilon]$ with $s > \frac{1}{\|u_0\|_\varepsilon}$, we get

$$\mathbf{G}_P(su_0) \geq \mathbf{G}_P\left(\frac{u_0}{\|u_0\|_\varepsilon}\right)\|u_0\|_\varepsilon^\theta s^\theta.$$

Therefore

$$I_\varepsilon(su_0) \leq C_1 s^2 - C_2 s^\theta \quad \text{for} \quad s > \frac{1}{\|u_0\|_\varepsilon}\cdot$$

Since $\theta > 2$, the conclusion (2) follows easily by taking $e = su_0$ with s large enough. □

Applying the Mountain Pass theorem without (PS) condition, we know there is a $(PS)_{c_\varepsilon}$ sequence $(u_n) \subset E$, *i.e.*

$$I'_\varepsilon(u_n) \to 0, \quad I_\varepsilon(u_n) \to c_\varepsilon,$$

where c_ε defined by

$$0 < c_\varepsilon := \inf_{u \in E\setminus\{0\}} \max_{t \geq 0} I_\varepsilon(tu) \tag{2.4}$$

and moreover there is a constant $c > 0$ independent of ε such that $c_\varepsilon > c > 0$. Using assumptions (h_4) and (f_4), for each $u \in E\setminus\{0\}$, there is a unique $t = t(u)$ such that

$$I_\varepsilon(t(u)u) = \max_{s \geq 0} I_\varepsilon(su) \ \text{ and } \ t(u)u \in \mathcal{N}_\varepsilon.$$

Then it is standard to see that the Minimax value c_ε can be characterized by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u). \tag{2.5}$$

Lemma 2.3. *Suppose that conditions (h'_1)–(h_5) and (f'_1)–(f_4) are satisfied. Let c_ε be the minimax value defined in (2.4), then there holds*

$$c_\varepsilon < \frac{(2-\mu)(\theta-2)}{8\theta}\cdot$$

Furthermore, the $(PS)_{c_\varepsilon}$ sequence (u_n) also satisfies

$$\limsup_{n\to\infty} \|u_n\|_\varepsilon^2 < \frac{2-\mu}{4}\cdot$$

Proof. Let $u_p \in E$ be the positive radial function such that S_p is achieved. By assumption (h_5), it is easy to see that

$$\begin{aligned}
c_\varepsilon &= \inf_{u \in E\setminus\{0\}} \max_{t \geq 0} I_\varepsilon(tu) \\
&\leq \max_{t \geq 0} I_\varepsilon(tu_p) \\
&\leq \max_{t \geq 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^2}(|\nabla u_p|^2 + \kappa_{\max}|u_p|^2) - \frac{t^{2p}\tau_{\min}^2 C_p^2}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * |u_p|^p\right]|u_p|^p\right\} \\
&= \frac{(p-1)S_p^{\frac{2p}{p-1}}}{2p^{\frac{p}{p-1}}(\tau_{\min}C_p)^{\frac{2}{p-1}}} \\
&< \frac{(2-\mu)(\theta-2)}{8\theta}\cdot
\end{aligned}$$

From assumptions (h_3) and (f_3), there holds

$$c_\varepsilon = \lim_{n\to\infty}\left(I_\varepsilon(u_n) - \frac{1}{\theta}\langle I'_\varepsilon(u_n), u_n\rangle\right) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\limsup_{n\to\infty}\|u_n\|_\kappa^2,$$

consequently, we know

$$\limsup_{n\to\infty}\|u_n\|_\varepsilon^2 < \frac{2-\mu}{4}.$$

□

Next, we are ready to prove some boundedness Lemmas.

Lemma 2.4. *Suppose that conditions $(h'_1), (h_2)$ (f'_1) and (f_2) are satisfied. Let (u_n) be a bounded sequence such that $\limsup_{n\to\infty}\|u_n\|_\varepsilon^2 < \frac{(2-\mu)}{4}$. Then there exists C such that*

$$\left|\frac{1}{|x|^\mu} * G(u_n)\right|_{L^\infty(\mathbb{R}^2)} < C, \quad \forall n \in \mathbb{N}.$$

Proof. For $\beta > 1$, there exists $C_0 > 0$ such that

$$G(s) \leq C_0\Big(|s|^{\frac{4-\mu}{2}} + |s|\big[\mathrm{e}^{\beta 4\pi s^2} - 1\big]\Big), \forall s \in \mathbb{R},$$

and so

$$\begin{aligned}\left|\frac{1}{|x|^\mu} * G(u_n)\right| &= \left|\int_{\mathbb{R}^2}\frac{G(u_n)}{|x-y|^\mu}\right| \\ &= \left|\int_{|x-y|\leq 1}\frac{G(u_n)}{|x-y|^\mu}\right| + C\left|\int_{|x-y|\geq 1}\frac{G(u_n)}{|x-y|^\mu}\right| \\ &\leq \int_{|x-y|\leq 1}\frac{|u_n|^{\frac{4-\mu}{2}} + |u_n|\big[\mathrm{e}^{\beta 4\pi|u_n|^2} - 1\big]}{|x-y|^\mu} \\ &\quad + C\int_{|x-y|\geq 1}\left(|u_n|^{\frac{4-\mu}{2}} + |u_n|\left[\mathrm{e}^{\beta 4\pi|u_n|^2} - 1\right]\right).\end{aligned}$$

Since

$$\frac{1}{|y|^\mu} \in L^{\frac{2+\delta}{\mu}}(B_1^c(0)) \ \ \forall \ \ \delta > 0,$$

take $\delta \approx 0^+$ such that

$$q_{1,\delta} = \frac{4-\mu}{2}\,\frac{2+\delta}{(2+\delta)-\mu} > 2.$$

Using the Hölder inequality, we get

$$\int_{|x-y|\geq 1}\frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^\mu} \leq C_0\left(\int_{|x-y|\geq 1}|u_n|^{q_{1,\delta}}\right)^{\frac{(2+\delta)-\mu}{2+\delta}} = C_1.$$

Since

$$\limsup_{n\to\infty}\|u_n\|_\varepsilon^2 < \frac{2-\mu}{4},$$

by the Trudinger-Moser inequality, fixing $\beta > 1$ close to 1, there exists C_2 such that

$$\int_{|x-y|\geq 1}|u_n|\big[\mathrm{e}^{\beta 4\pi u_n^2} - 1\big] \leq |u_n|_2\int_{\mathbb{R}^2}\left(\left[\mathrm{e}^{2\beta m 4\pi\frac{u_n^2}{\|u_n\|_\varepsilon^2}} - 1\right]\right)^{\frac{1}{2}} \leq C_2.$$

Choosing $t \in \left(\frac{2}{2-\mu}, +\infty\right)$, since $\frac{(4-\mu)t}{2} > 2$ and $1 - \frac{t\mu}{t-1} > -1$, it follows from the Hölder inequality that

$$\begin{aligned}\int_{|x-y|\le 1} \frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^\mu} &\le \left(\int_{|x-y|\le 1} |u_n|^{\frac{(4-\mu)t}{2}}\right)^{\frac{1}{t}} \left(\int_{|x-y|\le 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}}\right)^{\frac{t-1}{t}} \\ &\le C_2 \left(\int_{|r|\le 1} |r|^{1-\frac{t\mu}{t-1}} \mathrm{d}r\right)^{\frac{t-1}{t}} \\ &\le C_3\end{aligned}$$

for some C_3. For $t > \frac{2}{2-\mu}$ and close to $\frac{2}{2-\mu}$, using again the Trudinger-Moser inequality, we know there is C_4 such that

$$\begin{aligned}\int_{|x-y|\le 1} \frac{|u_n|\left[\mathrm{e}^{\beta 4\pi u_n^2} - 1\right]}{|x-y|^\mu} &\le \left(\int_{|x-y|\le 1} |u_n\left[\mathrm{e}^{\beta 4\pi u_n^2} - 1\right]|^t\right)^{\frac{1}{t}} \left(\int_{|x-y|\le 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}}\right)^{\frac{t-1}{t}} \\ &\le \left(\int_{|x-y|\le 1} |u_n|^{2t}\right)^{\frac{1}{2t}} \left(\int_{|x-y|\le 1} \left[\mathrm{e}^{2\beta t\|u_n\|_\varepsilon^2 4\pi \frac{u_n^2}{\|u_n\|_\varepsilon^2}} - 1\right]\right)^{\frac{1}{2t}} \left(\int_{|r|\le 1} |r|^{1-\frac{t\mu}{t-1}} \mathrm{d}r\right)^{\frac{t-1}{t}} \\ &\le C_4.\end{aligned}$$

Thus the conclusion is proved. □

Lemma 2.5. *Suppose that conditions $(h_1'), (h_2)$, (f_1') and (f_2) are satisfied. Let (u_n) be a bounded sequence with $\limsup_{n\to\infty} \|u_n\|_\varepsilon^2 < \dfrac{(2-\mu)}{4}$, then there exists C such that*

$$|g(u_n)|_{\frac{4}{2-\mu}} \le C, \quad |G(u_n)|_{\frac{4}{4-\mu}} \le C, \qquad \forall n \in \mathbb{N}.$$

Proof. Since conditions $(h_1'), (h_2)$, (f_1') and (f_2) are satisfied, for any $\delta > 0$ and $\beta > 1$, there exist $C(\delta, \beta) > 0$ such that

$$|g(s)| \le \delta |s|^{\frac{2-\mu}{2}} + C(\delta,\beta)\left[\mathrm{e}^{\beta 4\pi s^2} - 1\right], \forall s \in \mathbb{R},$$

and

$$|G(s)| \le \delta |s|^{\frac{4-\mu}{2}} + C(\delta,\beta)|s|\left[\mathrm{e}^{\beta 4\pi s^2} - 1\right], \forall s \in \mathbb{R}.$$

Then

$$\begin{aligned}|g(u_n)|_{\frac{4}{2-\mu}} &\le \delta |u_n|_2^{\frac{2-\mu}{2}} + C(\delta,\beta)|\mathrm{e}^{\beta 4\pi u_n^2} - 1|_{\frac{4r}{4-\mu}} \\ &\le C_1 \|u_n\|_\varepsilon^{\frac{2-\mu}{2}} + C_1 \left(\int_{\mathbb{R}^2} \left[\mathrm{e}^{\left(\frac{4\beta}{2-\mu}\|u_n\|_\varepsilon^2 4\pi \frac{u_n^2}{\|u_n\|_\varepsilon^2}\right)} - 1\right]\right)^{\frac{2-\mu}{4}}.\end{aligned}$$

Since $\limsup \|u_n\|_\varepsilon^2 < \frac{(2-\mu)}{4}$, taking $\beta > 1$ sufficiently close to 1, by the Trudinger-Moser inequality, we know there exist C_2 such that

$$|g(u_n)|_{\frac{4}{2-\mu}} \le C_2.$$

Similarly,

$$\begin{aligned}|G(u_n)|_{\frac{4}{4-\mu}} &\le \delta |u_n|_2^{\frac{4-\mu}{2}} + C(\delta,\beta)\left|u_n \mathrm{e}^{\beta 4\pi u_n^2} - 1\right|_{\frac{4}{4-\mu}} \\ &\le \delta \|u_n\|_\varepsilon^{\frac{4-\mu}{2}} + C(\delta,\beta)|u_n|_2 \left(\int_{\mathbb{R}^2} [\mathrm{e}^{\frac{4\beta}{2-\mu}\|u_n\|_\varepsilon^2 4\pi \frac{u_n^2}{\|u_n\|_\varepsilon^2}} - 1]\right)^{\frac{2-\mu}{4}} \\ &\le C_3 \|u_n\|_\varepsilon^{\frac{4-\mu}{2}} + C_3 \|u_n\|_\varepsilon \left(\int_{\mathbb{R}^2} [\mathrm{e}^{(\frac{4\beta m}{2-\mu} 4\pi \frac{u_n^2}{\|u_n\|_\varepsilon^2})} - 1]\right)^{\frac{2-\mu}{4}},\end{aligned}$$

by using again the Trudinger-Moser inequality, there exists $C_5 > 0$ such that

$$|G(u_n)|_{\frac{4}{4-\mu}} \leq C_5. \qquad \square$$

Lemma 2.6. *Suppose that conditions $(h_1'), (h_2)$, (f_1') and (f_2) are satisfied. Let $(u_n) \subset E$ be the bounded $(PS)_{c_\varepsilon}$ sequence with weak limit u_ε in E, then u_ε satisfies $I_\varepsilon'(u_\varepsilon) = 0$.*

Proof. We need only to prove that, for any $\varphi \in E$, there holds

$$\mathbf{G}_Q'(u_n)[\varphi] \to \mathbf{G}_Q'(u_\varepsilon)[\varphi]. \tag{2.6}$$

In fact, for any $\varphi \in E$,

$$\begin{aligned}|\mathbf{G}_Q'(u_n)[\varphi]-\mathbf{G}_Q'(u_\varepsilon)[\varphi]| &= \left|\int_{\mathbb{R}^2}\left\{\left[\frac{1}{|x|^\mu} * \big(Q(\varepsilon x)G(u_n)\big)\right]Q(\varepsilon x)g(u_n) - \left[\frac{1}{|x|^\mu} * \big(Q(\varepsilon x)G(u_\varepsilon)\big)\right]Q(\varepsilon x)g(u_\varepsilon)\right\}\varphi\right| \\ &\leq C\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * G(u_n)\right]\Big(g(u_n) - g(u_\varepsilon)\Big)\varphi\right| + C\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(G(u_n) - G(u_\varepsilon)\big)\right]g(u_\varepsilon)\varphi\right|.\end{aligned} \tag{2.7}$$

For the first term, since $Q(x)$ is bounded, it follows from Lemma 2.4 that

$$\left|\frac{1}{|x|^\mu} * G(u_n)\right|_{L^\infty(\mathbb{R}^2)} < C, \quad \forall n \in \mathbb{N}.$$

Then,

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * G(u_n)\right]\big(g(u_n) - g(u_\varepsilon)\big)\varphi\right| \leq C\left|\int_{\mathbb{R}^2}\big(g(u_n) - g(u_\varepsilon)\big)\varphi\right|.$$

Since $u_n(x) \to u_\varepsilon(x)$ a.e. in $\mathbb{R}^2$, the continuity of g implies that $g(u_n(x)) \to g(u_\varepsilon(x))$ a.e. in $\mathbb{R}^2$. Moreover, by Lemma 2.5, $(g(u_n))$ is bounded in $L^{\frac{4}{2-\mu}}(\mathbb{R}^2)$, from which it follows that

$$g(u_n) \rightharpoonup g(u_\varepsilon) \quad \text{in} \quad L^{\frac{4}{2-\mu}}(\mathbb{R}^2).$$

Consequently,

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * G(u_n)\right]\big(g(u_n) - g(u_\varepsilon)\big)\varphi\right| \to 0 \tag{2.8}$$

for any $\varphi \in E$.

For the second term,

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(G(u_n) - G(u_\varepsilon)\big)\right]g(u_\varepsilon)\varphi\right| = \left|\int_{\mathbb{R}^2}\big(G(u_n) - G(u_\varepsilon)\big)\left[\frac{1}{|x|^\mu} * \big(g(u_\varepsilon)\varphi\big)\right]\right|.$$

Since $u_n(x) \to u_\varepsilon(x)$ a.e. in $\mathbb{R}^2$, the continuity of G implies that $G(u_n(x)) \to G(u_\varepsilon(x))$ a.e. in $\mathbb{R}^2$. Moreover, by Lemma 2.5 that $(G(u_n))$ is bounded in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$, we get

$$G(u_n) \rightharpoonup G(u_\varepsilon) \quad \text{in} \quad L^{\frac{4}{4-\mu}}(\mathbb{R}^2).$$

Since

$$\frac{1}{|x|^\mu} * \big(g(u_\varepsilon)\varphi\big) \in L^{\frac{4}{\mu}}(\mathbb{R}^2),$$

we obtain,

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(G(u_n) - G(u_\varepsilon)\big)\right]g(u_\varepsilon)\varphi\right| \to 0, \tag{2.9}$$

for any $\varphi \in E$. From (2.8) and (2.9), we know (2.6) holds. Thus, we immediately obtain that

$$\langle I'_\varepsilon(u_\varepsilon), \varphi\rangle = 0,$$

for any $\varphi \in E$. This means that the weak limit u_ε is a critical point of I_ε. □

Remark 2.7. From Lemma 2.6, we know that u_ε is a critical point of I_ε. However, it is still not clear whether it is nontrivial and c_ε is achieved, in order to prove that c_ε is attained by a nontrivial function, we will use a cutting-off arguments for the potentials in Section 4.

In the following we prove a nonlocal version of the Brezis−Lieb Lemma for nonlinearities of critical exponential growth.

Lemma 2.8. *Suppose that conditions* (h_1), (h_2), (f_1) *and* (f_2) *are satisfied and let* (u_n) *be a bounded sequence in* E *satisfying* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty} \|u_n\|_\varepsilon^2 < \frac{2-\mu}{4}\cdot$$

Then

$$\boldsymbol{G}_P(u_n) - \boldsymbol{G}_P(u_n - u) - \boldsymbol{G}_P(u) = o_n(1) \tag{2.10}$$

and

$$\boldsymbol{G}'_P(u_n)[\varphi] - \boldsymbol{G}'_P(u_n - u)[\varphi] - \boldsymbol{G}'_P(u)[\varphi] = o_n(1) \tag{2.11}$$

uniformly for $\varphi \in E$, $\|\varphi\|_\varepsilon \le 1$.

Proof. We only prove the conclusion (2.10) for $\mathbf{G}_Q$ under assumption (h'_1) and (f'_1) as an example, the stronger assumption (h_1) and (f_1) are only used to prove the conclusion (2.11) for $\mathbf{G}'_P$. Set $v_n = u_n - u$, we know

$$G(u_n) - G(v_n) = G(v_n + u) - G(v_n) = g(v_n + t_n u)u,$$

$0 < t_n < 1$. Set $w_n = v_n + t_n u$, from a result due to Brezis and Lieb [11], we get

$$\|u_n\|_\varepsilon^2 = \|v_n\|_\varepsilon^2 + \|u\|_\varepsilon^2 + o_n(1),$$

therefore

$$\|w_n\|_\varepsilon^2 \le 2\|u_n\|_\varepsilon^2,$$

and consequently,

$$\limsup_{n\to\infty} \|w_n\|_\varepsilon^2 \le \frac{2-\mu}{2}\cdot \tag{2.12}$$

Given $\delta > 0$, $q > 1$, and $\beta > 1$, we get

$$G(v_n + u) - G(v_n) \le \delta |v_n + t_n u|^{\frac{2-\mu}{2}} |u| + C_\delta |v_n + t_n u|^q |u| H_n,$$

where $H_n = \mathrm{e}^{\beta 4\pi w_n^2} - 1$. Since $0 < t_n < 1$ and for $a, b > 0$ and $0 < s < 1$ there holds $(a+b)^s < a^s + b^s$, we know

$$G(v_n + u) - G(v_n) \le \delta |v_n|^{\frac{2-\mu}{2}} |u| + \delta |u|^{\frac{4-\mu}{2}} + C|v_n|^q |u| H_n + C|u|^{q+1} H_n,$$

and then

$$|G(v_n + u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} \le \big(\delta C |v_n|^2 + C_1 |u|^2 + C_1 |v_n|^{\frac{4q}{4-\mu}} |u|^{\frac{4}{4-\mu}} \tilde{H}_n + C_1 |u|^{q+1} \tilde{H}_n + C_1 |u|^{\frac{4q}{4-\mu}} H\big),$$

where $\tilde{H}_n = \mathrm{e}^{\frac{4\beta}{4-\mu}4\pi w_n^2} - 1$ and $H = \mathrm{e}^{\frac{4\beta}{4-\mu}4\pi u^2} - 1$. Using Young's inequality, we have

$$\begin{aligned}&|G(v_n+u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} \\ &\quad \leq \big(\delta C|v_n|^2 + C_1|u|^2 + \delta C_1|v_n|^{\frac{4qr_1}{4-\mu}}\tilde{H}_n^{r_1} + C_2|u|^{\frac{4r_1'}{4-\mu}} + \delta\tilde{H}_n^{r_2} + C_3|u|^{(q+1)r_2'} + C_1|u|^{\frac{4q}{4-\mu}}H\big),\end{aligned} \tag{2.13}$$

where $r_1, r_1', r_2, r_2' > 1$ satisfies $\dfrac{1}{r_1} + \dfrac{1}{r_1'} = 1$ and $\dfrac{1}{r_2} + \dfrac{1}{r_2'} = 1$. From (2.12), taking $\beta, r > 1$ and close to 1, by the Trudinger-Moser inequality, we get

$$\int_{\mathbb{R}^2}\left[\mathrm{e}^{(\frac{4\beta r}{4-\mu}\|w_n\|_\varepsilon^2 4\pi\frac{w_n^2}{\|w_n\|_\varepsilon^2})} - 1\right] < C. \tag{2.14}$$

Define

$$\Psi_{\delta,n} = \max\Big\{|G(v_n+u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} - \Big(\delta|v_n|^2 + \delta C_1|v_n|^{\frac{4qr_1}{4-\mu}}\tilde{H}_n^{r_1} + \delta\tilde{H}_n^{r_2}\Big), 0\Big\},$$

from (2.13), it is easy to see

$$\Psi_{\delta,n} \to 0, \text{a.e. in } \ \mathbb{R}^2 \ \text{ and } \ \Psi_{\delta,n} \leq \big(C_1|u|^2 + C_3|u|^{(q+1)r_2'} + C_1|u|^{\frac{4q}{4-\mu}}H\big) \in \ \ L^1(\mathbb{R}^2).$$

Then the Lebesgue's theorem implies that

$$\int_{\mathbb{R}^2}\Psi_{\delta,n} \to 0, \quad \text{as} \quad n \to \infty. \tag{2.15}$$

Using (2.14) we know

$$|G(v_n+u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} \leq \Big(\delta|v_n|^2 + \delta C_1|v_n|^{\frac{4qr_1}{4-\mu}}\tilde{H}_n^{r_1} + \delta\tilde{H}_n^{r_2}\Big) + \Psi_{\delta,n},$$

we can obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}|G(v_n+u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} \leq C\delta,$$

by the arbitrariness of δ, we know

$$\lim_{n\to+\infty}\sup\int_{\mathbb{R}^2}|G(v_n+u) - G(v_n) - G(u)|^{\frac{4}{4-\mu}} = 0 \tag{2.16}$$

which means

$$G(u_n) - G(u_n - u) \to G(u) \ \text{ in } \ L^{\frac{4}{4-\mu}}(\mathbb{R}^2).$$

Using an equivalent form of the Hardy−Littlewood−Sobolev inequality, which says if $w \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ there holds

$$\left|\frac{1}{|x|^\mu} * w\right|_{\frac{\mu}{4}} \leq C_\mu|w|_{\frac{4}{4-\mu}}.$$

Consequently,

$$\frac{1}{|x|^\mu} * \Big(P(\varepsilon x)\big(G(u_n) - G(u_n - u)\big)\Big) \to \frac{1}{|x|^\mu} * \big(P(\varepsilon x)\big(G(u)\big) \ \text{ in } \ L^{\frac{4}{\mu}}(\mathbb{R}^2),$$

therefore

$$\frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \Big(P(\varepsilon x)\big(G(u_n) - G(u_n - u)\big)\Big)\right]P(\varepsilon x)\big(G(u_n) - G(u_n - u)\big) \to \mathbf{G}_P(u) \tag{2.17}$$

and

$$\int_{\mathbb{R}^2}\left[\frac{1}{|x|^{\mu}}*\Big(P(\varepsilon x)(G(u_n)-G(u_n-u))\Big)\right]P(\varepsilon x)G(u_n-u)\to 0, \tag{2.18}$$

since $G(u_n-u)\rightharpoonup 0$ in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$. Notice that

$$\begin{aligned}\mathbf{G}_P(u_n)-\mathbf{G}_P(u_n-u)-\int_{\mathbb{R}^2}\left[\frac{1}{|x|^{\mu}}*\Big(P(\varepsilon x)(G(u_n)-G(u_n-u))\Big)\right]P(\varepsilon x)G(u_n-u)\\ =\frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^{\mu}}*\Big(P(\varepsilon x)\big(G(u_n)-G(u_n-u)\big)\Big)\right]P(\varepsilon x)\big(G(u_n)-G(u_n-u)\big),\end{aligned} \tag{2.19}$$

we obtain from (2.17)-(2.19) the conclusion (1) for $\mathbf{G}_P$. □

3. An autonomous problem

To study the existence and concentration of solutions of equation (1.6), we need to establish some comparison lemmas for the Mountain-Pass levels of the critical equation with different coefficients. The existence of ground state solution for the nonlocal Hartree equation in $\mathbb{R}^2$ with critical exponential growth was obtained in a recent paper by Alves and Yang in [4, 6], there the authors studied the existence of ground state solutions for the following equation with periodic nonlinearity

$$-\Delta u+V(x)u=\left[\frac{1}{|x|^{\mu}}*F(x,u)\right]f(x,u)\quad\text{in}\quad\mathbb{R}^2. \tag{3.1}$$

In the present paper we will consider the autonomous equation of the form

$$\begin{cases}-\Delta u+\kappa u=\left[\frac{1}{|x|^{\mu}}*[\tau H(u)+\nu F(u)]\right][\tau h(u)+\nu f(u)], & \text{in } \mathbb{R}^2,\\ u\in H^1(\mathbb{R}^2),\end{cases} \tag{3.2}$$

where $\kappa\in[\kappa_{\min},\kappa_{\max}],\tau\in[\tau_{\min},\tau_{\max}]$ and $\nu\in[\tau_{\min},\tau_{\max}]$ are three positive constants.

The energy functional associated to equation (3.2) is defined by

$$\Phi_{\kappa,\tau,\nu}(u)=\frac{1}{2}\|u\|_{\kappa}^2-\mathbf{G}_{\tau\nu}(u),$$

where

$$\mathbf{G}_{\tau\nu}(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^{\mu}}*[\tau H(u)+\nu F(u)]\right][\tau H(u)+\nu F(u)]$$

and

$$\|u\|_{\kappa}:=\left(\int_{\mathbb{R}^2}\left(|\nabla u|^2+\kappa|u|^2\right)\right)^{1/2}$$

is an equivalent norm on E. The Mountain Pass value $m_{\kappa,\tau,\nu}$ of $\Phi_{\kappa,\tau,\nu}$ on E is defined by

$$m_{\kappa,\tau,\nu}:=\inf_{u\in E\backslash\{0\}}\max_{t\geq 0}\Phi_{\kappa,\tau,\nu}(tu). \tag{3.3}$$

Denote by $\mathcal{N}_{\kappa,\tau,\nu}$ the Nehari manifold of $\Phi_{\kappa,\tau,\nu}$, then $m_{\kappa,\tau,\nu}$ can also be characterized by

$$m_{\kappa,\tau,\nu}=\inf_{u\in\mathcal{N}_{\kappa,\tau,\nu}}\Phi_{\kappa,\tau,\nu}(u). \tag{3.4}$$

The same arguments in Lemma 2.3 implies that the bounded (PS) sequence satisfies

$$\limsup_{n\to\infty}\|u_n\|_{\kappa}^2\leq\frac{2\theta}{\theta-2}m_{\kappa,\tau,\nu}<\frac{2-\mu}{4}. \tag{3.5}$$

Following [6], we give a sketch proof of the existence of ground states for the completeness.

Theorem 3.1. *Suppose that conditions* $(f_1)-(f_5)$ *are satisfied. Then, for any* $\kappa \in [\kappa_{\min}, \kappa_{\max}], \tau \in [\tau_{\min}, \tau_{\max}]$, $\nu \in [\tau_{\min}, \tau_{\max}]$, *the equation* (3.2) *has a ground state solution* $u_{\kappa,\tau,\nu}$ *with* $\Phi_{\kappa,\tau,\nu}(u_{\kappa,\tau,\nu}) = m_{\kappa,\tau,\nu}$.

Proof. Let (u_n) be a $(PS)_{m_{\kappa,\tau,\nu}}$ sequence. Since (u_n) is bounded with $\limsup \|u_n\|_\kappa^2 < \frac{2-\mu}{4}$, we have either (u_n) is vanishing, *i.e.*, there exists $r > 0$ such that

$$\limsup_{y\in\mathbb{R}^2} \int_{B_r(y)} |u_n|^2 = 0$$

or non-vanishing, *i.e.*, there exist $r, \delta > 0$ and a sequence $(y_n) \subset \mathbb{Z}^2$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \geq \delta.$$

If (u_n) is vanishing, then by Lions' result (see for example, Lem. I.21 of [35]), we know

$$u_n \to 0 \text{ in } L^s(\mathbb{R}^2), \quad 2 < s < +\infty.$$

Using the Hardy−Littlewood−Sobolev inequality and the Hölder's inequality, we know

$$\left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \leq C|F(u_n)|_{\frac{4}{4-\mu}} |f(u_n)u_n|_{\frac{4}{4-\mu}} \leq C|f(u_n)u_n|^2_{\frac{4}{4-\mu}}.$$

Since for any $\delta > 0$, $p > 1$ and $\beta > 1$, there exists $C(\delta, p, \beta) > 0$ such that

$$f(s) \leq \delta s^{\frac{2-\mu}{2}} + C(\delta,p,\beta)s^{p-1}\left[\mathrm{e}^{\beta 4\pi s^2} - 1\right], \ \ \forall s \in \mathbb{R},$$

then,

$$|f(u_n)u_n|_{\frac{4}{4-\mu}} \leq \delta|u_n|_2^{\frac{4-\mu}{2}} + C(\delta,p,\beta)|u_n|^{\frac{4-\mu}{4t'}}_{\frac{4pt'}{4-\mu}} \left(\int_{\mathbb{R}^2}\left[\mathrm{e}^{\left(\frac{4\beta t}{4-\mu}\|u_n\|_\kappa^2 4\pi \frac{u_n^2}{\|u_n\|_\kappa^2}\right)} - 1\right]\right)^{\frac{4-\mu}{4t}}$$

where $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$. Taking $\beta, t > 1$ sufficiently close to 1, by the Trudinger-Moser inequality, we know there exist C_1 such that

$$\left(\int_{\mathbb{R}^2}\left[\mathrm{e}^{\left(\frac{4\beta t}{4-\mu}\|u_n\|^2 4\pi \frac{u_n^2}{\|u_n\|_\kappa^2}\right)} - 1\right]\right)^{\frac{4-\mu}{4t}} \leq C_1,$$

thus

$$\left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \leq \varepsilon|u_n|_2^{\frac{4-\mu}{2}} + C_2|u_n|^{\frac{4-\mu}{4t'}}_{\frac{4pt'}{4-\mu}}.$$

Since $t > 1$ is close to 1, we know $\frac{4pt'}{4-\mu} > 2$, then

$$\left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * F(u_n)\right] f(u_n)u_n\right| \to 0, \quad n \to \infty,$$

similarly we have

$$\left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * H(u_n)\right] h(u_n)u_n\right| \to 0, \quad n \to \infty$$

and

$$\left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * H(u_n)\right] h(u_n)u_n\right| \to 0, \quad n \to \infty.$$

Consequently, we have

$$u_n \to 0 \ \text{ in } \ E, \quad n \to \infty,$$

which yields a contradiction immediately. Thereby, vanishing case does not hold.

Let us now define $v_n = u_n(\cdot - y_n)$, then $\|v_n\|_\kappa = \|u_n\|_\kappa$ and

$$\int_{B_r(0)} |v_n|^2 \geq \delta.$$

Since $\Phi_{\kappa,\tau,\nu}$ and $\Phi'_{\kappa,\tau,\nu}$ are both invariant by $\mathbb{Z}^2$-translation, we can deduce that

$$\Phi_{\kappa,\tau,\nu}(v_n) \to c_V \ \text{ and } \ \Phi'_{\kappa,\tau,\nu}(v_n) \to 0.$$

Since (v_n) is also bounded, we may assume $v_n \rightharpoonup v$. Once that $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^2)$, we have $v \neq 0$. To show that $\Phi'_{\kappa,\tau,\nu}(v) = 0$, we can repeat the same arguments in Lemma 2.6. □

In the following, we will study the property of the ground state solution obtained in Theorem 3.1. For simplicity, we will denote the solution $u_{\kappa,\tau,\nu}$ by u.

Lemma 3.2. *Assume that conditions (h_1)–(h_5) and (f_1)–(f_4) are satisfied. If u is the ground state solution obtained in Theorem 3.1 then the positive solution u belongs to $C^2(\mathbb{R}^2)$ and decays to zero as $|x| \to \infty$. Moreover, there exist C, $\beta > 0$ such that the ground state solution satisfies*

$$|u(x)| \leq C\exp(-\beta|x|), \forall x \in R^2.$$

Proof. Define

$$K(x) := \frac{1}{|x|^\mu} * [\tau H(u) + \nu F(u)],$$

and following the steps in Lemma 2.4, there exists $C > 0$ such that

$$|K(x)| \leq C. \tag{3.6}$$

If u is a solution

$$-\Delta u + \kappa u = K(x)[\tau h(u) + \nu f(u)], \quad \text{in} \quad \mathbb{R}^2$$

with $K \in L^\infty(\mathbb{R}^2)$. By the Trundiger-Moser inequality, $h(u), f(u) \in L^q(\mathbb{R}^2)$ for q large enough, adapting the Moser iteration arguments found in [3], we can show that there exists $C > 0$ such that

$$|u|_\infty < C,$$

and u decays to zero as $|x| \to \infty$. The regularity theory of elliptic equation implies $u \in C^2_{\text{loc}}(\mathbb{R}^2)$. Applying Harnack's inequality, we can conclude that $u(x) > 0$ in $\mathbb{R}^2$.

The property of exponential decay at infinity follows from a standard comparison arguments. Notice that (h_1) and (f_1) implies that

$$\lim_{s\to 0} \frac{h(s)}{s} = 0, \quad \lim_{s\to 0} \frac{f(s)}{s} = 0.$$

The fact that the solution u decay uniformly to zero as $|x| \to +\infty$, we can take $\rho_0 > 0$ such that

$$K(x)\frac{[\tau h(u) + \nu f(u)]}{u(x)} \leq \frac{\kappa}{2}$$

for all $|x| \geq \rho_0$. Consequently,

$$-\Delta u(x) + \frac{\kappa}{2}u(x) = K(x)\frac{[\tau h(u) + \nu f(u)]}{u(x)} - \frac{\kappa}{2}u(x) \leq 0$$

for all $|x| \geq \rho_0$. Let s and T be positive constants such that

$$s^2 < \tfrac{\kappa}{2} \text{ and } u(x) \le T\exp(-s\rho_0) \text{ for all } |x| = \rho_0.$$

Hence, the function $\psi(x) = T\exp(-s|x|)$ satisfies

$$-\Delta\psi + \frac{\kappa}{2}\psi \ge \left(\frac{\kappa}{2} - s^2\right)\psi > 0$$

for all $x \neq 0$. Therefore, taking $\eta = \max\{u - \psi, 0\} \in H_0^1(|x| > \rho_0)$ as a test function, we get

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^2} \left(\nabla u \nabla \eta + \frac{\kappa}{2} u\eta\right) \\ &\ge \int_{\mathbb{R}^2} \left[(\nabla u - \nabla\psi)\nabla\eta + \frac{\kappa}{2}(u - \psi)\eta\right] \\ &\ge \frac{\kappa}{2}\int_{\{x\in\mathbb{R}^2 : u \ge \psi\}} |u - \psi|^2 \ge 0 \end{aligned}$$

for all $|x| > \rho_0$. Consequently, the set $\Omega := \{x \in \mathbb{R}^2 : |x| > \rho_0 \text{ and } u \ge \psi(x)\}$ is empty, proving the Lemma. □

The following lemma describes a comparison between the Mountain Pass values for different parameters $\kappa, \tau, \nu > 0$, which will play an important role in proving the existence results in Section 4.

Lemma 3.3. *Let* $\kappa_i \in [\kappa_{\min}, \kappa_{\max}], \tau_i \in [\tau_{\min}, \tau_{\max}], \nu_i \in [\tau_{\min}, \tau_{\max}]$, $i = 1, 2$, *with* $\min\{\kappa_2 - \kappa_1, \tau_1 - \tau_2, \nu_1 - \nu_2\} \ge 0$. *Then* $m_{\kappa_1,\tau_1,\nu_1} \le m_{\kappa_2,\tau_2,\nu_2}$. *If additionally* $\max\{\kappa_2 - \kappa_1, \tau_1 - \tau_2, \nu_1 - \nu_2\} > 0$ *then* $m_{\kappa_1,\tau_1,\nu_1} < m_{\kappa_2,\tau_2,\nu_2}$.

Proof. We prove $m_{\kappa_1,\tau_1,\nu_1} \le m_{\kappa_2,\tau_2,\nu_2}$ for example. From Theorem 3.1, choose u be a solution of problem (3.2) with coefficients κ_2, τ_2, ν_2 such that $\Phi_{\kappa_2,\tau_2,\nu_2}(u) = m_{\kappa_2,\tau_2,\nu_2}$. There holds

$$\Phi_{\kappa_2,\tau_2,\nu_2}(u) = \max_{t\ge 0} \Phi_{\kappa_2,\tau_2,\nu_2}(tu)$$

and there exists $t_0 > 0$ such that $\Phi_{\kappa_1,\tau_1,\nu_1}(t_0 u) = \max_{t\ge 0} \Phi_{\kappa_1,\tau_1,\nu_1}(tu)$. Then

$$\begin{aligned} m_{\kappa_1,\tau_1,\nu_1} &= \inf_{w\in E\setminus\{0\}} \max_{t\ge 0} \Phi_{\kappa_1,\tau_1,\nu_1}(tw) \\ &\le \max_{t\ge 0} \Phi_{\kappa_1,\tau_1,\nu_1}(tu) \\ &= \Phi_{\kappa_1,\tau_1,\nu_1}(t_0 u) \\ &\le \Phi_{\kappa_2,\tau_2,\nu_2}(t_0 u) \\ &\le \Phi_{\kappa_2,\tau_2,\nu_2}(u) \\ &= m_{\kappa_2,\tau_2,\nu_2}. \end{aligned}$$

□

4. Cutting-off functionals

In order to prove that c_ε in (2.4) can be attained by a nontrivial function, we need to introduce some cutting-off techniques for problem (2.1). For any $\kappa_{\min} \le a \le \kappa_\infty$, $\tau_\infty \le b \le \tau_{\max}$, we set

$$V^a(\varepsilon x) := \max\{a, V(\varepsilon x)\}, \quad P^b(\varepsilon x) := \min\{b, P(\varepsilon x)\}$$

and consider the auxiliary equation

$$-\Delta u + V^a(\varepsilon x)u = \left[\frac{1}{|x|^\mu} * \left(P^b(\varepsilon x)G(u)\right)\right] P^b(\varepsilon x)g(u).$$

Using the notations above, let us investigate the functional

$$I_\varepsilon^{ab}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V^a(\varepsilon x)|u|^2\right) - \mathbf{G}_P^b(u),$$

where

$$\mathbf{G}_P^b(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \left(P^b(\varepsilon x)G(u)\right)\right] P^b(\varepsilon x)G(u).$$

The associated Nehari manifold will be denoted by

$$\mathcal{N}_\varepsilon^{ab} = \left\{u \in E : u \neq 0, \langle I_\varepsilon^{ab\prime}(u), u\rangle = 0\right\}$$

while the corresponding least energy will denoted by c_ε^{ab} .

Lemma 4.1. *Suppose that conditions (h_1)–(h_5) and (f_1)–(f_4) are satisfied. Then,*

(1) $m_{a,b,b} \leq c_\varepsilon^{ab}$*;*
(2) $\limsup_{\varepsilon\to 0} c_\varepsilon^{ab} \leq m_{V^a(0),P^b(0),P^b(0)}$. *If $V(0) \leq a$ and $P(0) \geq b$, then*

$$\limsup_{\varepsilon\to 0} c_\varepsilon^{ab} = m_{a,b,b}.$$

Proof.

(1) Observe that

$$I_\varepsilon^{ab}(u) = \Phi_{a,b,b}(u) + \frac{1}{2}\int_{\mathbb{R}^2}(V^a(\varepsilon x) - a)|u|^2 + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left(b^2 - P_\varepsilon^b(y)P^b(\varepsilon x)\right)G(u(y))G(u(x))}{|x-y|^\mu}.$$

Then, the same arguments explored in Lemma 3.3 lead to

$$m_{a,b,b} \leq c_\varepsilon^{ab}.$$

(2) Let u be a solution of equation (3.2) with coefficients $\kappa = V^a(0), \tau = P^b(0), \nu = P^b(0)$ such that $\Phi_{V^a(0),P^b(0),P^b(0)}(u) = m_{V^a(0),P^b(0),P^b(0)}$. Then there is an unique $t_\varepsilon := t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon^{ab}$. Thus

$$0 < c_\varepsilon^{ab} \leq I_\varepsilon^{ab}(t_\varepsilon u) = \max_{s\geq 0} I_\varepsilon^{ab}(su).$$

From the boundedness of V, P and Q, applying the same arguments explored in Lemma 2.2, we know that there exists $C_1, C_2 > 0$ such that

$$I_\varepsilon^{ab}(su) \leq C_1 s^2 - C_2 s^\theta \ \text{ for } \ s > \frac{1}{\|u\|_\varepsilon}.$$

Thereby, there exists $T > 0$ independent of ε such that $I_\varepsilon^{ab}(su) < 0$ for all $s \geq T$. Consequently, $t_\varepsilon < T$ and we may assume that $t_\varepsilon \to t_0$.

Observe that

$$\begin{aligned} I_\varepsilon^{ab}(t_\varepsilon u) =& \Phi_{V^a(0),P^b(0),P^b(0)}(t_\varepsilon u) + \frac{1}{2}\int_{\mathbb{R}^2}(V^a(\varepsilon x) - V^a(0))|t_\varepsilon u|^2 \\ &+ \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left(P^b(0)^2 - P_\varepsilon^b(y)P^b(\varepsilon x)\right)G(t_\varepsilon u(y))G(t_\varepsilon u(x))}{|x-y|^\mu}. \end{aligned}$$

Once that P is bounded, G is increasing and $t_\varepsilon \to t_0$, Lebesgue's theorem implies

$$\int_{\mathbb{R}^2} \Big(V^a(\varepsilon x) - V^a(0)\Big)|t_\varepsilon u|^2 \to 0$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\Big(P^b(0)^2 - P^b_\varepsilon(y)P^b(\varepsilon x)\Big)G(t_\varepsilon u(y))G(t_\varepsilon u(x))}{|x-y|^\mu} \to 0,$$

as $\varepsilon \to 0$. Consequently, there holds

$$\begin{aligned}\limsup_{\varepsilon\to 0} c^{ab}_\varepsilon &\le \limsup_{\varepsilon\to 0} I^{ab}_\varepsilon(t_\varepsilon u)\\ &= \limsup_{\varepsilon\to 0}\Big(\Phi_{V^a(0),P^b(0),P^b(0)}(t_\varepsilon u) + o_\varepsilon(1)\Big)\\ &= \Phi_{V^a(0),P^b(0),P^b(0)}(t_0 u)\\ &\le \Phi_{V^a(0),P^b(0),P^b(0)}(u),\end{aligned}$$

finishing the proof. □

Without loss of generality, in the following we may assume that $x^* = 0 \in \mathcal{P}$ in $(VP1)$ or $x^* = 0 \in \mathcal{V}\cap\mathcal{P}$ if $\mathcal{V}\cap\mathcal{P} \ne \emptyset$. Let

$$e := V(0) = \min_{x\in\mathcal{P}} V(x) \le V(x) \quad \text{for all} \quad |x| \ge R. \tag{4.1}$$

We have an important upper bound for the Mountain Pass level c_ε defined in (2.4).

Lemma 4.2. *There holds*

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le m_{e,\tau_{\max},\tau_{\max}}.$$

Proof. Since $V^a_\varepsilon(x) = \max\{a, V(\varepsilon x)\}$, $P^b_\varepsilon(x) = \min\{b, P(\varepsilon x)\}$, if we choose $a = \kappa_{\min}$ and $b = \tau_{\max}$, then $V^a_\varepsilon(x) = V(\varepsilon x)$ and $P^b_\varepsilon(x) = P(\varepsilon x)$. Consequently, by the definition of I^{ab}_ε, there holds

$$c^{ab}_\varepsilon = c_\varepsilon.$$

From Lemmas 4.1, we know

$$\limsup_{\varepsilon\to 0} c_\varepsilon \le m_{e,\tau_{\max},\tau_{\max}}.$$ □

To consider the existence of semiclassical states concentrating at the nonlinear potential, we need to introduce the second auxiliary problem. In order to do so, we will partially truncate the nonlinear potential $P(x)$, *i.e.* we will cut off the potential in front of the subcritical term only. For $d \in [\tau_{\min}, \tau_{\max})$, we still set

$$P^d(\varepsilon x) := \min\{d, P(\varepsilon x)\}$$

and consider

$$-\Delta u + V^e(\varepsilon x)u = \left(\frac{1}{|x|^\mu} * [P^d(\varepsilon x)H(u) + P(\varepsilon x)F(u)]\right)[P^d(\varepsilon x)h(u) + P(\varepsilon x)f(u)].$$

The associated energy functional is defined by

$$\tilde{I}^{ed}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V^e(\varepsilon x)|u|^2\right) - \int_{\mathbb{R}^2}\left(\frac{1}{|x|^\mu} * [P^d(\varepsilon x)H(u) + P(\varepsilon x)F(u)]\right)[P^d(\varepsilon x)H(u) + P(\varepsilon x)F(u)],$$

the corresponding Nehari manifold is $\tilde{\mathcal{N}}^{ed}_\varepsilon$ and the least energy is $\tilde{c}^{ed}_\varepsilon$.

We have an important lower bound for the least energy $\tilde{c}^{ed}_\varepsilon$.

Lemma 4.3.

$$\tilde{c}_{\varepsilon}^{ed} \geq m_{e,\tau_{\max},d}.$$

Proof. Since $V_{\varepsilon}^{e}(x) \geq e, \quad P^{d}(\varepsilon x) \leq d$ and $P_{\varepsilon}(x) \leq \tau_{\max}$, from the characterization of the value $m_{e,\tau_{\max},d}$, we know

$$\inf_{u \in E} \max_{t \geq 0} \tilde{I}_{\varepsilon}^{ed}(tu) \geq \inf_{u \in E} \max_{t \geq 0} \Phi_{e,\tau_{\max},d}(tu),$$

i.e.

$$\tilde{c}_{\varepsilon}^{ed} \geq m_{e,\tau_{\max},d}.$$ □

5. Proof of the main results

In this part, we will prove the existence and concentration of the ground states in Theorem 1.5.

Lemma 5.1. *The minimax value c_{ε} is achieved if ε is small enough. Hence, problem (2.1) has a solution of least energy if ε is small enough.*

Proof. Since the least energy c_{ε} can be characterized by

$$c_{\varepsilon} = \inf_{u \in \mathcal{N}_{\varepsilon}} I_{\varepsilon}(u),$$

we can choose a minimizing sequence $(u_n) \subset \mathcal{N}_{\varepsilon}$ of I_{ε} such that $I_{\varepsilon}(u_n) \to c_{\varepsilon}$. By Ekeland's variational principle [35], we may also assume that it is a bounded (PS) sequence at c_{ε}. Once that we are assuming that $g(s) = 0$ for all $s \leq 0$, a simple calculus gives $\|u_n^{-}\| \to 0$, then we can assume that $u_n \geq 0$ for all $n \in \mathbb{N}$. From Lemma 2.6 it follows that $I_{\varepsilon}'(u_{\varepsilon}) = 0$. To complete the proof, we need to show that $u_{\varepsilon} \neq 0$ if ε is small enough.

Assume, by the contrary, there exists a sequence $\varepsilon_j \to 0$ with $u_{\varepsilon_j} = 0$. For each fixed j, let $(u_n) \subset \mathcal{N}_{\varepsilon_j}$ be the (PS) sequence at c_{ε_j} such that $u_n \rightharpoonup u_{\varepsilon_j} = 0$ in E. Select $d \in (\tau_{\infty}, \tau_{\max})$ and consider the functional $\tilde{I}_{\varepsilon_j}^{ed}$ with constant e defined in (4.1). Note that for each u_n there is a unique t_n such that $t_n u_n \in \tilde{\mathcal{N}}_{\varepsilon_j}^{ed}$ and a unique $\tilde{t}_n$ such that $\tilde{t}_n u_n \in \mathcal{N}_{\varepsilon_j}^{ed}$, *i.e.*

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V^e(\varepsilon_j x)|u_n|^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[P^d(\varepsilon_j y) H(t_n u_n) + P(\varepsilon_j y) F(t_n u_n)][P^d(\varepsilon_j x) h(t_n u_n) + P(\varepsilon_j x) f(t_n u_n)] t_n u_n}{t_n^2 |x-y|^{\mu}}$$

and

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V^e(\varepsilon_j x)|u_n|^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{P^d(\varepsilon_j y)[H(\tilde{t}_n u_n) + F(\tilde{t}_n u_n)] P^d(\varepsilon_j x)[h(\tilde{t}_n u_n) + f(\tilde{t}_n u_n)] \tilde{t}_n u_n}{\tilde{t}_n^2 |x-y|^{\mu}}.$$

Obviously, from $P^d(\varepsilon x) := \min\{d, P(\varepsilon x)\}$, we know

$$t_n \leq \tilde{t}_n.$$

We first claim that the sequence $\tilde{t}_n$ is bounded. Since (u_n) is bounded with $\|u_n\|_{\varepsilon_j}^2 \geq \alpha$, we claim that there exist $(y_n) \subset \mathbb{R}^2$ and $R, \delta > 0$ such that

$$\int_{B_R(y_n)} |u_n|^2 \geq \delta, \quad n \in \mathbb{N}. \tag{5.1}$$

Thus, the sequence $v_n(x) = u_n(x + y_n)$ is bounded in E and its weak limit $v \in E$ is not zero, *i.e.*, $v \neq 0$. Hence, there is $\Omega \subset \mathbb{R}^2$ with $|\Omega| > 0$ such that

$$v(x) > 0 \quad \forall x \in \Omega.$$

Since (u_n) is bounded and $\inf_{x\in\mathbb{R}^2} P(x) > 0$, there is a positive constant C_1 satisfying

$$C_1 \geq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{G(\tilde{t}_n u_n(y))g(\tilde{t}_n u_n(x))u_n(x)}{\tilde{t}_n|x-y|^\mu} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{G(\tilde{t}_n v_n(y))g(\tilde{t}_n v_n(x))v_n(x)}{\tilde{t}_n|x-y|^\mu} \quad \forall n\in\mathbb{N},$$

hence

$$C_1 \geq \int_\Omega\int_\Omega \frac{G(\tilde{t}_n v_n(y))g(\tilde{t}_n v_n(x))\tilde{t}_n v_n(x)}{\tilde{t}_n^2|x-y|^\mu} \quad \forall n\in\mathbb{N}.$$

If $\tilde{t}_n \to \infty$ as $n\to\infty$, by Fatou's lemma and the fact that

$$\lim_{s\to+\infty}\frac{G(s)}{s} = \lim_{s\to+\infty} g(s) = +\infty,$$

we get

$$C_1 \geq \liminf_{n\to+\infty}\int_\Omega\int_\Omega \frac{G(\tilde{t}_n v_n(y))g(\tilde{t}_n v_n(x))v_n(x)}{\tilde{t}_n|x-y|^\mu} = +\infty$$

which is absurd. Thereby, $(\tilde{t}_n)$ is a bounded sequence.

In order to prove (5.1), we suppose by contradiction that

$$u_n \to 0 \text{ in } L^s(\mathbb{R}^2), \quad 2 < s < +\infty.$$

Since for any $\delta > 0$, $p > 1$ and $\beta > 1$, there exists $C(\delta,p,\beta) > 0$ such that

$$g(s) \leq \delta s^{\frac{2-\mu}{2}} + C(\delta,p,\beta)s^{p-1}\left[e^{\beta 4\pi s^2} - 1\right], \ \forall s\in\mathbb{R}.$$

Then,

$$|g(u_n)u_n|_{\frac{4}{4-\mu}} \leq \delta|u_n|_2^{\frac{4-\mu}{2}} + C(\delta,p,\beta)|u_n|_{\frac{4pt'}{4-\mu}}^{\frac{4-\mu}{4t'}}\left(\int_{\mathbb{R}^2}\left[e^{\left(\frac{4\beta t}{4-\mu}\|u_n\|_{\varepsilon_j}^2 4\pi\frac{u_n^2}{\|u_n\|_{\varepsilon_j}^2}\right)} - 1\right]\right)^{\frac{4-\mu}{4t}}$$

where $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$. Taking $\beta, t > 1$ sufficiently close to 1, by the Trudinger-Moser inequality, we know there exist C_1 such that

$$\left(\int_{\mathbb{R}^2}\left[e^{\left(\frac{4\beta t}{4-\mu}\|u_n\|_{\varepsilon_j}^2 4\pi\frac{u_n^2}{\|u_n\|_{\varepsilon_j}^2}\right)} - 1\right]\right)^{\frac{4-\mu}{4t}} \leq \left(\int_{\mathbb{R}^2}\left[e^{\left(\frac{4\beta\|u_n\|_{\varepsilon_j}^2 t}{4-\mu} 4\pi\frac{u_n^2}{\|u_n\|_{\varepsilon_j}^2}\right)} - 1\right]\right)^{\frac{4-\mu}{4t}} \leq C_1,$$

where we used the fact that

$$\limsup_{n\to\infty}\|u_n\|_{\varepsilon_j}^2 < \frac{2-\mu}{4}.$$

Then,

$$\begin{aligned}\left|\mathbf{G}'_P(u_n)[u_n]\right| &= \left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * (P(\varepsilon x)G(u_n))\right]P(\varepsilon x)g(u_n)u_n\right| \\ &\leq C|g(u_n)u_n|_{\frac{4}{4-\mu}}^2 \\ &\leq \delta|u_n|_2^{4-\mu} + C_2|u_n|_{\frac{4pt'}{4-\mu}}^{\frac{4-\mu}{2t'}}.\end{aligned}$$

Since $t > 1$ is close to 1, we know $\frac{4pt'}{4-\mu} > 2$, consequently

$$\mathbf{G}'_P(u_n)[u_n] \to 0, \quad n\to\infty.$$

However,

$$\alpha \leq \|u_n\|_{\varepsilon_j}^2 = \mathbf{G'}_P(u_n)[u_n]$$

which leads to a contradiction.

Furthermore we can prove that the sequence $(\tilde{t}_n)$ satisfies

$$\limsup_{n\to\infty} \tilde{t}_n \leq 1.$$

In fact, suppose by contradiction that there exist $\delta > 0$ and a subsequence of $(\tilde{t}_n)$, still denoted by itself, such that

$$\tilde{t}_n \geq 1+\delta \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $(u_n) \subset \mathcal{N}_{\varepsilon_j}$ and $\tilde{t}_n u_n \in \mathcal{N}_{\varepsilon_j}^{ed}$, we have

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(\varepsilon_j x)|u_n|^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{P(\varepsilon_j y)G(u_n)P(\varepsilon_j x)g(u_n)u_n}{|x-y|^\mu} \tag{5.2}$$

and

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V^e(\varepsilon_j x)|u_n|^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{P^d(\varepsilon_j y)G(\tilde{t}_n u_n)P^d(\varepsilon_j x)g(\tilde{t}_n u_n)\tilde{t}_n u_n}{\tilde{t}_n^2|x-y|^\mu}. \tag{5.3}$$

Notice that $u_n \rightharpoonup u_{\varepsilon_j} = 0$ in E and $u_n \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q \geq 1$, $u_n \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q \geq 1$. On one hand, it is easy to see that

$$\int_{\mathbb{R}^2} (V^e(\varepsilon_j x) - V(\varepsilon_j x))|u_n|^2 = \int_{\{x:V(\varepsilon_j x)\leq e\}} (e - V(\varepsilon_j x))|u_n|^2 = o_n(1), \tag{5.4}$$

since $\{x : V(\varepsilon_j x) \leq e\}$ is bounded. On the other hand, one can see

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{P(\varepsilon_j y)G(u_n)P(\varepsilon_j x)g(u_n)u_n}{|x-y|^\mu} - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{P^d(\varepsilon_j y)G(u_n)P^d(\varepsilon_j x)g(u_n)u_n}{|x-y|^\mu}\right| \\
&\quad\leq \left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \big(P(\varepsilon_j x)G(u_n)\big)\right]\big(P(\varepsilon_j x) - P^d(\varepsilon_j x)\big)g(u_n)u_n\right| \\
&\qquad + \left|\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \big(P^d(\varepsilon_j x)g(u_n)u_n\big)\right]\big(P(\varepsilon_j x) - P^d(\varepsilon_j x)\big)G(u_n)\right|.
\end{aligned} \tag{5.5}$$

Since for any $\delta > 0$, $\alpha > 0$, there exists $C(\delta, \alpha) > 0$ such that

$$G(s) \leq \delta s^{\frac{4-\mu}{2}} + C(\delta,\alpha)s\big[\mathrm{e}^{\alpha s^2} - 1\big], \ \ \forall s \in \mathbb{R}.$$

Recall that (u_n) is bounded with

$$\limsup_{n\to\infty} \|u_n\|_{\varepsilon_j}^2 < \frac{2-\mu}{4},$$

it follows that there exists C such that

$$|P(\varepsilon_j x)g(u_n)u_n|_{\frac{4}{4-\mu}} \leq C.$$

Since $u_n \rightharpoonup u_{\varepsilon_j} = 0$ in E and $u_n \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q \geq 1$, we know

$$|(P(\varepsilon_j x) - P^b(\varepsilon_j x))G(t_n u_n)|_{\frac{4}{4-\mu}} = \int_{\{x:P(\varepsilon_j x)\geq b\}} |P(\varepsilon_j x) - b|^{\frac{4}{4-\mu}} |G(t_n u_n)|^{\frac{4}{4-\mu}} = o_n(1),$$

since $\{x : P(\varepsilon_j x) \geq b\}$ is bounded. Therefore we can conclude that

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \left(P^d(\varepsilon_j x) g(u_n) u_n\right)\right]\left(P(\varepsilon_j x) - P^d(\varepsilon_j x)\right) G(u_n)\right| = o_n(1), \tag{5.6}$$

similarly, we know

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \left(P(\varepsilon_j x) G(u_n)\right)\right]\left(P(\varepsilon_j x) - P^d(\varepsilon_j x)\right) g(u_n) u_n\right| = o_n(1). \tag{5.7}$$

From (5.6) and (5.7), we know

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{P(\varepsilon_j y) G(u_n) P(\varepsilon_j x) g(u_n) u_n}{|x-y|^\mu} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{P^d(\varepsilon_j y) G(u_n) P^d(\varepsilon_j x) g(u_n) u_n}{|x-y|^\mu} + o_n(1). \tag{5.8}$$

Thus, from (5.2), (5.3), (5.4) and (5.8), we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{P^d(\varepsilon_j y) G(\tilde{t}_n u_n) P^d(\varepsilon_j x) g(\tilde{t}_n u_n)\tilde{t}_n u_n}{\tilde{t}_n^2|x-y|^\mu} - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{P^d(\varepsilon_j y) G(u_n) P^d(\varepsilon_j x) g(u_n) u_n}{|x-y|^\mu} = o_n(1).$$

Recall that the sequence $v_n(x) = u_n(x+y_n)$ is bounded in E and its weak limit $v \in E$ is not zero, *i.e.*, $v \neq 0$. Hence, there is $\Omega \subset \mathbb{R}^2$ with $|\Omega| > 0$ such that $v(x) > 0 \quad \forall x \in \Omega$. Consequently, from (h_4) and (f_4), we get

$$\begin{aligned}
0 &< \int_\Omega\int_\Omega \frac{|v_n(y)||v_n(x)|}{|x-y|^\mu}\left[\frac{G((1+\delta)v_n(y)) g((1+\delta)v_n(x))(1+\delta)v_n(x)}{(1+\delta)|v_n(y)|(1+\delta)|v_n(x)|} - \frac{G(v_n(y)) g(v_n(x)) v_n(x)}{|v_n(y)||v_n(x)|}\right] \\
&= \int_\Omega\int_\Omega \left[\frac{G((1+\delta)v_n(y)) g((1+\delta)v_n(x))(1+\delta)v_n(x)}{(1+\delta)^2|x-y|^\mu} - \frac{G(v_n(y)) g(v_n(x)) v_n(x)}{|x-y|^\mu}\right] \\
&\leq o_n(1).
\end{aligned}$$

Let $n \to \infty$ in the last inequality and apply Fatou's lemma, we know that

$$0 < \int_\Omega\int_\Omega \frac{G((1+\delta)v(y)) g((1+\delta)v(x))(1+\delta)v(x)}{(1+\delta)^2|x-y|^\mu} - \frac{G(v(y)) g(v(x)) v(x)}{|x-y|^\mu} = 0$$

which is absurd, thus

$$\limsup_{n\to\infty} \tilde{t}_n \leq 1$$

and consequently,

$$\limsup_{n\to\infty} t_n \leq 1.$$

In what follows, we assume that $t_n \to t_0 \leq 1$, as $n \to \infty$. By (5.4), we have

$$\int_{\mathbb{R}^2}(V^e_{\varepsilon_j}(x) - V(\varepsilon_j x))|t_n u_n|^2 = \int_{\{x: V(\varepsilon_j x) \leq e\}}(e - V(\varepsilon_j x))|t_n u_n|^2 = o_n(1).$$

By using the fact that $t_n \to t_0 \leq 1$ and $\limsup_{n\to\infty}\|u_n\|^2_{\varepsilon_j} < \dfrac{2-\mu}{4}$, we know that there exists C such that

$$|P(\varepsilon_j y) F(t_n u_n(x))|_{\frac{4}{4-\mu}} \leq C.$$

Since

$$\begin{aligned}
\left(\mathbf{G}_P(t_n u_n) - \mathbf{G}^d_P(t_n u_n)\right) = {}& 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{P(\varepsilon_j y) F(t_n u_n(x))[P(\varepsilon_j x) - P^d(\varepsilon_j x)] H(t_n u_n(x))}{|x-y|^\mu} \\
&+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{[P(\varepsilon_j y) P(\varepsilon_j x) - P^d(\varepsilon_j y) P^d(\varepsilon_j x)] H(t_n u_n(x)) H(t_n u_n(y))}{|x-y|^\mu},
\end{aligned}$$

by repeating the arguments in proving (5.7), we can get

$$\mathbf{G}_P(t_nu_n) - \mathbf{G}_P^d(t_nu_n) = o_n(1).$$

Notice that

$$\tilde{c}_{\varepsilon_j}^{ed} \le \tilde{I}_{\varepsilon_j}^{ed}(t_nu_n) = I_{\varepsilon_j}(t_nu_n) + \frac{1}{2}\int_{\mathbb{R}^2}(V^e(\varepsilon_j x) - V(\varepsilon_j x))|t_nu_n|^2 + \mathbf{G}_P(t_nu_n) - \mathbf{G}_P^d(t_nu_n),$$

we have

$$\begin{aligned}\tilde{c}_{\varepsilon_j}^{ed} &\le I_{\varepsilon_j}(t_nu_n) + o_n(1)\\ &\le I_{\varepsilon_j}(u_n) + o_n(1),\end{aligned}$$

hence $\tilde{c}_{\varepsilon_j}^{ed} \le c_{\varepsilon_j}$ as $n \to \infty$. However, from Lemma 4.3, there holds

$$m_{e,\tau_{\max},d} \le \tilde{c}_{\varepsilon_j}^{ed},$$

leading to

$$m_{e,\tau_{\max},d} \le c_{\varepsilon_j}.$$

Taking limit $j \to +\infty$ and using Lemma 4.2, we get

$$m_{e,\tau_{\max},d} \le m_{e,\tau_{\max},\tau_{\max}},$$

this is a contradiction by Lemma 3.3, since $d < \tau_{\max}$. □

Lemma 5.2. *Let u_{ε_n} be the solution obtained in Lemma 5.1. Then, there is $y_n \in \mathbb{R}^2$ with $\varepsilon_n y_n \to y_0 \in \mathcal{A}_P$, i.e.,*

$$\lim_{\varepsilon_n \to 0} \operatorname{dist}(\varepsilon_n y_n, \mathcal{A}_P) = 0,$$

such that the sequence $v_n(x) := u_{\varepsilon_n}(x + y_n)$ converges strongly in E to a ground state solution v of

$$-\Delta v + V(y_0)v = P^2(y_0)\left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

In particular, if $\mathcal{V} \cap \mathcal{P} \neq \emptyset$, it follows that $\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathcal{V} \cap \mathcal{P}) = 0$, and up to subsequences, v_n converges in E to a ground state solution v of

$$-\Delta v + \kappa_{\min} v = \tau_{\max}^2\left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

Proof. Let (u_{ε_n}), $\varepsilon_n \to 0$, be the sequence of solutions of equation (2.1) obtained in Lemma 5.1, it is easy to see (u_{ε_n}) is bounded in E. Moreover, there exist $r, \delta > 0$ and a sequence $(y_n) \subset \mathbb{R}^2$ such that

$$\liminf_{n \to \infty} \int_{B_r(y_n)} |u_n|^2 \ge \delta. \tag{5.9}$$

By setting $v_n(x) := u_{\varepsilon_n}(x + y_n)$, $\tilde{V}_n(x) = V(\varepsilon_n(x + y_n))$,$\tilde{P}_n(x) = P(\varepsilon_n(x + y_n))$, we see that v_n solves the below problem

$$-\Delta u + \tilde{V}_n(x)u = \left[\frac{1}{|x|^\mu} * \left(\tilde{P}_n(x)G(u)\right)\right] \tilde{P}_n(x)g(u), \tag{5.10}$$

with energy functional given by

$$\tilde{I}_{\varepsilon_n}(v_n) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla v_n|^2 + \tilde{V}_n(x)|v_n|^2) - \mathbf{G}_{\tilde{P}_n}(v_n),$$

with

$$\mathbf{G}_{\tilde{P}_n}(v_n) = \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(\tilde{P}_n(x)G(v_n)\big)\right]\tilde{P}_n(x)G(v_n).$$

Since $v_n(x) = u_{\varepsilon_n}(x+y_n)$ is also bounded, from (5.9), we may assume that $v_n \rightharpoonup v$ in E with $v \neq 0$ and $v \geq 0$.

Claim 1. The sequence $(\varepsilon_n y_n)$ must be bounded.

Otherwise if $\varepsilon_n y_n \to \infty$, as $n \to \infty$, then we may suppose that $V(\varepsilon_n y_n) \to V_0 \geq e$, $P(\varepsilon_n y_n) \to P_0 < \tau_{\max}$. Since

$$\langle \tilde{I}'_{\varepsilon_n}(v_n), \varphi\rangle = 0$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we must have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^2}(\nabla v_n \nabla\varphi + \tilde{V}_n(x)v_n\varphi) - \mathbf{G}'_{\tilde{P}_n}(v_n)[\varphi] \\
&= \int_{\mathbb{R}^2}(\nabla v_n\nabla\varphi + V_0 v_n\varphi) - P_0^2\mathbf{G}'(v_n)[\varphi] \\
&\quad + \int_{\mathbb{R}^2}(\tilde{V}_{\varepsilon_n}(x) - V_0)v_n\varphi + P_0^2\mathbf{G}'(v_n)[\varphi] - \mathbf{G}'_{\tilde{P}_n}(v_n)[\varphi].
\end{aligned} \tag{5.11}$$

Since V is a continuous function and $\varphi \in C_0^\infty(\mathbb{R}^2)$, it follows that

$$\int_{\mathbb{R}^2}(\tilde{V}_n(x) - V_0)v_n\varphi = o_n(1).$$

One observes that

$$\begin{aligned}
\mathbf{G}'_{\tilde{P}_n}(v_n)[\varphi] - P_0^2\mathbf{G}'(v)[\varphi] &= \int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(\tilde{P}_n(x)G(v_n)\big)\right]\Big(\tilde{P}_n(x)g(v_n) - P_0 g(v)\Big)\varphi \\
&\quad + P_0\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(\tilde{P}_{\varepsilon_n}(x)G(v_n) - P_0G(v)\big)\right]f(v)\varphi.
\end{aligned} \tag{5.12}$$

Since

$$\begin{aligned}
K(v_n) &:= \int_{\mathbb{R}^2}(|\nabla v_n|^2 + \tilde{V}_n(x)|v_n|^2) \\
&= \frac{2\theta}{\theta-2}\left(\tilde{I}_{\varepsilon_n}(v_n) - \frac{1}{\theta}\tilde{I}'_n(v_n)v_n\right) \\
&= \frac{2\theta}{\theta-2}c_{\varepsilon_n} < \frac{2-\mu}{4},
\end{aligned}$$

by repeating the arguments in Lemma 2.4, we know

$$\left|\frac{1}{|x|^\mu} * \big(\tilde{P}_n(x)G(v_n)\big)\right| < C,$$

hence, by estimating the first term in (5.12), we have

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu} * \big(\tilde{P}_n(x)G(v_n)\big)\right]\Big(\tilde{P}_n(x)g(v_n) - P_0g(v)\Big)\varphi\right| \leq C\left|\int_{\mathbb{R}^2}\Big(\tilde{P}_n(x)g(v_n) - P_0g(v)\Big)\varphi\right|.$$

Since $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^2$, the continuity of g implies that $g(v_n(x)) \to g(v(x))$ a.e. in $\mathbb{R}^2$ it follows that

$$g(v_n) \rightharpoonup g(v) \quad\text{in}\quad L^{\frac{4}{2-\mu}}(\mathbb{R}^2),$$

therefore

$$\left|\int_{\mathbb{R}^2}\Big(\tilde{P}_n(x)g(v_n)-P_0g(v)\Big)\varphi\right|\to 0,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2),$$

and then

$$\left|\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*\big(\tilde{P}_n(x)G(v_n)\big)\right]\Big(\tilde{P}_n(x)g(v_n)-P_0g(v)\Big)\varphi\right|\to 0,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2). \tag{5.13}$$

For the second term, since G is continuous, then for any x, $\tilde{P}_n(x)G(v_n(x))\to P_0G(v(x))$, therefore we have $\tilde{P}_n(x)G(v_n(x))$ converges weakly to $P_0G(v)$ in $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$. Recall that

$$\frac{1}{|x|^\mu}*w(x)\in L^{\frac{4}{\mu}}(\mathbb{R}^2)$$

for all $w(x)\in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$, we know that the convolution defines a linear bounded operator from $L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$ to $L^{\frac{4}{\mu}}(\mathbb{R}^2)$, thus

$$\frac{1}{|x|^\mu}*\big(\tilde{P}_n(x)G(v_n)\big)\rightharpoonup P_0\frac{1}{|x|^\mu}*\big(G(v_n)\big)\ \text{ in }\ L^{\frac{4}{\mu}}(\mathbb{R}^2).$$

Therefore, we have

$$\int_{\mathbb{R}^2}\left[\frac{1}{|x|^\mu}*\big(\tilde{P}_n(x)G(v_n)-P_0G(v)\big)\right]g(v)\varphi\to 0,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2). \tag{5.14}$$

From (5.13) and (5.14), we obtain

$$\mathbf{G}'_{\tilde{P}_n}(v_n)[\varphi]-P_0^2\mathbf{G}'(v)[\varphi]\to 0,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2).$$

Then by taking a limit in (5.11), we get

$$\int_{\mathbb{R}^2}(\nabla v\nabla\varphi+V_0v\varphi)-P_0^2\mathbf{G}'(v)[\varphi]=0$$

for any $\varphi\in C_0^\infty(\mathbb{R}^2)$, which shows that v is nothing but a solution of the equation

$$-\Delta v+V_0v=P_0^2\Big[\frac{1}{|x|^\mu}*G(v)\Big]g(v).$$

Observe that

$$I_{\varepsilon_n}(u_n)=\tilde{I}_{\varepsilon_n}(v_n).$$

By Fatou's Lemma and Lemma 3.3, we see that

$$\begin{aligned}
m_{e,\tau_{\max},\tau_{\max}} &< m_{V_0,P_0,P_0}\\
&\le \Phi_{V_0,P_0,P_0}(v)\\
&= \frac{P_0^2}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{G(v(y))g(v(x))v(x)-G(v(y))G(v(x))}{|x-y|^\mu}\\
&\le \liminf_{n\to\infty}\tilde{I}_{\varepsilon_n}(v_n)\\
&= \liminf_{n\to\infty}c_{\varepsilon_n}
\end{aligned}$$

which contradicts to Lemma 4.2 which says

$$\limsup_{n\to\infty}c_{\varepsilon_n}\le m_{e,\tau_{\max},\tau_{\max}}.$$

Thus $(\varepsilon_n y_n)$ is bounded, and we may assume $\varepsilon_n y_n \to y_0$.

Claim 2. $y_0 \in \mathcal{A}_P := \{x \in \mathcal{P} : V(x) = V(0)\} \cup \{x \notin \mathcal{P} : V(x) < V(0)\}$.

Following the arguments in the proof of Claim 1, we know v is a solution of the equation

$$-\Delta v + V(y_0)v = P^2(y_0)\left[\frac{1}{|x|^\mu} * G(v)\right] g(v), \quad \text{in } \mathbb{R}^2. \tag{5.15}$$

If $y_0 \notin \mathcal{A}_P$, then it is easy to see $m_{e,\tau_{\max},\tau_{\max}} < m_{V(y_0),P(y_0),P(y_0)}$. Repeat the arguments of Claim 1 again, we arrive at a contradiction by Lemma 4.2,

$$\limsup_{n\to\infty} c_{\varepsilon_n} \le m_{e,\tau_{\max},\tau_{\max}} < m_{V(y_0),P(y_0),P(y_0)} \le \liminf_{n\to\infty} c_{\varepsilon_n}.$$

Therefore $y_0 \in \mathcal{A}_P$, which means $\lim_{n\to\infty} \operatorname{dist}(\varepsilon_n y_n, \mathcal{A}_P) = 0$. In particular, if $\mathcal{V} \cap \mathcal{P} \neq \emptyset$, then $\lim_{n\to\infty} \operatorname{dist}(\varepsilon_n y_n, \mathcal{V} \cap \mathcal{P}) = 0$.

Repeat the arguments in Lemma 4.1, we get

$$\lim_{n\to\infty} \tilde{I}_{\varepsilon_n}(v_n) \le m_{V(y_0),P(y_0),P(y_0)},$$

consequently,

$$\Phi_{V(y_0),P(y_0),P(y_0)}(v) = m_{V(y_0),P(y_0),P(y_0)},$$

and so and up to subsequences, (v_n) converges weakly in $H^1(\mathbb{R}^2)$ to a ground state solution v of

$$-\Delta v + \kappa_{\min} v = \tau_{\max}^2 \left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

Claim 3. (v_n) converges strongly to v in E.

On one hand, since V, P are continuous functions, we know

$$\int_{\mathbb{R}^2} \tilde{V}_n(x)|v|^2 = \int_{\mathbb{R}^2} V(y_0)|v|^2 + o_n(1),$$

$$\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * \tilde{P}_n(x)G(v)\right] \tilde{P}_n(x)G(v) = P(y_0)^2 \int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * G(v)\right] G(v) + o_n(1).$$

On the other hand, using the nonlocal Brezis−Lieb type results in Lemma 2.8, we have

$$\mathbf{G}_{\tilde{P}_n}(v_n) - \mathbf{G}_{\tilde{P}_n}(v_n - v) - \mathbf{G}_{\tilde{P}_n}(v) = o_n(1),$$

$$\mathbf{G}'_{\tilde{P}_n}(v_n)[\varphi] - \mathbf{G}'_{\tilde{P}_n}(v_n - v)[\varphi] - \mathbf{G}'_{\tilde{P}_n}(v)[\varphi] = o_n(1)$$

uniformly for $\varphi \in E, \|\varphi\| \le 1$. Thus we can derive

$$\tilde{I}_{\varepsilon_n}(v_n - v) = \tilde{I}_{\varepsilon_n}(v_n) - \Phi_{V(y_0),P(y_0),P(y_0)}(v) + o_n(1).$$

Since

$$\lim_{n\to\infty} \tilde{I}_{\varepsilon_n}(v_n) = \Phi_{V(y_0),P(y_0),P(y_0)}(v),$$

it follows that

$$\lim_{n\to\infty} \tilde{I}_{\varepsilon_n}(v_n - v) = 0.$$

Similarly, we drive that

$$\tilde{I}'_{\varepsilon_n}(v_n - v) \to 0,$$

which implies

$$\lim_{n\to\infty} \langle \tilde{I}'_{\varepsilon_n}(v_n - v), (v_n - v)\rangle = 0.$$

Hence,

$$\|v_n - v\|^2 \le C \lim_{n\to\infty} \Big(\tilde{I}_{\varepsilon_n}(v_n - v) - \frac{1}{\theta}\langle \tilde{I}'_{\varepsilon_n}(v_n - v), (v_n - v)\rangle\Big) = 0,$$

showing that $v_n \to v$ in E. □

Lemma 5.3. *There exists $C > 0$ such that $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Furthermore*

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}$$

and there exist C, $\beta > 0$ such that

$$|v_n(x)| \le C\exp(-\beta|x|), \quad \forall x \in \mathbb{R}^2.$$

Proof. From Lemma 5.2, we know $\varepsilon_n y_{\varepsilon_n} \to y_0 \in \mathcal{A}_P$, as $n \to \infty$ and $v_{\varepsilon_n}(x) := u_{\varepsilon_n}(x + y_{\varepsilon_n})$ converges strongly in E to a ground state solution v of

$$-\Delta v + V_0 v = P_0^2\Big[\frac{1}{|x|^\mu} * G(v)\Big]g(v).$$

Define

$$W_n^1(x) := \frac{1}{|x|^\mu} * \tilde{P}_n(x)G(v).$$

Notice that

$$\int_{\mathbb{R}^2} (|\nabla v_n|^2 + \tilde{V}_n(x)|v_n|^2) < \frac{2-\mu}{4},$$

we know there is $C > 0$ such that

$$|W_n^1(x)| \le C, \quad \forall n \in \mathbb{N}. \tag{5.16}$$

For any $R > 0$, $0 < r \le \frac{R}{2}$, let $\eta \in C^\infty(\mathbb{R}^2)$, $0 \le \eta \le 1$ with $\eta(x) = 1$ if $|x| \ge R$ and $\eta(x) = 0$ if $|x| \le R - r$ and $|\nabla\eta| \le \frac{2}{r}$. For $L > 0$, let

$$v_{L,n} = \begin{cases} v_n(x), & v(x) \le L \\ L, & v_n(x) \ge L, \end{cases}$$

and

$$z_{L,n} = \eta^2 v_{L,n}^{2(\gamma-1)} v_n \quad \textit{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\gamma-1}$$

with $\gamma > 1$ to be determined later. Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2 = &-2(\gamma-1)\int_{\mathbb{R}^2} v_n v_{L,n}^{2\gamma-3}\eta^2\nabla v_n \nabla v_{L,n} + \int_{\mathbb{R}^2} W_n^1(x)\tilde{P}_n(x)g(v_n)\eta^2 v_n v_{L,n}^{2(\gamma-1)} \\ &- \int_{\mathbb{R}^2} \tilde{V}_{\varepsilon_n}(x)|v_n|^2\eta^2 v_{L,n}^{2(\gamma-1)} - 2\int_{\mathbb{R}^2} \eta v_{L,n}^{2(\gamma-1)} v_n \nabla v_n \nabla\eta.\end{aligned} \tag{5.17}$$

Take $\beta, s > 1$ such that $\beta s K(v_n) < 1$, by Trudinger-Moser inequality, there exists C such that

$$\int_{\mathbb{R}^2} \big[e^{\beta 4\pi v_n^2} - 1\big]^s \le \int_{\mathbb{R}^2} \Big[e^{\beta s 4\pi K(v_n)\frac{v_n^2}{K(v_n)}} - 1\Big] < C. \tag{5.18}$$

Let $t=\sqrt{s}$, $p>\frac{2t}{t-1}>2$ and $\gamma=\frac{p(t-1)}{2t}$, for any $\delta>0$, there exists $C(\delta,p,\beta)>0$ such that

$$G(u)\le \delta u^2+C(\delta,p,\beta)u^{p-1}\big[\mathrm{e}^{\beta 4\pi|u|^2}-1\big],\ \ \forall u\in\mathbb{R}.$$

Thus for δ sufficiently small, from (5.16), (5.17) and Young's inequality, we get

$$\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\le C\int_{\mathbb{R}^2}v_n^p\eta^2 v_{L,n}^{2(\gamma-1)}\big[\mathrm{e}^{\beta 4\pi v_n^2}-1\big]+C\int_{\mathbb{R}^2}v_n^2 v_{L,n}^{2(\gamma-1)}|\nabla\eta|^2. \tag{5.19}$$

Using this fact, the estimates in [3] shows that

$$|w_{L,n}|_p^2\le C\gamma^2\left(C'+\left[\int_{|x|\ge R-r}{v_n}^{(p-2)t}\big[\mathrm{e}^{\beta 4\pi v_n^2}-1\big]^t\right]^{\frac{1}{t}}\right)\left[\int_{|x|\ge R-r}{v_n}^{\frac{2\gamma t}{t-1}}\right]^{\frac{t-1}{t}}.$$

By (5.18) and Hölder's inequality, we know

$$|w_{L,n}|_p^2\le C\gamma^2\left[\int_{|x|\ge R-r}{v_n}^{\frac{2\gamma t}{t-1}}\right]^{\frac{t-1}{t}}.$$

Now, following the same iteration arguments explored in [3], we find

$$|v_n|_{L^\infty(|x|\ge R)}\le C|v_n|_{p(|x|\ge R/2)}. \tag{5.20}$$

For $x_0\in B_R$, we can use the same argument taking $\eta\in C_0^\infty(\mathbb{R}^2,[0,1])$ with $\eta(x)=1$ if $|x-x_0|\le\rho'$ and $\eta(x)=0$ if $|x-x_0|>2\rho'$ and $|\nabla\eta|\le\frac{2}{\rho'}$, to prove that

$$|v_n|_{L^\infty(|x-x_0|\le\rho')}\le C|v_n|_{p(|x|\le 2\rho')}. \tag{5.21}$$

With (5.20) and (5.21), by a standard covering argument we can show that there exists $C>0$ such that

$$|v_n|_\infty<C.$$

Then, by regularity theory, we know $v_n\in C^2$ are classical solutions. Using again the convergence of (v_n) to v in E in (5.20), for each $\delta>0$ fixed, there exists $R>0$ such that $|v_n|_{L^\infty(|x|\ge R)}<\delta,\forall n\in N$. Thus,

$$\lim_{|x|\to\infty}v_n(x)=0\quad\text{uniformly in}\quad n\in\mathbb{N}.$$

The exponential decay property follows from a standard comparison arguments, we can repeat the arguments in Lemma 3.2. □

The next lemma is due to [7].

Lemma 5.4. *There exists $\delta_0>0$ such that $|v_n|_\infty\ge\delta_0$ for all $n\in\mathbb{N}$.*

Concentration behavior. If u_{ε_n} is a solution of problem (2.1), then $v_n(x)=u_{\varepsilon_n}(x+y_n)$ is a solution of problem

$$\begin{cases}-\triangle v_n+\tilde V_n(x)v_n=\left[\dfrac{1}{|x|^\mu}*\tilde P_n(x)G(v_n)\right]\tilde P_n(x)g(v_n)\\ v_n\in E, v_n(x)>0,\ \forall x\in\mathbb{R}^2,\end{cases}$$

with $\tilde V_n(x)=V(\varepsilon_n x+\varepsilon_n y_n)$, $\tilde P_n(x)=P(\varepsilon_n x+\varepsilon_n y_n)$ and $(y_n)\subset\mathbb{R}^2$ given in Lemma 5.2. Moreover, up to a subsequence,

$$v_n\to v\quad\text{in}\quad E,\quad \tilde y_n\to y_0\in\mathcal{A}_P,$$

where $\tilde{y}_n = \varepsilon_n y_n$. If b_n denotes a maximum point of v_n, from Lemmas 5.3 and 5.4, we know it is a bounded sequence in $\mathbb{R}^2$. Thus, there is $R > 0$ such that $b_n \in B_R(0)$. Thereby, the global maximum of u_{ε_n} is $z_{\varepsilon_n} = b_n + y_n$ and

$$\varepsilon_n z_{\varepsilon_n} = \varepsilon_n b_n + \varepsilon_n y_n = \varepsilon_n b_n + \tilde{y}_n.$$

From boundedness of (b_n), we get the limit

$$\lim_{n\to\infty} \varepsilon_n z_{\varepsilon_n} = y_0,$$

which together with the continuity of V gives

$$\lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V(y_0) \quad \text{and} \quad \lim_{n\to\infty} P(\varepsilon_n z_{\varepsilon_n}) = P(y_0).$$

We also point out that for any $\varepsilon > 0$ the sequence $\varepsilon z_\varepsilon$ is bounded, where z_ε is the maximum point of the solution u_ε obtained in Lemma 5.1. In fact, if there exists $\varepsilon_j \to 0$ and z_{ε_j} of u_{ε_j} such that $\varepsilon_j z_{\varepsilon_j} \to \infty$. However, from the above arguments, we know

$$\varepsilon_j z_{\varepsilon_j} = \varepsilon_j b_{\varepsilon_j} + \varepsilon_j y_{\varepsilon_j},$$

where y_{ε_j} is obtained in (5.9) by non-vanishing argument with $\varepsilon_j y_{\varepsilon_j}$ bounded, and b_{ε_j} is the maximum point of of $v_{\varepsilon_j} = u_{\varepsilon_j}(x + y_{\varepsilon_j})$. Consequently, $\varepsilon_j z_{\varepsilon_j} - \varepsilon_j y_{\varepsilon_j} = \varepsilon_j b_{\varepsilon_j} \to \infty$. which contradicts with the fact b_{ε_j} lies in a ball $B_R(0)$.

Proof of Theorem 1.5. From Lemma 5.1, there is a positive solution for equation (2.1) for $\varepsilon > 0$ small enough. Therefore, the function $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of equation (1.6). Thus, the maximum points x_ε and z_ε of w_ε and u_ε respectively, satisfy the equality $x_\varepsilon = \varepsilon z_\varepsilon$. Setting $v_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$, $\varepsilon \to 0$, it follows Lemma 5.2 that,

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{A}_P) = 0$$

and v_ε converges in E to a ground state solution v of

$$-\Delta v + V(x_0)v = P^2(x_0)\left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

From Lemma 5.3, for some $c, C > 0$,

$$|w_\varepsilon(x)| \le C\exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right).$$

In particular if $\mathcal{V} \cap \mathcal{P} \ne \emptyset$, then $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{V} \cap \mathcal{P}) = 0$ and up to subsequences, v_ε converges in E to a ground state solution v of

$$-\Delta v + \kappa_{\min} v = \tau_{\max}^2 \left[\frac{1}{|x|^\mu} * G(v)\right] g(v).$$

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