ESAIM: COCV 23 (2017) 1751–1765
DOI: 10.1051/cocv/2016072

**ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

# ON CONTACT SUB-PSEUDO-RIEMANNIAN ISOMETRIES

Marek Grochowski[1] and Wojciech Kryński[2]

**Abstract.** We study isometries in contact sub-pseudo-Riemannian geometry. In particular we give an upper bound on the dimension of the isometry group of a general sub-pseudo-Riemannian manifold and prove that the maximal dimension is attained for the left invariant structures on the Heisenberg group.

**Mathematics Subject Classification.** 53C17, 34H05.

Received October 22, 2015. Revised September 14, 2016. Accepted November 10, 2016.

## 1. Introduction

### 1.1. Results

Let $M$ be a smooth connected manifold. A *sub-pseudo-Riemannian structure* on $M$ is a pair $(D, g)$ made up of a smooth bracket generating distribution $D$ of constant rank and a smooth pseudo-Riemannian metric $g$ on $D$. At each point $q \in M$, $g$ can be represented as a diagonal matrix

$$\mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$$

with, say, $l$ minuses. Clearly, by continuity, the number $l$ does not depend on a point $q$. It will be denoted $\mathrm{ind}(g)$ and called *the index of the metric* $(D, g)$.

A triple $(M, D, g)$ is called a *sub-pseudo-Riemannian manifold*. In particular, if $\mathrm{ind}(g) = 0$ then $(M, D, g)$ is called a *sub-Riemannian manifold*. This case is best known and there are a lot of papers and books devoted to the sub-Riemannian geometry (see [1–4, 15] and references therein). If $\mathrm{ind}(g) = 1$ then $(M, D, g)$ is called a *sub-Lorentzian manifold* (see [6, 8, 9, 11]). The sub-pseudo-Riemannian structures can be interpreted as control systems [1, 5]. In particular the sub-Lorentzian structures give rise to a class of control-affine systems (*cf.* [5, 6]).

In sub-pseudo-Riemannian geometry we can ask the same questions as in the classical pseudo-Riemannian geometry. One of the most fundamental problems considered in the pseudo-Riemannian geometry is connected to calculations of the isometry group of a given pseudo-Riemannian manifold. We shall consider a generalisation of this problem to the sub-pseudo-Riemannian case.

*Keywords and phrases.* Contact structure, sub-Riemannian geometry, sub-Lorentzian geometry, Heisenberg group, isometry group, control-affine systems.

[1] Faculty of Mathematics and Natural Sciences, Cardinal Wyszyński University, 01-938 Warszawa, Poland.
m.grochowski@uksw.edu.pl

[2] Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland. w.krynski@impan.pl

Article published by EDP Sciences 

**Definition 1.1.** Fix a sub-pseudo-Riemannain manifold $(M, D, g)$. A diffeomorphism $f\colon M \to M$ is called an isometry if

(D1) $f$ preserves the distribution, *i.e.* $f_*(D) = D$;
(D2) $f_*\colon D_q \to D_{f(q)}$ is a linear isometry for every $q \in M$, *i.e.* $g(f_*(v), f_*(w)) = g(v, w)$ for all $v, w \in D_q$.

The set of all isometries is a group (in fact a Lie group as it will become clear soon) and will be denoted $\mathfrak{I}(M, D, g)$. The component of the identity of this group is $\mathfrak{I}_0(M, D, g)$. Clearly $\dim \mathfrak{I}(M, D, g) = \dim \mathfrak{I}_0(M, D, g)$. We shall assume that $D$ is a contact distribution meaning that it is locally given by the kernel of a contact one-form $\alpha$ satisfying

$$(\mathrm{d}\alpha)^{\wedge n} \wedge \alpha \neq 0, \tag{1.1}$$

where $\dim M = 2n + 1$. In this case $(M, D, g)$ will be referred to as a *contact sub-pseudo-Riemannian manifold*. Our main result is the following theorem.

**Theorem 1.2.** *Let $(M, D, g)$ be a contact sub-pseudo-Riemannian manifold. If $\operatorname{ind}(g)$ is even or $\operatorname{ind}(g) = \frac{1}{2} \operatorname{rk} D$ then*

$$\dim \mathfrak{I}(M, D, g) \leq \dim M + \left(\frac{1}{2} \operatorname{rk} D\right)^2 \cdot \tag{1.2}$$

*If $\operatorname{ind}(g)$ is odd and $\operatorname{ind}(g) \neq \frac{1}{2} \operatorname{rk} D$ then*

$$\dim \mathfrak{I}(M, D, g) \leq \dim M + \left(\frac{1}{2} \operatorname{rk} D - 1\right)^2 + 1. \tag{1.3}$$

In Section 3, Proposition 3.6, we will show that the maximal dimension in (1.2) and (1.3) is attained by a left-invariant structure on the Heisenberg group. More precisely we will show that for any value of $\operatorname{ind}(g) \in \{0, 1, \ldots, \operatorname{rk} D\}$ and any $t \leq \min\{\operatorname{ind}(g), \operatorname{rk} D - \operatorname{ind}(g)\}$ such that $\operatorname{ind}(g) - t$ is even there is a left-invariant structure such that

$$\dim \mathfrak{I}(M, D, g) = \dim M + \left(\frac{1}{2} \operatorname{rk} D - t\right)^2 + t^2. \tag{1.4}$$

In particular the maximal dimension in Theorem 1.2 is attained for $t = 0$ if $\operatorname{ind}(g)$ is even, for $t = \operatorname{ind}(g)$ if $\operatorname{ind}(g) = \frac{1}{2} \operatorname{rk} D$ and for $t = 1$ if $\operatorname{ind}(g)$ is odd and not equal $\frac{1}{2} \operatorname{rk} D$.

Let us point out here that invariants for the contact sub-pseudo-Riemannian structures has been recently constructed in [7] (see also [1] for the sub-Riemannian case). The invariants vanish for the left-invariant structures satisfying (1.4).

### 1.2. Connections with control systems.

Suppose that $(\Sigma)$ $\dot{q} = f(q, u)$ is a control system on a manifold $M$. By a symmetry of $(\Sigma)$ we mean a diffeomorphism of $M$ which maps the trajectories of $(\Sigma)$ onto trajectories of $(\Sigma)$. It turns out that the described results concerning isometry groups of sub-pseudo-Riemannian manifolds can be formulated in terms of symmetries of certain control systems.

Indeed, suppose that $(M, D, g)$ is a sub-pseudo-Riemannian manifold with $\operatorname{ind}(g) = l$ and $\operatorname{rk} D = k$. By a timelike (resp. spacelike) curve on $(M, D, g)$ we mean an absolutely continuous curve $\gamma\colon [a, b] \to M$ such that $\dot{\gamma} \in D_{\gamma(t)}$ and moreover $g(\dot{\gamma}(t), \dot{\gamma}(t)) < 0$ (resp. $g(\dot{\gamma}(t), \dot{\gamma}(t)) > 0$) for almost every $t \in [a, b]$. Suppose that $X_1, \ldots, X_k$ is an orthonormal basis for $(D, g)$ defined on an open set $U$ such that $g(X_i, X_i) = -1$ for $i = 1, \ldots, l$ and $g(X_i, X_i) = 1$ for $i = l + 1, \ldots, k$. Timelike (resp. spacelike) curves in $U$ with unit speed parametrization can be represented as solutions to the following control system

$$\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \tag{1.5}$$

with the set of control parameters equal to

$$\mathcal{U} = \left\{ (u_1, \ldots, u_k) \mid -\sum_{i=1}^{l} u_i^2 + \sum_{i=l+1}^{k} u_i^2 = -1 \right\},$$

or

$$\mathcal{U} = \left\{ (u_1, \ldots, u_k) \mid -\sum_{i=1}^{l} u_i^2 + \sum_{i=l+1}^{k} u_i^2 = 1 \right\},$$

respectively, where controls are supposed to be measurable and essentially bounded. Now, it is easy to show that in both cases the symmetries of (1.5) coincide with the isometry group $\mathfrak{I}(U, D_{|U}, g)$. One can also consider the sets of null, or nonspecelike curves defined by similar control systems. However in the latter cases the isometry group $\mathfrak{I}(U, D_{|U}, g)$ is only a subgroup of all symmetries.

In the sub-Lorentzian setting the future directed nonspacelike curves can be described by a control-affine system. To be precise, by a time orientation of a sub-Lorentzian manifold $(M, D, g)$ we understand a timelike vector field $X$ on $M$ (*i.e.* $X(q) \in D_q$ and $g(X(q), X(q)) < 0$ for every $q \in M$). A nonspacelike curve $\gamma\colon [a, b] \to M$ is said to be future directed if $g(\dot\gamma(t), X(\gamma(t))) < 0$ a.e. on $[a, b]$ (*cf.* [5,6]). Suppose that $X$ is a fixed time orientation and $U$ is an open set on which there exist spacelike vector fields $X_2, \ldots, X_k$ such that $X, X_2, \ldots, X_k$ form an orthonormal basis for $(D, g)$ over $U$. As it is explained in [5] every nonspacelike future directed curve in $U$ is, up to a reparameterization, a trajectory of the control-affine system

$$\dot q = X + \sum_{i=2}^{k} u_i X_i(q), \tag{1.6}$$

where the set of control parameters equals the unit ball in $\mathbb{R}^{k-1}$ centered at zero. Now it is clear that $\mathfrak{I}(U, D_{|U}, g)$ is a group of symmetries of the system (1.6). We refer to [1,5,6] for more information on the mentioned control systems, the corresponding reachable sets and optimal solutions to the control problems.

### 1.3. The content of the paper.

The paper is organised as follows. In Section 2 we formulate and explain basic facts and assumptions that we use later on. We show that $g$ can be extended to a metric on $TM$ in a canonical way and exploit this fact to prove that $\mathfrak{I}(M, D, g)$ is a Lie group (Thm. 2.2). Moreover, we introduce a canonical symplectic structure on $D$.

Sections 3 and 4 are devoted to special classes of sub-pseudo-Riemannian metrics. In Section 3 we consider sub-pseudo-Riemannian structures satisfying an additional compatibility condition. In the Riemannian signature the condition guarantees that $D$ caries an almost complex structure. In Section 4 we consider so-called regular structures, which include all sub-Riemannian and sub-Lorentzian metrics in neighbourhoods of generic points. We estimate from above dimensions of the isometry groups for these spacial classes of structures (Thms. 3.3 and 4.2). Moreover, we construct examples with isometry groups of dimension given by formula (1.4).

Section 5 contains the proof of Theorem 1.2. The main idea relies on the calculation of the Tanaka prolongations of certain graded Lie algebras and on the results of Kruglikov [12,13] that extend Tanaka's theory to the case of non-constant symbol algebras.

## 2. Contact sub-pseudo-Riemannian structures

### 2.1. Extended metric

Let $(M, D, g)$ be a contact sub-pseudo-Riemannian manifold of dimension $2n + 1$. Fix $q \in M$ and assume that $D = \ker\alpha$ in a neighbourhood of $q$, where $\alpha$ satisfies (1.1). The contact form $\alpha$ defines the Reeb vector field $X_\alpha$ by the conditions

$$X_\alpha \in \ker \mathrm{d}\alpha, \qquad \alpha(X_\alpha) = 1. \tag{2.1}$$

It follows that $X_\alpha$ is transverse to $D$. Clearly $X_\alpha$ depends essentially on the choice of $\alpha$ and the one-form is not unique. However it can be normalised in the following way. Let $(X_1,\dots,X_{2n})$ be an orthonormal frame of $D$ in a neighbourhood of $q$. Then, multiplying $\alpha$ by a smooth function, we can impose the condition

$$|(\mathrm{d}\alpha)^{\wedge n}(X_1,\dots,X_{2n})| = 1, \tag{2.2}$$

which does not depend on the choice of an orthonormal frame. As a result, we get a canonical contact form $\alpha$ given up to a multiplication by $\pm 1$ in the neighbourhood of $q\in M$. We shall see later that for oriented structures one can get rid of this ambiguity and get a unique canonical global contact form $\alpha$ on $M$. However, we do not need the uniqueness at this point and using the two normalised contact forms in a neighbourhood of any point $q\in M$ we are able to extend $g$ from $D$ to a metric $G$ on $TM$. Indeed, we set

$$G|_{D\times D} = g$$

and

$$G(X_\alpha,X_\alpha)=1, \qquad G(X_\alpha,D)=0,$$

where $\alpha$ is a contact form satisfying (2.2) and $X_\alpha$ is the Reeb vector field corresponding to $\alpha$. Since $\alpha$ is given up to a sign, we conclude that $X_\alpha$ is given up to a sign too. However, $G$ does not depend on the sign and we obtain unique $G$ in a neighbourhood of each point $q\in M$. The uniqueness implies that $G$ must coincide on overlaps of neighbourhoods of different points. Thus, we get a globally defined metric $G$ on $M$ which is canonically determined by the structure $(D,g)$. Since any isometry preserves the form $\alpha$ up to a sign, we have proved the following

**Proposition 2.1.** *If $f\colon M\to M$ is an isometry of a contact sub-pseudo-Riemannian structure $(D,g)$ then $f^*G=G$. Thus $f$ is an isometry of $G$, too.*

We shall denote by $\mathfrak{I}(M,G)$ the group of isometries of $(M,G)$. We refer to [7] for more detailed discussion on the possible extensions of $g$.

Let $O_G(M)$ be the orthonormal frame bundle for $G$. We define $O_{D,g}(M)$, the *orthonormal frame bundle* of $(D,g)$, as a sub-bundle of $O_G(M)$ consisting of points $(q;v_1,\dots,v_{2n},v_0)$ such that $(v_1,\dots,v_{2n})$ is an orthonormal basis of $D_q$. In particular, it follows that $v_0=X_\alpha(q)$ where $\alpha$ is one of the two contact forms normalised by (2.2) in a neighbourhood of $q$. Now, any pseudo-Riemannian isometry $f\in\mathfrak{I}(M,G)$ is uniquely determined by the values of $f(q)$ and $f_*(q)$ where $q$ is an arbitrary fixed point in $M$ [10]. Since $\mathfrak{I}(M,D,g)$ is a closed subgroup of $\mathfrak{I}(M,G)$ we get

**Theorem 2.2.** *$\mathfrak{I}(M,D,g)$ is a Lie group with respect to the open-compact topology. Moreover any contact sub-pseudo-Riemannian isometry $f\in\mathfrak{I}(M,D,g)$ is uniquely determined by two values: $f(q)$ and $f_*(q)$, where $q\in M$ is an arbitrarily fixed point. Additionally, fixing an arbitrary point $(q;v_1,\dots,v_{2n},v_0)\in O_{D,g}(M)$, the mapping*

$$f\longmapsto (f(q);f_*(v_1),\dots,f_*(v_{2n}),f_*(v_0)) \tag{2.3}$$

*defines an embedding of $\mathfrak{I}(M,D,g)$ to $O_{D,g}(M)$.*

*Proof.* Follows from the fact that $\mathfrak{I}(M,G)$ is a Lie group [10] (note that although [10] deals with Riemannian metrics only, the proof remains unchanged for metrics of arbitrary index) and its subgroup $\mathfrak{I}(M,D,g)$ is closed in $\mathfrak{I}(M,G)$. $\square$

## 2.2. Orientation

Let $(M,D,g)$ be a contact sub-pseudo-Riemannian manifold of dimension $2n+1$. We shall say that the structure is *oriented* if the two vector bundles $TM$ and $D$ are oriented (see [6] for various notions of orientations related to the casual decomposition of the distribution under consideration). We shall prove that the structure is oriented if and only if there is a global contact form annihilating $D$. There are two cases depending on the parity of $n$.

If $n$ is even then $(\mathrm{d}\alpha)^{\wedge n}$ is independent of the sign of $\alpha$. Conversely, the sign of $\mathrm{d}\alpha^{\wedge n}\wedge\alpha$ changes if the sign of $\alpha$ changes. Thus, on the one hand, $D$ is canonically oriented, because fixing an open cover $\{U_s\}_{s\in\Sigma}$ of $M$ and local contact forms $\{\alpha_s\}_{s\in\Sigma}$ annihilating $D$ on $U_s$ we can rescale the forms such that $(\mathrm{d}\alpha_s)^{\wedge n}$ glue to a global $2n$-form non-degenerate on $D$. On the other hand, $M$ is oriented if and only if there is a global contact form annihilating $D$. Indeed, if $\alpha$ is a global contact form then $\mathrm{d}\alpha^{\wedge n}\wedge\alpha$ defines an orientation of $M$. Conversely, if an orientation of $M$ is given then we can rescale local contact forms $\{\alpha_s\}_{s\in\Sigma}$ annihilating $D$ such that $\mathrm{d}\alpha_s^{\wedge n}\wedge\alpha_s$ agree with the orientation. Clearly, such rescaled one-forms must coincide on the intersections of domains $U_s$. Thus, they define a global one-form on $M$.

If $n$ is odd then $(\mathrm{d}\alpha)^{\wedge n}\wedge\alpha$ is independent of the sign of $\alpha$. Conversely, the sign of $\mathrm{d}\alpha^{\wedge n}$ changes if the sign of $\alpha$ changes. Thus, similarly to the case of even $n$, we deduce that on the one hand $M$ is canonically oriented, and, on the other hand, $D$ is oriented if and only if there is global contact form annihilating $D$.

Suppose that $(M,D,g)$ is oriented. In view of the discussion above we can assume that the orientation of $M$ is given by $\mathrm{d}\alpha^{\wedge n}\wedge\alpha$ and the orientation of $D$ is given by $\mathrm{d}\alpha^{\wedge n}$, where $\alpha$ is a global contact form. Then $\alpha$ is given up to a multiplication by a positive function. However, we can choose the unique one which satisfies the normalisation condition (2.2). We shall call this form the *canonical contact form* of an oriented contact structure. The canonical contact form satisfies

$$(\mathrm{d}\alpha)^{\wedge n}(X_1,\ldots,X_{2n})=1. \tag{2.4}$$

where $(X_1,\ldots,X_{2n})$ is an arbitrary positively oriented orthonormal frame of $D$.

If $(M,D,g)$ is oriented then we shall consider isometries preserving the orientation.

### 2.3. Symplectic structure

Assume that $(M,D,g)$ is an oriented contact sub-pseudo-Riemannian manifold and let $\alpha$ be the canonical contact form. We introduce

$$\omega=-\mathrm{d}\alpha|_D.$$

Then $\omega$ is a symplectic structure on $D$ canonically defined by $\alpha$.

**Proposition 2.3.** *If $f\colon M\to M$ is an isometry of an oriented contact sub-pseudo-Riemannian structure then $f^*\omega=\omega$.*

The pair $(g,\omega)$ defines the operator $J\colon D\to D$ by the formula

$$\omega_q(v,w)=g(J_q(v),w),\qquad q\in M,\quad v,w\in D_q. \tag{2.5}$$

The eigenvalues of $J_q$ are basic invariants of the structure $(D,g)$ at each point $q\in M$. A real invariant subspace of $D_q$ corresponding to an eigenvalue $\lambda$ of $J_q$ will be denoted $D_\lambda$ (if $\lambda$ is complex then $D_\lambda=D_{\bar\lambda}$). Each $D_\lambda$ can decompose further into a number of real invariant subspaces corresponding to different Jordan blocks of $J_q$. The following properties of $J$ will be used later (we give a proof for completeness, however the properties can be also extracted from the Kronecker theorem on normal forms of pencils of matrices applied to the pair $(\omega,g)$, see [14, 17]).

**Proposition 2.4.** *Let $(M,D,g)$ be an oriented contact sub-pseudo-Riemannian manifold and let $q\in M$. Then*

- (P1) *If $\lambda$ is an eigenvalue of $J_q$ then also $-\lambda$ is and $D_\lambda$ and $D_{-\lambda}$ are of the same dimension.*
- (P2) *Any two invariant subspaces $D_{\lambda_1}$ and $D_{\lambda_2}$ are orthonormal unless $\lambda_1=-\lambda_2$. In particular, if $\lambda$ has non-zero real part then $g|_{D_\lambda}=0$.*

*Proof.* Let $S$ be the skew-symmetric matrix of $\omega$ written in an orthonormal basis of $D$ with respect to $g$. Then $SI_l$ is the matrix of $J$, where $l = \operatorname{ind}(g)$ and

$$I_l = \operatorname{diag}\left(\underbrace{-1,\dots,-1}_{l},+1,\dots,+1\right).$$

Let $J^*\colon D^* \to D^*$ be the map dual to $J$. Then $J^*$ written in the dual orthonormal basis has the matrix $(SI_l)^T = -I_lS$. Hence, $I_l$ maps $D_\lambda$ to the invariant subspace of $J^*$ corresponding to the eigenvalue $-\lambda$. Indeed, if $J(X) = \lambda X$ then, since $I_l^2 = \mathrm{Id}$, $J^*(I_lX) = -I_lSI_lX = -I_lJ(X) = -\lambda I_lX$, and similarly if $J(X) = \lambda X + Y$ then $J^*(X) = -\lambda I_lX - I_lY$. On the other hand $J^*$ has the same decomposition to the Jordan blocks as $J$. This proves (P1).

In order to prove (P2) assume first that $\lambda_1$ and $\lambda_2$ are real eigenvalues of $J$. Let $X_1^0$ and $X_2^0$ be the corresponding eigenvectors and let $X_1^i$ and $X_2^j$ satisfy $J(X_1^i) = \lambda_1 X_1^i + X_1^{i-1}$ and $J(X_2^j) = \lambda_2 X_2^j + X_2^{j-1}$ for $i = 1,\dots,s_1$ and $j = 1,\dots,s_2$. Then

$$\begin{aligned}0 &= \omega(X_1^i, X_2^j) + \omega(X_2^j, X_1^i) = g(J(X_1^i), X_2^j) + g(J(X_2^j), X_1^i)\\ &= (\lambda_1+\lambda_2)g(X_1^i, X_2^j) + g(X_1^{i-1}, X_2^j) + g(X_1^i, X_2^{j-1}),\end{aligned}$$

where the two last terms are 0 if $i = 0$ or $j = 0$, respectively. This, by induction, proves $g(X_1^i, X_2^j) = 0$ for all $i = 0,\dots,s_1$ and $j = 0,\dots,s_2$, provided that $\lambda_1 + \lambda_2 \neq 0$.

Now, assume that both $\lambda_1$ and $\lambda_2$ are complex eigenvalues of $J$. Let $Z_1^0 = X_1^0 + \mathrm{i}Y_1^0$ and $Z_2^0 = X_2^0 + \mathrm{i}Y_2^0$ be the corresponding eigenvectors and let $Z_1^i = X_1^i + \mathrm{i}Y_1^i$ and $Z_2^j = X_2^j + \mathrm{i}Y_2^j$ satisfy $J(Z_1^i) = \lambda_1 Z_1^i + Z_1^{i-1}$ and $J(Z_2^j) = \lambda_2 Z_2^j + Z_2^{j-1}$ for $i = 1,\dots,s_1$ and $j = 1,\dots,s_2$. Then similarly to the real case we get (proceeding by induction and omitting terms involving $Z_1^{i-1}$ and $Z_2^{j-1}$)

$$\begin{aligned}0 &= \omega(Z_1^i, Z_2^j) + \omega(Z_2^j, Z_1^i)\\ &= (\lambda_1+\lambda_2)(g(X_1^i, X_2^j) - g(Y_1^i, Y_2^j) + \mathrm{i}(g(X_1^i, Y_2^j) + g(Y_1^i, X_2^j))),\end{aligned}$$

and

$$\begin{aligned}0 &= \omega(\bar Z_1^i, Z_2^j) + \omega(Z_2^j, \bar Z_1^i)\\ &= (\bar\lambda_1+\lambda_2)(g(X_1^i, X_2^j) + g(Y_1^i, Y_2^j) + \mathrm{i}(g(X_1^i, Y_2^j) - g(Y_1^i, X_2^j))),\end{aligned}$$

which imply that $g(X_1^i, X_2^j) = 0$, $g(Y_1^i, X_2^j) = 0$, $g(X_1^i, Y_2^j) = 0$ and $g(Y_1^i, Y_2^j) = 0$ for all $i = 0,\dots,s_1$ and $j = 0,\dots,s_2$, provided that $\lambda_1 + \lambda_2 \neq 0$ and $\bar\lambda_1 + \lambda_2 \neq 0$.

Finally, if $\lambda_1$ is complex and $\lambda_2$ is real and $Z_1^0 = X_1^0 + \mathrm{i}Y_1^0$ and $X_2^0$ are the corresponding eigenvectors and $Z_1^i = X_1^i + \mathrm{i}Y_1^i$ and $Z_2^j$ satisfy $J(Z_1^i) = \lambda_1 Z_1^i + Z_1^{i-1}$ and $J(X_2^j) = \lambda_2 X_2^j + X_2^{j-1}$ for $i = 1,\dots,s_1$ and $j = 1,\dots,s_2$, then exactly the same argument proves that $g(X_1^i, X_2^j) = 0$, $g(Y_1^i, X_2^j) = 0$. This completes the proof of (P2). □

We get that at each $q \in M$

$$D_q = \hat D_q \oplus \tilde D_q \tag{2.6}$$

where $\hat D$ is as a sum of eigenspaces corresponding to purely imaginary eigenvalues of $J$ and $\tilde D$ is a sum of eigenspaces corresponding to eigenvalues of $J$ with non-zero real part. The last one are null with respect to $g$ and appear in pairs $D_\lambda \oplus D_{-\lambda}$.

**Corollary 2.5.** $\operatorname{ind}(g|_{\tilde D}) = \frac{1}{2}\dim\tilde D$ *and* $\operatorname{ind}(g|_{\hat D})$ *is even. In particular,* $D = \hat D$ *in the sub-Riemannian case, and* $\dim\tilde D = 2$ *in the sub-Lorentzian case.*

*Proof.* Note that (P2) in Proposition 2.4 imply that $g$ restricted to any $D_\lambda \oplus D_{-\lambda}$, where $\lambda$ has non-trivial real part, is of the form $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ where $A$ is certain $s\times s$ matrix, $s = \dim D_\lambda$. This implies that $\mathrm{ind}(g|_{D_\lambda \oplus D_{-\lambda}}) = s$ and consequently $\mathrm{ind}(g|_{\tilde{D}}) = \frac{1}{2}\dim \tilde{D}$. On the other hand, if $\lambda$ is purely imaginary then, proceeding as in the proof of Proposition 2.4, one gets $g(Z^i, Z^j) = 0$ where $Z^1, \ldots, Z^s$ are complex vectors spanning the complexification of $D_\lambda$. This implies that $g$ restricted to $D_\lambda$ is composed from $2 \times 2$ blocks of the form $\begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix}$, where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ and $b_{ii} = 0$. Consequently $\mathrm{ind}(g|_{\hat{D}})$ is even. $\square$

Let $\hat{J} = J|_{\hat{D}}$ and $\tilde{J} = J|_{\tilde{D}}$. Then, according to (2.6), we can write

$$J = \hat{J} \oplus \tilde{J}.$$

Note that if all Jordan blocks corresponding to purely imaginary eigenvalues of $J$ are 2-dimensional then

$$\hat{J} = \begin{pmatrix} 0 & -b_1 & \ldots & 0 & 0 \\ b_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -b_s \\ 0 & 0 & \ldots & b_s & 0 \end{pmatrix}, \tag{2.7}$$

for some $(b_1, \ldots, b_s)$, where $s = \frac{1}{2}\dim \hat{D}$. It is easy to prove that $\hat{J}$ is necessarily of this form provided that $g$ is definite on each $D_\lambda$. In particular this happens if $g$ is sub-Riemannian (see [1]) or sub-Lorentzian (due to Cor. 2.5). The numbers $(b_1, \ldots, b_s)$ are called frequencies in this case.

Similarly, if all eigenvalues of $\tilde{J}$ are real and the corresponding Jordan blocks are 1-dimensional then

$$\tilde{J} = \begin{pmatrix} 0 & c_1 & \ldots & 0 & 0 \\ c_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & c_t \\ 0 & 0 & \ldots & c_t & 0 \end{pmatrix} \tag{2.8}$$

for some $(c_1, \ldots, c_t)$ where $t = \dfrac{1}{2}\dim \tilde{D}$. If (2.7) and (2.8) mutually hold then

$$\left(\prod_{i=1}^{s} b_i\right)\left(\prod_{i=1}^{t} c_i\right) = 1,$$

due to (2.4).

## 2.4. Reduction

Let $(D, g)$ be an oriented sub-pseudo-Riemannian contact structure on $M$. Then the symplectic structure $\omega$ reduces the full frame bundle of $M$ to the set of frames $O_{D,g,\omega}(M)$ consisting of $(q; v_1, \ldots, v_{2m}, v_0)$ such that $(v_1, \ldots, v_{2m})$ put $(g_q, \omega_q)$ into the canonical Kronecker form defined, for instance, in ([14], Thm. 12.1) and $v_0 = X_\alpha(q)$, where $X_\alpha$ is the Reeb vector field corresponding to the canonical contact form. Then the following group acts freely and transitively on $O_{D,g,\omega}(M)_q$

$$\mathcal{G}_{g,\omega}(q) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in O(g_q) \cap Sp(\omega_q) \right\}, \tag{2.9}$$

where $O(g_q)$ is the subgroup of $GL(D_q)$ preserving $g_q$ and $Sp(\omega_q)$ is the subgroup of $GL(D_q)$ preserving $\omega_q$. Of course $O(g_q) \simeq O(l, 2n-l)$, where $l = \mathrm{ind}(g)$ and $O(l, 2n-l)$ is the standard group of matrices preserving a metric of index $l$ and $Sp(\omega_q) \simeq Sp(2n)$, where $Sp(2n)$ is the group of matrices preserving the standard symplectic form given by

$$\Omega = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}. \tag{2.10}$$

Note that automatically $O(g_q) \cap Sp(\omega_q) \subset SO(g_q)$, because the orientation is defined in terms of $\omega_q$.

The intersection $O(g_q) \cap Sp(\omega_q)$ essentially depends on $g$ and $\omega$ at a given point and the groups $\mathcal{G}_{g,\omega}(q)$ may be not isomorphic for different $q$ (in particular $O_{D,g,\omega}(M)$ is not necessarily a fiber bundle). Actually, we shall show later that the dimension of $\mathcal{G}_{g,\omega}(q)$ depends on the decomposition of $J_q$ into the sum of eigenspaces.

## 3. Compatibility condition

### 3.1. Isometries of compatible structures

We will consider a particular class of oriented contact sub-pseudo-Riemannian structures such that $g$ and $\omega$ are compatible. One expects that the most symmetric structures are among this class.

**Definition 3.1.** Let $(M, D, g)$ be an oriented sub-pseudo-Riemannian manifold and let $\omega$ be the corresponding symplectic structure on $D$. Then $g$ and $\omega$ are *compatible* if in a neighbourhood of any $q \in M$ there is an orthonormal frame with respect to $g$ which is, after possible reordering, symplectic with respect to $\omega$. The sub-pseudo-Riemannian structure satisfies the *compatibility condition* if $g$ and $\omega$ are compatible.

Note that in the case of compatible structures with $g$ being Riemannian, $J$ is an almost complex structure on $D$. Similarly, in the case of compatible structures with $\mathrm{ind}(g) = \frac{1}{2}\,\mathrm{rk}\, D$, $J$ is a para-CR structure, provided that there are no purely imaginary eigenvalues of $J$. In general, the compatibility condition can be expressed in terms of frequencies.

**Proposition 3.2.** *An oriented contact sub-pseudo-Riemannian structure satisfies the compatibility condition if and only if $\hat{J}$ is of the form (2.7) with $b_i = 1$, $i = 1, \ldots, s$ and $\tilde{J}$ is of the form (2.8) with $c_i = 1$, $i = 1, \ldots, t$.*

*Proof.* Follows directly from the definition. □

The bundle $O_{D,g,\omega}(M)$ for a structure $(D, g)$ satisfying the compatibility condition is the bundle of frames that are mutually orthonormal with respect to $g$ and symplectic with respect to $\omega$. Proposition 3.2 implies that under the compatibility condition all $\mathcal{G}_{g,\omega}(q)$, $q \in M$, are isomorphic, because $J_q$ depends smoothly on $q$ and $M$ is connected. Thus $O_{D,g,\omega}(M)$ is a principal bundle with the structure group isomorphic to $\mathcal{G}_{g,\omega}(q)$ for any fixed $q \in M$. The structure group will be simply denoted $\mathcal{G}_{g,\omega}$. Moreover, Proposition 2.3 implies that the embedding (2.3) restricted to the component of identity $\mathfrak{I}_0(M, D, g)$ takes values in $O_{D,g,\omega}(M)$. Precisely, fixing $(q; v_1, \ldots, v_{2n}, v_0) \in O_{D,g,\omega}$ we get that

$$f \longmapsto (f(q); f_*(v_1), \ldots, f_*(v_{2n}), f_*(v_0)) \tag{3.1}$$

defines an embedding of $\mathfrak{I}_0(M, D, g)$ in $O_{D,g,\omega}(M)$. This embedding permits to prove

**Theorem 3.3.** *Let $(M, D, g)$ be an oriented contact sub-pseudo-Riemannian manifold satisfying the compatibility condition. Then*

$$\dim \mathfrak{I}(M, D, g) \leq 2n + 1 + s^2 + (n - s)^2,$$

*where $\dim M = 2n + 1$ and $s = \frac{1}{2}\,\mathrm{rk}\,\hat{D}$ is the multiplicity of $\mathrm{i} = \sqrt{-1}$ as an eigenvalue of the endomorphism $J$. Moreover, the parity of $n - s$ equals to the parity of $\mathrm{ind}(g)$.*

*Proof.* We recall that $g$ restricted to any two-dimensional component in the decomposition (2.7) is definite (compare Cor. 2.5). Additionally $g$ restricted to $\tilde{D}$ has index equal to $\frac{1}{2}\operatorname{rk}\tilde{D}$. Thus

$$\operatorname{ind}(g) = \frac{1}{2}\operatorname{rk}\tilde{D} \mod 2$$

and since $\frac{1}{2}\operatorname{rk}\tilde{D} = n - s$ the last statement of the Theorem follows.

Therefore, it is sufficient to compute the dimension of $\mathcal{G}_{g,\omega}$ in order to complete the proof, because the existence of the embedding (3.1) implies

$$\dim \mathfrak{I}_0(M, D, g) \leq \dim M + \dim \mathcal{G}_{g,\omega},$$

and $\dim \mathfrak{I}(M, D, g) = \dim \mathfrak{I}_0(M, D, g)$. The result follows from the following general Lemma that will be also used later in the proof of Theorem 1.2.

**Lemma 3.4.** *Let $s = \frac{1}{2}\operatorname{rk}\hat{D}$ and $t = \frac{1}{2}\operatorname{rk}\tilde{D}$, where $\hat{D}$ and $\tilde{D}$ are defined by the decomposition (2.6) of the operator $J$ for a pair $(g, \omega)$ of arbitrary non-degenerate symmetric and skew-symmetric bi-linear forms on $D$, $\operatorname{rk} D = 2n$. Then*

$$\dim (O(g) \cap Sp(\omega)) \leq s^2 + t^2.$$

*Moreover, if $g$ and $\omega$ are compatible then the equality holds.*

*Proof.* We shall consider the Lie algebra $\mathfrak{g}$ of $O(g) \cap Sp(\omega)$, because $\dim \mathfrak{g} = \dim (O(g) \cap Sp(\omega))$. Since any element of $O(g) \cap Sp(\omega)$ preserves the invariant subspaces $D_\lambda$ of $J$ we get that any $A \in \mathfrak{g}$, according to (2.6), decomposes into the following block form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where $B$ is of dimension $2s \times 2s$ and $C$ is of dimension $2t \times 2t$. Thus we shall estimate the possible number of independent entries of $B$ and $C$.

Let us consider $B$ first. In order to get an estimate we can assume that $\hat{J}$ is of the form (2.7) with all $b_i = 1$. Otherwise $B$ would decompose into smaller blocks. So, we can choose a basis in $\hat{D}$ such that $g$ is diagonal and $\omega$ is a standard symplectic form. Then, on the one hand $B$ is completely determined by entries above the diagonal, because $B \in \mathfrak{so}(\hat{l}, 2s - \hat{l})$, where $\hat{l} = \operatorname{ind} g|_{\hat{D}}$. On the other hand $B$ is completely determined by the entries above the anti-diagonal (including the anti-diagonal itself), because $B \in \mathfrak{sp}(2s)$. Thus, $B$ has exactly $s^2$ independent entries.

Now, let us consider $C$. According to Proposition 2.4 we write $\tilde{D} = \tilde{D}^+ \oplus \tilde{D}^-$ where $\tilde{D}^\pm$ are $t$-dimensional subspaces of $\tilde{D}$ corresponding to eigenvalues of $J$ with positive and negative real part, respectively. Moreover, Proposition 2.4 implies

$$g|_{\tilde{D}} = \begin{pmatrix} 0 & G^T \\ G & 0 \end{pmatrix}, \qquad \omega|_{\tilde{D}} = \begin{pmatrix} 0 & S^T \\ -S & 0 \end{pmatrix},$$

for certain $t \times t$-dimensional matrices $G$ and $S$. It follows that

$$C = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},$$

where all $E$ and $F$ are of dimension $t \times t$ and satisfy $E = -G^{-1}FG$ and $E = -S^{-1}FS$. In particular $E$ is completely determined by $F$. Hence, $C$ has at most $t^2$ independent entries (if $S = G$ then $C$ has exactly $t^2$ independent entries). $\square$

### 3.2. Left invariant structures on the Heisenberg group

We will show that the upper bound on the dimension of the group of isometries from Theorem 3.3 is attained. In particular, taking into account the parity of $\mathrm{ind}(g)$, we will show that there are structures with the isometry groups of dimensions as in Theorem 1.2 and formula (1.4).

To this end we consider left-invariant structures on the Heisenberg group. We recall that the Heisenberg group is realised as the space $\mathbb{R}^{2n+1}$ with the contact distribution $D$ defined as follows. Suppose we have coordinates $x_1,\dots,x_n,y_1,\dots,y_n,z$ on $\mathbb{R}^{2n+1}$ which will be denote by $(x,y,z)$ for short. Let

$$X_i=\frac{\partial}{\partial x_i}+\frac{1}{2}y_i\frac{\partial}{\partial z},\qquad Y_i=\frac{\partial}{\partial y_i}-\frac{1}{2}x_i\frac{\partial}{\partial z}, \tag{3.2}$$

$i=1,\dots,n$. Define $D$ to be

$$D=\mathrm{span}\{X_1,Y_1,\dots,X_n,Y_n\}.$$

We equip $(\mathbb{R}^{2n+1},D)$ with metric $g$ by declaring the frame $(X_1,Y_1,\dots,X_n,Y_n)$ to be orthonormal and such that

$$g(X_i,X_i)=t_i,\qquad g(Y_i,Y_i)=s_i,$$

where $t_i,s_i\in\{-1,1\}$ depending on the signature of $g$. The vector fields (3.2) are left invariant fields with respect to the standard multiplication on the Heisenberg group

$$\begin{aligned}(x_1,\dots,x_n,y_1,\dots,y_n,z)&*(x_1',\dots,x_n',y_1',\dots,y_n',z')\\ &=\left(x_1+x_1',\dots x_n+x_n',y_1+y_1',\dots,y_n+y_n',z+z'+\frac{1}{2}\sum_{i=11}^{n}(y_ix_i'-y_i'x_i)\right).\end{aligned} \tag{3.3}$$

The symplectic structure on $D$ is the standard one

$$\omega=\sum_{i=1}^{n}\mathrm{d}x_i\wedge\mathrm{d}y_i.$$

Take a matrix $\sigma\in Sp(\omega)\cap O(g)$. We will show that the map $f_\sigma\colon\mathbb{R}^{2n+1}\to\mathbb{R}^{2n+1}$ defined by

$$f_\sigma(x,y,z)=(\sigma\cdot(x,y)^T,z) \tag{3.4}$$

is an isometry. Denote $f_\sigma=(f_\sigma^1,\dots,f_\sigma^{2n},f_\sigma^{2n+1})$. Then

$$f_\sigma^i(x,y,z)=\sum_{j=1}^{n}(\sigma_{i,j}x_j+\sigma_{i,n+j}y_j) \tag{3.5}$$

for $i=1,\dots,2n$. First we have

**Lemma 3.5.** *For any $\sigma\in Sp(\omega)$*

$$f_{\sigma*}(X_i)(x,y,z)=\sum_{j=1}^{n}\sigma_{j,i}X_j(f_\sigma(x,y,z))+\sum_{j=1}^{n}\sigma_{n+j,i}Y_j(f_\sigma(x,y,z))$$

*and*

$$f_{\sigma*}(Y_i)(x,y,z)=\sum_{j=1}^{n}\sigma_{j,n+i}X_j(f_\sigma(x,y,z))+\sum_{j=1}^{n}\sigma_{n+j,n+i}Y_j(f_\sigma(x,y,z)).$$

*In particular, $f_\sigma$ preserves $D$.*

*Proof.* We will prove the first equality only. Using (3.5) we directly compute

$$f_{\sigma *}(X_i) = \sum_{j=1}^{n} \sigma_{j,i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \sigma_{n+j,i} \frac{\partial}{\partial y_j} + \frac{1}{2} y_i \frac{\partial}{\partial z}.$$

Now, it is enough to show that

$$\sum_{j=1}^{n} \sigma_{j,i} f_\sigma^{n+j}(x,y,z) - \sum_{j=1}^{n} \sigma_{n+j,i} f_\sigma^{j}(x,y,z) = y_i.$$

However, using (3.5) again, we have

$$\sum_{j=1}^{n} \sigma_{j,i} f_\sigma^{n+j}(x,y,z) - \sum_{j=1}^{n} \sigma_{n+j,i} f_\sigma^{j}(x,y,z) = \sum_{j,k=1}^{n} (\sigma_{n+j,k}\sigma_{j,i} - \sigma_{j,k}\sigma_{n+j,i}) x_k + \sum_{j,k=1}^{n} (\sigma_{n+j,n+k}\sigma_{j,i} - \sigma_{j,n+k}\sigma_{n+j,i}) y_k$$

and the lemma follows from the fact that $\omega$ is the standard symplectic form, *i.e.* $\sigma\,\Omega\,\sigma^T = \Omega$, where $\Omega$ is given by (2.10). □

Now, we can prove the following

**Proposition 3.6.** *The group of orientation preserving isometries of the left-invariant contact sub-pseudo-Riemannian structure defined above on the Heisenberg group is isomorphic to*

$$\mathbb{R}^{2n+1} \ltimes \left(Sp(\omega) \cap O(g)\right). \tag{3.6}$$

*Proof.* If $\sigma \in Sp(\omega) \cap O(g)$ then the formulae for $f_{\sigma *}(X_i)$ and $f_{\sigma *}(Y_i)$ in Lemma 3.5 imply that $f_\sigma$ is an isometry. Thus any $\sigma \in Sp(\omega) \cap O(g)$ defines an isometry of $(D,g)$ and we get the second factor in (3.6). The first factor in (3.6) comes from left translations. There can not be more isometries due to the embedding (3.1). □

**Remark 3.7.** Let us remark that the full group of isometries is isomorphic to the product $\mathbb{R}^{2n+1} \ltimes \left(\tilde{Sp}(\omega) \cap O(g)\right)$ where $\tilde{Sp}(\omega)$ is the group preserving $\omega$ up to the sign.

# 4. Regularity condition

## 4.1. Isometries of regular structures

Before proceeding to the general case announced in Theorem 1.2 we will describe a class of sub-pseudo-Riemannian structures which generalize those satisfying the compatibility condition but, at the same time, simple enough so that the isometry groups can be explicitly computed.

**Definition 4.1.** Let $(M,D,g)$ be a contact sub-pseudo-Riemannian manifold of dimension $2n+1$. The metric $(D,g)$ is said to satisfy *the regularity condition* if there exists a global orthonormal frame $X_1,\ldots,X_{2n}$ with respect to which the symplectic form $\omega$ on $D$ can be written as

$$\omega = \sum_{i=1}^{n} b_i \alpha^i \wedge \alpha^{n+i}$$

where $\alpha^1,\ldots,\alpha^{2n}$ is the co-frame dual to $X_1,\ldots,X_{2n}$, and $b_1,\ldots,b_n$ are smooth real-valued functions such that there exist positive integers $k_1,\ldots,k_r$, $k_1+\ldots+k_r = n$, for which

$$b_1 = \ldots = b_{k_1} < b_{k_1+1} = \ldots = b_{k_1+k_2} < \ldots < b_{k_1+\ldots+k_{r-1}+1} = \ldots = b_n \tag{4.1}$$

holds on the whole of $M$.

Note that any sub-Riemannian or sub-Lorentzian structure fulfills the regularity condition at least on an open subset of $M$. Clearly, the functions $b_i$ are related to either real or purely imaginary eigenvalues of the operator $J$. In fact, if $(M, D, g)$ is regular then $\tilde{J}$ has necessarily form (2.8). Let

$$D^i = \operatorname{span}\{X_j, X_{n+j} \mid k_1 + \ldots + k_{i-1} + 1 \leq j \leq k_1 + \ldots + k_i\}.$$

Then all $D^i$, $i = 1, \ldots, r$, are invariant with respect to $J$ and $D$ splits into the Whitney sum

$$D = D^1 \oplus \ldots \oplus D^r.$$

Moreover, the groups $\mathcal{G}_{g,\omega}(q)$, $q \in M$, split into the direct product

$$\mathcal{G}_{g,\omega}(q) \simeq \left(Sp(\omega|_{D^1_q}) \cap O(g|_{D^1_q})\right) \oplus \ldots \oplus \left(Sp(\omega|_{D^r_q}) \cap O(g|_{D^r_q})\right). \tag{4.2}$$

All groups $\mathcal{G}_{g,\omega}(q)$ are isomorphic under the regularity condition and will be shortly denoted $\mathcal{G}_{g,\omega}$. Consequently, the bundle $O_{D,g,\omega}(M)$ admits a reduction to a $\mathcal{G}_{g,\omega}$-structure which can be realized as the set of all such frames $(q; v_1, \ldots, v_{2n}, v_0) \in O_{D,g,\omega}$ that

$$v_j, v_{n+j} \in D^i_q, \qquad k_1 + \ldots + k_{i-1} + 1 \leq j \leq k_1 + \ldots + k_i,$$

for $i = 1, \ldots, r$. The presented considerations lead to the following

**Theorem 4.2.** *Let $(M, D, g)$ be an oriented contact sub-pseudo-Riemannian manifold satisfying the regularity condition. Then*

$$\dim \mathfrak{I}(M, D, g) \leq 2n + 1 + s_1^2 + (k_1 - s_1)^2 + \ldots + s_r^2 + (k_r - s_r)^2,$$

*where $s_i = \frac{1}{2}\operatorname{rk}(D^i \cap \hat{D})$.*

*Proof.* Indeed, $\dim \mathfrak{I}(M, D, g) \leq \dim M + \dim \mathcal{G}_{g,\omega}$ and the result follows from Lemma 3.4 applied to each factor of $\mathcal{G}_{g,\omega}$ separately. □

## 4.2. Left invariant regular structures

Now we are going to show that the upper bound on the dimension of the isometry group given in Theorem 4.2 is attained. To this end, fix positive real numbers $b_i$, $i = 1, \ldots, n$, as in (4.1) and define the following multiplication on $\mathbb{R}^{2n+1}$

$$\begin{aligned}(x_1, \ldots, x_n, y_1, \ldots, y_n, z) &* (x_1', \ldots, x_n', y_1', \ldots, y_n', z') \\ &= \left(x_1 + x_1', \ldots, x_n + x_n', y_1 + y_1', \ldots, y_n + y_n', z + z' + \frac{1}{2}\sum_{i=1}^n b_i(y_i x_i' - y_i' x_i)\right).\end{aligned} \tag{4.3}$$

The multiplication (4.3) can be treated as a deformation of the standard multiplication (3.3). Now it is not difficult to see that the left invariant vector fields with respect to this multiplication are given by formulae

$$X_i = \frac{\partial}{\partial x_i} + \frac{b_i}{2} y_i \frac{\partial}{\partial z}, \qquad Y_i = \frac{\partial}{\partial y_i} - \frac{b_i}{2} x_i \frac{\partial}{\partial z}. \tag{4.4}$$

Let $D = \operatorname{span}\{X_1, Y_1, \ldots, X_n, Y_n\}$ and define metric $g$ by declaring the basis $X_1, Y_1, \ldots, X_n, Y_n$ to be orthonormal with

$$g(X_i, X_i) = p_i, \qquad g(Y_i, Y_i) = r_i,$$

where $p_i, r_i \in \{-1, +1\}$ depending on the index of the metric, $i = 1, \ldots, n$. It clear that the canonical contact form is

$$\alpha = \mathrm{d}z - \sum_{i=1}^n (1/2) b_i (y_i \mathrm{d}x_i - x_i \mathrm{d}y_i)$$

and

$$\omega = \sum_{i=1}^{n} b_i \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

It follows from the construction that the left translations with respect to (4.3) are isometries of $(\mathbb{R}^{2n+1}, D, g)$, because vector fields (4.4) are left invariant. Moreover, any $\sigma \in \mathcal{G}_{g,\omega}$ decomposes according to the splitting (4.2). Performing similar calculations as in Lemma 3.5 for each factor of this decomposition one can prove

**Proposition 4.3.** *The group of orientation preserving isometries of the left invariant contact sub-pseudo-Riemannian structure $(D, g)$ constructed above on $\mathbb{R}^{2n+1}$ is isomorphic to*

$$\mathbb{R}^{2n+1} \ltimes \mathcal{G}_{g,\omega},$$

*where $\mathcal{G}_{g,\omega}$ is given by (4.2).*

## 5. General case

### 5.1. Symbol algebra

Let $(M, D, g)$ be an oriented contact sub-pseudo-Riemannian manifold. Let $\mathfrak{g}(D)(q)$ be the symbol algebra of $D$ at point $q \in M$. It is a two-step nilpotent graded Lie algebra

$$\mathfrak{g}(D)(q) = \mathfrak{g}_{-1}(q) \oplus \mathfrak{g}_{-2}(q)$$

where

$$\mathfrak{g}_{-1}(q) = D_q, \qquad \mathfrak{g}_{-2}(q) = T_qM/D_q.$$

The Lie bracket $\mathfrak{g}_{-1}(q) \wedge \mathfrak{g}_{-1}(q) \to \mathfrak{g}_{-2}(q)$ is defined in terms of the Lie bracket of vector fields on $M$ as follows. Let $v, w \in \mathfrak{g}_{-1}(q)$ and let $X_v$ and $X_w$ be two extensions of $v$ and $w$, respectively, to sections of $D$ in a neighborhood of $q$. Then

$$[v, w] = [X_v, X_w](q) \mod D_q$$

does not depend on the extension and defines the Lie bracket in $\mathfrak{g}(D)(q)$. Clearly, the Lie algebra $\mathfrak{g}(D)(q)$ does not depend on $q$. Moreover the dual space $\mathfrak{g}_{-2}(q)^*$ can be identified with $D_q^{\perp} \subset T_q^*M$ spanned by the contact form $\alpha_q$. It follows that

$$\alpha_q([v, w]) = \omega_q(v, w),$$

*i.e.* the Lie algebra structure is determined by the symplectic form $\omega$.

The symbol algebra $\mathfrak{g}(D, g)(q)$ of $D$ equipped with $g$ at point $q \in M$ is defined as follows

$$\mathfrak{g}(D, g)(q) = \mathfrak{g}(D)(q) \oplus \mathfrak{g}_0(q)$$

where $\mathfrak{g}_0(q)$ is the algebra of matrices $A \in \mathfrak{gl}(\mathfrak{g}(D)(q))$ preserving the metric $g$, *i.e.*

$$g(Av, w) + g(v, Aw) = 0$$

and the Lie bracket on $\mathfrak{g}_{-1}(q)$, *i.e.*

$$[Av, w] + [v, Aw] = A[v, w].$$

Since the Lie bracket is encoded in terms of $\omega$ it follows that $\mathfrak{g}_0(q)$ is the Lie algebra of the Lie group $\mathcal{G}_{g,\omega}(q)$ and actually can be thought of as a sub-algebra of $\mathfrak{gl}(\mathfrak{g}_{-1}(q))$. Defining

$$[A, v] = Av$$

for $v \in \mathfrak{g}_{-1}(q)$ we get that $\mathfrak{g}(D, g)(q)$ is a graded Lie algebra. We refer to [16] for more information on the symbol algebras of distributions and related structures.

## 5.2. Prolongation

The first prolongation of $\mathfrak{g}(D,g)(q)$ is defined as

$$\mathrm{pr}_1(\mathfrak{g}(D,g)(q)) = \mathfrak{g}(D,g)(q) \oplus \mathfrak{g}_1(q),$$

where $\mathfrak{g}_1(q)$ is the set of all Lie algebra derivations $\mathfrak{g}(D) \to \mathfrak{g}(D,g)$ increasing the gradation by 1, *i.e.* any $A \in \mathfrak{g}_1(q)$ maps $\mathfrak{g}_{-1}(q)$ to $\mathfrak{g}_0(q)$ and $\mathfrak{g}_{-2}(q)$ to $\mathfrak{g}_{-1}(q)$ such that

$$A([v,w]) = A(v)w - A(w)v \tag{5.1}$$

for all $v, w \in \mathfrak{g}_{-1}(q)$. Note that $\dim \mathfrak{g}_{-2} = 1$ thus for any $A \in \mathfrak{g}_1(q)$ the image $A(\mathfrak{g}_{-2})$ is a one- or zero-dimensional subspace of $\mathfrak{g}_{-2}$.

Higher prolongations of $\mathfrak{g}(D,g)(g)$ are defined by induction, similarly to the first prolongation, as Lie algebra derivations increasing the gradation by $k \in \mathbb{N}$. We get

$$\mathrm{pr}\, \mathfrak{g}(D,g)(q) = \mathfrak{g}(D,g)(q) \oplus \bigoplus_{k\in\mathbb{N}} \mathfrak{g}_k(q)$$

and one equips $\mathrm{pr}\, \mathfrak{g}(D,g)(q)$ with the structure of a graded Lie algebra in a natural way. However we shall not describe the structure in detail because we have the following

**Lemma 5.1.** *The first prolongation of* $\mathfrak{g}(D,g)(q)$ *is trivial. Consequently*

$$\mathrm{pr}\, \mathfrak{g}(D,g)(q) = \mathfrak{g}(D,g)(q)$$

*Proof.* Let $\alpha_q^* \in \mathfrak{g}_{-2}(q)$ be a vector dual to the contact form $\alpha_q$, *i.e.* $\alpha_q(\alpha_q^*) = 1$. Choose $A \in \mathfrak{g}_1(q)$ and denote $v_A = A(\alpha_q^*)$. Let $(v_1, \ldots, v_{2n})$ be an orthonormal basis of $D_q$. Then (5.1) reads

$$A(v_i)v_j - A(v_j)v_i = g(J(v_i), v_j)v_A. \tag{5.2}$$

Since $(v_1, \ldots, v_{2n})$ is orthonormal it follows that all $A(v_i)$, $i = 1, \ldots, 2n$, are orthonormal matrices in $\mathfrak{so}(l, 2n-l)$. Now, for a fixed value of $v_A$ there is unique $A$ that solves (5.2) in $\mathfrak{so}(l, 2n-l)$, where $l = \mathrm{ind}(g)$. This follows from the uniqueness of the Levi−Civita connection of a pseudo-Riemannian metric which is equivalent to the algebraic fact that the system

$$A(v_i)v_j - A(v_j)v_i = 0 \tag{5.3}$$

has unique solution $A = 0$ in the algebra $\mathfrak{so}(l, 2n-l)$. The unique solution to (5.2) is of the form

$$A = \frac{1}{2}\sum_{i=1}^{2k} (v_A \cdot J(v_i)^T) v_i^*$$

where $(v_1^*, \ldots, v_{2n}^*)$ are dual to $(v_1, \ldots, v_{2n})$ with respect to $g$ and $v_A \cdot J(v_i)^T = A(v_i)$ is a rank-one square matrix $A(v_i) = (a_{jk}^i)_{j,k=1,\ldots,2n}$ with entries $a_{jk}^i = v_j^*(v_A)v_k^*(J(v_i))$. Now, since all $A(v_i)$ are orthonormal it follows that $v_j^*(v_A)v_j^*(J(v_i)) = 0$ for any $j = 1, \ldots, 2n$. But, for any $i$ there is $j$ such that $v_j^*(J(v_i)) \neq 0$. Thus we get that $v_j^*(v_A) = 0$, for $j = 1, \ldots, 2n$. Consequently, $v_A = 0$. This reduces (5.2) to (5.3). Hence $A = 0$, because this is the unique solution to (5.3) as was explained above. □

Now we are able to apply Theorem 1 of [13] and get

**Theorem 5.2.** *Let* $(M, D, g)$ *be an oriented contact sub-pseudo-Riemannian manifold. Then the dimension of the algebra of the infinitesimal symmetries of* $(M, D, g)$ *is estimated from above by*

$$\dim M + \inf_{q\in M} \dim \mathcal{G}_{g,\omega}(q).$$

*Proof.* We have $\mathrm{pr}\, \mathfrak{g}(D,g)(q) = \mathfrak{g}(D,g)(q)$. Thus $\dim \mathrm{pr}\, \mathfrak{g}(D,g)(q) = \dim M + \dim \mathcal{G}_{g,\omega}(q)$ since $\mathfrak{g}_0(q)$ is the Lie algebra of $\mathcal{G}_{g,\omega}(q)$. All the prolongations are finite. Therefore, by [13], we have that the dimension of the algebra of infinitesimal symmetries is estimated from above by $\inf_{q\in M} \dim \mathrm{pr}\, \mathfrak{g}(D,g)(q)$. □

### 5.3. Proof of Theorem 1.2

If $(M, D, g)$ is an oriented sub-pseudo-Riemannian manifold then it suffices to consider isometries preserving the orientation because $\dim \mathfrak{I}(M, D, g) = \dim \mathfrak{I}_0(M, D, g)$. The dimension of $\mathfrak{I}_0(M, D, g)$ equals to the dimension of the algebra of infinitesimal isometries. Therefore we can apply Theorem 5.2. The maximal possible dimension of $\mathcal{G}_{g,\omega}$ is computed in Lemma 3.4.

If $(M, D, g)$ is not oriented then we consider a double cover $\tilde{M}$ of $M$ consisting of pairs $(q, \alpha_q)$ where $q \in M$ and $\alpha_q$ is one of the two normalized co-vectors in $T_q^*M$ annihilating $D(q)$. Then $\tilde{M}$ carries a canonical lift $(\tilde{D}, \tilde{g})$ of the structure $(D, g)$ and the structure $(\tilde{M}, \tilde{D}, \tilde{g})$ is oriented, because $(q, \alpha_q) \mapsto \alpha_q$ defines a global contact form on $\tilde{M}$ annihilating $\tilde{D}$. Moreover, any isometry of the original structure $(M, D, g)$ defines an isometry of $(\tilde{M}, \tilde{D}, \tilde{g})$ and thus $\dim \mathfrak{I}(M, D, g) \leq \dim \mathfrak{I}(\tilde{M}, \tilde{D}, \tilde{g})$. Therefore, the estimate in the not oriented case follows from the estimate in the oriented case.

*Acknowledgements.* The work of Wojciech Kryński has been partially supported by the Polish National Science Centre grant DEC-2011/03/D/ST1/03902. We thank the anonymous reviewer for important comments that improved the paper essentially.

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